

Error estimates of a regularized finite difference method for the logarithmic Schrödinger equation

Weizhu Bao, Rémi Carles, Chunmei Su, Qinglin Tang

▶ **To cite this version:**

Weizhu Bao, Rémi Carles, Chunmei Su, Qinglin Tang. Error estimates of a regularized finite difference method for the logarithmic Schrödinger equation. SIAM Journal on Numerical Analysis, 2019, 57 (2), pp.657-680. 10.1137/18M1177445 . hal-01745366

HAL Id: hal-01745366
https://hal.science/hal-01745366

Submitted on 28 Mar 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ERROR ESTIMATES OF A REGULARIZED FINITE DIFFERENCE METHOD FOR THE LOGARITHMIC SCHRÖDINGER EQUATION*

WEIZHU BAO†, RÉMI CARLES‡, CHUNMEI SU§, AND QINGLIN TANG¶

Abstract. We present a regularized finite difference method for the logarithmic Schrödinger equation (LogSE) and establish its error bound. Due to the blow-up of the logarithmic nonlinearity, i.e. $\ln\rho \to -\infty$ when $\rho \to 0^+$ with $\rho = |u|^2$ being the density and u being the complex-valued wave function or order parameter, there are significant difficulties in designing numerical methods and establishing their error bounds for the LogSE. In order to suppress the round-off error and to avoid blow-up, a regularized logarithmic Schrödinger equation (RLogSE) is proposed with a small regularization parameter $0 < \varepsilon \ll 1$ and linear convergence is established between the solutions of RLogSE and LogSE in term of ε. Then a semi-implicit finite difference method is presented for discretizing the RLogSE and error estimates are established in terms of the mesh size h and time step τ as well as the small regularization parameter ε. Finally numerical results are reported to confirm our error bounds.

Key words. Logarithmic Schrödinger equation, logarithmic nonlinearity, regularized logarithmic Schrödinger equation, semi-implicit finite difference method, error estimates, convergence rate.

AMS subject classifications. 35Q40, 35Q55, 65M15, 81Q05

1. Introduction.

We consider the logarithmic Schrödinger equation (LogSE) which arises in a model of nonlinear wave mechanics (cf. [7]),

$$\begin{cases} i\partial_t u(\mathbf{x},t) + \Delta u(\mathbf{x},t) = \lambda u(\mathbf{x},t)\,\ln(|u(\mathbf{x},t)|^2), \quad \mathbf{x}\in\Omega, \quad t>0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}, \end{cases} \tag{1.1}$$

where t is time, $\mathbf{x}\in\mathbb{R}^d$ $(d=1,2,3)$ is the spatial coordinate, $\lambda\in\mathbb{R}\backslash\{0\}$ measures the force of the nonlinear interaction, $u := u(\mathbf{x},t)\in\mathbb{C}$ is the dimensionless wave function or order parameter and $\Omega=\mathbb{R}^d$ or $\Omega\subset\mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition[1] fixed on the boundary. It admits applications to quantum mechanics [7,8], quantum optics [9,20], nuclear physics [17], transport and diffusion phenomena [16, 22], open quantum systems [18, 26], effective quantum gravity [27], theory of superfluidity and Bose-Einstein condensation [3]. The logarithmic Schrödinger equation enjoys three conservation laws, *mass*, *momentum* and *energy* [12,13], like in the case of the nonlinear Schrödinger equation with a

*This work was partially supported by the Ministry of Education of Singapore grant R-146-000-223-112 (MOE2015-T2-2-146) (W. Bao).

†Department of Mathematics, National University of Singapore, Singapore 119076 (matbaowz@nus.edu.sg, URL: http://www.math.nus.edu.sg/~bao/)

‡CNRS, Institut Montpelliérain Alexander Grothendieck, Univ. Montpellier, France (Remi.Carles@math.cnrs.fr, URL: http://carles.perso.math.cnrs.fr)

§Department of Mathematics, University of Innsbruck, Innsbruck 6020, Austria (sucm13@163.com)

¶School of Mathematics, Sichuan University, Chengdu 610064 (qinglin_tang@scu.edu.cn)

[1]Whenever we consider this case, it is assumed that the boundary is Lipschitz continuous.

power-like nonlinearity (e.g. cubic):

$$
\begin{aligned}
M(t) :&= \|u(\cdot,t)\|^2_{L^2(\Omega)} = \int_\Omega |u(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x} = M(0),\\
P(t) :&= \operatorname{Im} \int_\Omega \overline{u}(\mathbf{x},t)\nabla u(\mathbf{x},t) d\mathbf{x} \equiv \operatorname{Im} \int_\Omega \overline{u_0}(\mathbf{x})\nabla u_0(\mathbf{x}) d\mathbf{x} = P(0), \quad t \geq 0,\\
E(t) :&= \int_\Omega \left[|\nabla u(\mathbf{x},t)|^2 d\mathbf{x} + \lambda F(|u(\mathbf{x},t)|^2)\right] d\mathbf{x}\\
&\equiv \int_\Omega \left[|\nabla u_0(\mathbf{x})|^2 + \lambda F(|u_0(\mathbf{x})|^2)\right] d\mathbf{x} = E(0),
\end{aligned}
\tag{1.2}
$$

where $\operatorname{Im} f$ and $\overline{f}$ denote the imaginary part and complex conjugate of f, respectively, and

$$
F(\rho) = \int_0^\rho \ln(s) ds = \rho \ln \rho - \rho, \qquad \rho \geq 0. \tag{1.3}
$$

On a mathematical level, the logarithmic nonlinearity possesses several features that make it quite different from more standard nonlinear Schrödinger equations. First, the nonlinearity is not locally Lipschitz continuous because of the behavior of the logarithm function at the origin. Note that in view of numerical simulation, this singularity of the "nonlinear potential" $\lambda \ln(|u(\mathbf{x},t)|^2)$ makes the choice of a discretization quite delicate. The second aspect is that whichever the sign of λ, the nonlinear potential energy in E has no definite sign. In fact, whether the nonlinearity is repulsive/attractive (or defocusing/focusing) depends on both λ and the value of the density $\rho := \rho(\mathbf{x},t) = |u(\mathbf{x},t)|^2$. When $\lambda > 0$, then the nonlinearity $\lambda\rho \ln \rho$ is repulsive when $\rho > 1$; and respectively, it is attractive when $0 < \rho < 1$. On the other hand, when $\lambda < 0$, then the nonlinearity $\lambda\rho \ln \rho$ is attractive when $\rho > 1$; and respectively, it is repulsive when $0 < \rho < 1$. Therefore, solving the Cauchy problem for (1.1) is not a trivial issue, and constructing solutions which are defined for all time requires some work; see [10,13,15]. Essentially, the outcome is that if u_0 belongs to (a subset of) $H^1(\Omega)$, (1.1) has a unique, global solution, regardless of the space dimension d (see also Theorem 2.2 below).

Next, the large time behavior reveals new phenomena. A first remark suggests that nonlinear effects are weak. Indeed, unlike what happens in the case of a homogeneous nonlinearity (classically of the form $\lambda|u|^p u$), replacing u with ku ($k \in \mathbb{C} \setminus \{0\}$) in (1.1) has only little effect, since we have

$$
i\partial_t(ku) + \Delta(ku) = \lambda ku \ln\left(|ku|^2\right) - \lambda(\ln|k|^2)ku\,.
$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$
ku(\mathbf{x},t)e^{-it\lambda \ln|k|^2}
$$

solves (1.1) (with initial datum ku_0). In particular, the size of the initial datum does not influence the dynamics of the solution. In spite of this property which is reminiscent of linear equations, nonlinear effects are stronger in (1.1) than in, say, cubic Schrödinger equations in several respects. For $\Omega = \mathbb{R}^d$, it was established in [11] that in the case $\lambda < 0$, no solution is dispersive (not even for small data, in view of the above remark), while if $\lambda > 0$, the results from [10] show that every solution disperses, at a faster rate than for the linear equation.

In view of the gauge invariance of the nonlinearity, for $\Omega = \mathbb{R}^d$, (1.1) enjoys the standard Galilean invariance: if $u(\mathbf{x}, t)$ solves (1.1), then, for any $\mathbf{v} \in \mathbb{R}^d$, so does

$$u(\mathbf{x} - 2\mathbf{v}t, t)e^{i\mathbf{v}\cdot\mathbf{x} - i|\mathbf{v}|^2 t}.$$

A remarkable feature of (1.1) is that it possesses a large set of explicit solutions. In the case $\Omega = \mathbb{R}^d$: if u_0 is Gaussian, $u(\cdot, t)$ is Gaussian for all time, and solving (1.1) amounts to solving ordinary differential equations [7]. For simplicity of notation, we take the one-dimensional case as an example. If the initial data in (1.1) with $\Omega = \mathbb{R}$ is taken as

$$u_0(x) = b_0 e^{-\frac{a_0}{2}x^2 + ivx}, \qquad x \in \mathbb{R},$$

where $a_0, b_0 \in \mathbb{C}$ and $v \in \mathbb{R}$ are given constants satisfying $\alpha_0 := \operatorname{Re} a_0 > 0$ with $\operatorname{Re} f$ denoting the real part of f, then the solution of (1.1) is given by [2, 10]

$$u(x,t) = \frac{b_0}{\sqrt{r(t)}} e^{i(vx - v^2 t) + Y(x - 2vt, t)}, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.4}$$

with

$$Y(x,t) = -i\phi(t) - \alpha_0 \frac{x^2}{2r(t)^2} + i\frac{\dot{r}(t)}{r(t)}\frac{x^2}{4}, \qquad x \in \mathbb{R}, \quad t \geq 0, \tag{1.5}$$

where $\phi := \phi(t) \in \mathbb{R}$ and $r := r(t) > 0$ solve the ODEs [2, 10]

$$\begin{aligned} &\dot{\phi} = \frac{\alpha_0}{r^2} + \lambda \ln |b_0|^2 - \lambda \ln r, \quad \phi(0) = 0, \\ &\ddot{r} = \frac{4\alpha_0^2}{r^3} + \frac{4\lambda\alpha_0}{r}, \quad r(0) = 1, \ \dot{r}(0) = -2 \operatorname{Im} a_0. \end{aligned} \tag{1.6}$$

In the case $\lambda < 0$, the function r is (time) periodic (in agreement with the absence of dispersive effects). In particular, if $a_0 = -\lambda > 0$, it follows from (1.6) that $r(t) \equiv 1$ and $\phi(t) = \phi_0 t$ with $\phi_0 = \lambda\big[\ln(|b_0|^2) - 1\big]$, which generates the uniformly **moving Gausson** as [2, 10]

$$u(x,t) = b_0 e^{\frac{\lambda}{2}(x - 2vt)^2 + i(vx - (\phi_0 + v^2)t)}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.7}$$

As a very special case with $b_0 = e^{1/2}$ and $v = 0$ such that $\phi_0 = 0$, one can get the **static Gausson** as

$$u(x,t) = e^{1/2} e^{\lambda |x|^2/2}, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{1.8}$$

This special solution is orbitally stable [11, 14]. On the other hand, in the case $\lambda > 0$, it is proven in [10] that for general initial data (not necessarily Gaussian), there exists a universal dynamics. For extensions to higher dimensions, we refer to [2, 10] and references therein. Therefore, (1.1) possesses several specific features, which make it quite different from the nonlinear Schrödinger equation.

Different numerical methods have been proposed and analyzed for the nonlinear Schrödinger equation with smooth nonlinearity (e.g. cubic nonlinearity) in the literature, such as the finite difference methods [4, 5], finite element methods [1, 19] and the time-splitting pseudospectral methods [6, 24]. However, they cannot be applied to the LogSE (1.1) directly due to the blow-up of the logarithmic nonlinearity, i.e.

$\ln \rho \to -\infty$ when $\rho \to 0^+$. The main aim of this paper is to present a regularized finite difference method for the LogSE (1.1) by introducing a proper regularized logarithmic Schrödinger equation (RLogSE) and then discretizing the RLogSE via a semi-implicit finite difference method. Error estimates will be established between the solutions of LogSE and RLogSE as well as their numerical approximations.

