INFINITE TRANSITIVITY, FINITE GENERATION, AND DEMAZURE ROOTS

M Zaidenberg, I Arzhantsev, K Kuyumzhiyan

INFINITE TRANSITIVITY, FINITE GENERATION, AND DEMAZURE ROOTS

I. ARZHANTSEV, K. KUYUMZHIYAN, AND M. ZAIDENBERG

Abstract. An affine algebraic variety X of dimension ≥ 2 is called *flexible* if the subgroup $\mathrm{SAut}(X) \subset \mathrm{Aut}(X)$ generated by the one-parameter unipotent subgroups acts m-transitively on $\mathrm{reg}\,(X)$ for any $m \geq 1$. In a preceding paper ([4]) we proved that any nondegenerate toric affine variety X is flexible. Here we show that if such a toric variety X is smooth in codimension 2 then one can find a subgroup of $\mathrm{SAut}(X)$ generated by a finite number of one-parameter unipotent subgroups which has the same transitivity property. In fact, four such subgroups are enough for $X = \mathbb{A}^n$ if $n \geq 3$, and just three if $n = 2$.

Contents

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let X be an affine variety over $\mathbb{K}$. A one-parameter subgroup H of $\mathrm{Aut}(X)$ isomorphic as an algebraic group to the additive group $\mathbb{G}_a$ of the base field $\mathbb{K}$ is called a *unipotent* one-parameter subgroup, or a $\mathbb{G}_a$*-subgroup*, for short. One can consider the subgroup $\mathrm{SAut}(X) \subset \mathrm{Aut}(X)$ generated by all the one-parameter unipotent subgroups. It is known ([4, Thm. 2.1]) that for a toric variety X with no toric factor (that is, a nondegenerate toric variety) the group $\mathrm{SAut}(X)$ acts infinitely transitively in the smooth locus $\mathrm{reg}(X)$, that is, m-transitively for any $m \geq 1$. Varieties X with this property are called *flexible* ([2], [4]). Actually, the simple transitivity of $\mathrm{SAut}(X)$ in

The first author was supported by the grant RCF-DFG 16-41-01013.

This work was done during a stay of the first and the second authors at the Institut Fourier, Grenoble, France. They would like to thank this institution for hospitality, support, and excellent working conditions.

2010 *Mathematics Subject Classification:* 14R20, 32M17.

Key words: affine variety, toric variety, group action, one-parameter subgroup, Demazure root, transitivity.

$\mathrm{reg}(X)$ already guarantees that X is flexible ([2, Thm. 0.1]). The same is true for quasi-affine varieties ([5], [19]). In turn, the flexibility implies several other useful properties, for instance, the unirationality (see, e.g., [2], [11], [42]). The flexibility has found important applications, e.g., to the Zariski cancellation problem ([20]). It is known ([19, Thm. 1.1]) that the flexibility survives upon passing to the complement of a subvariety of codimension at least 2. Different flexibility properties are intensively studied in complex analytic geometry, see, e.g., surveys [2], [21], [28].

The toric affine varieties with no toric factors are flexible ([4, Thm. 2.1]). There are several other interesting classes of flexible affine varieties (see, e.g., [2], [4], [5], [6], [16], [38], [39], [40], [42], [43], [45]).

In fact, for a flexible X certain proper subgroups G of $\mathrm{SAut}(X)$ act also infinitely transitively in $\mathrm{reg}(X)$, or at least in a Zariski dense open subset of $\mathrm{reg}(X)$. This is the case for a subgroup G generated by a sufficiently rich family of $\mathbb{G}_a$-subgroups of $\mathrm{SAut}(X)$, see [2, Thm. 2.2]. Let us stay on this in more detail.

Let $\mathrm{LND}(X)$ stand for the set of all nonzero locally nilpotent derivations (LNDs, for short) of the structure algebra $\mathcal{O}_X(X)$. Any $\partial \in \mathrm{LND}(X)$ is a generator of a $\mathbb{G}_a$-subgroup $H = \exp(\mathbb{K}\partial)$ of $\mathrm{Aut}(X)$, and any $\mathbb{G}_a$-subgroup $H \subset \mathrm{Aut}(X)$ has the form $H = \exp(\mathbb{K}\partial)$ for some $\partial \in \mathrm{LND}\,(X)$. If $a \in \ker \partial$ then $a\partial$ is again an LND called a *replica* of ∂. The subgroup $H(a) = \exp(\mathbb{K}a\partial)$ is called a *replica* of H.

A family $\mathfrak{F}$ of $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(X)$ is called *saturated* if
(i) any replica of $H \in \mathfrak{F}$ belongs to $\mathfrak{F}$, and
(ii) $\mathfrak{F}$ is closed under conjugation by the elements of the subgroup $G = G(\mathfrak{F})$ generated by the members of $\mathfrak{F}$ ([2, Def. 2.1]).
Notice ([20, Lem. 4.6]) that for any family $\mathfrak{F}$ which verifies (i) there exists a larger family $\mathfrak{F}'$ satisfying both (i) and (ii) such that $G(\mathfrak{F}) = G(\mathfrak{F}')$. If $G(\mathfrak{F})$ has an open orbit in X and $\mathfrak{F}$ is saturated then the action of G on this orbit is infinitely transitive ([2, Thm. 2.2]). By [19, Prop. 2.15] one can find such a family $\mathfrak{F}$ composed by the replicas of just two LNDs.

We say that X is *generically flexible* if $\mathrm{SAut}(X)$ acts on X with an open orbit. For instance ([26]), any Gizatullin surface X is generically flexible. Notice that the open orbit of $\mathrm{Aut}(X)$ can be smaller than $\mathrm{reg}(X)$, see, e.g., [30] for examples of Gizatullin surfaces with this property.

