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# SOME PROPERTIES AND AN APPLICATION OF MULTIVARIATE EXPONENTIAL POLYNOMIALS

FENG QI, DA-WEI NIU, DONGKYU LIM, AND BAI-NI GUO

ABSTRACT. In the paper, the authors introduce a notion “multivariate exponential polynomials” which generalize exponential numbers and polynomials, establish explicit formulas, inversion formulas, and recurrence relations for multivariate exponential polynomials in terms of the Stirling numbers of the first and second kinds with the help of the Faà di Bruno formula, two identities for the Bell polynomials of the second kind, and the inversion theorem for the Stirling numbers of the first and second kinds, construct some determinantal inequalities and product inequalities for multivariate exponential polynomials with the aid of some properties of completely monotonic functions and other known results, derive the logarithmic convexity and logarithmic concavity for multivariate exponential polynomials, and finally find an application of multivariate exponential polynomials to white noise distribution theory by confirming that multivariate exponential polynomials satisfy conditions for sequences required in white noise distribution theory.

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*Key words and phrases*. multivariate exponential polynomial; Bell number; exponential polynomial; explicit formula; recurrence relation; inversion theorem; Bell polynomial of the second kind; Stirling number; determinantal inequality; product inequality; completely monotonic function; white noise distribution theory.

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## 1. MOTIVATIONS

In combinatorial mathematics, the Bell numbers  $B_n$  for  $n \geq 0$  count the number of ways a set with  $n$  elements can be partitioned into disjoint and nonempty subsets. These numbers have been studied by mathematicians since the 19th century, and their roots go back to medieval Japan, but they are named after Eric Temple Bell, a Scottish-born mathematician and science fiction writer, who lived in the United States for most of his life and wrote about  $B_n$  in the 1930s. The Bell numbers  $B_n$  for  $n \geq 0$  can be generated by

$$e^{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \frac{877}{5040}t^7 + \dots$$

and the first twelve Bell numbers  $B_n$  for  $0 \leq n \leq 11$  are positive integers

$$1, \quad 1, \quad 2, \quad 5, \quad 15, \quad 52, \quad 203, \quad 877, \quad 4140, \quad 21147, \quad 115975, \quad 678570.$$

For detailed information on the Bell numbers  $B_n$ , please refer to [15, 17, 20, 23, 36, 40, 49] and plenty of references therein.

As a generalization of the Bell numbers  $B_n$  for  $n \geq 0$ , the Touchard polynomials  $T_n(x)$  for  $n \geq 0$  can be generated by

$$\begin{aligned} e^{x(e^t-1)} &= \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = 1 + xt + \frac{1}{2}x(x+1)t^2 + \frac{1}{6}x(x^2+3x+1)t^3 \\ &+ \frac{1}{24}x(x^3+6x^2+7x+1)t^4 + \frac{1}{120}x(x^4+10x^3+25x^2+15x+1)t^5 + \dots \end{aligned}$$

and the first seven Touchard polynomials  $T_n(x)$  for  $0 \leq n \leq 6$  are positive integer polynomials

$$\begin{aligned} &1, \quad x, \quad x(x+1), \quad x(x^2+3x+1), \quad x(x^3+6x^2+7x+1), \\ &x(x^4+10x^3+25x^2+15x+1), \quad x(x^5+15x^4+65x^3+90x^2+31x+1). \end{aligned}$$

It is obvious that  $T_n(1) = B_n$  for  $n \geq 0$ . Occasionally the polynomials  $T_n(x)$  are also called [30] the Bell polynomials and denoted by  $B_n(x)$ . There have been researches on applications of the Touchard polynomials  $T_n(x)$  in nonlinear Fredholm-Volterra integral equations [30] and soliton theory in [27, 28, 29], including connections with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. Therefore, applications of the Touchard polynomials  $T_n(x)$  to integrable nonlinear equations are greatly expected and any amendment on multilinear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the community. For more information about the Touchard polynomials  $T_n(x)$ , please refer to [24, 44, 45, 57] and closely related references therein.

For  $k \geq 2$ , let

$$\exp_k(t) = \overbrace{\exp(\exp(\exp \cdots \exp(\exp(t)) \cdots))}^k = \sum_{n=0}^{\infty} B_k(n) \frac{t^n}{n!}.$$

In the papers [1, Section 3, pp. 84–87], [2, 3, 4], [16, Section 7.3, pp. 63–66], and [25, Section 4, pp. 329–334], the quantities

$$b_k(n) = \frac{B_k(n)}{\exp_k(0)}, \quad n \geq 0$$

were introduced, were called the Bell numbers of order  $k \geq 2$  or the  $k$ th order Bell's numbers, and were applied as an important example to white noise distribution theory [26]. The quantities  $b_k(n)$  satisfy some conditions for sequences required in white noise distribution theory on a CKS-space. For details, please refer to Section 5 in this paper. Since  $b_2(n) = B_n$ , the quantities  $b_k(n)$  are a generalization of the Bell numbers  $B_n$ . By virtue of the software MATHEMATICA, we can obtain the power series

$$\exp_3(t) = e^e + e^{1+e}t + \frac{1}{2}e^{1+e}(2+e)t^2 + \frac{1}{6}e^{1+e}(5+6e+e^2)t^3 + \cdots.$$

This implies that the first four Bell numbers  $b_3(n)$  of order 3 for  $0 \leq n \leq 3$  are 1,  $e$ ,  $e(2+e)$ , and  $e(5+6e+e^2)$ , which are not all positive integers. Hence, we can regard that the Bell numbers  $b_k(n)$  of order  $k$  are not a good generalization of the Bell numbers  $B_n = b_2(n)$  for  $n \geq 0$ .

In this paper, we will introduce a notion “multivariate exponential polynomials”, denoted by  $Q_{m,n}(\mathbf{x}_m)$ , which generalize exponential numbers  $B_n$  and exponential polynomials  $T_n(x)$ , establish explicit formulas, inversion formulas, and recurrence relations for  $Q_{m,n}(\mathbf{x}_m)$  in terms of the Stirling numbers of the first and second kinds  $s(n, k)$  and  $S(n, k)$  with the help of the Faà di Bruno formula, two identities for the Bell polynomials of the second kind  $B_{n,k}(\mathbf{x}_{n-k+1})$ , and the inversion theorem for  $s(n, k)$  and  $S(n, k)$ , construct some determinantal inequalities and product inequalities for  $Q_{m,n}(\mathbf{x}_m)$  with the aid of some properties of completely monotonic functions, derive the logarithmic convexity and logarithmic concavity for the sequences  $\{Q_{m,n}(\mathbf{x}_m)\}_{n \geq 0}$  and  $\{\frac{Q_{m,n}(\mathbf{x}_m)}{n!}\}_{n \geq 0}$  respectively, and finally find an application of multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  to white noise distribution theory by confirming that the polynomial sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  satisfies conditions for sequences required in white noise distribution theory.

## 2. MULTIVARIATE EXPONENTIAL POLYNOMIALS

Now we generalize exponential numbers  $B_n$  and exponential polynomials  $T_n(x)$  by introducing the notion “multivariate exponential polynomial”.