The rest of the paper is organized as follows. In Section 2, we propose a regularized version of (1.1) with a small regularization parameter $0 < \varepsilon \ll 1$, and analyze its properties, as well as the convergence of its solution to the solution of (1.1). In Section 3, we introduce a semi-implicit finite difference method for discretizing the regularized logarithmic Schrödinger equation, and prove an error estimate, in which the dependence of the constants with respect to the regularization parameter ε is tracked very explicitly. Finally, numerical results are provided in Section 4 to confirm our error bounds and to demonstrate the efficiency and accuracy of the proposed numerical method.

Throughout the paper, we use $H^m(\Omega)$ and $\|\cdot\|_{H^m(\Omega)}$ to denote the standard Sobolev spaces and their norms, respectively. In particular, the norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_{L^2(\Omega)}$ and $(\cdot,\cdot)$, respectively. Moreover, we adopt $A \lesssim B$ to mean that there exists a generic constant $C > 0$ independent of the regularization parameter ε, the time step τ and the mesh size h such that $A \le C\,B$, and $\lesssim_c$ means the constant C depends on c.

2. A regularized logarithmic Schrödinger equation. It turns out that a direct simulation of the solution of (1.1) is very delicate, due to the singularity of the logarithm at the origin, as discussed in the introduction. Instead of working directly with (1.1), we shall consider the following regularized logarithmic Schrödinger equation (RLogSE) with a samll regularized parameter $0 < \varepsilon \ll 1$ as

$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) + \Delta u^\varepsilon(\mathbf{x},t) = \lambda u^\varepsilon(\mathbf{x},t)\,\ln\left(\varepsilon + |u^\varepsilon(\mathbf{x},t)|\right)^2, \quad \mathbf{x}\in\Omega, \quad t>0, \\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}. \end{cases} \tag{2.1}$$

2.1. Conserved quantities. For the RLogSE (2.1), it can be similarly deduced that the mass, momentum, and energy are conserved.

PROPOSITION 2.1. *The mass, momentum, and 'regularized' energy are formally conserved for the RLogSE* (2.1)*:*

$$\begin{aligned} M^\varepsilon(t) &:= \int_\Omega |u^\varepsilon(\mathbf{x},t)|^2 d\mathbf{x} \equiv \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x} = M(0), \\ P^\varepsilon(t) &:= \mathrm{Im}\int_\Omega \overline{u^\varepsilon}(\mathbf{x},t)\nabla u^\varepsilon(\mathbf{x},t) d\mathbf{x} \equiv \mathrm{Im}\int_\Omega \overline{u_0}(\mathbf{x})\nabla u_0(\mathbf{x}) d\mathbf{x} = P(0), \quad t\ge 0, \\ E^\varepsilon(t) &:= \int_\Omega \left[|\nabla u^\varepsilon(\mathbf{x},t)|^2 + \lambda F_\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)\right](\mathbf{x},t) d\mathbf{x} \\ &\equiv \int_\Omega \left[|\nabla u_0(\mathbf{x})|^2 + \lambda F_\varepsilon(|u_0(\mathbf{x})|^2)\right] d\mathbf{x} = E^\varepsilon(0), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} F_\varepsilon(\rho) &= \int_0^\rho \ln(\varepsilon+\sqrt{s})^2 ds \\ &= \rho\ln\left(\varepsilon+\sqrt{\rho}\right)^2 - \rho + 2\varepsilon\sqrt{\rho} - \varepsilon^2\ln\left(1+\sqrt{\rho}/\varepsilon\right)^2, \qquad \rho\ge 0. \end{aligned} \tag{2.3}$$

Proof. The conservation for mass and momentum is standard, and relies on the fact that the right hand side of (2.1) involves u^ε multiplied by a *real* number. For the energy $E^\varepsilon(t)$, we compute

$$\begin{aligned}\frac{d}{dt}E^\varepsilon(t) &= 2\,\mathrm{Re}\int_\Omega \left[\nabla u^\varepsilon\cdot\nabla\partial_t\overline{u^\varepsilon} + \lambda u^\varepsilon\partial_t\overline{u^\varepsilon}\ln(\varepsilon+|u^\varepsilon|)^2 - \lambda u^\varepsilon\partial_t\overline{u^\varepsilon}\right](\mathbf{x},t)d\mathbf{x}\\ &\quad + 2\lambda\int_\Omega \partial_t|u^\varepsilon|\left[\varepsilon+\frac{|u^\varepsilon|^2-\varepsilon^2}{\varepsilon+|u^\varepsilon|}\right](\mathbf{x},t)d\mathbf{x}\\ &= 2\,\mathrm{Re}\int_\Omega \left[\partial_t\overline{u^\varepsilon}\left(-\Delta u^\varepsilon+\lambda u^\varepsilon\ln(\varepsilon+|u^\varepsilon|)^2\right)\right](\mathbf{x},t)d\mathbf{x}\\ &= 2\,\mathrm{Re}\int_\Omega i|\partial_t u^\varepsilon|^2(\mathbf{x},t)d\mathbf{x} = 0, \quad t\geq 0,\end{aligned}$$

which completes the proof. □

Note however that since the above 'regularized' energy involves L^1-norm of u^ε for any $\varepsilon>0$, E^ε is obviously well-defined for $u_0\in H^1(\Omega)$ when Ω has finite measure, but not when $\Omega=\mathbb{R}^d$. This aspect is discussed more into details in Subsections 2.3.3 and 2.4.

2.2. The Cauchy problem. For $\alpha>0$ and $\Omega=\mathbb{R}^d$, denote by L^2_α the weighted L^2 space

$$L^2_\alpha := \{v\in L^2(\mathbb{R}^d),\quad \mathbf{x}\longmapsto\langle\mathbf{x}\rangle^\alpha v(\mathbf{x})\in L^2(\mathbb{R}^d)\},$$

where $\langle\mathbf{x}\rangle := \sqrt{1+|\mathbf{x}|^2}$, with norm

$$\|v\|_{L^2_\alpha} := \|\langle\mathbf{x}\rangle^\alpha v(\mathbf{x})\|_{L^2(\mathbb{R}^d)}.$$

In the case where Ω is bounded, we simply set $L^2_\alpha=L^2(\Omega)$. Regarding the Cauchy problems (1.1) and (2.1), we have the following result.

THEOREM 2.2. *Let $\lambda\in\mathbb{R}$ and $\varepsilon>0$. Consider* (1.1) *and* (2.1) *on $\Omega=\mathbb{R}^d$, or bounded Ω with homogeneous Dirichlet or periodic boundary condition. Consider an initial datum $u_0\in H^1_0(\Omega)\cap L^2_\alpha$, for some $0<\alpha\leq 1$.*

- *There exists a unique, global solution $u\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1_0(\Omega)\cap L^2_\alpha)$ to* (1.1)*, and a unique, global solution $u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1_0(\Omega)\cap L^2_\alpha)$ to* (2.1)*.*
- *If in addition $u_0\in H^2(\Omega)$, then $u,u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^2(\Omega))$.*
- *In the case $\Omega=\mathbb{R}^d$, if in addition $u_0\in H^2\cap L^2_2$, then $u,u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^2\cap L^2_2)$.*

Proof. This result can be proved by using more or less directly the arguments invoked in [10]. First, for fixed $\varepsilon>0$, the nonlinearity in (2.1) is locally Lipschitz, and grows more slowly than any power for large $|u^\varepsilon|$. Therefore, the standard Cauchy theory for nonlinear Schrödinger equations applies (see in particular [12, Corollary 3.3.11 and Theorem 3.4.1]), and so if $u_0\in H^1_0(\Omega)$, then (2.1) has a unique solution $u^\varepsilon\in L^\infty_{\mathrm{loc}}(\mathbb{R};H^1_0(\Omega))$. Higher Sobolev regularity is propagated, with controls depending on ε in general.

A solution u of (1.1) can be obtained by compactness arguments, by letting $\varepsilon\to 0$ in (2.1), provided that we have suitable bounds independent of $\varepsilon>0$. We have

$$i\partial_t\nabla u^\varepsilon+\Delta\nabla u^\varepsilon = 2\lambda\ln\left(\varepsilon+|u^\varepsilon|\right)\nabla u^\varepsilon + 2\lambda\frac{u^\varepsilon}{\varepsilon+|u^\varepsilon|}\nabla|u^\varepsilon|.$$

The standard energy estimate (multiply the above equation by $\nabla\overline{u^\varepsilon}$, integrate over Ω and take the imaginary part) yields, when $\Omega = \mathbb{R}^d$ or when periodic boundary conditions are considered,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^\varepsilon\|^2_{L^2(\Omega)} \leq 2|\lambda| \int_\Omega \frac{|u^\varepsilon|}{\varepsilon + |u^\varepsilon|} \, |\nabla |u^\varepsilon|| \, |\nabla u^\varepsilon| d\mathbf{x} \leq 2|\lambda| \|\nabla u^\varepsilon\|^2_{L^2(\Omega)}.$$

Gronwall lemma yields a bound for u^ε in $L^\infty(0,T;H^1(\Omega))$, uniformly in $\varepsilon > 0$, for any given $T > 0$. Indeed, the above estimate uses the property

$$\mathrm{Im} \int_\Omega \nabla\overline{u^\varepsilon} \cdot \Delta \nabla u^\varepsilon \, d\mathbf{x} = 0,$$

which needs not be true when Ω is bounded and u^ε satisfies homogeneous Dirichlet boundary conditions. In that case, we use the conservation of the energy E^ε (Proposition 2.1), and write

$$\begin{aligned}
\|\nabla u^\varepsilon(t)\|^2_{L^2(\Omega)} &\leq E^\varepsilon(u_0) + 2|\lambda| \int_\Omega |u^\varepsilon(\mathbf{x},t)|^2 \, |\ln(\varepsilon + |u^\varepsilon(\mathbf{x},t)|)| \, d\mathbf{x} \\
&\quad + 2\varepsilon|\lambda| \|u^\varepsilon(t)\|_{L^1(\Omega)} + 2|\lambda|\varepsilon^2 \int_\Omega |\ln(1 + |u^\varepsilon(\mathbf{x},t)|/\varepsilon)| \, d\mathbf{x} \\
&\lesssim 1 + \varepsilon|\Omega|^{1/2} \|u^\varepsilon(t)\|_{L^2(\Omega)} + \int_\Omega |u^\varepsilon(\mathbf{x},t)|^2 \, |\ln(\varepsilon + |u^\varepsilon(\mathbf{x},t)|)| \, d\mathbf{x} \\
&\lesssim 1 + \int_\Omega |u^\varepsilon(\mathbf{x},t)|^2 \, |\ln(\varepsilon + |u^\varepsilon(\mathbf{x},t)|)| \, d\mathbf{x}, \quad t \geq 0,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the conservation of the mass $M^\varepsilon(t)$. Writing, for $0 < \eta \ll 1$,

$$\begin{aligned}
&\int_\Omega |u^\varepsilon|^2 \, |\ln(\varepsilon + |u^\varepsilon|)| \, d\mathbf{x} \\
&\lesssim \int_{\varepsilon + |u^\varepsilon| > 1} |u^\varepsilon|^2 \, (\varepsilon + |u^\varepsilon|)^\eta \, d\mathbf{x} + \int_{\varepsilon + |u^\varepsilon| < 1} |u^\varepsilon|^2 \, (\varepsilon + |u^\varepsilon|)^{-\eta} \, d\mathbf{x} \\
&\lesssim \|u^\varepsilon\|_{L^2(\Omega)} + \|u^\varepsilon\|^{2+\eta}_{L^{2+\eta}(\Omega)} + \|u^\varepsilon\|^{2-\eta}_{L^{2-\eta}(\Omega)} \lesssim 1 + \|\nabla u^\varepsilon\|^{d\eta/2}_{L^2(\Omega)},
\end{aligned}$$

where we have used the interpolation inequality (see e.g. [23])

$$\|u\|_{L^p(\Omega)} \lesssim \|u\|^{1-\alpha}_{L^2(\Omega)} \|\nabla u\|^\alpha_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}, \quad p = \frac{2d}{d-2\alpha}, \ 0 \leq \alpha < 1,$$

we obtain again that u^ε is bounded in $L^\infty(0,T;H^1(\Omega))$, uniformly in $\varepsilon > 0$, for any given $T > 0$.