The assumption of saturation could be too restrictive in applications. The aim of the present paper is to elaborate more moderate conditions on the family $\mathfrak{F}$ which still guarantee the infinite transitivity of $G(\mathfrak{F})$ on the open orbit. In Section 2 we formulate such conditions, see Theorem 2.2. It occurs that for a generically flexible variety already a countable family of $\mathbb{G}_a$-subgroups generates a group acting infinitely transitively on its open orbit, see Corollary 2.8. A useful observation in Section 3 consists in the fact that the orbits and transitivity of an algebraically generated group $G \subset \mathrm{Aut}(X)$ are not affected upon passing to the closure $\overline{G}$, see Proposition 3.4. The remaining part of the paper appeared as a result of our discussions on the following

Conjecture 1.1. *Any generically flexible affine variety X admits a finite collection $\{H_1, \ldots, H_N\}$ of $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(X)$ such that the group $G = \langle H_1, \ldots, H_N \rangle$ acts infinitely transitively on its open orbit.*

In Section 5 we fix this conjecture for toric affine varieties under a certain mild restriction. In Section 4 we recall some basics on toric varieties and Cox rings. We deal there with the LNDs of the structure ring of a toric affine variety X which are normalized by the acting torus $\mathbb{T}$. The degree e of such an LND ∂ is a lattice vector called a *Demazure root*. The corresponding $\mathbb{G}_a$-subgroup $H_e \subset \mathrm{Aut}(X)$ normalized by $\mathbb{T}$ is called a (*Demazure*) *root subgroup*.

Our approach exploits hardly the following phenomenon. Consider a group G acting effectively on a toric affine variety X and generated by its unipotent subgroups. The closure $\overline{G}$ of G with respect to the ind-topology could contain more Demazure root subgroups than the group G itself. However, the multiple transitivity of $\overline{G}$ on its orbit is inherited by the group G, see Proposition 3.4.

To describe some extra Demazure root subgroups contained in $\overline{G}$ we develop in Section 4.2 certain degeneration techniques. Given an LND $\partial \in \mathrm{Der}(\mathcal{O}_X(X))$ generating a one-parameter unipotent subgroup of G we define its Newton polytope $N(\partial)$ with respect to the acting torus. The extremal points of this polytope correspond to one-parameter unipotent root subgroups which belong to $\overline{G}$, see Proposition 4.16. To find a convenient (non-root) LND ∂ we conjugate one Demazure root subgroup by a second one which does not centralize the first. The Newton polytope $N(\partial)$ of the resulting LND ∂ occurs to be a segment whose one end is a desired extra Demazure root. This segment can be found explicitly by using a version of the Baker-Campbell-Hausdorff formula, see Corollary 4.14.

The simplest toric affine varieties are the affine spaces $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$. In this case both the affine group Aff_n and the group $\mathrm{SL}(n,\mathbb{K})$ extended by just one root subgroup act infinitely transitively on their open orbits, see, e.g., Corollary 5.3. This is based on the results of Bodnarchuk ([9, 10]), Edo ([17]), and Furter ([24]) concerning cotame automorphisms of the affine spaces, see Definition 5.9. We establish the following facts, see Theorems 5.16 and 5.17.

Theorem 1.2. *For any $n \geq 3$ one can find four $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ which generate a subgroup acting infinitely transitively on $\mathbb{A}^n$. The same is true for some $n+2$ Demazure root subgroups. For $n = 2$, the latter holds with just three Demazure root subgroups.*

Our main result in this direction for toric affine varieties (see Theorem 5.19) is the following

Theorem 1.3. *For any toric affine variety X of dimension at least 2, with no toric factor, and smooth in codimention 2 one can find a finite collection of Demazure root subgroups such that the group generated by these ones acts infinitely transitively in the smooth locus $\mathrm{reg}(X)$.*

2. Infinite transitivity on the open orbit

We are working over an algebraically closed field $\mathbb{K}$ of characteristic zero. We let $\mathbb{A}^n$ stand for the affine space of dimension n over $\mathbb{K}$, and $\mathbb{G}_a$ and $\mathbb{G}_m$ for the additive and the multiplicative groups of $\mathbb{K}$, respectively, viewed as algebraic groups.

2.1. Let X be an affine variety over $\mathbb{K}$ of dimension $n \geq 2$. Consider a finite collection of pairwise non-collinear locally nilpotent derivations $\partial_1, \partial_2, \ldots, \partial_k$ of $\mathcal{O}_X(X)$ which contains a subset of n linearly independent derivations. For each $i = 1, \ldots, k$ fix a finitely generated subalgebra $A_i \subset \ker \partial_i$ such that the fraction field $\mathrm{Frac}\,(A_i)$ has finite index in $\mathrm{Frac}\,(\ker \partial_i)$. Consider the following possibilities:

(α) $\mathcal{O}_X(X)$ is generated by $A_1, \ldots, A_k$;

(β) $[\mathrm{Frac}\,(\ker\partial_i):\mathrm{Frac}\,(A_i)]=1$ for some value of i;
(γ) $[\mathrm{Frac}\,(\ker\partial_i):\mathrm{Frac}\,(A_i)]>1$ for all $i=1,\dots,k$. In the latter case we fix an extra element $b_1\in\ker\partial_1$ such that $\mathrm{Frac}\,(\ker\partial_1)$ is generated by b_1 and $\mathrm{Frac}\,(A_1)$. In cases (α) and (β) one might consider $b_1=0$.

Let G be the subgroup of $\mathrm{SAut}(X)$ generated by the $\mathbb{G}_a$-subgroups

$$H_0=\exp(\mathbb{K}b_1\partial_1)\ \text{ and }\ H_i(a_i)=\exp(\mathbb{K}a_i\partial_i)\quad\text{where}\quad a_i\in A_i,\ \ i=1,\dots,k\,.$$

Notice that G acts on X with an open orbit $\mathscr{O}_G$, see [2, Corollary 1.11a].

