A theory of thin layers in electrical engineering; application to eddy-current calculation inside a shell using the BIE software Phi3D

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A THEORY OF THIN LAYERS IN ELECTRICAL ENGINEERING; APPLICATION TO EDDY-CURRENT CALCULATION INSIDE A SHELL USING THE BIE SOFTWARE PHI3D.

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Abstract: During last years, severall numerical formulations have been developped by us to modelize physical problems like: conducting film effects over the surface of insulators (pollution) [1], high frequency eddy-currents [2], earth field effect on the hull of a ship [3].
The physical effects being at stake are completly different, but in each of these examples, they originate from a region *thin in regard with* the other geometrical dimensions. An efficient numerical approach consists on using a surfacic representation with special boundary conditions expressing the solution inside the thin region.
From the synthesis of these models proposed in past times, we built a *theory of thin shells* which may be used to solve novel field problems.
As an example, we apply that theory to get the boundary conditions for eddy currents flowing inside a shell of thickness e (roughly: e = 0.1 delta to 10 delta). The practical applications may concern the optimisation of the induction heating of pans (french *art culinaire*) or the computation of losses into the carcase of an electrical engine.

Equation of continuity for a shell.

Introduction [4]
The fondamental property we will use here is the continuity of vector B, mathematically expressed by the equation of continuity, which is valid for each non-divergent quantity:

$$\text{div}\,\mathbf{B} = 0 \qquad (1)$$

This equation expresses the concept of tubes of induction wich cannot converge upon a point without again diverging therefrom.
Note that, on the contrary, some meaning vector quantities are divergent. The typical example is the electric induction D:

$$\text{div}\,\mathbf{D} = q \qquad (2)$$

div **D** is not everywhere equal to zero and it represents the distribution of electric load: in other words, the righ-hand-term of the equation of continuity symbolises the source of the flux of the considered vectorial quantity.

Particular expression for a shell (fig. 1)
In a first time, let us consider that **B** is everywhere tangent to the surface of the shell. Then, the normal flux is zero, and the tangential induction is non-divergent:

$$\text{div}_S\mathbf{B} = 0 \qquad (3)$$

In fact, we have to take into account the depth e of the shell, and the equation of continuity is only valid for the equivalent shell flux density **F** defined by the integral value:

$$\text{div}_S[\underbrace{\int_e \mathbf{B}_{tg}.dz}_{\mathbf{F}}] = 0 \qquad (4)$$

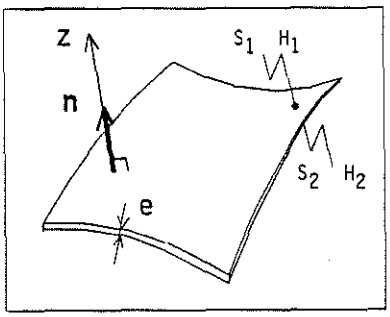

Fig. 1

What happens in a second time if we add normal flux densities B_{1n} and B_{2n} on the surfaces S_1 and S_2? These distributions are increasing or decreasing the internal flux **F**; by the fact, they are the actual sources of **F**, like q is the source of the electrical induction **D**.

Then **F** becomes divergent:

$$\boxed{\mathrm{div}_s[\int_e \mathbf{B}_{tg}.dz] = B_{1n} - B_{2n}} \qquad (5)$$

This is the expression we were looking for. We will now apply this particular form of the equation of continuity to three magnetic or electromagnetic configurations.

Thin ferromagnetic shell, static.

We considere here the effect of a static magnetic field on a thin ferromagnetic shell, for example the carcase of an electrical engine.
An hypothesis has to be done to evaluate the shell flux density **F** as a function of **B** on S: it seems adequat to assign a constant value to the tangential component of **B** through the depth.
Then: $\mathbf{F} = e \,.\, \mathbf{B}_{s,tg} = e.\mu_r \,.\, \mathbf{H}_{s,tg}$ and (with e constant): (6)

(5) => $\boxed{e.\mu_r \,.\, \mathrm{div}_s \mathbf{H}_s = H_{1n} - H_{2n}}$ (7)

with: μ_r: relative permeability of the shell. (8)

Note that, if e leads to zero, the left-hand-term in (7) is zero and we find again the classical interface condition between two ferromagnetic materials. If e is not a constant, a term $\mathbf{H}_s.\mathbf{grad}_s e$ has to be introduced in (7).