In the case where Ω is bounded, compactness arguments show that u^ε converges to a solution u to (1.1); see [12,13]. When $\Omega = \mathbb{R}^d$, compactness in space is provided by multiplying (2.1) with $\langle \mathbf{x} \rangle^{2\alpha} \overline{u^\varepsilon}$ and integrating in space:

$$\frac{d}{dt}\|u^\varepsilon\|^2_{L^2_\alpha} = 4\alpha \, \mathrm{Im} \int \frac{\mathbf{x} \cdot \nabla u^\varepsilon}{\langle \mathbf{x} \rangle^{2-2\alpha}} \, \overline{u^\varepsilon}(t) \, d\mathbf{x} \lesssim \| \langle \mathbf{x} \rangle^{2\alpha-1} u^\varepsilon \|_{L^2(\Omega)} \|\nabla u^\varepsilon\|_{L^2(\Omega)},$$

where we have used Cauchy-Schwarz inequality. Recalling that $0 < \alpha \leq 1$,

$$\| \langle \mathbf{x} \rangle^{2\alpha-1} u^\varepsilon \|_{L^2(\Omega)} \leq \| \langle \mathbf{x} \rangle^{\alpha} u^\varepsilon \|_{L^2(\Omega)} = \|u^\varepsilon\|_{L^2_\alpha},$$

and we obtain a bound for u^ε in $L^\infty(0,T;H^1(\Omega)\cap L^2_\alpha)$ which is uniform in ε. Uniqueness of such a solution for (1.1) follows from the arguments of [13], involving a specific algebraic inequality, generalized in Lemma 2.4 below. Note that at this stage, we know that u^ε converges to u by compactness arguments, so we have no convergence estimate. Such estimates are established in Subsection 2.3.

To prove the propagation of the H^2 regularity, we note that differentiating twice the nonlinearity in (2.1) makes it unrealistic to expect direct bounds which are uniform in ε. To overcome this difficulty, the argument proposed in [10] relies on Kato's idea: instead of differentiating the equation twice in space, differentiate it once in time, and use the equation to infer H^2 regularity. This yields the second part of the theorem.

To establish the last part of the theorem, we prove that $u \in L^\infty_{\rm loc}(\mathbb{R};L^2_2)$ and the same approach applies to u^ε. It follows from (1.1) that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{L^2_2} &= -2\,\mathrm{Im}\int_{\mathbb{R}^d}\langle\mathbf{x}\rangle^4\overline{u(\mathbf{x},t)}\Delta u(\mathbf{x},t)d\mathbf{x}\\ &= 8\,\mathrm{Im}\int_{\mathbb{R}^d}\langle\mathbf{x}\rangle^2\overline{u(\mathbf{x},t)}\,\mathbf{x}\cdot\nabla u(\mathbf{x},t)d\mathbf{x} \le 8\,\|u(t)\|_{L^2_2}\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{2.4}$$

By Cauchy-Schwarz inequality and integration by parts, we have

$$\begin{aligned}&\|\mathbf{x}\cdot\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}\\ &\quad\le\sum_{j=1}^d\sum_{k=1}^d\int_{\mathbb{R}^d}x_j^2\frac{\partial u(\mathbf{x},t)}{\partial x_k}\frac{\partial\overline{u(\mathbf{x},t)}}{\partial x_k}d\mathbf{x}\\ &\quad=-2\int_{\mathbb{R}^d}\overline{u(\mathbf{x},t)}\,\mathbf{x}\cdot\nabla u(\mathbf{x},t)d\mathbf{x}-\int_{\mathbb{R}^d}|\mathbf{x}|^2\overline{u(\mathbf{x},t)}\Delta u(\mathbf{x},t)d\mathbf{x}\\ &\quad\le\frac12\|\mathbf{x}\cdot\nabla u(t)\|^2_{L^2(\mathbb{R}^d)}+2\|u(t)\|^2_{L^2(\mathbb{R}^d)}+\frac12\|u(t)\|^2_{L^2_2}+\frac12\|\Delta u(t)\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

which yields directly that

$$\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)}\le 2\|u(t)\|_{L^2(\mathbb{R}^d)}+\|u(t)\|_{L^2_2}+\|\Delta u(t)\|_{L^2(\mathbb{R}^d)}.$$

This together with (2.4) gives that

$$\frac{d}{dt}\|u(t)\|_{L^2_2}\le 4\|\mathbf{x}\cdot\nabla u(t)\|_{L^2(\mathbb{R}^d)}\le 4\|u(t)\|_{L^2_2}+8\|u(t)\|_{L^2(\mathbb{R}^d)}+4\|\Delta u(t)\|_{L^2(\mathbb{R}^d)}.$$

Since we already know that $u\in L^\infty_{\rm loc}(\mathbb{R};H^2(\mathbb{R}^d))$, Gronwall lemma completes the proof. □

REMARK 2.1. *We emphasize that if $u_0\in H^k(\mathbb{R}^d)$, $k\ge 3$, we cannot guarantee in general that this higher regularity is propagated in* (1.1)*, due to the singularities stemming from the logarithm. Still, this property is fulfilled in the case where u_0 is Gaussian, since then u remains Gaussian for all time. However, our numerical tests, in the case where the initial datum is chosen as the dark soliton of the cubic Schrödinger equation multiplied by a Gaussian, suggest that even the H^3 regularity is not propagated in general.*

2.3. Convergence of the regularized model. In this subsection, we show the approximation property of the regularized model (2.1) to (1.1).

2.3.1. A general estimate. We prove:

LEMMA 2.3. *Suppose the equation is set on Ω, where $\Omega = \mathbb{R}^d$, or $\Omega \subset \mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition, then we have the general estimate:*

$$\frac{d}{dt}\|u^\varepsilon(t) - u(t)\|_{L^2(\Omega)}^2 \le 4|\lambda| \left(\|u^\varepsilon(t) - u(t)\|_{L^2(\Omega)}^2 + \varepsilon \|u^\varepsilon(t) - u(t)\|_{L^1(\Omega)} \right). \tag{2.5}$$

Before giving the proof of Lemma 2.3, we introduce the following lemma, which is a variant of [12, Lemma 9.3.5], established initially in [13, Lemme 1.1.1].

LEMMA 2.4. *Let $\varepsilon \ge 0$ and denote $f_\varepsilon(z) = z \ln(\varepsilon + |z|)$, then we have*

$$|\mathrm{Im}\left((f_\varepsilon(z_1) - f_\varepsilon(z_2))\,(\overline{z_1} - \overline{z_2}) \right)| \le |z_1 - z_2|^2, \quad z_1, z_2 \in \mathbb{C}.$$

Proof. Notice that

$$\mathrm{Im}\left[(f_\varepsilon(z_1) - f_\varepsilon(z_2))\,(\overline{z_1} - \overline{z_2}) \right] = \frac{1}{2}\left[\ln(\varepsilon + |z_1|) - \ln(\varepsilon + |z_2|) \right] \mathrm{Im}(\overline{z_1} z_2 - z_1 \overline{z_2}).$$

Supposing, for example, $0 < |z_2| \le |z_1|$, we can obtain that

$$|\ln(\varepsilon + |z_1|) - \ln(\varepsilon + |z_2|)| = \ln\left(1 + \frac{|z_1| - |z_2|}{\varepsilon + |z_2|} \right) \le \frac{|z_1| - |z_2|}{\varepsilon + |z_2|} \le \frac{|z_1 - z_2|}{|z_2|},$$

and

$$|\mathrm{Im}(\overline{z_1} z_2 - z_1 \overline{z_2})| = |z_2(\overline{z_1} - \overline{z_2}) + \overline{z_2}(z_2 - z_1))| \le 2|z_2|\,|z_1 - z_2|.$$

Otherwise the result follows by exchanging z_1 and z_2. ∎

Proof. (Proof of Lemma 2.3) Subtracting (1.1) from (2.1), we see that the error function $e^\varepsilon := u^\varepsilon - u$ satisfies

$$i\partial_t e^\varepsilon + \Delta e^\varepsilon = \lambda \left[u^\varepsilon \ln(\varepsilon + |u^\varepsilon|)^2 - u \ln(|u|^2) \right].$$

Multiplying the error equation by $\overline{e^\varepsilon(t)}$, integrating in space and taking the imaginary parts, we can get by using Lemma 2.4 that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e^\varepsilon(t)\|_{L^2(\Omega)}^2 &= 2\lambda\,\mathrm{Im} \int_\Omega \left[u^\varepsilon \ln(\varepsilon + |u^\varepsilon|) - u \ln(|u|) \right] (\overline{u^\varepsilon} - \overline{u})(\mathbf{x}, t) d\mathbf{x} \\
&\le 2|\lambda| \|e^\varepsilon(t)\|_{L^2(\Omega)}^2 + 2|\lambda| \left| \int_\Omega \overline{e^\varepsilon} u \left[\ln(\varepsilon + |u|) - \ln(|u|) \right] (\mathbf{x}, t) d\mathbf{x} \right| \\
&\le 2|\lambda| \|e^\varepsilon(t)\|_{L^2(\Omega)}^2 + 2\varepsilon|\lambda| \|e^\varepsilon(t)\|_{L^1(\Omega)},
\end{aligned}$$

where we have used the general estimate $0 \le \ln(1 + |x|) \le |x|$. ∎

2.3.2. Convergence for bounded domain. If Ω has finite measure, then we can have the following convergence behavior.

PROPOSITION 2.5. *Assume that Ω has finite measure, and let $u_0 \in H^2(\Omega)$. For any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \le C_1 \varepsilon, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\Omega))} \le C_2 \varepsilon^{1/2}, \tag{2.6}$$

where C_1 depends on $|\lambda|$, T, $|\Omega|$ and C_2 depends on $|\lambda|$, T, $|\Omega|$ and $\|u_0\|_{H^2(\Omega)}$.

Proof. Note that $\|e^\varepsilon(t)\|_{L^1(\Omega)} \leq |\Omega|^{1/2}\|e^\varepsilon(t)\|_{L^2(\Omega)}$, then it follows from (2.5) that

$$\frac{d}{dt}\|e^\varepsilon(t)\|_{L^2(\Omega)} \leq 2|\lambda|\|e^\varepsilon(t)\|_{L^2(\Omega)} + 2\varepsilon|\lambda||\Omega|^{1/2}.$$

Applying Gronwall's inequality, we immediately get that

$$\|e^\varepsilon(t)\|_{L^2(\Omega)} \leq \left(\|e^\varepsilon(0)\|_{L^2(\Omega)} + \varepsilon|\Omega|^{1/2}\right)e^{2|\lambda|t} = \varepsilon|\Omega|^{1/2}e^{2|\lambda|t}.$$

The convergence rate in H^1 follows from the property $u^\varepsilon, u \in L^\infty_{\rm loc}(\mathbb{R}; H^2(\Omega))$ and the Gagliardo-Nirenberg inequality [21],

$$\|\nabla v\|_{L^2(\Omega)} \lesssim \|v\|^{1/2}_{L^2(\Omega)}\|\Delta v\|^{1/2}_{L^2(\Omega)},$$

which completes the proof. ∎

REMARK 2.2. *The weaker rate in the H^1 estimate is due to the fact that Lemma 2.3 is not easily adapted to H^1 estimates, because of the presence of the logarithm. Differentiating* (1.1) *and* (2.1) *makes it hard to obtain the analogue in Lemma 2.3. This is why we bypass this difficulty by invoking boundedness in H^2 and interpolating with the error bound at the L^2 level. If we have u^ε, $u \in L^\infty_{\rm loc}(\mathbb{R}; H^k(\Omega))$ for $k > 2$, then the convergence rate in $H^1(\Omega)$ can be improved as*

$$\|e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \lesssim \varepsilon^{\frac{k-1}{k}},$$

by using the inequality (see e.g. [23]):

$$\|v\|_{H^1(\Omega)} \lesssim \|v\|^{1-1/k}_{L^2(\Omega)}\|v\|^{1/k}_{H^k(\Omega)}.$$

2.3.3. Convergence for the whole space. In order to prove the convergence rate of the regularized model (2.1) to (1.1) for the whole space, we need the following lemma.