The following theorem is the main result of this section.

Theorem 2.2. *The action of G on $\mathscr{O}_G$ is infinitely transitive.*

This theorem is actually a refined version of Theorem 2.2 in [2]. Its proof follows, with some modifications, the lines of the proof of Theorem 2.2 in [2].

Lemma 2.3. *Let $\Omega\subset\mathscr{O}_G$ be a dense open subset. Then for any finite collection of distinct points $Q_1,Q_2,\dots,Q_m\in\mathscr{O}_G$ there exists $g\in G$ such that $g(Q_i)\in\Omega$ for every $i=1,2,\dots,m$.*

Proof. By [2, Prop. 1.5] there is a finite collection of $\mathbb{G}_a$-subgroups $U_1,U_2,\dots,U_N$ in G such that for any $x\in\mathscr{O}_G$ we have $\mathscr{O}_G=(U_1\cdot\ldots\cdot U_N).x$. This gives a surjective morphism

$$\varphi_x\colon\mathbb{A}^N\to\mathscr{O}_G,\quad (t_1,\dots,t_N)\mapsto(U_1(t_1)\cdot\ldots\cdot U_N(t_N)).x\,.$$

Letting $\varphi_i=\varphi_{Q_i}$, $i=1,\dots,m$ consider the dense open subset

$$\omega=\bigcap_{i=1}^m\varphi_i^{-1}(\Omega)\subset\mathbb{A}^N\,.$$

Pick up a point $(t_1,t_2,\dots,t_N)\in\omega$, and let $g=U_1(t_1)\cdot\ldots\cdot U_N(t_N)\in G$. Then for any $i=1,\dots,m$ one has $g(Q_i)\in\Omega$. □

In the sequel we use the following notation.

Notation 2.4. Let $H_i=H_i(1)$, $i=1,\dots,k$. Letting $W_i=\mathrm{Spec}\,A_i$ consider the morphism $\pi_i\colon X\to W_i$ induced by the inclusion $A_i\hookrightarrow\mathcal{O}_X(X)$. There is a Zariski open, dense subset $\omega_i\subset W_i$ such that on $U_i:=\pi_i^{-1}(\omega_i)$ there exists the geometric quotient U_i/H_i. The inclusion $A_i\subset\mathcal{O}_{U_i}(U_i)$ induces a generically finite morphism $q_i\colon U_i/H_i\to W_i$. Due to our assumption, ω_i can be chosen so that q_i is a finite morphism onto its image, of degree $d_i:=[\mathrm{Frac}\,(\ker\partial_i):\mathrm{Frac}\,(A_i)]$. One can find a dense open subset $\Omega\subset\mathscr{O}_G$ such that

(i) in each point $x\in\Omega$ the vectors $\partial_1(x),\dots,\partial_k(x)$ are pairwise non-collinear and generate the tangent space T_xX;
(ii) for $i=1,\dots,k$ one has $\pi_i(\Omega)\subset\mathrm{reg}(W_i)$ and the restriction $\pi_i|_\Omega\colon\Omega\to W_i$ is a smooth morphism;
(iii) for $i=1,\dots,k$ the fiber of $\pi_i|_\Omega$ over any point $w\in\pi_i(\Omega)$ is a dense open subset of the union of d_i orbits of H_i;
(iv) in case (γ) for $i=1$ these orbits are separated by the function $b_1\in\ker\partial_1$ as in 2.1.

For each $i=1,\dots,k$ there is a factorization

$$\pi_i\colon\Omega\xrightarrow{p_i}\Omega/H_i\xrightarrow{q_i}W_i\,.$$

We have the following analogue of Lemma 2.10 in [2].

Lemma 2.5. *For any pair of distinct points $Q_1, Q_2 \in \mathscr{O}_G$ there exists $g \in G$ such that for every $i = 1, \dots, k$ the points $g.Q_1$ and $g.Q_2$ are separated by A_i for $i = 1, \dots, k$.*

Proof. By Lemma 2.3 one may suppose that $Q_1, Q_2 \in \Omega$ where $\Omega \subset X$ is as in 2.4. Assume first that for some $i \in \{1, \dots, k\}$ the algebra A_i separates Q_1 and Q_2. This is so in cases (α) and (β); anyway, one may consider that $i = 1$. Let $a_1 \in A_1$ be such that $a_1(Q_1) = 0$ and $a_1(Q_2) = 1$. Then $H_1(a_1)$ fixes Q_1 and moves Q_2 along its H_1-orbit. Let

$$Q_2(t) = \exp(t\partial_1)(Q_2), \quad t \in \mathbb{K}.$$

Given $i \geq 2$ the condition

$$Q_2(t) \in \Omega \quad \text{and} \quad \pi_i(Q_2(t)) \neq \pi_i(Q_1) \tag{1}$$

is an open condition in $t \in \mathbb{K}$. Since $Q_2 \in \Omega$, by (i) and (ii) the image $\pi_i(H_1(Q_2))$ in W_i is one-dimensional. It follows that (1) holds on a dense open subset in $\mathbb{K}$. Moreover, the latter is true simultaneously for all $i = 2, \dots, k$, as required.

Now one may restrict to case (γ). Suppose that A_1 does not separate Q_1 and Q_2. Assume further that $H_1(Q_1) \neq H_1(Q_2)$. Then b_1 separates Q_1 and Q_2 due to (iv).