**Definition 2.1.** For  $m \in \mathbb{N}$  and  $t, x_k \in \mathbb{R}$  with  $1 \leq k \leq m$ , denote  $\mathbf{x}_m = (x_1, \dots, x_m)$ . Define the quantities  $Q_{m,n}(\mathbf{x}_m)$  by

$$G(t; \mathbf{x}_m) = e \left( \begin{array}{c} x_1 \\ e \left( \begin{array}{c} x_2 \\ e \left( \begin{array}{c} \dots \\ e \left( \begin{array}{c} x_{m-1} (e^{x_m (e^t - 1)} - 1) \\ -1 \end{array} \right) \\ -1 \end{array} \right) \\ -1 \end{array} \right) \\ -1 \end{array} \right) = \sum_{n=0}^{\infty} Q_{m,n}(\mathbf{x}_m) \frac{t^n}{n!}. \quad (2.1)$$

Let  $g(t) = e^t - 1$  for  $t \in \mathbb{R}$ . Then the generating function  $G(t; \mathbf{x}_m)$  of the quantities  $Q_{m,n}(\mathbf{x}_m)$  for  $n \geq 0$  is an  $m$ -time self-composite function of  $g(t)$  in the manner

$$\begin{aligned} G(t; \mathbf{x}_m) &= \exp(x_1 [\exp(x_2 [\exp(\dots x_{m-1} [\exp(x_m [\exp(t) - 1]) - 1] \dots) - 1]) - 1]) \\ &= \exp(x_1 g(x_2 g(\dots x_{m-1} g(x_m g(t)) \dots))) \\ &= g(x_1 g(x_2 g(\dots x_{m-1} g(x_m g(t)) \dots))) + 1. \end{aligned} \quad (2.2)$$

In other words, the generating function  $G(t; \mathbf{x}_m) = G(t; x_1, x_2, \dots, x_m)$  satisfies

$$G(t; x_1, x_2, \dots, x_m) = \exp(x_1 G(t; x_2, \dots, x_m) - 1) \quad \text{and} \quad G(t; x) = e^{x(e^t - 1)}.$$

It is easy to see that  $Q_{1,n}(1) = B_n$  and  $Q_{1,n}(x) = T_n(x)$  for  $n \geq 0$ ,  $G(0; \mathbf{x}_m) = 1$  and  $Q_{m,0}(\mathbf{x}_m) = 1$  for  $m \in \mathbb{N}$ , and, by virtue of MATHEMATICA,

$$\begin{aligned} Q_{2,0}(x, y) &= 1, & Q_{2,1}(x, y) &= xy, & Q_{2,2}(x, y) &= xy(1 + y + xy), \\ Q_{2,3}(x, y) &= xy(1 + 3y + y^2 + 3xy + 3xy^2 + x^2y^2), & Q_{3,0}(x, y, z) &= 1, \\ Q_{3,1}(x, y, z) &= xyz, & Q_{3,2}(x, y, z) &= xyz(1 + z + yz + xyz). \end{aligned} \quad (2.3)$$

All of these are positive integer polynomials of degree  $m \times n$ .

For conveniently referring to  $Q_{m,n}(\mathbf{x}_m)$ , we recommend the names “multi-order exponential polynomials”, “higher order exponential polynomials”, “multivariate exponential polynomials”, “exponential polynomials of order  $m$ ”, “exponential polynomials of  $m$  variables  $x_1, x_2, \dots, x_m$ ”, or “ $m$ -variate exponential polynomials” alternatively for  $Q_{m,n}(\mathbf{x}_m)$ .

When  $x_1 = x_2 = \dots = x_m = 1$ , we denote  $Q_{m,n}(1, 1, \dots, 1)$  by  $Q_{m,n}$  and call them “higher order exponential numbers”, “multi-order exponential numbers”, or “exponential numbers of order  $m$ ” alternatively. By MATHEMATICA, we can obtain the first thirty-two exponential numbers  $Q_{m,n}$  for  $2 \leq m \leq 5$  and  $0 \leq n \leq 7$ , all of which are positive integers, listed in Table 1.

On 24 October 2017, we found in [34, Section 5] two tables in which concrete numbers  $Q_{m,n}$  for  $2 \leq m \leq 5$  and  $1 \leq n \leq 21$  are listed.

TABLE 1. The first few  $Q_{m,n}$  for  $2 \leq m \leq 5$  and  $0 \leq n \leq 7$ 

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$m = 2$	1	1	3	12	60	358	2471	19302
$m = 3$	1	1	4	22	154	1304	12915	146115
$m = 4$	1	1	5	35	315	3455	44590	660665
$m = 5$	1	1	6	51	561	7556	120196	2201856

### 3. AN EXPLICIT FORMULA, AN IDENTITY, AND THREE RECURRENCE RELATIONS FOR MULTIVARIATE EXPONENTIAL POLYNOMIALS

In this section, by virtue of the Faà di Bruno formula, two identities of the Bell polynomials of the second kind  $B_{n,k}(\mathbf{x}_{n-k+1})$ , and the inversion theorem for the Stirling numbers of the first and second kinds  $s(n, k)$  and  $S(n, k)$ , we will find explicit formula, an identity, and three recurrence relations for multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$ .

Roughly judging from those expressions in (2.3), we guess that multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  for  $m \in \mathbb{N}$  and  $n \geq 0$  should be positive integer polynomials of degree  $m \times n$ . This guess will be verified by the following theorem.

**Theorem 3.1.** *Let  $m \in \mathbb{N}$  and  $n \geq 0$ . Multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  can be computed explicitly by*

$$Q_{m,n}(\mathbf{x}_m) = \left( \prod_{q=1}^m \sum_{\ell_q=0}^{\ell_{q-1}} \right) \left[ \prod_{q=1}^m S(\ell_{q-1}, \ell_q) \right] \left( \prod_{q=1}^m x_q^{\ell_{m-q+1}} \right), \quad (3.1)$$

where  $\ell_0 = n$  and  $S(n, k)$  for  $n \geq k \geq 0$ , which can be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!},$$

stand for the Stirling numbers of the second kind. Consequently, multi-order exponential numbers  $Q_{m,n}$  can be computed explicitly by

$$Q_{m,n} = \left( \prod_{q=1}^m \sum_{\ell_q=0}^{\ell_{q-1}} \right) \left[ \prod_{q=1}^m S(\ell_{q-1}, \ell_q) \right]. \quad (3.2)$$

*Proof.* In combinatorics, the Bell polynomials of the second kind  $B_{n,k}(\mathbf{x}_{n-k+1})$  are defined by

$$B_{n,k}(\mathbf{x}_{n-k+1}) = \sum_{\substack{1 \leq i \leq n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i \ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{\ell_i} \quad (3.3)$$

for  $n \geq k \geq 0$ , see [17, p. 134, Theorem A], and satisfy two identities

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(\mathbf{x}_{n-k+1}) \quad (3.4)$$

and

$$B_{n,k}(1, 1, \dots, 1) = S(n, k) \quad (3.5)$$

in [17, p. 135], where  $a$  and  $b$  are any complex numbers. The Faà di Bruno formula for computing higher order derivatives of composite functions can be described in terms of the Bell polynomials of the second kind  $B_{n,k}(\mathbf{x}_{n-k+1})$  by

$$\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=0}^n f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \dots, h^{(n-k+1)}(x)), \quad (3.6)$$

see [17, p. 139, Theorem C] or [51, Section 2]. Therefore, making use of (3.6), (3.4), and (3.5) in sequence and considering the composite relation (2.2) inductively yield