To use this equation as interface condition in a BEM program, it remains to express it in terms of scalar potential, using for example the reduced scalar magnetic potential V:

$$\mathbf{H} = \mathbf{H}_0 - \mathbf{grad}V \qquad (9)$$

$\mathbf{H}_0$: source field. $d_n = \mathbf{n}.\mathbf{grad}$

(5) => $\boxed{e.\mu_r.(\mathrm{div}_s \mathbf{H}_0 - \mathrm{lap}_s V) = (H_{on} - d_n V_1) - (H_{on} - d_n V_2)}$ (10)

The boundary condition (10) is not trivial because of the Laplacian operator. In fact, it is a surfacic differencial equation, to be solved together with the volumic Boundary Integral Equation using for example the FEM. Both are discretized and solved using usual numerical procedures.

Eddy currents, high frrequency [2]

In that case, the object is not geometrically a shell, but a massive conducting body in a high frequency source field. For a sufficiently high frequency, the field refuses to penetrate deep into the conducting material.
This penetration is characterized by the skin depth delta. If delta is little in comparison with the geometrical dimensions of the conductor, the following well known analytical solution for the penetration of **B** is valid:

$$\mathbf{B}_{tg}(z) = \mathbf{B}_s.\exp[-(1+j).z/\mathrm{delta}] \qquad (11)$$

We can use it to calculate the equivalent flux density **F** as a function of **B** on S:

$$\mathbf{F} = \int_e \mathbf{B}_{tg}(z).dz = (1-j)/2 \,.\, \mathrm{delta} \,.\, \mathbf{B}_{s,tg} \qquad (12)$$

This expression is completly similar to (6), but the complex factor. The related expressions of the equation of continuity are following (compare to (7) and (10)):

(5) => $\boxed{(1-j)/2 \,.\, \mathrm{delta}.\mu_r \,.\, \mathrm{div}_s \mathbf{H}_s = H_n}$ (13)

or: $\boxed{(1-j)/2 \,.\, \mathrm{delta}.\mu_r \,.\, (\mathrm{div}_s \mathbf{H}_0 - \mathrm{lap}_s V) = (H_{on} - d_n V)}$ (14)

We find again by a different way the result of a previous paper [2]. Note the limit case at very high frequency, when delta - and B_n - lead to zero: equation (14) becomes the simple boundary condition

(Neuman, homogenous) of the *infinit frequency* formulation:

$$(5) => \quad \boxed{d_n V = H_{on}} \tag{15}$$

That means that the eddy currents constitute a perfect magnetic screen.

Synthesis

A common proceeding was used for both previous examples:

1- Choice of a non-divergent vector quantity (**B**).
2- Choice of a particular distribution of this vector through the shell (constant, exponential, ...).
3- Related particular expression of the equivalent flux density **F** (eq. 6 and 12) and of the eq. of continuity (eq. 7 and 13).
4- The same, but in term of scalar potential (eq. 10 and 14).

The strong point is that two completly different shell problems have been solved using exactly the same proceeding, only the distribution of $\mathbf{B}_{tg}$ through the shell was changed.
We will now use this procedure once more to get a general formulation for the magnetic field in a ferromagnetic and conducting shell, valid for any frequency, including previous *static*, *high frequency* and *infinit frequency* models.

Magnetic field into a shell: the general analytical model

In that general case, we have to considere the values of **B** on both sides (index 1 and 2) of the shell, described using 4 complexe surfacic distributions: reduced potentials V_1, V_2; normal derivatives $d_n V_1$ and $d_n V_2$. The volumic equation or BIE gives us two relations, the analytical model of the shell has to give two more.