Consider the flow

$$\phi_t = \exp(t(b_1 - b_1(Q_1))\partial_1) \subset H_0 \cdot H_1 \subset G,$$

and let $Q_2(t) = \phi_t(Q_2)$. Then ϕ_t fixes Q_1 and moves Q_2 along its H_1-orbit. Now the same argument as before applies and proves that the algebra A_i, $i = 2, \dots, k$ separates Q_1 and $Q_2(t)$ for a general $t \in \mathbb{K}$. Since by our assumptions $k \geq n \geq 2$, one may interchange now the role of A_1 and A_k and achieve as before that A_1 separates the images of Q_1 and Q_2 under the action of $\exp(ta_k\partial_k)(Q_1)$ for a suitable $a_k \in A_k$ and a general $t \in \mathbb{K}$. This gives the result.

Suppose further that $H_1(Q_1) = H_1(Q_2)$. We claim that for every $i = 2, \dots, k$ and for a general $t \in \mathbb{K}$ the points $Q_1(t)$ and $Q_2(t)$ are separated by A_i. Indeed, assume to the contrary that $\pi_i(Q_1(t)) = \pi_i(Q_2(t))$ for some $i \geq 2$ and for all $t \in \mathbb{K}$. Since the image $\pi_i(H_1(Q_1))$ in W_i is one-dimensional there exists $a_i \in A_i$ such that the restriction $a_i|_{H_1(Q_1)}$ defines a non-constant polynomial $p_i \in \mathbb{K}[t]$. Since $Q_2 = Q_1(\tau)$ for some nonzero $\tau \in \mathbb{K}$ one has $Q_2(t) = Q_1(t + \tau)$. It follows that $p_i(t) = p_i(t + \tau)$ for any $t \in \mathbb{K}$, a contradiction. The proof ends by the argument used in the previous case for $i = 1$. □

Lemma 2.6. *For any finite collection of distinct points $Q_1, \dots, Q_m \in \mathscr{O}_G$ there exists an element $g \in G$ such that the points $g(Q_1), \dots, g(Q_m)$ are separated by A_i for $i = 1, \dots, k$.*

Proof. By Lemma 2.3 we may assume that $Q_j \in \Omega$ $\forall i = 1, \dots, m$. We proceed by induction on m. For $m = 1$ the assertion is evidently true. Assume that the points $Q_1, \dots, Q_{m-1}$ are already separated by A_i for $i = 1, \dots, k$. Applying Lemma 2.5 and its proof to Q_m and Q_1 one may replace the cortege $(Q_1, \dots, Q_m)$ by a new one $(Q_1^{(1)}(t), \dots, Q_m^{(1)}(t))$ so that the separation still holds for $Q_1^{(1)}(t), \dots, Q_{m-1}^{(1)}(t)$ with a generic $t \in \mathbb{K}$, and in addition it holds for $Q_1^{(1)}(t)$ and $Q_m^{(1)}(t)$. Fixing such a value $t_1 \in \mathbb{K}$ one may apply the same procedure to obtain a new cortege $(Q_j^{(2)}(t_2))_{j=1,\dots,m}$ preserving the former property and adding the separation of $Q_2^{(2)}(t_2)$ and $Q_m^{(2)}(t_2)$, and so for. Finally one arrives at a cortege with the desired separation property. □

Lemma 2.7. *For any finite collection of distinct points $Q_1, \dots, Q_m \in \mathscr{O}_G$ the stabilizer $\mathrm{Stab}_{Q_1,\dots,Q_m}(G)$ acts transitively on $\mathscr{O}_G \setminus \{Q_1, \dots, Q_m\}$.*

Proof. We proceed by induction on m. The assertion is evidently true for $m = 0$. Assuming that it holds for a given $m \geq 0$ consider a collection of $m+1$ distinct points $Q_1, \ldots, Q_m, Q_{m+1} \in \mathscr{O}_G$. By Lemma 2.3 one may assume that these points lie in Ω. Applying Lemma 2.6 one may suppose that for $i = 1, \ldots, k$ the images $\pi_i(Q_j) \in W_i$, $j = 1, \ldots, m+1$ are all distinct. Then for every $i = 1, \ldots, k$ there exists $a_i \in A_i$ which vanishes in $Q_1, \ldots, Q_m$ and does not vanish in Q_{m+1}.

Consider the $\mathbb{G}_a$-subgroups

$$H_i(a_i) \subset \mathrm{Stab}_G(Q_1, \ldots, Q_m), \quad i = 1, \ldots, k\,.$$

The orbit of Q_{m+1} under the action of the stabilizer $\mathrm{Stab}_G(Q_1, \ldots, Q_m)$ is locally closed (see, e.g., Proposition 1.3 in [2]) and contains the one-dimensional H_i-orbits of Q_{m+1}, $i = 1, \ldots, k$. By (i) the tangent vectors to these orbits at Q_{m+1} span the tangent space $T_{Q_{m+1}}X$. It follows that the orbit $\mathrm{Stab}_G(Q_1, \ldots, Q_m)(Q_{m+1})$ is open in X whatever is the point $Q_{m+1} \in \mathscr{O}_G \setminus \{Q_1, \ldots, Q_m\}$. Since an open dense orbit is unique one has

$$\mathrm{Stab}_G(Q_1, \ldots, Q_m)(Q_{m+1}) = \mathscr{O}_G \setminus \{Q_1, \ldots, Q_m\}\,.$$

□

Proof of Theorem 2.2. We have to show that for any two ordered corteges $(Q_1, \ldots, Q_m)$ and $(Q'_1, \ldots, Q'_m)$ in $\mathscr{O}_G$ there is $g \in G$ such that $g.Q_j = Q'_j$, $j = 1, \ldots, m$. Assuming by induction that $Q_i = Q'_i$, $i = 1, \ldots, m-1$, by Lemma 2.7 one can find $g \in \mathrm{Stab}_G(Q_1, \ldots, Q_{m-1})$ such that $g.Q_m = Q'_m$, as required. □

Corollary 2.8. *Let X be a generically flexible affine variety* [1] *of dimension $n \geq 2$. Then there exists a countable collection of $\mathbb{G}_a$-subgroups $\{H_1, \ldots, H_n, \ldots\}$ such that the subgroup*

$$G = \langle H_i \,|\, i \in \mathbb{N} \rangle \subset \mathrm{SAut}(X)$$

acts on X with an open orbit $\mathscr{O}_G$ and is infinitely transitive on $\mathscr{O}_G$.