$$\begin{aligned} \frac{\partial^n G(t; \mathbf{x}_m)}{\partial t^n} &= \sum_{\ell_1=0}^n \frac{\partial^{\ell_1} G(u_1; \mathbf{x}_{m-1})}{\partial^{\ell_1} u_1} B_{n,\ell_1}(x_m e^t, \dots, x_m e^t) \\ &= \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \frac{\partial^{\ell_2} G(u_2; \mathbf{x}_{m-2})}{\partial^{\ell_2} u_2} B_{\ell_1, \ell_2}(x_{m-1} e^{u_1}, \dots, x_{m-1} e^{u_1}) x_m^{\ell_1} e^{\ell_1 t} S(n, \ell_1) \\ &= \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \frac{\partial^{\ell_2} G(u_2; \mathbf{x}_{m-2})}{\partial^{\ell_2} u_2} x_{m-1}^{\ell_2} e^{\ell_2 u_1} S(\ell_1, \ell_2) x_m^{\ell_1} e^{\ell_1 t} S(n, \ell_1) \\ &= \dots \\ &= \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{m-2}=0}^{\ell_{m-3}} \sum_{\ell_{m-1}=0}^{\ell_{m-2}} \frac{\partial^{\ell_{m-1}} G(u_{m-1}; x_1)}{\partial^{\ell_{m-1}} u_{m-1}} B_{\ell_{m-2}, \ell_{m-1}}(x_2 e^{u_{m-2}}, \dots, \\ & x_2 e^{u_{m-2}}) x_3^{\ell_{m-2}} e^{\ell_{m-2} u_{m-3}} S(\ell_{m-3}, \ell_{m-2}) \dots x_{m-1}^{\ell_2} e^{\ell_2 u_1} S(\ell_1, \ell_2) x_m^{\ell_1} e^{\ell_1 t} S(n, \ell_1) \\ &= \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{m-2}=0}^{\ell_{m-3}} \sum_{\ell_{m-1}=0}^{\ell_{m-2}} \sum_{\ell_m=0}^{\ell_{m-1}} \frac{d^{\ell_m} e^{u_m}}{d^{\ell_m} u_m} B_{\ell_{m-1}, \ell_m}(x_1 e^{u_{m-1}}, \dots, x_1 e^{u_{m-1}}) \\ & \times x_2^{\ell_{m-1}} e^{\ell_{m-1} u_{m-2}} S(\ell_{m-2}, \ell_{m-1}) x_3^{\ell_{m-2}} e^{\ell_{m-2} u_{m-3}} S(\ell_{m-3}, \ell_{m-2}) \\ & \dots x_{m-1}^{\ell_2} e^{\ell_2 u_1} S(\ell_1, \ell_2) x_m^{\ell_1} e^{\ell_1 t} S(n, \ell_1) \\ &= \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \dots \sum_{\ell_{m-2}=0}^{\ell_{m-3}} \sum_{\ell_{m-1}=0}^{\ell_{m-2}} \sum_{\ell_m=0}^{\ell_{m-1}} e^{u_m} x_1^{\ell_m} e^{\ell_m u_{m-1}} S(\ell_{m-1}, \ell_m) \\ & \times x_2^{\ell_{m-1}} e^{\ell_{m-1} u_{m-2}} S(\ell_{m-2}, \ell_{m-1}) x_3^{\ell_{m-2}} e^{\ell_{m-2} u_{m-3}} S(\ell_{m-3}, \ell_{m-2}) \\ & \dots x_{m-1}^{\ell_2} e^{\ell_2 u_1} S(\ell_1, \ell_2) x_m^{\ell_1} e^{\ell_1 t} S(n, \ell_1) \\ &= \left( \prod_{q=1}^m \sum_{\ell_q=0}^{\ell_{q-1}} \right) e^{u_m} \exp \left( \sum_{q=1}^m \ell_q u_{q-1} \right) \left[ \prod_{q=1}^m S(\ell_{q-1}, \ell_q) \right] \left( \prod_{q=1}^m x_q^{\ell_{m-q+1}} \right), \end{aligned}$$

where  $u_0 = u_0(t) = t$  and  $u_q = u_q(u_{q-1}) = x_{m-q+1}(e^{u_{q-1}} - 1)$  for  $1 \leq q \leq m$ . When  $t \rightarrow 0$ , it follows that  $u_q \rightarrow 0$  for all  $0 \leq q \leq m$ . As a result, by the definition in (2.1), we have

$$Q_{m,n}(\mathbf{x}_m) = \lim_{t \rightarrow 0} \frac{\partial^n G(t; \mathbf{x}_m)}{\partial^n t} = \left( \prod_{q=1}^m \sum_{\ell_q=0}^{\ell_{q-1}} \right) \left[ \prod_{q=1}^m S(\ell_{q-1}, \ell_q) \right] \left( \prod_{q=1}^m x_q^{\ell_{m-q+1}} \right).$$

The formula (3.1) is thus proved.

The formula (3.2) follows from taking  $x_1 = x_2 = \dots = x_m = 1$  in (3.1). The proof of Theorem 3.1 is complete.  $\square$

*Remark 3.1.* When letting  $m = 1, 2, 3$  in (3.1), we can recover and find explicit formulas

$$T_n(x) = \sum_{k=0}^n S(n, k)x^k, \quad Q_{2,n}(x, y) = \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} S(n, \ell_1)S(\ell_1, \ell_2)x^{\ell_2}y^{\ell_1}, \quad (3.7)$$

and

$$Q_{3,n}(x, y, z) = \sum_{\ell_1=0}^n \sum_{\ell_2=0}^{\ell_1} \sum_{\ell_3=0}^{\ell_2} S(n, \ell_1)S(\ell_1, \ell_2)S(\ell_2, \ell_3)x^{\ell_3}y^{\ell_2}z^{\ell_1} \quad (3.8)$$

for  $n \geq 0$ . The first formula in (3.7) was also recovered in [39, Theorem 3.1]. The second formula in (3.7) and the formula (3.8) coincide with those special values in (2.3). This convinces us that Theorem 3.1 and its proof in this paper are correct.

**Theorem 3.2.** *Let  $m \in \mathbb{N}$  and  $n \geq 0$ . Multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  satisfy the identity*

$$\left( \prod_{q=1}^m \left[ \frac{1}{x_q^{\ell_{q-1}}} \sum_{\ell_q=0}^{\ell_{q-1}} s(\ell_{q-1}, \ell_q) \right] \right) Q_{m,\ell_m}(\mathbf{x}_m) = 1, \quad (3.9)$$

where  $\ell_0 = n$  and  $s(n, k)$  for  $n \geq k \geq 0$ , which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind. Consequently, multi-order exponential numbers  $Q_{m,n}$  satisfy the identity

$$\left( \prod_{q=1}^m \left[ \sum_{\ell_q=0}^{\ell_{q-1}} s(\ell_{q-1}, \ell_q) \right] \right) Q_{m,\ell_m} = 1. \quad (3.10)$$

*Proof.* The formula (3.1) can be rearranged as

$$Q_{m,n}(\mathbf{x}_m) = \sum_{\ell_1=0}^n S(n, \ell_1)x_m^{\ell_1} \sum_{\ell_2=0}^{\ell_1} S(\ell_1, \ell_2)x_{m-1}^{\ell_2}$$



$$\cdots \sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2}, \ell_{m-1}) x_2^{\ell_{m-1}} \sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1}, \ell_m) x_1^{\ell_m}. \quad (3.11)$$

In [52, p. 171, Theorem 12.1], it is stated that, if  $b_\alpha$  and  $a_k$  are a collection of constants independent of  $n$ , then

$$a_n = \sum_{\alpha=0}^n S(n, \alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^n s(n, k) a_k. \quad (3.12)$$

Applying this inversion theorem to (3.11) consecutively leads to

$$\begin{aligned} & \sum_{\ell_2=0}^n S(n, \ell_2) x_{m-1}^{\ell_2} \cdots \sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2}, \ell_{m-1}) x_2^{\ell_{m-1}} \sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1}, \ell_m) x_1^{\ell_m} \\ &= \frac{1}{x_m^n} \sum_{\ell_1=0}^n s(n, \ell_1) Q_{m, \ell_1}(\mathbf{x}_m), \\ & \frac{1}{x_{m-1}^n} \sum_{\ell_2=0}^n s(n, \ell_2) \frac{1}{x_m^{\ell_2}} \sum_{\ell_1=0}^{\ell_2} s(\ell_2, \ell_1) Q_{m, \ell_1}(\mathbf{x}_m) \\ &= \sum_{\ell_3=0}^n S(n, \ell_3) \cdots \sum_{\ell_{m-1}=0}^{\ell_{m-2}} S(\ell_{m-2}, \ell_{m-1}) x_2^{\ell_{m-1}} \sum_{\ell_m=0}^{\ell_{m-1}} S(\ell_{m-1}, \ell_m) x_1^{\ell_m}, \end{aligned}$$