LEMMA 2.6. *For $d = 1, 2, 3$, if $v \in L^2(\mathbb{R}^d) \cap L^2_2$, then we have*

$$\|v\|_{L^1(\mathbb{R}^d)} \leq C\|v\|^{1-d/4}_{L^2(\mathbb{R}^d)}\|v\|^{d/4}_{L^2_2}, \tag{2.7}$$

where $C > 0$ depends on d.

Proof. Applying the Cauchy-Schwarz inequality, we can get for fixed $r > 0$,

$$\begin{aligned}
\|v\|_{L^1(\mathbb{R}^d)} &= \int_{|\mathbf{x}|\leq r}|v(\mathbf{x})|d\mathbf{x} + \int_{|\mathbf{x}|\geq r}\frac{|\mathbf{x}|^2|v(\mathbf{x})|}{|\mathbf{x}|^2}d\mathbf{x} \\
&\lesssim r^{d/2}\left(\int_{|\mathbf{x}|\leq r}|v(\mathbf{x})|^2d\mathbf{x}\right)^{\frac{1}{2}} + \left(\int_{|\mathbf{x}|\geq r}|\mathbf{x}|^4|v(\mathbf{x})|^2d\mathbf{x}\right)^{\frac{1}{2}}\left(\int_{|\mathbf{x}|\geq r}\frac{1}{|\mathbf{x}|^4}d\mathbf{x}\right)^{\frac{1}{2}} \\
&\lesssim r^{d/2}\|v\|_{L^2(\mathbb{R}^d)} + r^{d/2-2}\|v\|_{L^2_2}.
\end{aligned}$$

Then (2.7) can be obtained by setting $r = \left(\|v\|_{L^2_2}/\|v\|_{L^2(\mathbb{R}^d)}\right)^{1/2}$. ∎

PROPOSITION 2.7. *Assume that $\Omega = \mathbb{R}^d$, $1 \le d \le 3$, and let $u_0 \in H^2(\mathbb{R}^d) \cap L^2_2$. For any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \le C_1 \varepsilon^{\frac{4}{4+d}}, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\mathbb{R}^d)))} \le C_2 \varepsilon^{\frac{2}{4+d}},$$

where C_1 depends on d, $|\lambda|$, T, $\|u_0\|_{L^2_2}$ and C_2 depends on additional $\|u_0\|_{H^2(\mathbb{R}^d)}$.

Proof. Applying (2.7) and the Young's inequality, we deduce that

$$\varepsilon\|e^\varepsilon(t)\|_{L^1(\mathbb{R}^d)} \le \varepsilon C_d \|e^\varepsilon(t)\|^{1-d/4}_{L^2(\mathbb{R}^d)} \|e^\varepsilon(t)\|^{d/4}_{L^2_2} \le C_d \left(\|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{\frac{8}{4+d}} \|e^\varepsilon(t)\|^{\frac{2d}{4+d}}_{L^2_2} \right),$$

which together with (2.5) gives that

$$\frac{d}{dt}\|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} \le 4|\lambda|(1+C_d)\|e^\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} + 4C_d|\lambda|\varepsilon^{\frac{8}{4+d}} \|e^\varepsilon(t)\|^{\frac{2d}{4+d}}_{L^2_2}.$$

Gronwall lemma yields

$$\|e^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \le \varepsilon^{\frac{4}{4+d}} \|e^\varepsilon(t)\|^{\frac{d}{4+d}}_{L^2_2} e^{tC_{d,|\lambda|}}.$$

The proposition follows by recalling that u^ε, $u \in L^\infty_{\rm loc}(\mathbb{R}; H^2(\mathbb{R}^d) \cap L^2_2)$. □

REMARK 2.3. *If we have $u^\varepsilon, u \in L^\infty_{\rm loc}(\mathbb{R}; L^2_m)$ for $m > 2$, then by applying the inequality*

$$\varepsilon\|v\|_{L^1(\mathbb{R}^d)} \lesssim \varepsilon\|v\|^{1-\frac{d}{2m}}_{L^2(\mathbb{R}^d)} \|v\|^{\frac{d}{2m}}_{L^2_m} \lesssim \|v\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{\frac{4m}{2m+d}} \|v\|^{\frac{2d}{2m+d}}_{L^2_m},$$

which can be proved like above, the convergence rate can be improved as

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \lesssim \varepsilon^{\frac{2m}{2m+d}}.$$

REMARK 2.4. *If in addition $u^\varepsilon, u \in L^\infty_{\rm loc}(\mathbb{R}; H^s(\mathbb{R}^d))$ for $s > 2$, then the convergence rate in $H^1(\mathbb{R}^d)$ can be improved as*

$$\|e^\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{R}^d)))} \le C\varepsilon^{\frac{2m}{2m+d}\frac{s-1}{s}},$$

by using the Gagliardo-Nirenberg inequality:

$$\|\nabla v\|_{L^2(\mathbb{R}^d)} \le C\|v\|^{1-1/s}_{L^2(\mathbb{R}^d)} \|\nabla^s v\|^{1/s}_{L^2(\mathbb{R}^d)}.$$

The previous two remarks apply typically in the case of Gaussian initial data.

2.4. Convergence of the energy. In this subsection we will show the convergence of the energy $E^\varepsilon(u_0) \to E(u_0)$.

PROPOSITION 2.8. *For $u_0 \in H^1(\Omega) \cap L^1(\Omega)$, the energy $E^\varepsilon(u_0)$ converges to $E(u_0)$ with*

$$|E^\varepsilon(u_0) - E(u_0)| \le 4\,\varepsilon|\lambda|\|u_0\|_{L^1(\Omega)}.$$

Proof. It can be deduced from the definition that

$$\begin{aligned}|E^\varepsilon(u_0) - E(u_0)| = 2|\lambda| \Big| &\varepsilon \|u_0\|_{L^1(\Omega)} + \int_\Omega |u_0(\mathbf{x})|^2 \left[\ln(\varepsilon + |u_0(\mathbf{x})| - \ln(|u_0(\mathbf{x})|)\right] d\mathbf{x} \\ &-\varepsilon^2 \int_\Omega \ln\left(1 + |u_0(\mathbf{x})|/\varepsilon\right) d\mathbf{x}\Big| \\ &\leq 4\,\varepsilon|\lambda| \|u_0\|_{L^1(\Omega)},\end{aligned}$$

which completes the proof. □

REMARK 2.5. *If Ω is bounded, then $H^1(\Omega) \subseteq L^1(\Omega)$. If $\Omega = \mathbb{R}^d$, then Lemma 2.6 (and its natural generalizations) shows that $H^1(\mathbb{R}) \cap L_1^2 \subseteq L^1(\mathbb{R})$, and if $d = 2, 3$, $H^1(\mathbb{R}^d) \cap L_2^2 \subseteq L^1(\mathbb{R}^d)$.*

REMARK 2.6. *This regularization is reminiscent of the one considered in [10] in order to prove (by compactness arguments) that* (1.1) *has a solution,*

$$i\partial_t u^\varepsilon(\mathbf{x}, t) + \Delta u^\varepsilon(\mathbf{x}, t) = \lambda u^\varepsilon(\mathbf{x}, t)\, \ln\left(\varepsilon + |u^\varepsilon(\mathbf{x}, t)|^2\right), \quad \mathbf{x} \in \Omega, \quad t > 0. \tag{2.8}$$

With that regularization, it is easy to adapt the error estimates established above for (2.1). *Essentially, ε must be replaced by $\sqrt{\varepsilon}$ (in Lemma 2.3, and hence in its corollaries).*

3. A regularized semi-implicit finite difference method. In this section, we study the approximation properties of a finite difference method for solving the regularized model (2.1). For simplicity of notation, we set $\lambda = 1$ and only present the numerical method for the RLogSE (2.1) in 1D, as extensions to higher dimensions are straightforward. When $d = 1$, we truncate the RLogSE on a bounded computational interval $\Omega = (a, b)$ with homogeneous Dirichlet boundary condition (here $|a|$ and b are chosen large enough such that the truncation error is negligible):

$$\begin{cases} i\partial_t u^\varepsilon(x, t) + \partial_{xx} u^\varepsilon(x, t) = u^\varepsilon(x, t)\, \ln(\varepsilon + |u^\varepsilon(x, t)|)^2, \quad x \in \Omega, \quad t > 0, \\ u^\varepsilon(x, 0) = u_0(x), \quad x \in \overline{\Omega}; \qquad u^\varepsilon(a, t) = u^\varepsilon(b, t) = 0, \quad t \geq 0, \end{cases} \tag{3.1}$$

3.1. A finite difference scheme and main results on error bounds. Choose a mesh size $h := \Delta x = (b - a)/M$ with M being a positive integer and a time step $\tau := \Delta t > 0$ and denote the grid points and time steps as

$$x_j := a + jh, \quad j = 0, 1, \cdots, M; \quad t_k := k\tau, \quad k = 0, 1, 2, \ldots$$

Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1, 2, \cdots, M - 1\}, \quad \mathcal{T}_M^0 = \{j \mid j = 0, 1, \cdots, M\}.$$

Let $u_j^{\varepsilon,k}$ be the approximation of $u^\varepsilon(x_j, t_k)$, and denote $u^{\varepsilon,k} = (u_0^{\varepsilon,k}, u_1^{\varepsilon,k}, \ldots, u_M^{\varepsilon,k})^T \in \mathbb{C}^{M+1}$ as the numerical solution vector at $t = t_k$. Define the standard finite difference operators

$$\delta_t^c u_j^k = \frac{u_j^{k+1} - u_j^{k-1}}{2\tau}, \quad \delta_x^+ u_j^k = \frac{u_{j+1}^k - u_j^k}{h}, \quad \delta_x^2 u_j^k = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2}.$$

Denote

$$X_M = \left\{ v = (v_0, v_1, \ldots, v_M)^T \mid v_0 = v_M = 0 \right\} \subseteq \mathbb{C}^{M+1},$$

equipped with inner products and norms defined as (recall that $u_0 = v_0 = u_M = v_M = 0$ by Dirichlet boundary condition)

$$(u,v) = h\sum_{j=1}^{M-1} u_j\overline{v_j}, \quad \langle u,v\rangle = h\sum_{j=0}^{M-1} u_j\overline{v_j}, \quad \|u\|_\infty = \sup_{j\in\mathcal{T}_M^0}|u_j|; \tag{3.2}$$
$$\|u\|^2 = (u,u), \quad |u|_{H^1}^2 = \left\langle \delta_x^+ u, \delta_x^+ u\right\rangle, \quad \|u\|_{H^1}^2 = \|u\|^2 + |u|_{H^1}^2.$$

Then we have for $u,\, v \in X_M$,

$$(-\delta_x^2 u, v) = \left\langle \delta_x^+ u, \delta_x^+ v\right\rangle = (u, -\delta_x^2 v). \tag{3.3}$$

Consider a semi-implicit finite difference (SIFD) discretization of (3.1) as following

$$i\delta_t^c u_j^{\varepsilon,k} = -\frac{1}{2}\delta_x^2(u_j^{\varepsilon,k+1} + u_j^{\varepsilon,k-1}) + u_j^{\varepsilon,k}\ln(\varepsilon + |u_j^{\varepsilon,k}|)^2, \quad j\in\mathcal{T}_M, \quad k\geq 1. \tag{3.4}$$

The boundary and initial conditions are discretized as

$$u_0^{\varepsilon,k} = u_M^{\varepsilon,k} = 0, \quad k\geq 0; \quad u_j^{\varepsilon,0} = u_0(x_j), \quad j\in\mathcal{T}_M^0. \tag{3.5}$$

In addition, the first step $u_j^{\varepsilon,1}$ can be obtained via the Taylor expansion as

$$u_j^{\varepsilon,1} = u_j^{\varepsilon,0} + \tau u_1(x_j), \quad j\in\mathcal{T}_M^0, \tag{3.6}$$

where

$$u_1(x) := \partial_t u^\varepsilon(x,0) = i\left[u_0''(x) - u_0(x)\ln(\varepsilon + |u_0(x)|)^2\right], \quad a\leq x\leq b.$$

Let $0 < T < T_{\max}$ with $T_{\max}$ the maximum existence time of the solution u^ε to the problem (3.1) for a fixed $0\leq\varepsilon\ll 1$. By using the standard von Neumann analysis, we can show that the discretization (3.4) is conditionally stable under the stability condition

$$0 < \tau \leq \frac{1}{2\max\{|\ln\varepsilon|, \ln(\varepsilon + \max\limits_{j\in\mathcal{T}_M}|u_j^{\varepsilon,k}|)\}}, \qquad 0\leq k\leq \frac{T}{\tau}. \tag{3.7}$$

Define the error functions $e^{\varepsilon,k}\in X_M$ as

$$e_j^{\varepsilon,k} = u^\varepsilon(x_j,t_k) - u_j^{\varepsilon,k}, \quad j\in\mathcal{T}_M^0, \quad 0\leq k\leq\frac{T}{\tau}, \tag{3.8}$$

where u^ε is the solution of (3.1). Then we have the following error estimates for (3.4) with (3.5) and (3.6).