Proof. The generic flexibility of X implies that there is a collection of n linearly independent LNDs $\partial_1, \ldots, \partial_n \in \mathrm{LND}(X)$. Letting $H_i = \exp(\mathbb{K}\partial_i)$ choose for any $i = 1, \ldots, n$ a finitely generated subalgebra $A_i \subset \ker \partial_i$ separating the general H_i-orbits, and let $\{a_{i,j}\}_{j \in \mathbb{N}}$ be a countable Hamel basis of A_i viewed as a vector space over $\mathbb{K}$. Letting $H_{i,j} = \exp(\mathbb{K}a_{i,j}\partial_i)$ consider the group

$$G = \langle H_{i,j} \,|\, i = 1, \ldots, n,\ j \in \mathbb{N} \rangle \subset \mathrm{SAut}(X)\,.$$

Clearly, G acts on X with an open orbit $\mathscr{O}_G$ and satisfies condition (β) of 2.1. Therefore, Theorem 2.2 applies and gives the desired conclusion. □

Example 2.9. Let X be a toric affine variety of dimension ≥ 2 with no toric factor. Then the subgroup $G \subset \mathrm{SAut}(X)$ generated by all the root subgroups H_e, where e runs over the (countable) set of all the Demazure roots of X (see Section 4) acts infinitely transitively in $\mathrm{reg}(X)$. This follows from Corollary 2.8 or, alternatively, from the proof of Theorem 2.1 in [4].

[1] That is, the group $\mathrm{SAut}(X)$ acts on X with an open orbit.

3. Group closures and orbits

We gather some facts that will be used in the next section.

3.1. Recall that $\mathrm{Aut}(X)$ has a structure of an affine ind-group; see, e.g., [33] for generalities. In more detail, following [25] or [31, Prop. 2.1] we fix an embedding $X \hookrightarrow \mathbb{A}^n$ and introduce in $\mathcal{O}_X(X)$ a (positive) degree function. For $\alpha \in \mathrm{Aut}(X)$ one defines $\deg(\alpha)$ to be the maximum of the degrees of components of α. One can write $\mathrm{Aut}(X) = \varinjlim \Sigma_s$ where

- for $s \geq 1$, $\Sigma_s := \{\alpha \in \mathrm{Aut}(X) \mid \deg(\alpha), \deg(\alpha^{-1}) \leq s\}$ is a closed subvariety of the affine variety Σ_{s+1};
- for any $r, s \geq 1$ the composition yields a morphism $\Sigma_r \times \Sigma_s \to \Sigma_{rs}$;
- the inversion yields an automorphism of Σ_s.

The Zariski closure of a subset $F \subset \mathrm{Aut}(X)$ can be defined as

$$\overline{F} = \varinjlim \overline{(F \cap \Sigma_s)}$$

where the overline stands for the Zariski closure in Σ_s. The Zariski closure of F is a closed ind-subvariety of the ind-variety $\mathrm{Aut}(X)$. An *algebraic subgroup* of $\mathrm{Aut}(X)$ is a subgroup which is a closed subvariety of some Σ_s.

Lemma 3.2. (a) *The closure $\overline{G}$ of a subgroup $G \subset \mathrm{Aut}(X)$ is a closed ind-subgroup of $\mathrm{Aut}(X)$.*

(b) *If $\rho\colon \mathbb{A}^1 \to \mathrm{Aut}(X)$ is a morphism such that $\rho(t) \in G$ for $t \neq 0$ then $\rho(0) \in \overline{G}$.*

(c) *Any G-invariant closed subset $Y \subset X$ is $\overline{G}$-invariant.*

(d) *If G acts on X with an open orbit $\mathcal{O}_G$ then $\mathcal{O}_G$ coincides with the open orbit $\mathcal{O}_{\overline{G}}$ of $\overline{G}$.*

(e) *If a normal subgroup $G \subset \mathrm{Aut}(X)$ acts on X with an open orbit $\mathcal{O}_G$ then $\mathcal{O}_G = \mathcal{O}_{\mathrm{Aut}(X)}$.*

Proof. (a) Let $G_s = G \cap \Sigma_s$. By definition, $\overline{G}_s = \overline{G} \cap \Sigma_s$. Since $G_r \cdot G_s \subset G_{rs}$ and $\Sigma_r \times \Sigma_s \to \Sigma_{rs}$ is a morphism then $\overline{G}_r \times \overline{G}_s \to \overline{G}_{rs}$ is a morphism. Since $\Sigma_s^{-1} = \Sigma_s$ and the inversion is an automorphism of Σ_s one has $G_s^{-1} = G_s$ and the inversion $G_s \to \Sigma_s$ extends to a morphism $\overline{G}_s \to \Sigma_s$ which is still the inversion with values in $\overline{G}_s$. Now (a) follows.

(b) One has $\rho(\mathbb{A}^1) \subset \Sigma_s$ for some $s \geq 1$. Hence $\rho(\mathbb{A}^1 \setminus \{0\}) \subset G_s$, and so, $\rho(0) \in \overline{G}_s$.

(c) The statement follows immediately from the fact that the action map $\Sigma_s \times X \to X$ is a morphism for any $s \geq 1$.