and, inductively,

$$\begin{aligned} & \sum_{\ell_{m-1}=0}^n s(n, \ell_{m-1}) \frac{1}{x_3^{\ell_{m-1}}} \sum_{\ell_{m-2}=0}^{\ell_{m-1}} \cdots \frac{1}{x_{m-1}^{\ell_3}} \sum_{\ell_2=0}^{\ell_3} s(\ell_3, \ell_2) \frac{1}{x_m^{\ell_2}} \sum_{\ell_1=0}^{\ell_2} s(\ell_2, \ell_1) Q_{m, \ell_1}(\mathbf{x}_m) \\ &= x_2^n \sum_{\ell_m=0}^n S(n, \ell_m) x_1^{\ell_m} \end{aligned}$$

which can be further rewritten as

$$\begin{aligned} x_1^n &= \sum_{\ell_1=0}^n \frac{s(n, \ell_1)}{x_2^{\ell_1}} \sum_{\ell_2=0}^{\ell_1} \frac{s(\ell_1, \ell_2)}{x_3^{\ell_2}} \sum_{\ell_3=0}^{\ell_2} \frac{s(\ell_2, \ell_3)}{x_4^{\ell_3}} \\ & \quad \cdots \frac{s(\ell_{m-3}, \ell_{m-2})}{x_{m-1}^{\ell_{m-2}}} \sum_{\ell_{m-1}=0}^{\ell_{m-2}} \frac{s(\ell_{m-2}, \ell_{m-1})}{x_m^{\ell_{m-1}}} \sum_{\ell_m=0}^{\ell_{m-1}} s(\ell_{m-1}, \ell_m) Q_{m, \ell_m}(\mathbf{x}_m) \end{aligned}$$

and the identity (3.9).

The identity (3.10) follows from taking  $x_1 = x_2 = \cdots = x_m = 1$  in (3.9). The proof of Theorem 3.2 is complete.  $\square$

*Remark 3.2.* When letting  $m = 1, 2, 3$  in (3.9), we can recover and derive

$$\sum_{k=0}^n s(n, k) T_k(x) = x^n, \quad \sum_{\ell_1=0}^n \frac{s(n, \ell_1)}{y^{\ell_1}} \sum_{\ell_2=0}^{\ell_1} s(\ell_1, \ell_2) Q_{2, \ell_2}(x, y) = x^n, \quad (3.13)$$

and

$$\sum_{\ell_1=0}^n \frac{s(n, \ell_1)}{y^{\ell_1}} \sum_{\ell_2=0}^{\ell_1} \frac{s(\ell_1, \ell_2)}{z^{\ell_2}} \sum_{\ell_3=0}^{\ell_2} s(\ell_2, \ell_3) Q_{3, \ell_3}(x, y, z) = x^n \quad (3.14)$$

for  $n \geq 0$ . The first identity (3.13) was also obtained in [39, Theorem 3.1]. The second identity in (3.13) for  $n = 0, 1, 2, 3$  and the identity (3.14) for  $n = 0, 1, 2$  can be easily verified by special values in (2.3).

**Theorem 3.3.** *Let  $m \in \mathbb{N}$  and  $2 \leq k \leq m - 1$ . Then multivariate exponential polynomials  $Q_{m, n}(\mathbf{x}_m)$  satisfy*

$$Q_{m, n}(\mathbf{x}_m) = \sum_{\ell=0}^n Q_{k-1, \ell}(\mathbf{x}_{k-1}) x_k^\ell \times B_{n, \ell}(Q_{m-k, 1}(x_{k+1}, \dots, x_m), \dots, Q_{m-k, n-\ell+1}(x_{k+1}, \dots, x_m)). \quad (3.15)$$

Consequently, multi-order exponential numbers  $Q_{m, n}$  satisfy

$$Q_{m, n} = \sum_{\ell=0}^n Q_{k-1, \ell} B_{n, \ell}(Q_{m-k, 1}, \dots, Q_{m-k, n-\ell+1}). \quad (3.16)$$

*Proof.* The generating function in (2.1) can be rewritten as

$$\begin{aligned} G(t; x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_{m-1}, x_m) &= \exp(x_1 g(x_2 g(\dots x_{k-1} g(x_k [\exp(x_{k+1} g(\dots x_{m-1} g(x_m g(t)) \dots)) - 1]) \dots))) \\ &= \exp(x_1 g(x_2 g(\dots x_{k-1} g(x_k [G(t; x_{k+1}, \dots, x_{m-1}, x_m) - 1]) \dots))) \\ &= \exp(x_1 g(x_2 g(\dots x_{k-1} [\exp(x_k [G(t; x_{k+1}, \dots, x_{m-1}, x_m) - 1]) - 1] \dots))) \\ &= \exp(x_1 g(x_2 g(\dots x_{k-1} [\exp(v) - 1] \dots))) \\ &= G(x_k [G(t; x_{k+1}, \dots, x_{m-1}, x_m) - 1]; x_1, \dots, x_{k-1}), \end{aligned}$$

where  $2 \leq k \leq m - 1$  and  $v = v(t) = x_k [G(t; x_{k+1}, \dots, x_m) - 1]$ . By the definition in (2.1) and the Faà di Bruno formula (3.6), we can obtain

$$\frac{\partial^n G(t; \mathbf{x}_m)}{\partial t^n} = \sum_{\ell=0}^n \frac{\partial^\ell G(v; \mathbf{x}_{k-1})}{\partial v^\ell} B_{n, \ell}(v'(t), \dots, v^{(n-\ell+1)}(t)),$$

where

$$v^{(p)}(t) = x_k \frac{\partial^p G(t; x_{k+1}, \dots, x_m)}{\partial t^p}, \quad 1 \leq p \leq n - \ell + 1.$$

Further taking  $t \rightarrow 0$  yields

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\partial^n G(t; \mathbf{x}_m)}{\partial t^n} &= Q_{m, n}(\mathbf{x}_m), \quad \lim_{t \rightarrow 0} v^{(p)}(t) = x_k Q_{m-k, p}(x_{k+1}, \dots, x_m), \\ \lim_{t \rightarrow 0} \frac{\partial^\ell G(v; \mathbf{x}_{k-1})}{\partial v^\ell} &= \lim_{v \rightarrow 0} \frac{\partial^\ell G(v; \mathbf{x}_{k-1})}{\partial v^\ell} = Q_{k-1, \ell}(\mathbf{x}_{k-1}), \end{aligned}$$

and, consequently,

$$\begin{aligned} Q_{m,n}(\mathbf{x}_m) &= \sum_{\ell=0}^n Q_{k-1,\ell}(\mathbf{x}_{k-1}) B_{n,\ell}(x_k Q_{m-k,1}(x_{k+1}, \dots, x_m), \dots, \\ &\quad x_k Q_{m-k,n-\ell+1}(x_{k+1}, \dots, x_m)) \\ &= \sum_{\ell=0}^n Q_{k-1,\ell}(\mathbf{x}_{k-1}) x_k^\ell B_{n,\ell}(Q_{m-k,1}(x_{k+1}, \dots, x_m), \\ &\quad \dots, Q_{m-k,n-\ell+1}(x_{k+1}, \dots, x_m)), \end{aligned}$$

where in the last step we used the identity (3.4). Hence, the identity (3.15) follows.

The identity (3.16) can be derived from (3.15) by taking  $x_1 = x_2 = \dots = x_m = 1$ . The proof of Theorem 3.3 is complete.  $\square$

*Remark 3.3.* In recent years, there have been some literature such as [34, 41, 42, 43, 50, 61, 62, 63, 64] devoted to deep investigation and extensive applications of the Bell polynomials of the second kind  $B_{n,k}(\mathbf{x}_{n-k+1})$ .