THEOREM 3.1 (**Main result**). *Assume that the solution u^ε is smooth enough over $\Omega_T := \Omega\times[0,T]$, i.e.*

$$(A) \qquad u^\varepsilon \in C\left([0,T];H^5(\Omega)\right)\cap C^2\left([0,T];H^4(\Omega)\right)\cap C^3\left([0,T];H^2(\Omega)\right),$$

and there exist $\varepsilon_0 > 0$ and $C_0 > 0$ independent of ε such that

$$\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} + \|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq C_0,$$

uniformly in $0 \le \varepsilon \le \varepsilon_0$. *Then there exist* $h_0 > 0$ *and* $\tau_0 > 0$ *sufficiently small with* $h_0 \sim \sqrt{\varepsilon} e^{-CT|\ln(\varepsilon)|^2}$ *and* $\tau_0 \sim \sqrt{\varepsilon} e^{-CT|\ln(\varepsilon)|^2}$ *such that, when* $0 < h \le h_0$ *and* $0 < \tau \le \tau_0$ *satisfying the stability condition* (3.7), *we have the following error estimates*

$$\begin{aligned} &\|e^{\varepsilon,k}\| \le C_1(\varepsilon,T)(h^2+\tau^2), \qquad 0 \le k \le \frac{T}{\tau}, \\ &\|e^{\varepsilon,k}\|_{H^1} \le C_2(\varepsilon,T)(h^2+\tau^2), \quad \|u^{\varepsilon,k}\|_\infty \le \Lambda + 1, \end{aligned} \tag{3.9}$$

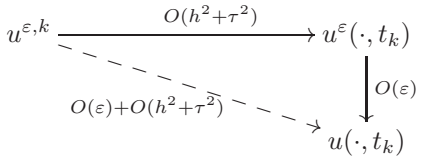
where $\Lambda = \|u^\varepsilon\|_{L^\infty(\Omega_T)}$, $C_1(\varepsilon,T) \sim e^{CT|\ln(\varepsilon)|^2}$, $C_2(\varepsilon,T) \sim \frac{1}{\varepsilon} e^{CT|\ln(\varepsilon)|^2}$ *and* C *depends on* C_0.

The error bounds in this Theorem show not only the quadratical convergence in terms of the mesh size h and time step τ but also how the explicit dependence on the regularization parameter ε. Here we remark that the Assumption (A) is valid at least in the case of taking Gaussian as the initial datum.

Define the error functions $\widetilde{e}^{\varepsilon,k} \in X_M$ as

$$\widetilde{e}_j^{\varepsilon,k} = u(x_j,t_k) - u_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}, \tag{3.10}$$

where u^ε is the solution of the LogSE (1.1) with $\Omega = (a,b)$. Combining Proposition 2.5 and Theorem 3.1, we immediately obtain (see an illustration in the following diagram):



Corollary 3.2. *Under the assumptions of Proposition 2.5 and Theorem 3.1, we have the following error estimates*

$$\begin{aligned} &\|\widetilde{e}^{\varepsilon,k}\| \le C_1\varepsilon + C_1(\varepsilon,T)(h^2+\tau^2), \\ &\|\widetilde{e}^{\varepsilon,k}\|_{H^1} \le C_2\varepsilon^{1/2} + C_2(\varepsilon,T)(h^2+\tau^2), \qquad 0 \le k \le \frac{T}{\tau}, \end{aligned} \tag{3.11}$$

where C_1 *and* C_2 *are presented as in Proposition 2.5, and* $C_1(\varepsilon,T)$ *and* $C_2(\varepsilon,T)$ *are given in Theorem 3.1.*

3.2. Error estimates. Define the local truncation error $\xi_j^{\varepsilon,k} \in X_M$ for $k \ge 1$ as

$$\begin{aligned} \xi_j^{\varepsilon,k} = {} & i\delta_t^c u^\varepsilon(x_j,t_k) + \frac{1}{2}\left(\delta_x^2 u^\varepsilon(x_j,t_{k+1}) + \delta_x^2 u^\varepsilon(x_j,t_{k-1})\right) \\ & - u^\varepsilon(x_j,t_k)\ln(\varepsilon + |u^\varepsilon(x_j,t_k)|)^2, \qquad j \in \mathcal{T}_M, \quad 1 \le k < \frac{T}{\tau}, \end{aligned} \tag{3.12}$$

then we have the following bounds for the local truncation error.

Lemma 3.3 (Local truncation error). *Under Assumption (A), we have*

$$\|\xi^{\varepsilon,k}\|_{H^1} \lesssim h^2 + \tau^2, \quad 1 \le k < \frac{T}{\tau}.$$

Proof. By Taylor expansion, we have

$$\xi_j^{\varepsilon,k} = \frac{i\tau^2}{4}\alpha_j^{\varepsilon,k} + \frac{\tau^2}{2}\beta_j^{\varepsilon,k} + \frac{h^2}{12}\gamma_j^{\varepsilon,k}, \tag{3.13}$$

where

$$\alpha_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)^2\partial_t^3 u^\varepsilon(x_j, t_k + s\tau)ds, \quad \beta_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)\partial_t^2 u_{xx}^\varepsilon(x_j, t_k + s\tau)ds,$$
$$\gamma_j^{\varepsilon,k} = \int_{-1}^{1}(1-|s|)^3\left(\partial_x^4 u^\varepsilon(x_j + sh, t_{k+1}) + \partial_x^4 u^\varepsilon(x_j + sh, t_{k-1})\right)ds.$$

By the Cauchy-Schwarz inequality, we can get that

$$\begin{aligned}
\|\alpha^{\varepsilon,k}\|^2 &= h\sum_{j=1}^{M-1}|\alpha_j^{\varepsilon,k}|^2 \le h\int_{-1}^{1}(1-|s|)^4 ds \sum_{j=1}^{M-1}\int_{-1}^{1}\left|\partial_t^3 u^\varepsilon(x_j, t_k + s\tau)\right|^2 ds \\
&= \frac{2}{5}\Big[\int_{-1}^{1}\|\partial_t^3 u^\varepsilon(\cdot, t_k + s\tau)\|_{L^2(\Omega)}^2 ds \\
&\quad - \int_{-1}^{1}\sum_{j=0}^{M-1}\int_{x_j}^{x_{j+1}}(|\partial_t^3 u^\varepsilon(x, t_k + s\tau)|^2 - |\partial_t^3 u^\varepsilon(x_j, t_k + s\tau)|^2)dxds\Big] \\
&= \frac{2}{5}\Big[\int_{-1}^{1}\|\partial_t^3 u^\varepsilon(\cdot, t_k + s\tau)\|_{L^2(\Omega)}^2 ds \\
&\quad - \int_{-1}^{1}\sum_{j=0}^{M-1}\int_{x_j}^{x_{j+1}}\int_{x_j}^{\omega}\partial_x|\partial_t^3 u^\varepsilon(x', t_k + s\tau)|^2 dx' d\omega ds\Big] \\
&\le \frac{2}{5}\int_{-1}^{1}\Big[\|\partial_t^3 u^\varepsilon(\cdot, t_k + s\tau)\|_{L^2(\Omega)}^2 \\
&\qquad + 2h\|\partial_t^3 u_x^\varepsilon(\cdot, t_k + s\tau)\|_{L^2(\Omega)}\|\partial_t^3 u^\varepsilon(\cdot, t_k + s\tau)\|_{L^2(\Omega)}\Big]ds \\
&\le \max_{0\le t\le T}\left(\|\partial_t^3 u^\varepsilon\|_{L^2(\Omega)} + h\|\partial_t^3 u_x^\varepsilon\|_{L^2(\Omega)}\right)^2,
\end{aligned}$$

which yields that when $h \le 1$,

$$\|\alpha^{\varepsilon,k}\| \le \|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}.$$

Applying the similar approach, it can be established that

$$\|\beta^{\varepsilon,k}\| \le 2\|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))}.$$

On the other hand, we can obtain that

$$\begin{aligned}
\|\gamma^{\varepsilon,k}\|^2 &\le h\int_{-1}^{1}(1-|s|)^6 ds \sum_{j=1}^{M-1}\int_{-1}^{1}\left|\partial_x^4 u^\varepsilon(x_j + sh, t_{k+1}) + \partial_x^4 u^\varepsilon(x_j + sh, t_{k-1})\right|^2 ds \\
&\le \frac{4h}{7}\sum_{j=1}^{M-1}\int_{-1}^{1}\left(\left|\partial_x^4 u^\varepsilon(x_j + sh, t_{k+1})\right|^2 + \left|\partial_x^4 u^\varepsilon(x_j + sh, t_{k-1})\right|^2\right)ds \\
&\le \frac{8}{7}\left(\|\partial_x^4 u^\varepsilon(\cdot, t_{k-1})\|_{L^2(\Omega)}^2 + \|\partial_x^4 u^\varepsilon(\cdot, t_{k+1})\|_{L^2(\Omega)}^2\right) \\
&\le 4\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}^2,
\end{aligned}$$

which implies that $\|\gamma^{\varepsilon,k}\| \le 2\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}$. Hence by Assumption (A), we get

$$\begin{aligned}\|\xi^{\varepsilon,k}\| &\lesssim \tau^2 \left(\|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))}\right) + h^2\|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}\\ &\lesssim_{C_0} \tau^2 + h^2.\end{aligned}$$

Applying δ_x^+ to $\xi^{\varepsilon,k}$ and using the same approach, we can get that

$$\begin{aligned}|\xi^{\varepsilon,k}|_{H^1} &\lesssim \tau^2 \left(\|\partial_t^3 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}\right) + h^2\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))}\\ &\lesssim_{C_0} \tau^2 + h^2,\end{aligned}$$

which completes the proof. □

For the first step, we have the following estimates.

LEMMA 3.4 (Error bounds for $k = 1$). *Under Assumption (A), the first step errors of the discretization* (3.6) *satisfy*

$$e^{\varepsilon,0} = 0, \quad \|e^{\varepsilon,1}\|_{H^1} \lesssim \tau^2.$$

Proof. By the definition of $u_j^{\varepsilon,1}$ in (3.6), we have

$$e_j^{\varepsilon,1} = \tau^2 \int_0^1 (1-s) u_{tt}^\varepsilon(x_j, s\tau) ds,$$

which implies that

$$\|e^{\varepsilon,1}\| \lesssim \tau^2 \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \lesssim \tau^2, \quad |e^{\varepsilon,1}|_{H^1} \lesssim \tau^2 \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \lesssim \tau^2,$$

and the proof is completed. □

Proof. [Proof of Theorem 3.1] We prove (3.9) by induction. It follows from Lemma 3.4 that (3.9) is true for $k = 0, 1$.