(d) Suppose to the contrary that $\mathcal{O}_G \subsetneq \mathcal{O}_{\overline{G}}$. Then $Y = \mathcal{O}_{\overline{G}} \setminus \mathcal{O}_G$ is a nonempty proper G-invariant closed subset of $\mathcal{O}_{\overline{G}}$. By (c), Y is $\overline{G}$-invariant, a contradiction.

The statement of (e) is a simple exercise. □

3.3. A subgroup $G \subset \mathrm{Aut}(X)$ generated by a family of connected algebraic subgroups of $\mathrm{Aut}(X)$ is called *algebraically generated* ([2]). The orbits of G are locally closed subsets of X in the Zariski toplogy; see [2, Prop. 1.3].

Proposition 3.4. *Let $G \subset \mathrm{Aut}(X)$ be an algebraically generated subgroup. Then the following hold.*

(a) *The orbits of G and of $\overline{G}$ in X are the same. In particular, if $\overline{G}$ acts on X with an open orbit $\mathcal{O}_{\overline{G}}$ then G does and $\mathcal{O}_G = \mathcal{O}_{\overline{G}}$.*

(b) *If $\overline{G}$ acts m-transitively on $\mathcal{O}_{\overline{G}}$ then also G does.*

(c) *If $\overline{G}$ acts infinitely transitively on $\mathcal{O}_{\overline{G}}$ then also G does.*

Proof. (a) Let $x \in X$, and let $Y = \overline{G.x}$. By Lemma 3.2(c), Y is $\overline{G}$-invariant. The orbits $G.x$ and $\overline{G}.x \supset G.x$ are both open and dense in Y. Suppose to the contrary that $\overline{G}.x \neq G.x$, and let $Z = \overline{G}.x \setminus G.x$. Then Z is a nonempty $\overline{G}$-invariant closed subset which meets the orbit $\overline{G}.x$ open in Y. Hence $Z \supset \overline{G}.x \supset G.x$. This is a contradiction.

(b) Choose a cortege $\mathcal{Q}$ of distinct points $Q_1, \ldots, Q_m \in \mathscr{O}_{\overline{G}}$. Consider the diagonal action of $\mathrm{Aut}(X)$ on X^m, and let $D \subset X^m$ be the union of the big diagonals. Assume that $\overline{G}$ acts m-transitively on $\mathscr{O}_{\overline{G}}$. Then the $\overline{G}$-orbit of $\mathcal{Q}$ in $X^m \setminus D$ coincides with $(\mathscr{O}_{\overline{G}})^m \setminus D$. In particular, it is open. The image of $\overline{G}$ in $\mathrm{Aut}(X^m)$ is contained in the closure of the image of G in $\mathrm{Aut}(X^m)$. Applying (a) one concludes that $G.\mathcal{Q} = \overline{G}.\mathcal{Q} = (\mathscr{O}_{\overline{G}})^m \setminus D$, that is, G acts m-transitively on $\mathscr{O}_{\overline{G}}$.

Finally, (c) is immediate from (b). □

4. Toric varieties, Cox rings, and derivations

4.1. Toric affine varieties and Demazure roots.

4.1. Recall the combinatorial description of a toric affine variety (see, e.g., [14, Ch. 1], [23, Sec. 1.3]). Let M be a lattice of rank n, let $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ be the associated vector space over $\mathbb{Q}$, and let

$$\sigma^\vee \subset M_{\mathbb{Q}}$$

be a rational convex cone with a nonempty interior (the *weight cone*). Fix a basis of M. For $m = (m_1, \ldots, m_n) \in M$ by χ^m one means a Laurent monomial $x_1^{m_1} \ldots x_n^{m_n}$. Consider the graded affine algebra

$$A = \bigoplus_{m \in M \cap \sigma^\vee} \mathbb{K}\chi^m .$$

Then $X = \operatorname{Spec} A$ is a toric affine variety of dimension n equipped with the n-torus action defined by the grading. In fact, any toric affine variety arises in this way. The acting algebraic torus is the torus of characters $\mathbb{T} = \mathrm{Hom}(M, \mathbb{G}_m)$. By duality, M is the character lattice of $\mathbb{T}$.

Consider the dual lattice $N = \mathrm{Hom}(M, \mathbb{Z})$ and the dual cone

$$\sigma \subset N_{\mathbb{Q}}, \quad \sigma = \{x \in N_{\mathbb{Q}} \,|\, \langle x, y\rangle \geq 0 \ \ \forall y \in \sigma^\vee\} .$$

A *ray generator* of σ is a primitive lattice vector on an extremal ray of σ. Let Ξ be the set of ray generators of σ. Assume that X has no toric factor, that is, X cannot be decomposed into a product $\mathbb{G}_m \times Y$ where Y is another toric variety. The latter is equivalent to the fact that the cone $\sigma^\vee$ is pointed, that is, contains no line, and also to the fact that σ is of full dimension, that is, Ξ contains a basis of $N_{\mathbb{Q}}$. To any vector $\rho \in N$ there corresponds a $\mathbb{G}_m$-subgroup $R_\rho \subset \mathbb{T}$ acting via

$$t.\chi^m = t^{\langle \rho, m\rangle}\chi^m, \quad t \in \mathbb{K}^*, \ m \in \sigma^\vee \cap M .$$

Definition 4.2 (*Demazure roots and Demazure facets*)**.** Let $X = \operatorname{Spec} A$ be a toric affine variety with no toric factor associated to a lattice cone $\sigma^\vee \subset M_{\mathbb{Q}}$, and let $\Xi = \{\rho_1, \ldots, \rho_k\}$ be the set of primitive ray generators of $\sigma \subset N_{\mathbb{Q}}$. A *Demazure root* which belongs to a primitive ray generator $\rho_i \in \Xi$ is a vector $e \in M$ such that

(i) $\langle \rho_i, e\rangle = -1$;
(ii) $\langle \rho_j, e\rangle \geqslant 0 \ \ \forall j \neq i$,

see [15, §3.1], [34], [35]. The rational convex polyhedron $\mathcal{S}_i$ defined in the affine hyperplane $\mathcal{H}_i = \{\langle \rho_i, e\rangle = -1\}$ by (ii) will be called a *Demazure facet* of $\sigma^\vee$. The Demazure roots belonging to the ray generator $\rho_i \in \Xi$ are the points in $\mathcal{S}_i \cap M$.