**Theorem 3.4.** *Let  $m \in \mathbb{N}$  and  $n \geq 0$ . Multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  satisfy the recurrence relation*

$$Q_{m,n}(\mathbf{x}_m) = \sum_{\ell=0}^n S(n, \ell) Q_{m-1,\ell}(\mathbf{x}_{m-1}) x_m^\ell. \quad (3.17)$$

Consequently, the explicit formulas (3.1) and (3.9) are valid and

$$Q_{m-1,n}(\mathbf{x}_{m-1}) = \frac{1}{x_m^n} \sum_{\ell=0}^n s(n, \ell) Q_{m,\ell}(\mathbf{x}_m). \quad (3.18)$$

*Proof.* The generating function in (2.1) can be rewritten as

$$G(t; \mathbf{x}_m) = G(u; \mathbf{x}_{m-1}), \quad u = u(t) = x_m(e^t - 1).$$

Making use of the Faà di Bruno formula (3.6) and the identity (3.4) gives

$$\begin{aligned} \frac{\partial^n G(t; \mathbf{x}_m)}{\partial t^n} &= \sum_{\ell=0}^n \frac{\partial^\ell G(u; \mathbf{x}_{m-1})}{\partial u^\ell} B_{n,\ell}(u'(t), \dots, u^{(n-\ell+1)}(t)) \\ &= \sum_{\ell=0}^n \frac{\partial^\ell G(u; \mathbf{x}_{m-1})}{\partial u^\ell} B_{n,\ell}(x_m e^t, \dots, x_m e^t) \\ &= \sum_{\ell=0}^n \frac{\partial^\ell G(u; \mathbf{x}_{m-1})}{\partial u^\ell} x_m^\ell e^{\ell t} B_{n,\ell}(1, \dots, 1). \end{aligned}$$

Employing the identity

$$B_{n,k}(1, 1, \dots, 1) = S(n, k)$$

in [17, p. 135] and letting  $t \rightarrow 0$  reveal the recurrence relation (3.17).

Using the recurrence relation (3.17) consecutively leads to (3.1).

Applying the inversion theorem (3.12) to (3.17) results in (3.18).

Utilizing the recurrence relation (3.18) inductively arrives at (3.9). The proof of Theorem 3.4 is complete.  $\square$

*Remark 3.4.* If taking  $x_1 = \cdots = x_m = 1$  in Theorem 3.4, we can recover (3.2) and (3.10) and derive recurrence relations

$$Q_{m,n} = \sum_{\ell=0}^n S(n, \ell) Q_{m-1, \ell} \quad \text{and} \quad Q_{m-1, n} = \sum_{\ell=0}^n s(n, \ell) Q_{m, \ell}.$$

#### 4. INEQUALITIES FOR MULTIVARIATE EXPONENTIAL POLYNOMIALS

It seems that there have been more identities than inequalities in combinatorial mathematics. In this section, we will construct some determinantal inequalities and product inequalities for multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  and derive the logarithmic convexity and logarithmic concavity for the sequences  $\{Q_{m,n}(x)\}_{n \geq 0}$  and  $\{\frac{Q_{m,n}(x)}{n!}\}_{n \geq 0}$  respectively.

**Theorem 4.1.** *Let  $q \geq 1$  be a positive integer, let  $|e_{ij}|_q$  denote a determinant of order  $q$  with elements  $e_{ij}$ , and let  $x_k > 0$  for  $1 \leq k \leq m$ .*

(1) *If  $a_i$  for  $1 \leq i \leq q$  are non-negative integers, then*

$$|(-1)^{a_i+a_j} Q_{m, a_i+a_j}(\mathbf{x}_m)|_q \geq 0 \quad \text{and} \quad |Q_{m, a_i+a_j}(\mathbf{x}_m)|_q \geq 0. \quad (4.1)$$

(2) *If  $a = (a_1, a_2, \dots, a_q)$  and  $b = (b_1, b_2, \dots, b_q)$  are non-increasing  $q$ -tuples of non-negative integers such that  $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$  for  $1 \leq k \leq q-1$  and  $\sum_{i=1}^q a_i = \sum_{i=1}^q b_i$ , then*

$$\prod_{i=1}^q Q_{m, a_i}(\mathbf{x}_m) \geq \prod_{i=1}^q Q_{m, b_i}(\mathbf{x}_m). \quad (4.2)$$

*Proof.* Recall from [33, Chapter XIII] and [60, Chapter IV] that a function  $f$  is said to be absolutely monotonic on an interval  $I$  if it has derivatives of all orders and  $f^{(k-1)}(t) \geq 0$  for  $t \in I$  and  $k \in \mathbb{N}$ . Recall from [33, Chapter XIII], [54, Chapter 1], and [60, Chapter IV] that an infinitely differentiable function  $f$  is said to be completely monotonic on an interval  $I$  if it satisfies  $(-1)^k f^{(k)}(x) \geq 0$  on  $I$  for all  $k \geq 0$ . Theorem 2b in [60, p. 145] reads that, if  $f_1(x)$  is absolutely monotonic and  $f_2(x)$  is completely monotonic on their defined intervals, then their composite function  $f_1(f_2(x))$  is completely monotonic on its defined interval. Therefore, since  $e^t$  and  $e^{-t}$  are respectively absolutely and completely monotonic on  $[0, \infty)$ , by induction, it follows that, when  $x_1, x_2, \dots, x_m > 0$ , the generating function  $G(-t; \mathbf{x}_m)$  is completely monotonic with respect to  $t \in [0, \infty)$ . Moreover, by (2.1), it is obvious that

$$Q_{m,n}(\mathbf{x}_m) = (-1)^n \lim_{t \rightarrow 0} \frac{\partial^n G(-t; \mathbf{x}_m)}{\partial^n t}.$$

In [32] and [33, p. 367], it was proved that if  $f(t)$  is completely monotonic on  $[0, \infty)$ , then

$$|f^{(a_i+a_j)}(t)|_q \geq 0 \quad \text{and} \quad |(-1)^{a_i+a_j} f^{(a_i+a_j)}(t)|_q \geq 0. \quad (4.3)$$

Applying  $f(t)$  to the generating function  $G(-t; \mathbf{x}_m)$  in (4.3) and taking the limit  $t \rightarrow 0^+$  give

$$\lim_{t \rightarrow 0^+} \left| [G(-t; \mathbf{x}_m)]_t^{(a_i+a_j)} \right|_q = |(-1)^{a_i+a_j} Q_{m, a_i+a_j}(\mathbf{x}_m)|_q \geq 0$$

and

$$\lim_{t \rightarrow 0^+} |(-1)^{a_i+a_j} [G(-t; \mathbf{x}_m)]_t^{(a_i+a_j)}|_q = |Q_{m, a_i+a_j}(\mathbf{x}_m)|_q \geq 0$$

The determinantal inequalities in (4.1) follow.

In [33, p. 367, Theorem 2], it was stated that if  $f(t)$  is a completely monotonic function on  $[0, \infty)$ , then

$$\prod_{i=1}^q [(-1)^{a_i} f^{(a_i)}(t)] \geq \prod_{i=1}^q [(-1)^{b_i} f^{(b_i)}(t)]. \quad (4.4)$$

Applying  $f(t)$  to the generating function  $G(-t; \mathbf{x}_m)$  in (4.4) and taking the limit  $t \rightarrow 0^+$  result in

$$\begin{aligned} \lim_{t \rightarrow 0^+} \prod_{i=1}^q [(-1)^{a_i} (G(-t; \mathbf{x}_m))_t^{(a_i)}] &= \prod_{i=1}^q Q_{m, a_i}(\mathbf{x}_m) \\ &\geq \lim_{t \rightarrow 0^+} \prod_{i=1}^q [(-1)^{b_i} (G(-t; \mathbf{x}_m))_t^{(b_i)}] = \prod_{i=1}^q Q_{m, b_i}(\mathbf{x}_m). \end{aligned}$$

The product inequality (4.2) follows. The proof of Theorem 4.1 is complete.  $\square$

**Corollary 4.1.** *Let  $x_k > 0$  for  $1 \leq k \leq m$ . If  $\ell \geq 0$  and  $q \geq k \geq 0$ , then*

$$[Q_{m, q+\ell}(\mathbf{x}_m)]^k [Q_{m, \ell}(\mathbf{x}_m)]^{q-k} \geq [Q_{m, k+\ell}(\mathbf{x}_m)]^q.$$