Assume (3.9) is valid for $k \le n \le \frac{T}{\tau} - 1$. Next we need to show that (3.9) still holds for $k = n + 1$. Subtracting (3.4) from (3.12), we get the error equations

$$i\delta_t^c e_j^{\varepsilon,m} = -\frac{1}{2}(\delta_x^2 e_j^{\varepsilon,m+1} + \delta_x^2 e_j^{\varepsilon,m-1}) + r_j^{\varepsilon,m} + \xi_j^{\varepsilon,m}, \quad j \in \mathcal{T}_M, \quad 1 \le m \le \frac{T}{\tau} - 1, \tag{3.14}$$

where $r^{\varepsilon,m} \in X_M$ represents the difference between the logarithmic nonlinearity

$$r_j^{\varepsilon,m} = u^\varepsilon(x_j, t_m)\ln(\varepsilon + |u^\varepsilon(x_j, t_m)|)^2 - u_j^{\varepsilon,m}\ln(\varepsilon + |u_j^{\varepsilon,m}|)^2, \quad 1 \le m \le \frac{T}{\tau} - 1. \tag{3.15}$$

Multiplying both sides of (3.14) by $2\tau\,(\overline{e_j^{\varepsilon,m+1} + e_j^{\varepsilon,m-1}})$, summing together for $j \in \mathcal{T}_M$ and taking the imaginary parts, we obtain for $1 \le m < T/\tau$,

$$\begin{aligned}\|e^{\varepsilon,m+1}\|^2 - \|e^{\varepsilon,m-1}\|^2 &= 2\tau\,\mathrm{Im}(r^{\varepsilon,m} + \xi^{\varepsilon,m}, e^{\varepsilon,m+1} + e^{\varepsilon,m-1})\\ &\le 2\tau\left(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2 + \|e^{\varepsilon,m-1}\|^2\right).\end{aligned} \tag{3.16}$$

Summing (3.16) for $m = 1, 2, \cdots, n$ ($n \le \frac{T}{\tau} - 1$), we obtain

$$\begin{aligned}\|e^{\varepsilon,n+1}\|^2 + \|e^{\varepsilon,n}\|^2 \le \|e^{\varepsilon,0}\|^2 + \|e^{\varepsilon,1}\|^2 + 2\tau\|e^{\varepsilon,n+1}\|^2 + 2\tau\sum_{m=0}^{n-1}(\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2)\\ + 2\tau\sum_{m=1}^{n}\left(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2\right).\end{aligned} \tag{3.17}$$

For $m \le n$, when $|u_j^{\varepsilon,m}| \le |u^\varepsilon(x_j,t_m)|$, we write $r_j^{\varepsilon,m}$ as

$$\begin{aligned}
|r_j^{\varepsilon,m}| &= \Big| e_j^{\varepsilon,m} \ln(\varepsilon + |u^\varepsilon(x_j,t_m)|)^2 + 2u_j^{\varepsilon,m} \ln\Big(\frac{\varepsilon + |u^\varepsilon(x_j,t_m)|}{\varepsilon + |u_j^{\varepsilon,m}|}\Big)\Big| \\
&\le 2\max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon+\Lambda)|\}|e_j^{\varepsilon,m}| + 2|u_j^{\varepsilon,m}| \ln\Big(1 + \frac{|u^\varepsilon(x_j,t_m)| - |u_j^{\varepsilon,m}|}{\varepsilon + |u_j^{\varepsilon,m}|}\Big) \\
&\le 2|e_j^{\varepsilon,m}|(1 + \max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon+\Lambda)|\}).
\end{aligned}$$

On the other hand, when $|u^\varepsilon(x_j,t_m)| \le |u_j^{\varepsilon,m}|$, we write $r_j^{\varepsilon,m}$ as

$$\begin{aligned}
|r_j^{\varepsilon,m}| &= \Big| e_j^{\varepsilon,m} \ln(\varepsilon + |u_j^{\varepsilon,m}|)^2 + 2u^\varepsilon(x_j,t_m) \ln\Big(\frac{\varepsilon + |u^\varepsilon(x_j,t_m)|}{\varepsilon + |u_j^{\varepsilon,m}|}\Big)\Big| \\
&\le 2\max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon+1+\Lambda)|\}|e_j^{\varepsilon,m}| \\
&\quad + 2|u^\varepsilon(x_j,t_m)| \ln\Big(1 + \frac{|u_j^{\varepsilon,m}| - |u^\varepsilon(x_j,t_m)|}{\varepsilon + |u^\varepsilon(x_j,t_m)|}\Big) \\
&\le 2|e_j^{\varepsilon,m}|(1 + \max\{\ln(\varepsilon^{-1}), |\ln(\varepsilon+1+\Lambda)|\}),
\end{aligned}$$

where we use the assumption that $\|u^{\varepsilon,m}\|_\infty \le \Lambda + 1$ for $m \le n$. Thus it follows that

$$\|r^{\varepsilon,m}\|^2 \lesssim |\ln(\varepsilon)|^2 \|e^{\varepsilon,m}\|^2,$$

when ε is sufficiently small. Thus when $\tau \le \frac{1}{2}$, by using Lemmas 3.3, 3.4 and (3.17), we have

$$\begin{aligned}
\|e^{\varepsilon,n+1}\|^2 + \|e^{\varepsilon,n}\|^2 &\lesssim \|e^{\varepsilon,0}\|^2 + \|e^{\varepsilon,1}\|^2 + \tau \sum_{m=0}^{n-1} (\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2) \\
&\quad + \tau \sum_{m=1}^{n} \big(\|r^{\varepsilon,m}\|^2 + \|\xi^{\varepsilon,m}\|^2\big) \\
&\lesssim (h^2+\tau^2)^2 + \tau |\ln(\varepsilon)|^2 \sum_{m=0}^{n-1} (\|e^{\varepsilon,m}\|^2 + \|e^{\varepsilon,m+1}\|^2).
\end{aligned}$$

We emphasize here that the implicit multiplicative constant in this inequality depends only on C_0, but not on n. Applying the discrete Gronwall inequality, we can conclude that

$$\|e^{\varepsilon,n+1}\|^2 \lesssim e^{CT|\ln(\varepsilon)|^2}(h^2+\tau^2)^2,$$

for some C depending on C_0, which gives the error bound for $\|e^{\varepsilon,k}\|$ with $k = n+1$ in (3.9) immediately.

To estimate $|e^{\varepsilon,n+1}|_{H^1}$, multiplying both sides of (3.14) by $2(\overline{e_j^{\varepsilon,m+1} - e_j^{\varepsilon,m-1}})$ for $m \le n$, summing together for $j \in \mathcal{T}_M$ and taking the real parts, we obtain

$$\begin{aligned}
&|e^{\varepsilon,m+1}|_{H^1}^2 - |e^{\varepsilon,m-1}|_{H^1}^2 \\
&\quad = -2\,\mathrm{Re}\big(r^{\varepsilon,m} + \xi^{\varepsilon,m}, e^{\varepsilon,m+1} - e^{\varepsilon,m-1}\big) \\
&\quad = 2\tau\,\mathrm{Im}\big(r^{\varepsilon,m} + \xi^{\varepsilon,m}, -\delta_x^2(e^{\varepsilon,m+1} + e^{\varepsilon,m-1})\big) \\
&\quad = 2\tau\,\mathrm{Im}\big\langle \delta_x^+(r^{\varepsilon,m} + \xi^{\varepsilon,m}), \delta_x^+(e^{\varepsilon,m+1} + e^{\varepsilon,m-1})\big\rangle \\
&\quad \le 2\tau \left(|r^{\varepsilon,m}|_{H^1}^2 + |\xi^{\varepsilon,m}|_{H^1}^2 + |e^{\varepsilon,m+1}|_{H^1}^2 + |e^{\varepsilon,m-1}|_{H^1}^2\right). \qquad (3.18)
\end{aligned}$$

To give the bound for $\delta_x^+ r^{\varepsilon,m}$, for simplicity of notation, denote

$$u_{j,\theta}^{\varepsilon,m} = \theta u^\varepsilon(x_{j+1},t_m) + (1-\theta)u^\varepsilon(x_j,t_m), \quad v_{j,\theta}^{\varepsilon,m} = \theta v_{j+1}^{\varepsilon,m} + (1-\theta)v_j^{\varepsilon,m},$$

for $j \in \mathcal{T}_M$ and $\theta \in [0,1]$. Then we have

$$\begin{aligned}
\delta_x^+ r_j^{\varepsilon,m} &= 2\delta_x^+(u^\varepsilon(x_j,t_m)\ln(\varepsilon+|u^\varepsilon(x_j,t_m)|)) - 2\delta_x^+(u_j^{\varepsilon,m}\ln(\varepsilon+|u_j^{\varepsilon,m}|)) \\
&= \frac{2}{h}\int_0^1 [u_{j,\theta}^{\varepsilon,m}\ln(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)]'(\theta)d\theta - \frac{2}{h}\int_0^1 [v_{j,\theta}^{\varepsilon,m}\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)]'(\theta)d\theta \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= 2\delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \ln(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)d\theta - 2\delta_x^+ u_j^{\varepsilon,m}\int_0^1 \ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)d\theta, \\
I_2 &:= \delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \frac{|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|}d\theta - \delta_x^+ u_j^{\varepsilon,m}\int_0^1 \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}d\theta, \\
I_3 &:= \delta_x^+ \overline{u^\varepsilon(x_j,t_m)}\int_0^1 \frac{(u_{j,\theta}^{\varepsilon,m})^2}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)}d\theta - \delta_x^+ \overline{u_j^{\varepsilon,m}}\int_0^1 \frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}d\theta.
\end{aligned}$$

Then we estimate I_1, I_2 and I_3, separately. Similar as before, we have

$$\begin{aligned}
|I_1| &\le 2|\delta_x^+ u^\varepsilon(x_j,t_m)|\int_0^1 \Big|\ln\Big(\frac{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}\Big)\Big|d\theta + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right|\int_0^1 \Big|\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)\Big|d\theta \\
&= 2|\delta_x^+ u^\varepsilon(x_j,t_m)|\int_0^1 \ln\Big(1+\frac{\Big||u_{j,\theta}^{\varepsilon,m}|-|v_{j,\theta}^{\varepsilon,m}|\Big|}{\varepsilon+\min\{|u_{j,\theta}^{\varepsilon,m}|,|v_{j,\theta}^{\varepsilon,m}|\}}\Big)d\theta \\
&\quad + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right|\int_0^1 \Big|\ln(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)\Big|d\theta \\
&\le \frac{2}{\varepsilon}|\delta_x^+ u^\varepsilon(x_j,t_m)|\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right) + 2\left|\delta_x^+ e_j^{\varepsilon,m}\right|\max\{\ln(\varepsilon^{-1}),|\ln(\varepsilon+1+\Lambda)|\} \\
&\lesssim \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right) + \ln(\varepsilon^{-1})\left|\delta_x^+ e_j^{\varepsilon,m}\right|,
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &= \Big|\delta_x^+ u^\varepsilon(x_j,t_m)\int_0^1 \Big(\frac{|u_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|u_{j,\theta}^{\varepsilon,m}|} - \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_{j,\theta}^{\varepsilon,m}|}\Big)d\theta + \delta_x^+ e_j^{\varepsilon,m}\int_0^1 \frac{|v_{j,\theta}^{\varepsilon,m}|}{\varepsilon+|v_j^{\varepsilon,m}|}d\theta\Big| \\
&\le |\delta_x^+ e_j^{\varepsilon,m}| + |\delta_x^+ u^\varepsilon(x_j,t_m)|\int_0^1 \frac{\varepsilon|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}d\theta \\
&\le |\delta_x^+ e_j^{\varepsilon,m}| + \frac{|\delta_x^+ u^\varepsilon(x_j,t_m)|}{\varepsilon}\int_0^1 |u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|d\theta \\
&\lesssim |\delta_x^+ e_j^{\varepsilon,m}| + \frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right).
\end{aligned}$$