Definition 4.3 (*Homogeneous derivations*)**.** A derivation $\partial \in \mathrm{Der}(A)$ is called *homogeneous* if ∂ respects the grading, that is, sends any graded piece to another one.

The following description of homogeneous derivations on toric affine varieties completes the one in [35, Sect. 2].

Proposition 4.4. (a) *Any homogeneous derivation ∂ has the form $\partial = \lambda\partial_{\rho,e}$ for some $\lambda \in \mathbb{K}$, $\rho \in N$, and $e \in M$ where*

$$\partial_{\rho,e}(\chi^m) = \langle \rho, m\rangle \chi^{m+e} \quad \forall m \in \sigma^\vee \cap M\,. \tag{2}$$

(The lattice vector e is called the *degree* of ∂.)

(b) *Let*

$$\Sigma^\vee = \sigma^\vee \cup \bigcup_{i=1}^{k} \mathcal{S}_i\,.$$

Then $\partial = \partial_{\rho,e}(A) \subset A$ if and only if $e \in \Sigma^\vee \cap M$ and, in the case where $e \in \mathcal{S}_i \cap M$, $\partial = \lambda\partial_{\rho_i,e}$ for some $\lambda \in \mathbb{K}$.

(c) ([35, Lem. 2.6 and Thm. 2.7]) *A homogeneous derivation $\partial \in \mathrm{Der}(A)$ is locally nilpotent if and only if $\partial = \lambda\partial_{\rho_i,e}$ for a Demazure root $e \in \mathcal{S}_i$ and for some $\lambda \in \mathbb{K}$.*

The proof is straightforward.

Remark 4.5. The kernel of $\partial_{\rho,e}$ is spanned by the characters χ^m where $m \in M$ belongs to the hyperplane section τ_ρ of $\sigma^\vee$ defined by $\langle \rho, m\rangle = 0$. If $e \in \mathcal{S}_i \cap M$ is a Demazure root then $\tau_i := \tau_{\rho_i} = \rho_i^\vee$ is a facet of $\sigma^\vee$. The affine hyperplane $\mathcal{H}_i$ spanned by the Demazure facet $\mathcal{S}_i$ is parallel to τ_i.

Definition 4.6. Given a Demazure root $e \in \mathcal{S}_i$ the associated one-parameter unipotent subgroup $H_e = \exp(\mathbb{K}\partial_{\rho_i,e}) \subset \mathrm{SAut}(X)$ is called a *root subgroup*.

The next lemma is mainly borrowed in [35, Lem. 1.10].

Lemma 4.7. (a) *Any derivation $\partial \in \mathrm{Der}(A)$ admits a decomposition*

$$\partial = \sum_{e \in \Sigma^\vee \cap M} \partial_e \tag{3}$$

where ∂_e is a homogeneous derivation of degree e.

(b) *The set $\{e \in \Sigma^\vee \cap M | \partial_e \neq 0\}$ is finite. Its convex hull $N(\partial)$ is a polytope (called the* Newton polytope of ∂*).*

(c) *If $\partial \in \mathrm{LND}\,(A)$ then for any face τ of $N(\partial)$ one has*

$$\partial_\tau := \sum_{e \in \tau \cap M} \partial_e \in \mathrm{LND}\,(A)\,.$$

In particular, for any vertex e of $N(\partial)$ one has $\partial_e \in \mathrm{LND}\,(A)$.

Proof. To show (c) it suffices to notice that

$$(\partial_\tau)^l(\chi^m) = (\partial^l)_{l\tau}(\chi^m) = 0 \quad \forall m \in \sigma^\vee \cap M \quad \text{and} \quad \forall l = l(m) \gg 1\,.$$

For the rest see [35, Lem. 1.10]. □

The following lemma is immediate.

Lemma 4.8. *The semigroup $\mathcal{S}_i \cap M$ is a finitely generated $(\tau_i \cap M)$-module. For any $f \in \tau_i \cap M$ one has*

$$\chi^f \partial_{\rho_i, e} = \partial_{\rho_i, e+f} \in \mathrm{LND}\,(A)\,.$$

4.1.1. *Iterated commutators on toric varieties.* For homogeneous derivations one has the following lemma (cf. [44, Prop. 1 and Lem. 2]).

Lemma 4.9. (a) *For two nonzero homogeneous derivations $\partial = \partial_{\rho,e}$ and $\partial' = \partial_{\rho',e'}$ one has*

$$[\partial, \partial'] = \partial_{\hat{\rho}, \hat{e}} \quad \text{where} \quad \hat{\rho} = \langle \rho, e' \rangle \rho' - \langle \rho', e \rangle \rho \in N \quad \text{and} \quad \hat{e} = e + e'\,. \tag{4}$$

(b) *If $\hat{\rho} \neq 0$ then $[\partial, \partial']$ is a homogeneous derivation of degree $e + e'$ with the linear form $\hat{\rho}$. In particular, $e + e' \in \Sigma^\vee \cap M$.*

(c) *$\hat{\rho} = 0$ (that is, ∂ and ∂' commute) if and only if one of the following holds:*
- *ρ and ρ' are collinear and $\langle \rho, e \rangle = \langle \rho, e' \rangle$ (this holds, in particular, if $e, e' \in \mathcal{S}_i$ for some $i \in \{1, \ldots, k\}$);*
- *ρ and ρ' are non-collinear and $\langle \rho', e \rangle = \langle \rho, e' \rangle = 0$.*

4.10. Given two derivations (or vector fields) $U = \partial_1$ and $V = \partial_2$ in $\mathrm{Der}(A)$ we let

$$\mathrm{ad}_U^m(V) = [U, [U, \ldots [U, V] \ldots]] = \sum_{i=0}^{m} \binom{m}{i} \partial_1^{m-i} \partial_2 (-\partial_1)^i \tag{5}$$

where U is repeated m times, see [37]. Thus, $\mathrm{ad}_U^m \in \mathrm{End}\,(\mathrm{Der}(A))$.