*Proof.* This follows from taking

$$a = (\overbrace{q+\ell, \dots, q+\ell}^k, \overbrace{\ell, \dots, \ell}^{q-k}) \quad \text{and} \quad b = (k+\ell, k+\ell, \dots, k+\ell)$$

in the inequality (4.2). The proof of Corollary 4.1 is complete.  $\square$

**Theorem 4.2.** *When  $x \geq 1$ , the sequence  $\{T_n(x)\}_{n \geq 0}$  is logarithmically convex and the sequence  $\{\frac{T_n(x)}{n!}\}_{n \geq 0}$  is logarithmically concave; consequently, for  $p, q \geq 0$  and  $x \geq 1$ ,*

$$T_p(x)T_q(x) \leq T_{p+q}(x) \leq \binom{p+q}{p} T_p(x)T_q(x). \quad (4.5)$$

*When  $x_1, \dots, x_{m-1} \geq 2$  and  $x_m \geq 1$  with  $m \geq 2$ , the sequence  $\{Q_{m, n}(\mathbf{x}_m)\}_{n \geq 0}$  is logarithmically convex and the sequence  $\{\frac{Q_{m, n}(\mathbf{x}_m)}{n!}\}_{n \geq 0}$  is logarithmically concave;*

consequently, for  $p, q \geq 0$ , when  $x_1, \dots, x_{m-1} \geq 2$  and  $x_m \geq 1$  with  $m \geq 2$ ,

$$Q_{m,p}(\mathbf{x}_m)Q_{m,q}(\mathbf{x}_m) \leq Q_{m,p+q}(\mathbf{x}_m) \leq \binom{p+q}{p} Q_{m,p}(\mathbf{x}_m)Q_{m,q}(\mathbf{x}_m). \quad (4.6)$$

*Proof.* In [33, p. 369] and [35, p. 429, Remark], it was obtained that if  $f(t)$  is a completely monotonic function such that  $f^{(k)}(t) \neq 0$  for  $k \geq 0$ , then the sequence

$$\ln[(-1)^{k-1}f^{(k-1)}(t)], \quad k \geq 1 \quad (4.7)$$

is convex. Applying this conclusion to the generating function  $G(-t; \mathbf{x}_m)$  figures out that the sequence

$$\ln[(-1)^{k-1}(G(-t; \mathbf{x}_m))_t^{(k-1)}] \rightarrow \ln Q_{m,k-1}(\mathbf{x}_m), \quad t \rightarrow 0^+$$

for  $k \geq 1$  is convex. Equivalently, the sequence  $\{Q_{m,n}(\mathbf{x}_m)\}_{n \geq 0}$  is logarithmically convex.

Alternatively, letting  $\ell \geq 1$ ,  $n = 2$ ,  $a_1 = \ell + 2$ ,  $a_2 = \ell$ , and  $b_1 = b_2 = \ell + 1$  in the inequality (4.2) leads to

$$Q_{m,\ell}(\mathbf{x}_m)Q_{m,\ell+2}(\mathbf{x}_m) \geq Q_{m,\ell+1}^2(\mathbf{x}_m)$$

which means that the sequence  $\{Q_{m,n}(\mathbf{x}_m)\}_{n \geq 1}$  is logarithmically convex.

If  $\{1, X_1, X_2, \dots\}$  is a logarithmically concave sequence of nonnegative real numbers and the sequences  $\{A_n\}_{n \geq 0}$  and  $\{P_n\}_{n \geq 0}$  are defined by

$$\sum_{n=0}^{\infty} A_n u^n = \sum_{n=0}^{\infty} \frac{P_n}{n!} u^n = \exp\left(\sum_{i=1}^{\infty} X_i \frac{u^i}{i}\right),$$

then it was proved in [7, p. 58, Theorem 1] that the sequence  $\{A_n\}_{n \geq 0}$  is logarithmically concave and the sequence  $\{P_n\}_{n \geq 0}$  is logarithmically convex. By definition, we see that

$$\sum_{n=0}^{\infty} \frac{Q_{1,n}(x)}{n!} t^n = e^{x(e^t-1)} = \exp\left[\sum_{n=1}^{\infty} \frac{x}{(n-1)!} \frac{t^n}{n}\right].$$

It is easy to verify that the sequence  $\{1, \frac{x}{(n-1)!}\}_{n \geq 1}$  is logarithmically concave if and only if  $x \geq 1$ . Therefore, when  $x \geq 1$ , the sequence  $\{T_n(x)\}_{n \geq 0}$  is logarithmically convex and the sequence  $\{\frac{T_n(x)}{n!}\}_{n \geq 0}$  is logarithmically concave.

Theorem 2 in [1] states that

(1) if  $\{\alpha_n\}_{n \geq 0}$  is logarithmically convex with  $\alpha_0 = 1$ , then

$$\alpha_p \alpha_q \leq \alpha_{p+q}, \quad p, q \geq 0;$$

(2) if  $\{\frac{\alpha_n}{n!}\}_{n \geq 0}$  is logarithmically concave with  $\alpha_0 = 1$ , then

$$\alpha_{p+q} \leq \binom{p+q}{p} \alpha_p \alpha_q, \quad p, q \geq 0.$$

Combining this theorem with the logarithmic convexity and logarithmic concavity of the sequences  $\{T_n(x), x \geq 1\}_{n \geq 0}$  and  $\{\frac{T_n(x)}{n!}, x \geq 1\}_{n \geq 0}$  respectively leads to the double inequality (4.5).

Lemma 1 in [1] reads that, if  $\{\frac{\beta_n}{n!}\}_{n \geq 0}$  is a logarithmically concave sequence and  $r \geq 0$  such that  $\beta_2 \leq r\beta_1^2$ , then the sequence  $\{1, \frac{r\beta_n}{(n-1)!}\}_{n \geq 1}$  is logarithmically concave. When applying  $\beta_n = Q_{1,n}(x_m)$  for  $x_m \geq 1$ , we have

$$\frac{Q_{1,0}(x_m) Q_{1,2}(x_m)}{0! 2!} \leq \left[ \frac{Q_{1,1}(x_m)}{1!} \right]^2$$

which can be simplified as  $Q_{1,2}(x_m) \leq 2[Q_{1,1}(x_m)]^2$ . Accordingly, the sequence  $\{1, \frac{rQ_{1,n}(x_m)}{(n-1)!}, x_m \geq 1, r \geq 2\}_{n \geq 1}$  is logarithmically concave. Combining this with the above mentioned [7, Theorem 1] and the fact that

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{2,n}(x_{m-1}, x_m) \frac{t^n}{n!} &= \exp[x_{m-1}(\exp(x_m(\exp(t) - 1)) - 1)] \\ &= \exp \left[ x_{m-1} \left( \sum_{n=0}^{\infty} Q_{1,n}(x_m) \frac{t^n}{n!} - 1 \right) \right] = \exp \left[ \sum_{n=1}^{\infty} \frac{x_{m-1} Q_{1,n}(x_m) t^n}{(n-1)! n} \right] \end{aligned}$$

reveals that, when  $x_m \geq 1$  and  $x_{m-1} \geq 2$ , the sequences  $\{Q_{2,n}(x_{m-1}, x_m)\}_{n \geq 0}$  and  $\{\frac{Q_{2,n}(x_{m-1}, x_m)}{n!}\}_{n \geq 0}$  are logarithmically convex and logarithmically concave respectively. Generally, since

$$\sum_{n=0}^{\infty} Q_{k,n}(x_{m-k+1}, \dots, x_m) \frac{t^n}{n!} = \exp \left[ \sum_{n=1}^{\infty} \frac{x_{m-k+1} Q_{k-1,n}(x_{m-k+2}, \dots, x_m) t^n}{(n-1)! n} \right]$$

for  $2 \leq k \leq m$ , by the above mentioned [1, Lemma 1] and [7, Theorem 1], we can inductively conclude that, when  $x_1, \dots, x_{m-1} \geq 2$  and  $x_m \geq 1$ , the sequences  $\{Q_{m,n}(\mathbf{x}_m)\}_{n \geq 0}$  and  $\{\frac{Q_{m,n}(\mathbf{x}_m)}{n!}\}_{n \geq 0}$  are logarithmically convex and logarithmically concave respectively.