In view of the inequality that

$$\begin{aligned}
&\Big|\frac{(u_{j,\theta}^{\varepsilon,m})^2}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)}-\frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\Big|\\
&=\Big|\frac{(u_{j,\theta}^{\varepsilon,m})^2-u_{j,\theta}^{\varepsilon,m}v_{j,\theta}^{\varepsilon,m}}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)}+\frac{u_{j,\theta}^{\varepsilon,m}v_{j,\theta}^{\varepsilon,m}}{|u_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)}-\frac{(v_{j,\theta}^{\varepsilon,m})^2}{|v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\Big|\\
&\le\frac{|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{\varepsilon}+\frac{\Big|u_{j,\theta}^{\varepsilon,m}(v_{j,\theta}^{\varepsilon,m})^2(\overline{u_{j,\theta}^{\varepsilon,m}}-\overline{v_{j,\theta}^{\varepsilon,m}})+\varepsilon v_{j,\theta}^{\varepsilon,m}(u_{j,\theta}^{\varepsilon,m}|v_{j,\theta}^{\varepsilon,m}|-|u_{j,\theta}^{\varepsilon,m}|v_{j,\theta}^{\varepsilon,m})\Big|}{|u_{j,\theta}^{\varepsilon,m}||v_{j,\theta}^{\varepsilon,m}|(\varepsilon+|u_{j,\theta}^{\varepsilon,m}|)(\varepsilon+|v_{j,\theta}^{\varepsilon,m}|)}\\
&\le\frac{4|u_{j,\theta}^{\varepsilon,m}-v_{j,\theta}^{\varepsilon,m}|}{\varepsilon},
\end{aligned}$$

we can obtain that

$$I_3\lesssim|\delta_x^+e_j^{\varepsilon,m}|+\frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right).$$

Thus we can conclude that

$$|\delta_x^+r_j^{\varepsilon,m}|\lesssim\frac{1}{\varepsilon}\left(|e_j^{\varepsilon,m}|+|e_{j+1}^{\varepsilon,m}|\right)+\ln(\varepsilon^{-1})\left|\delta_x^+e_j^{\varepsilon,m}\right|.$$

Summing (3.18) for $m=1,2,\cdots,n$ ($n\le\frac{T}{\tau}-1$), we obtain

$$\begin{aligned}
|e^{\varepsilon,n+1}|_{H^1}^2+|e^{\varepsilon,n}|_{H^1}^2\le|e^{\varepsilon,0}|_{H^1}^2+|e^{\varepsilon,1}|_{H^1}^2+\tau\sum_{m=1}^{n}\left(|r^{\varepsilon,m}|_{H^1}^2+|\xi^{\varepsilon,m}|_{H^1}^2\right)\\
+\tau|e^{\varepsilon,n+1}|_{H^1}^2+\tau\sum_{m=0}^{n-1}(|e^{\varepsilon,m}|_{H^1}^2+|e^{\varepsilon,m+1}|_{H^1}^2).
\end{aligned}$$

Thus when $\tau\le1/2$, by using Lemmas 3.3 and 3.4, we have

$$\begin{aligned}
|e^{\varepsilon,n+1}|_{H^1}^2+|e^{\varepsilon,n}|_{H^1}^2&\lesssim|e^{\varepsilon,0}|_{H^1}^2+|e^{\varepsilon,1}|_{H^1}^2+\tau\sum_{m=1}^{n}\left(\frac{1}{\varepsilon^2}|e^{\varepsilon,m}|_{H^1}^2+|\xi^{\varepsilon,m}|_{H^1}^2\right)\\
&\quad+\tau|\ln(\varepsilon)|^2\sum_{m=0}^{n-1}\left(|e^{\varepsilon,m}|_{H^1}^2+|e^{\varepsilon,m+1}|_{H^1}^2\right)\\
&\lesssim\frac{e^{CT|\ln(\varepsilon)|^2}}{\varepsilon^2}(h^2+\tau^2)^2+\tau|\ln(\varepsilon)|^2\sum_{m=0}^{n-1}(|e^{\varepsilon,m}|_{H^1}^2+|e^{\varepsilon,m+1}|_{H^1}^2).
\end{aligned}$$

Applying the discrete Gronwall's inequality, we can get that

$$|e^{\varepsilon,n+1}|_{H^1}^2\lesssim e^{CT|\ln(\varepsilon)|^2}(h^2+\tau^2)^2/\varepsilon^2,$$

which establishes the error estimate for $\|e^{\varepsilon,k}\|_{H^1}$ for $k=n+1$. Finally the boundedness for the solution $u^{\varepsilon,k}$ can be obtained by the triangle inequality

$$\|u^{\varepsilon,k}\|_\infty\le\|u^\varepsilon(\cdot,t_k)\|_{L^\infty(\Omega)}+\|e^{\varepsilon,k}\|_\infty,$$

and the inverse Sobolev inequality [25]

$$\|e^{\varepsilon,k}\|_\infty\lesssim\|e^{\varepsilon,k}\|_{H^1},$$

which completes the proof of Theorem 3.1. □

4. Numerical results. In this section, we test the convergence rate of the regularized model (2.1) and the SIFD (3.4). To this end, we take $d = 1$, $\Omega = \mathbb{R}$ and $\lambda = -1$ in the LogSE (1.1) and consider two different initial data:

Case I: A Gaussian initial data, i.e. u_0 in (1.1) is chosen as

$$u_0(x) = \sqrt[4]{-\lambda/\pi}\, e^{ivx+\frac{\lambda}{2}x^2}, \qquad x \in \mathbb{R}, \tag{4.1}$$

with $v = 1$. In this case, the LogSE (1.1) admits the moving Gausson solution (1.7) with $v = 1$ and $b_0 = \sqrt[4]{-\lambda/\pi}$ as the exact solution.

Case II: A general initial data, i.e. u_0 in (1.1) is chosen as

$$u_0(x) = \tanh(x)e^{-x^2}, \qquad x \in \mathbb{R}, \tag{4.2}$$

which is the multiplication of a dark soliton of the cubic nonlinear Schrödinger equation and a Gaussian. Notice that in this case, the logarithmic term $\ln|u_0|^2$ is singular at $x = 0$.

The RLogSE (2.1) is solved numerically by the SIFD (3.4) on domains $\Omega = [-12, 12]$ and $\Omega = [-16, 16]$ for Case I and II, respectively. To quantify the numerical errors, we introduce the following error functions:

$$\begin{aligned} &\widehat{e}^{\varepsilon}(t_k) := u(\cdot, t_k) - u^{\varepsilon}(\cdot, t_k), \qquad && e^{\varepsilon}(t_k) := u^{\varepsilon}(\cdot, t_k) - u^{\varepsilon,k}, \\ &\widetilde{e}^{\varepsilon}(t_k) := u(\cdot, t_k) - u^{\varepsilon,k}, && e^{\varepsilon}_E := |E(u) - E^{\varepsilon}(u^{\varepsilon})|. \end{aligned} \tag{4.3}$$

Here u and u^{ε} are the exact solutions of the LogSE (1.1) and RLogSE (2.1), respectively, while $u^{\varepsilon,k}$ is the numerical solution of the RLogSE (2.1) obtained by the SIFD (3.4). The 'exact' solution u^{ε} is obtained numerically by the SIFD (3.4) with a very small time step, e.g. $\tau = 0.01/2^9$ and a very fine mesh size, e.g. $h = 1/2^{15}$. Similarly, the 'exact' solution u in Case II is obtained numerically by the SIFD (3.4) with a very small time step and a very fine mesh size as well as a very small regularization parameter ε, e.g. $\varepsilon = 10^{-14}$. The energy is obtained by the trapezoidal rule for approximating the integrals in the energy (1.2) and (2.2).

4.1. Convergence rate of the regularized model. Here we consider the error between the solutions of the RLogSE (2.1) and the LogSE (1.1). Fig. 4.1 shows $\|\widehat{e}^{\varepsilon}\|$, $\|\widehat{e}^{\varepsilon}\|_{H^1}$, $\|\widehat{e}^{\varepsilon}\|_{\infty}$ (the definition of the norms is given in (3.2)) at time $t = 0.5$ for Cases I & II, while Fig. 4.2 depicts $e^{\varepsilon}_E(0.5)$ for Cases I & II and time evolution of $\widehat{e}^{\varepsilon}(t)$ with different ε for Case I. For comparison, similar to Fig. 4.1, Fig. 4.3 displays the convergent results from (2.8) to (1.1).

From Figs. 4.1, 4.2 & 4.3 and additional numerical results not shown here for brevity, we can draw the following conclusions: (i) The solution of the RLogSE (2.1) converges linearly to that of the LogSE (1.1) in terms of the regularization parameter ε in both L^2-norm and L^{∞}-norm, and respectively, the convergence rate becomes $O(\sqrt{\varepsilon})$ in H^1-norm for Case II. (ii) The regularized energy $E^{\varepsilon}(u^{\varepsilon})$ converges linearly to the energy $E(u)$ in terms of ε. (iii) The constant C in (2.6) may grow linearly with time T and it is independent of ε. (iv) The solution of (2.8) converges at $O(\sqrt{\varepsilon})$ to that of (1.1) in both L^2-norm and L^{∞}-norm, and respectively, the convergence rate becomes $O(\varepsilon^{1/4})$ in H^1-norm for Case II. Thus (2.1) is much more accurate than (2.8) for the regularization of the LogSE (1.1). (v) The numerical results agree and confirm our analytical results in Section 2.

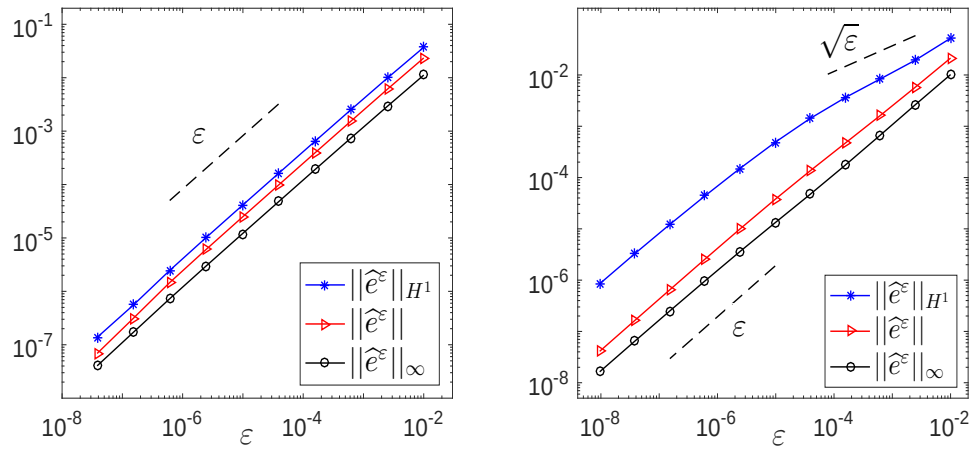


Fig. 4.1: Convergence of the RLogSE (2.1) to the LogSE (1.1), i.e. the error $\widehat{e}^{\varepsilon}(0.5)$ in different norms vs the regularization parameter ε for Case I (left) and Case II (right).

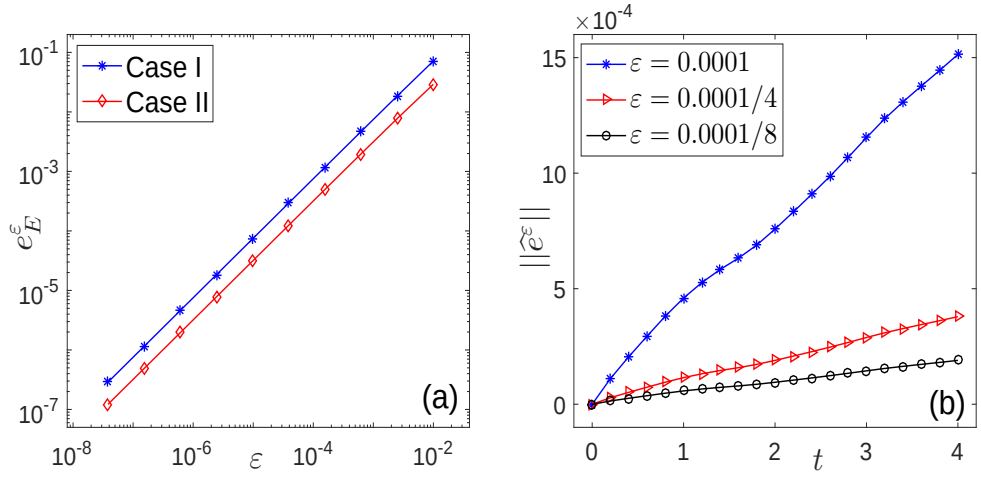


Fig. 4.2: Convergence of the RLogSE (2.1) to the LogSE (1.1): (a) error in energy $e_{\rm E}^{\varepsilon}(0.5)$ vs ε for Cases I & II, and (b) time evolution of $\|\widehat{e}^{\varepsilon}(t)\|$ vs time t under different ε for Case I.