First relation: divergence of the equivalent flux density.
The Maxwell's equations lead to the following variation of **B** through the shell:

$$\mathbf{B}(z) = \mathbf{b}.\exp(-\underline{a}.z) + \mathbf{b'}.\exp(+\underline{a}.z) \text{ with } \underline{a} = (1+j)/\text{delta} \tag{16}$$

The vector-coefficients **b** and **b'** depend on the surfacic values $\mathbf{B}_1$ and $\mathbf{B}_2$. The expresssion of **F** is obtained after some calculations:

$$\mathbf{F} = (1\text{-}j)/2\ .\ \text{delta}.\mu_r.\text{th}(\underline{a}.e/2)\ .\ (\mathbf{H}_{1tg}+\mathbf{H}_{2tg}) \text{ then:} \tag{17}$$

$$(5) => \quad \boxed{(1\text{-}j)/2\ .\ \text{delta}.\mu_r\ .\ \text{div}_s(\mathbf{H}_1+\mathbf{H}_2) = (H_{1n}-H_{2n})\ /\ \text{th}(\underline{a}.e/2)} \tag{18}$$

Second relation: Lenz's law.
The second relation is not obtained exactly by the same procedure; the general way is preserved, but starting with:

$$\text{curl}\ \mathbf{E} = -d\mathbf{B}/dt \tag{19}$$

The integration of **E** through the depth gives:

$$\text{sigma}.\int_e \mathbf{E}_{tg}.dz = \mathbf{n} \times (\mathbf{H}_1-\mathbf{H}_2) => \text{sigma}.\int_e \text{curl}\ \mathbf{E}.dz = \mathbf{n}.\text{div}_s(\mathbf{H}_1-\mathbf{H}_2) \text{ then:} \tag{20}$$

$$(19) => \quad \boxed{(1\text{-}j)/2\ .\ \text{delta}.\mu_r\ .\ \text{div}_s(\mathbf{H}_1-\mathbf{H}_2) = (H_{1n}+H_{2n})\ .\ \text{th}(\underline{a}.e/2)} \tag{21}$$

Relations (18) and (21) are similar, only signs have been changed and the th($\underline{a}$.e/2) factor jumps from numerator to denominator. The first relation lies the surfacic variations of the mean-value of $\mathbf{H}_{tg}$ into the shell (ie the flux) to its source, which is the resultant normal flux density. The second relation lies the surfacic variation of the resultant surface current density [2] to the mean-value of the normal flux density through the shell.

Limit cases: quasi-static and high frequency.
If the frequency leads to zero, the term th(a.e/2) leads to (a.e/2 then:

$$(17) => \quad \mathbf{F} \dashrightarrow e.mu_r . [(\mathbf{H}_{1,tg}+\mathbf{H}_{2,tg})/2] \tag{22}$$

which is to compare to (6). For the high frequencies, th(a.e/2) leads to 1, then:

$$(17) => \quad \mathbf{F} \dashrightarrow (1-j)/2 . delta.mu_r . (\mathbf{H}_{1,tg}+\mathbf{H}_{2,tg}) \tag{23}$$

which is to compare to (13).
By the fact, it becomes patent that the model of the shell presented here is very general; in particular, it includes both static and high frequency models.

The price to pay.
The following table showes the price to pay to use this more general formulation: for example, it needs 8 times more real unknowns as the infinit frequency formulation (for the same mesh): it has to be used with discrimination!

Formulation		Eq.#	Number of unknowns
Static	SF	7	2
General	GF	18+21	8 (4, complex)
High-Frequency	HFF	14	4 (2, complex)
Infinit-Frequency	IFF	15	1

These four shell formulations have been implemented in the research version of the industrial software PHI3D [5]. They will now be tested and compared.

Example

The example is a hollow sphere in a constant alternative source field (fig. 2). This is a simple configuration, but both ferromagnetic and eddy-currents effects are clearly shown (fig. 3): it has been solved successively using the Static Formulation (SF, eq.10), the High Frequency Form. (HFF, eq.14) and the General Form. (GF, eq.18 and 21), coupled with the Boundary Integral Equation Method for the infinit external region.

With given geometrical dimensions and physical properties, we can see that (fig. 5a-b):
- SF and GF give the same result for f<1Hz. With the choosen mesh (27 nodes, fig. 4), the error with the analytical value is less than 1%, that is the extent of the discretization error.
- GF and HFF are equivalent for f>250Hz, ie delta<e/2.
- GF allows the exact computation of the passage from static to high frequency.