Combining the above mentioned [1, Theorem 2] with the logarithmic convexity and logarithmic concavity of the sequences  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1, m \geq 2\}_{n \geq 0}$  and  $\{\frac{Q_{m,n}(\mathbf{x}_m)}{n!}, x_1, \dots, x_{m-1} \geq 2, x_m \geq 1, m \geq 2\}_{n \geq 0}$  respectively leads to the double inequality (4.6). The proof of Theorem 4.2 is complete.  $\square$

**Theorem 4.3.** *Let  $x_k > 0$  for  $1 \leq k \leq m$ . For  $q \geq 0$  and  $n \in \mathbb{N}$ , we have*

$$\left[ \prod_{\ell=0}^n Q_{m,q+2\ell+1}(\mathbf{x}_m) \right]^{1/(n+1)} \geq \left[ \prod_{\ell=0}^{n-1} Q_{m,q+2\ell+2}(\mathbf{x}_m) \right]^{1/n}. \quad (4.8)$$

*Proof.* If  $f(t)$  is a completely monotonic function on  $(0, \infty)$ , then, by the convexity of the sequence (4.7) and Nanson's inequality listed in [31, p. 205, 3.2.27],

$$\left[ \prod_{\ell=0}^n (-1)^{q+2\ell+1} f^{(q+2\ell+1)}(t) \right]^{1/(n+1)} \geq \left[ \prod_{\ell=1}^n (-1)^{q+2\ell} f^{(q+2\ell)}(t) \right]^{1/n}$$

for  $q \geq 0$ . Replacing  $f(t)$  by  $G(-t; \mathbf{x}_m)$  in the above inequality results in

$$\left[ \prod_{\ell=0}^n (-1)^{q+1} (G(-t; \mathbf{x}_m))_t^{(q+2\ell+1)} \right]^{1/(n+1)} \geq \left[ \prod_{\ell=1}^n (-1)^q (G(-t; \mathbf{x}_m))_t^{(q+2\ell)} \right]^{1/n}$$

for  $q \geq 0$ . Letting  $t \rightarrow 0^+$  in the above inequality leads to (4.8). The proof of Theorem 4.3 is complete.  $\square$

**Theorem 4.4.** *Let  $x_k > 0$  for  $1 \leq k \leq m$ . If  $\ell \geq 0$ ,  $n \geq k \geq q$ ,  $2k \geq n$ , and  $2q \geq n$ , then*

$$Q_{m,k+\ell}(\mathbf{x}_m) Q_{m,n-k+\ell}(\mathbf{x}_m) \geq Q_{m,q+\ell}(\mathbf{x}_m) Q_{m,n-q+\ell}(\mathbf{x}_m). \quad (4.9)$$

*Proof.* In [58, p. 397, Theorem D], it was recovered that, if  $f(t)$  is a completely monotonic function on  $(0, \infty)$  and if  $n \geq k \geq q$ ,  $k \geq n - k$ , and  $q \geq n - q$ , then

$$(-1)^n f^{(k)}(t) f^{(n-k)}(t) \geq (-1)^n f^{(q)}(t) f^{(n-q)}(t).$$

Replacing  $f(t)$  by the function  $(-1)^\ell [G(-t; \mathbf{x}_m)]_t^{(\ell)}$  in the above inequality leads to

$$\begin{aligned} (-1)^n [G(-t; \mathbf{x}_m)]_t^{(k+\ell)} [G(-t; \mathbf{x}_m)]_t^{(n-k+\ell)} \\ \geq (-1)^n [G(-t; \mathbf{x}_m)]_t^{(q+\ell)} [G(-t; \mathbf{x}_m)]_t^{(n-q+\ell)}. \end{aligned}$$

Further taking  $t \rightarrow 0^+$  finds the inequality (4.9). The proof of Theorem 4.4 is complete.  $\square$

**Theorem 4.5.** *Let  $x_k > 0$  for  $1 \leq k \leq m$ . For  $\ell \geq 0$  and  $q, n \in \mathbb{N}$ , let*

$$\begin{aligned} \mathcal{G}_{m,\ell,q,n} &= Q_{m,\ell+2q+n}(\mathbf{x}_m) [Q_{m,\ell}(\mathbf{x}_m)]^2 - Q_{m,\ell+q+n}(\mathbf{x}_m) Q_{m,\ell+q}(\mathbf{x}_m) Q_{m,\ell}(\mathbf{x}_m) \\ &\quad - Q_{m,\ell+n}(\mathbf{x}_m) Q_{m,\ell+2q}(\mathbf{x}_m) Q_{m,\ell}(\mathbf{x}_m) + Q_{m,\ell+n}(\mathbf{x}_m) [Q_{m,\ell+q}(\mathbf{x}_m)]^2, \\ \mathcal{H}_{m,\ell,q,n} &= Q_{m,\ell+2q+n}(\mathbf{x}_m) [Q_{m,\ell}(\mathbf{x}_m)]^2 - 2Q_{m,\ell+q+n}(\mathbf{x}_m) Q_{m,\ell+q}(\mathbf{x}_m) Q_{m,\ell}(\mathbf{x}_m) \\ &\quad + Q_{m,\ell+n}(\mathbf{x}_m) [Q_{m,\ell+q}(\mathbf{x}_m)]^2, \\ \mathcal{I}_{m,\ell,q,n} &= Q_{m,\ell+2q+n}(\mathbf{x}_m) [Q_{m,\ell}(\mathbf{x}_m)]^2 - 2Q_{m,\ell+n}(\mathbf{x}_m) Q_{m,\ell+2q}(\mathbf{x}_m) Q_{m,\ell}(\mathbf{x}_m) \\ &\quad + Q_{m,\ell+n}(\mathbf{x}_m) [Q_{m,\ell+q}(\mathbf{x}_m)]^2. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{G}_{m,\ell,q,n} &\geq 0, \quad \mathcal{H}_{m,\ell,q,n} \geq 0, \\ \mathcal{H}_{m,\ell,q,n} &\leq \mathcal{G}_{m,\ell,q,n} \quad \text{when } q \leq n, \\ \mathcal{I}_{m,\ell,q,n} &\geq \mathcal{G}_{m,\ell,q,n} \geq 0 \quad \text{when } n \geq q. \end{aligned} \quad (4.10)$$

*Proof.* In [59, Theorem 1 and Remark 2], it was obtained that, if  $f(t)$  is completely monotonic on  $(0, \infty)$  and

$$\begin{aligned} G_{q,n}(t) &= (-1)^n \{ f^{(n+2q)}(t) f^2(t) - f^{(n+q)}(t) f^{(q)}(t) f(t) \\ &\quad - f^{(n)}(t) f^{(2q)}(t) f(t) + f^{(n)}(t) [f^{(q)}(t)]^2 \}, \end{aligned}$$



$$\begin{aligned} H_{q,n}(t) &= (-1)^n \{ f^{(n+2q)}(t) f^2(t) - 2f^{(n+q)}(t) f^{(q)}(t) f(t) + f^{(n)}(t) [f^{(q)}(t)]^2 \}, \\ I_{q,n}(t) &= (-1)^n \{ f^{(n+2q)}(t) f^2(t) - 2f^{(n)}(t) f^{(2q)}(t) f(t) + f^{(n)}(t) [f^{(q)}(t)]^2 \} \end{aligned}$$

for  $n, q \in \mathbb{N}$ , then

$$\begin{aligned} G_{q,n}(t) &\geq 0, \quad H_{q,n}(t) \geq 0, \\ H_{q,n}(t) &\leq G_{q,n}(t) \quad \text{when } q \leq n, \\ I_{q,n}(t) &\geq G_{q,n}(t) \geq 0 \quad \text{when } n \geq q. \end{aligned} \tag{4.11}$$