4.2. Convergence rate of the finite difference method. Here we test the convergence rate of the SIFD (3.4) to the RLogSE (2.1) or the LogSE (1.1) in terms of mesh size h and time step τ under any fixed $0 < \varepsilon \ll 1$ for Case I. Fig. 4.4 shows the errors $\|e^{\varepsilon}(0.5)\|$ vs time step τ (with a fixed ratio between mesh size h and time step τ at $h = 75\tau/64$) under different ε. In addition, Table 4.1 displays $\|\widetilde{e}^{\varepsilon}(1)\|$ for varying ε and τ & h.

From Fig. 4.4, we can see that the SIFD (3.4) converges quadratically at $O(\tau^2 + h^2)$ to the RLogSE (2.1) for any fixed $\varepsilon > 0$, which confirms our error estimates in

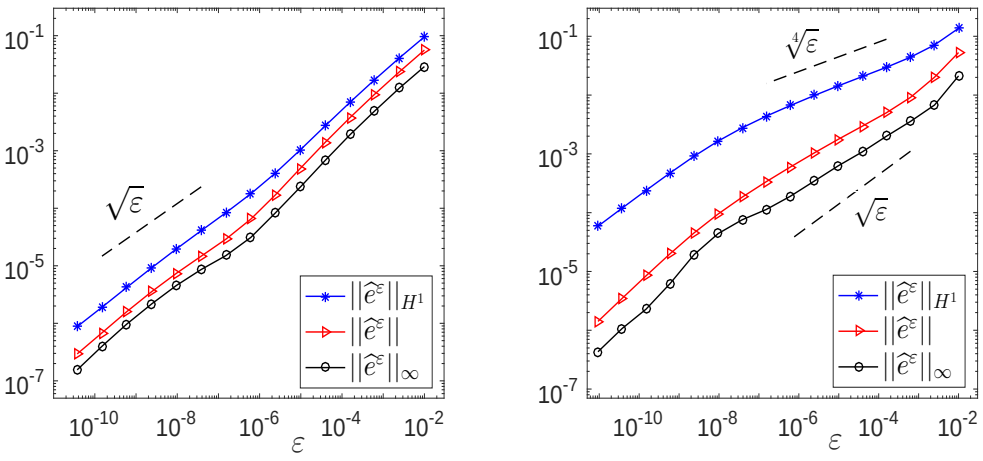


Fig. 4.3: Convergence of the RLogSE (2.8) to the LogSE (1.1), i.e. the error $\widehat{e}^{\varepsilon}(0.5)$ in different norms vs the regularization parameter ε for Case I (left) and Case II (right).

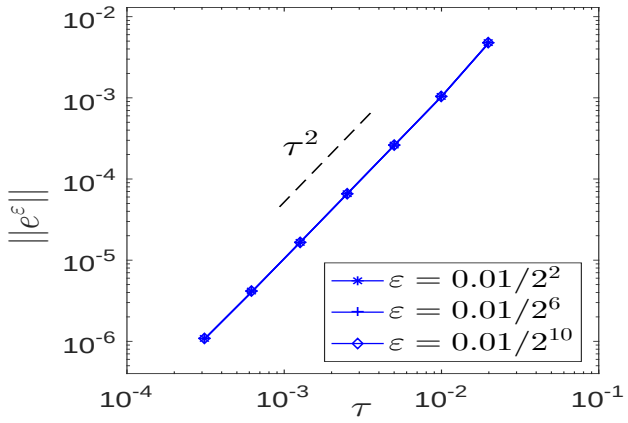


Fig. 4.4: Convergence of the SIFD (3.4) to the RLogSE (2.1), i.e. errors $\|e^{\varepsilon}(0.5)\|$ vs τ (with $h = 75\tau/64$) under different ε for Case I initial data.

Theorem 3.1. From Tab. 4.1, we can observe that: (i) the SIFD (3.4) converges quadratically at $O(\tau^2 + h^2)$ to the LogSE (1.1) only when ε is sufficiently small, e.g. $\varepsilon \lesssim h^2$ and $\varepsilon \lesssim \tau^2$ (cf. lower triangle below the diagonal in bold letter in Tab. 4.1), and (ii) when τ & h is sufficiently small, i.e., $\tau^2 \lesssim \varepsilon$ & $h^2 \lesssim \varepsilon$, the RLogSE (2.1) converge linearly at $O(\varepsilon)$ to the LogSE (1.1) (cf. each column in the right most of Table 4.1), which confirms the error bounds in Corollary 3.2.

5. Conclusion. In order to overcome the singularity of the log-nonlinearity in the logarithmic Schrödinger equation (LogSE), we proposed a regularized logarithmic Schrödinger equation (RLogSE) with a regularization parameter $0 < \varepsilon \ll 1$ and established linear convergence between RLogSE and LogSE in terms of the small regularization parameter. Then we presented a semi-implicit finite difference method

Table 4.1: Convergence of the SIFD (3.4) to the LogSE (1.1), i.e. $\|\widetilde{e}^{\varepsilon}(1)\|$ for different ε and τ & h for Case I.

	$h=0.1$	$h/2$	$h/2^2$	$h/2^3$	$h/2^4$	$h/2^5$	$h/2^6$	$h/2^7$	$h/2^8$	$h/2^9$
	$\tau=0.1$	$\tau/2$	$\tau/2^2$	$\tau/2^3$	$\tau/2^4$	$\tau/2^5$	$\tau/2^6$	$\tau/2^7$	$\tau/2^8$	$\tau/2^9$
ε=0.001	1.84E-1	**4.84E-2**	1.34E-2	5.96E-3	4.79E-3	4.62E-3	4.58E-3	4.57E-3	4.57E-3	4.57E-3
rate	–	**1.93**	1.85	1.17	0.31	0.05	0.01	0.00	0.00	0.00
$\varepsilon/4$	1.84E-1	4.75E-2	**1.19E-2**	3.36E-3	1.49E-3	1.20E-3	1.16E-3	1.15E-3	1.15E-3	1.15E-3
rate	–	1.96	**1.99**	1.83	1.17	0.31	0.05	0.01	0.00	0.00
$\varepsilon/4^2$	1.84E-1	4.73E-2	1.17E-2	**2.97E-3**	8.39E-4	3.74E-4	3.01E-4	2.90E-4	2.88E-4	2.88E-4
rate	–	1.96	2.01	**1.98**	1.83	1.17	0.31	0.05	0.01	0.00
$\varepsilon/4^3$	1.84E-1	4.72E-2	1.16E-2	2.91E-3	**7.43E-4**	2.10E-4	9.35E-5	7.54E-5	7.27E-5	7.21E-5
rate	–	1.96	2.02	2.00	**1.97**	1.83	1.16	0.31	0.05	0.01
$\varepsilon/4^4$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.27E-4	**1.86E-4**	5.24E-5	2.34E-5	1.89E-5	1.82E-5
rate	–	1.96	2.02	2.00	2.00	**1.97**	1.83	1.16	0.31	0.05
$\varepsilon/4^5$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.24E-4	1.82E-4	**4.64E-5**	1.31E-5	5.85E-6	4.72E-6
rate	–	1.96	2.02	2.01	2.00	1.99	**1.97**	1.83	1.16	0.31
$\varepsilon_0/4^6$	1.84E-1	4.72E-2	1.16E-2	2.90E-3	7.23E-4	1.81E-4	4.54E-5	**1.16E-5**	3.28E-6	1.47E-6
rate	–	1.96	2.02	2.01	2.00	2.00	1.99	**1.97**	1.83	1.16
$\varepsilon_0/4^7$	1.84E-1	4.72E-2	1.16E-2	2.89E-3	7.23E-4	1.81E-4	4.52E-5	1.14E-5	**2.90E-6**	8.22E-7
rate	–	1.96	2.02	2.01	2.00	2.00	2.00	2.00	**1.97**	1.82

for discretizing RLogSE and proved second-order convergence rates in terms of mesh size h and time step τ. Finally, we established error bounds of the semi-implicit finite difference method to LogSE, which depend explicitly on the mesh size h and time step τ as well as the small regularization parameter ε. Our numerical results confirmed our error bounds and demonstrated that they are sharp.

REFERENCES

[1] G. D. Akrivis, V. A. Dougalis, and O. A. Karakashian, *On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation*, Numer. Math., 59 (1991), pp. 31–53.

[2] A. H. Ardila, *Orbital stability of Gausson solutions to logarithmic Schrödinger equations*, Electron. J. Differ. Eq., 335 (2016), pp. 1–9.

[3] A. V. Avdeenkov and K. G. Zloshchastiev, *Quantum bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent*, J. Phys. B: Atomic, Molecular Optical Phys., 44 (2011), p. 195303.

[4] W. Bao and Y. Cai, *Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator*, SIAM J. Numer. Anal., 50 (2012), pp. 492–521.

[5] ———, *Optimal error estimates of finite difference methods for the Gross-Pitaevskii equation with angular momentum rotation*, Math. Comput., 82 (2013), pp. 99–128.

[6] W. Bao, D. Jaksch, and P. A. Markowich, *Numerical solution of the Gross–Pitaevskii equation for Bose–Einstein condensation*, J. Comput. Phys., 187 (2003), pp. 318–342.

[7] I. Białynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Ann. Physics, 100 (1976), pp. 62–93.

[8] ———, *Gaussons: Solitons of the logarithmic Schrödinger equation*, Special issue on solitons in physics, Phys. Scripta, 20 (1979), pp. 539–544.

[9] H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev, and D. Christodoulides, *Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media*, Phys. Rev. E, 68 (2003), p. 036607.

[10] R. Carles and I. Gallagher, *Universal dynamics for the defocusing logarithmic Schrödinger equation*, Duke Math. J., (2017). To appear. Archived as

https://hal.archives-ouvertes.fr/hal-01398526.

[11] T. Cazenave, *Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal., 7 (1983), pp. 1127–1140.

[12] ———, *Semilinear Schrödinger equations*, vol. 10 of Courant Lecture Notes in Mathematics, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[13] T. Cazenave and A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. (5), 2 (1980), pp. 21–51.

[14] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., 85 (1982), pp. 549–561.

[15] P. Guerrero, J. L. López, and J. Nieto, *Global H^1 solvability of the 3D logarithmic Schrödinger equation*, Nonlinear Anal. Real World Appl., 11 (2010), pp. 79–87.

[16] T. Hansson, D. Anderson, and M. Lisak, *Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis*, Phys. Rev. A, 80 (2009), p. 033819.

[17] E. F. Hefter, *Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics*, Phys. Rev. A, 32 (1985), pp. 1201–1204.

[18] E. S. Hernandez and B. Remaud, *General properties of Gausson-conserving descriptions of quantal damped motion*, Physica A, 105 (1980), pp. 130–146.

[19] O. Karakashian and C. Makridakis, *A space-time finite element method for the nonlinear Schrödinger equation: the continuous Galerkin method*, SIAM J. Numer. Anal., 36 (1999), pp. 1779–1807.

[20] W. Krolikowski, D. Edmundson, and O. Bang, *Unified model for partially coherent solitons in logaritmically nonlinear media*, Phys. Rev. E, 61 (2000), pp. 3122–3126.

[21] G. Leoni, *A first course in Sobolev spaces*, American Mathematical Soc., 2017.

[22] S. D. Martino, M. Falanga, C. Godano, and G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport*, Europhys. Lett., 63 (2003), pp. 472–475.

[23] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3), 13 (1959), pp. 115–162.

[24] T. R. Taha and M. I. Ablowitz, *Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, nonlinear Schrödinger equation*, J. Comput. Phys., 55 (1984), pp. 203–230.

[25] V. Thomée, *Galerkin finite element methods for parabolic problems*, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1997.

[26] K. Yasue, *Quantum mechanics of nonconservative systems*, Annals Phys., 114 (1978), pp. 479–496.

[27] K. G. Zloshchastiev, *Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences*, Grav. Cosmol., 16 (2010), pp. 288–297.