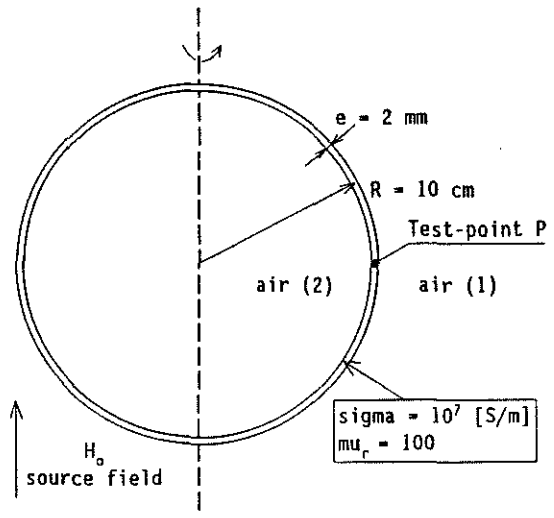

Fig. 2: definition of the test-problem:
the hollow ferromagnetic and conducting sphere
in a constant field

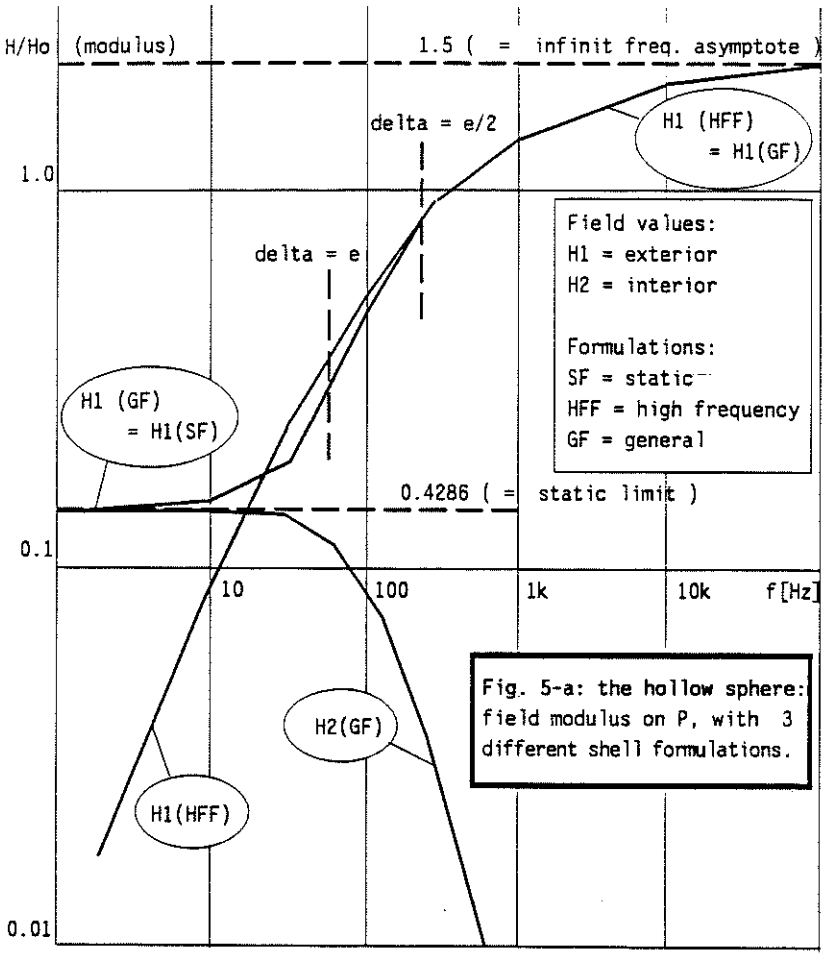

Fig. 5-a: the hollow sphere: field modulus on P, with 3 different shell formulations.