Replacing  $f(t)$  by  $(-1)^\ell [G(-t; \mathbf{x}_m)]_t^{(\ell)}$  in  $G_{q,n}(t)$ ,  $H_{q,n}(t)$ , and  $I_{q,n}(t)$  and simplifying produce

$$\begin{aligned} G_{q,n}(t) &= (-1)^{\ell+n} \{ [G(-t; \mathbf{x}_m)]_t^{(\ell+2q+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \}^2 \\ &\quad - [G(-t; \mathbf{x}_m)]_t^{(\ell+q+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+q)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \\ &\quad - [G(-t; \mathbf{x}_m)]_t^{(\ell+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+2q)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \\ &\quad + [G(-t; \mathbf{x}_m)]_t^{(\ell+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+q)} \}^2, \\ H_{q,n}(t) &= (-1)^{\ell+n} \{ [G(-t; \mathbf{x}_m)]_t^{(\ell+2q+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \}^2 \\ &\quad - 2[G(-t; \mathbf{x}_m)]_t^{(\ell+q+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+q)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \\ &\quad + [G(-t; \mathbf{x}_m)]_t^{(\ell+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+q)} \}^2, \\ I_{q,n}(t) &= (-1)^{\ell+n} \{ [G(-t; \mathbf{x}_m)]_t^{(\ell+2q+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \}^2 \\ &\quad - 2[G(-t; \mathbf{x}_m)]_t^{(\ell+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+2q)} [G(-t; \mathbf{x}_m)]_t^{(\ell)} \\ &\quad + [G(-t; \mathbf{x}_m)]_t^{(\ell+n)} [G(-t; \mathbf{x}_m)]_t^{(\ell+q)} \}^2. \end{aligned}$$

Further taking  $t \rightarrow 0^+$  reveals

$$\lim_{t \rightarrow 0^+} G_{q,n}(t) = \mathcal{G}_{m,\ell,q,n}, \quad \lim_{t \rightarrow 0^+} H_{q,n}(t) = \mathcal{H}_{m,\ell,q,n}, \quad \lim_{t \rightarrow 0^+} I_{q,n}(t) = \mathcal{I}_{m,\ell,q,n}.$$

Substituting these quantities into (4.11) and simplifying bring about inequalities in (4.10). The proof of Theorem 4.5 is complete.  $\square$

*Remark 4.1.* When taking  $x_1 = \dots = x_m = 1$ , all results in this section can become conclusions for multi-order exponential numbers  $Q_{m,n}$ .

## 5. AN APPLICATION TO WHITE NOISE DISTRIBUTION THEORY

In this section, we finally find an application of multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  to white noise distribution theory by confirming that the polynomial sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  satisfies conditions for sequences required in white noise distribution theory.

Let  $\{\alpha_n\}_{n \geq 0}$  be a sequence of positive numbers. In [1, 3, 4, 16, 25] and closely related references therein, for studying the spaces of test and generalized functions

and their characterization theorems in white noise distribution theory [26], the following conditions for the sequence  $\{\alpha_n\}_{n \geq 0}$  are required:

$$\alpha_0 = 1, \quad \inf_{n \geq 0} (\alpha_n \sigma^n) > 0, \quad \lim_{n \rightarrow \infty} \left( \frac{\alpha_n}{n!} \right)^{1/n} = 0, \quad \lim_{n \rightarrow \infty} \left( \frac{1}{n! \alpha_n} \right)^{1/n} = 0, \quad (5.1)$$

$$\limsup_{n \rightarrow \infty} \left[ \frac{n!}{\alpha_n} \inf_{x > 0} \frac{G_\alpha(x)}{x^n} \right]^{1/n} < \infty, \quad \limsup_{n \rightarrow \infty} \left[ n! \alpha_n \inf_{x > 0} \frac{G_{1/\alpha}(x)}{x^n} \right]^{1/n} < \infty, \quad (5.2)$$

$$\text{the sequence } \left\{ \frac{\alpha_n}{n!} \right\}_{n \geq 0} \text{ is logarithmically concave,} \quad (5.3)$$

$$\text{the sequence } \left\{ \frac{1}{n! \alpha_n} \right\}_{n \geq 0} \text{ is logarithmically concave,} \quad (5.4)$$

$$\text{the sequence } \{\alpha_n\}_{n \geq 0} \text{ is logarithmically convex,} \quad (5.5)$$

$$\text{there exists a constant } c_1 \text{ such that } \alpha_n \leq c_1^m \alpha_m \text{ for all } n \leq m, \quad (5.6)$$

$$\text{there exists a constant } c_2 \text{ such that } \alpha_{m+n} \leq c_2^{m+n} \alpha_m \alpha_n \text{ for all } m, n \geq 0, \quad (5.7)$$

$$\text{there exists a constant } c_3 \text{ such that } \alpha_m \alpha_n \leq c_3^{m+n} \alpha_{m+n} \text{ for all } m, n \geq 0, \quad (5.8)$$

where  $\sigma \geq 1$  is a constant,

$$G_\alpha(x) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} x^n, \quad G_{1/\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! \alpha_n}.$$

For details, please read [4, Appendix A] and closely related references therein.

Theorem 4.3 in [16] proved that the condition (5.3) implies the first one in (5.2). It is easy to check that the first two conditions in (5.1) implies the fourth one in (5.1). In [2], it was showed that the condition (5.4) implies the second one in (5.2), while (5.5) implies (5.4). In [25], it was pointed out that the condition (5.8) implies (5.6). In [4, p. 83], it was concluded that the essential conditions for distribution theory on a CKS-space are the first three in (5.1) and the conditions (5.3), (5.4), (5.7), and (5.8).

The sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  clearly satisfies the first two conditions in (5.1). Theorem 4.2 in this paper shows that the sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  satisfies the conditions (5.3) and (5.5). The left inequalities in (4.5) and (4.6) mean that taking  $c_3 = 1$  in (5.8) is sound. Since  $\binom{p+q}{p} \leq 2^{p+q}$  for  $p, q \geq 0$ , the right inequalities in (4.5) and (4.6) imply that the condition (5.7) applied to the sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  is valid for  $c_2 = 2$ . Since the generating function  $G(t; \mathbf{x}_m)$  of multivariate exponential polynomials  $Q_{m,n}(\mathbf{x}_m)$  is an entire function of  $t \in \mathbb{C}$ , by the root test, the sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  satisfies the third condition in (5.1). In conclusion, the sequence  $\{Q_{m,n}(\mathbf{x}_m), x_1, \dots, x_{m-1} \geq 2, x_m \geq 1\}_{n \geq 0}$  satisfies all the essential conditions for sequences required in distribution theory on a CKS-space.

## 6. MORE REMARKS

Finally we would like to list more remarks related to something in this paper.

*Remark 6.1.* In [37, 38], multivariate logarithmic polynomials and their generating function, the inverse of the generating function  $G(t; \mathbf{x}_m)$ , were investigated.

*Remark 6.2.* On 6 September 2017, Boyadzhiev wrote an e-mail to the first author and clarified the history of the Touchard polynomials  $T_n(x)$  as follows. The polynomials  $T_n(x)$  were used as early as 1843 in the works of Grunert (see [9, 10, 22]) and possibly could have been used before him. Bell [5, 6] called them “exponential polynomials”, so did Touchard [55, 56, 57]. They were called exponential polynomials also by Rota [53] and by Boyadzhiev [8, 9, 10, 11, 12, 13]. Touchard has no much contribution to the theory. Most properties were found by Grunert, Bell, and, for example, in the papers [8, 10, 14, 18, 19]. Using the name “Touchard polynomials” could be misleading.

*Remark 6.3.* One of anonymous referees pointed out that the exponential generating functions  $e^x$ ,  $e^{e^x-1}$ , and  $e^{e^{e^x-1}-1}$  have some combinatorial meanings. See [21, pp. 99 and 111], for example.

*Remark 6.4.* This paper is a revised version of the preprints [46, 47, 48].

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