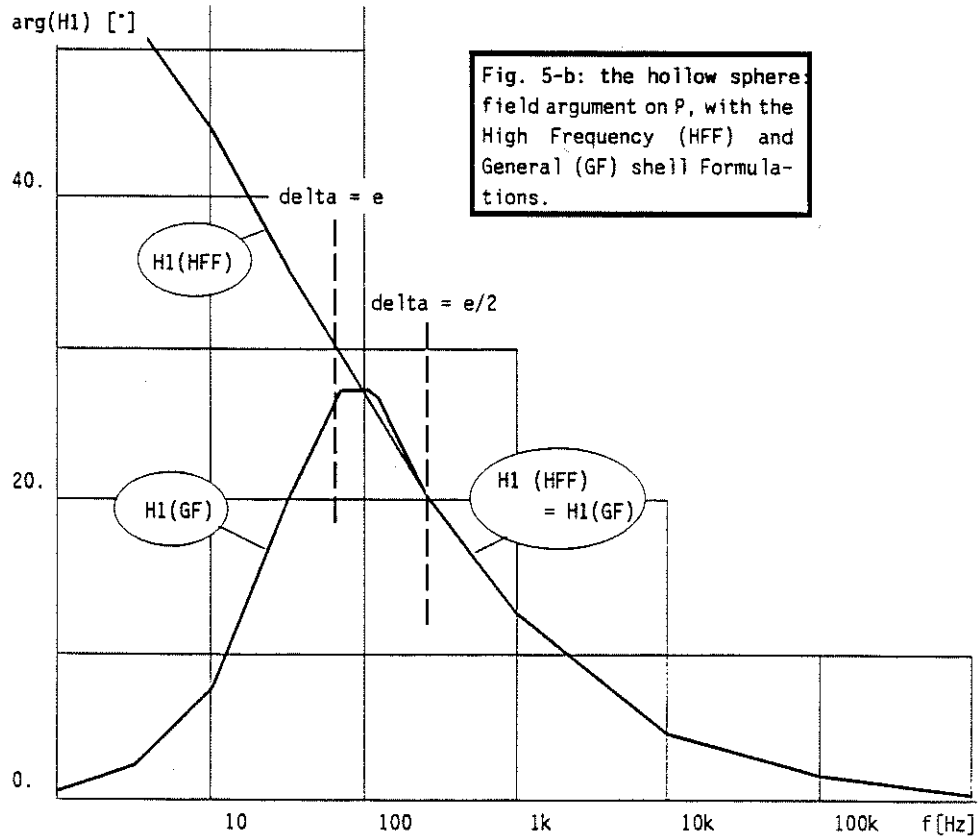

Fig. 5-b: the hollow sphere: field argument on P, with the High Frequency (HFF) and General (GF) shell Formulations.

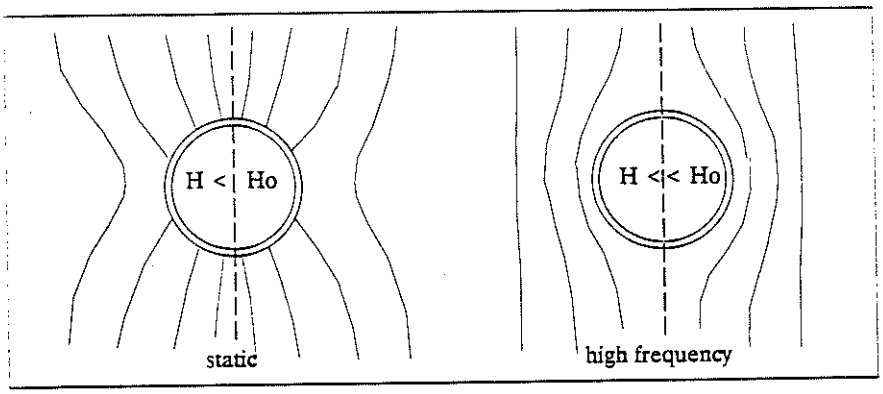

Fig. 3: Flux lines plot (schematic)

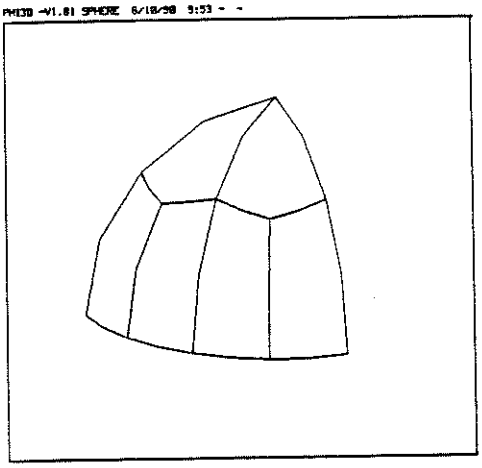

Fig. 4: Mesh of the 1/4 sphere.

Conclusion

The general formulation for the electromagnetic shell problems presented in this paper is well adapted to be coupled with the BIEM. The research works on, to take into account the saturation of the shell; to do that, we have to replace the analytical exponential solutions (11) and (16) by numerical, 1-dimensional non-linear solutions.

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