Lemma 4.11 (cf. [37])**.** *If $U \in \mathrm{LND}(A)$ then $\mathrm{ad}_U \in \mathrm{End}\,(\mathrm{Der}(A))$ is locally nilpotent, that is, for any $V \in \mathrm{Der}(A)$ there exists $c(V) > 0$ such that $\mathrm{ad}_U^m(V) = 0$ $\forall m > c(V)$.*

Proof. Let $A = \mathbb{K}[a_1, \ldots, a_k]$. There exists $N > 0$ such that $\partial_1^N(a_j) = 0$ $\forall j = 1, \ldots, k$. In the last sum in (5) applied to a_j certain members of the sum vanish so that it remains just N first members of the sum. For $m \gg 1$ the first N terms vanish as well, hence $\mathrm{ad}_U^m(V)(a_j) = 0$ for $j = 1, \ldots, k$, and so, $\mathrm{ad}_U^m(V) = 0$. □

4.12. We keep $U = \partial_1 \in \mathrm{Der}(A)$ being locally nilpotent. In the sequel we use the following version of the Baker-Campbell-Hausdorff formula, see, e.g., [12, Thm. 4.14]:

$$\mathrm{Ad}_{\exp(U)}(V) = \exp(\mathrm{ad}_U)(V) = V + \sum_{m=1}^{\infty} \frac{1}{m!} \mathrm{ad}_U^m(V)\,. \tag{6}$$

Due to Lemma 4.11, for a given $V = \partial_2$ the last sum is finite, hence the formula has sense. The second equality in (6) is just the usual exponential formula for the endomorphism ad_U of the vector space $\mathrm{Der}(A)$. Due to the first equality in (6), if V is locally nilpotent then $\exp(\mathrm{ad}_U)(V)$ is as well.

To apply this formula in our setting we need the following simple lemma.

Lemma 4.13. *Consider two nonzero homogeneous LNDs $U = \partial_{\rho_1, e_1}$ and $V = \partial_{\rho_2, e_2}$ in $\mathrm{Der}(A)$ where $e_i \in \mathcal{S}_i \cap M$, $i = 1, 2$. Letting*

$$c_2 = \langle \rho_2, e_1 \rangle, \quad d_1 = \langle \rho_1, e_2 \rangle, \quad \text{and} \quad \delta = d_1 + 1$$

and assuming that $c_2 \geq 1$ one has

$$\mathrm{ad}_U^m(V) = \partial_{r_m, f_m} \quad \forall m = 0, \ldots, d_1$$

where

$$r_m = \frac{d_1!}{(d_1-m)!}\rho_2 - mc_2\frac{d_1!}{(d_1-m+1)!}\rho_1 \quad \textit{and} \quad f_m = me_1 + e_2\,. \tag{7}$$

Furthermore, for any $d_1 \geq 0$ one has

$$\mathrm{ad}_U^m(V) = 0 \quad \forall m \geq \delta + 1 \tag{8}$$

and

$$\mathrm{ad}_U^\delta(V) = -c_2\delta!\partial_{\rho_1,f_\delta} \in \mathrm{LND}(A) \quad \textit{where} \quad f_\delta = \delta e_1 + e_2 \in \mathcal{S}_1 \cap M\,. \tag{9}$$

Proof. The assertions follow immediately from (4) by recursion on m. □

Corollary 4.14. *Under the assumptions of Lemma 4.13 let $\partial = \mathrm{Ad}_{\exp(U)}(V)$. Then the Newton polytope $N(\partial)$ is the segment $[e_2, e_2 + \delta e_1]$.*

Proof. Applying formula (6), by Lemma 4.13 one obtains:

$$\partial = V + \sum_{m=1}^{\delta} \frac{1}{m!}\partial_m$$

where $\partial_m = \partial_{r_m,f_m} \in \mathrm{Der}(A)$ is a homogeneous derivation of degree $f_m = me_1 + e_2$. Now the result follows. □

4.1.2. $\mathrm{Der}(A)$ *as a graded Lie algebra.* Given a lattice vector $e \in \Sigma^\vee$ consider the linear subspace

$$\mathcal{L}_e = \mathrm{span}\{\partial_{\rho,e}\}_{\rho\in N} \subset \mathrm{Der}(A)$$

generated by the homogeneous derivations of A of degree e. By Lemma 4.9 one has $[\mathcal{L}_e, \mathcal{L}_{e'}] \subset \mathcal{L}_{e+e'}$. Hence, the Lie algebra $\mathrm{Der}(A)$ is M-graded:

$$\mathrm{Der}(A) = \bigoplus_{e\in M} \mathcal{L}_e \quad \text{where} \quad \mathcal{L}_e \neq \{0\} \Leftrightarrow e \in \Sigma^\vee \cap M\,,$$

see Proposition 4.4(b) and Lemma 4.7(a).

Any subgroup T of the torus $\mathbb{T}$ acts by conjugation on $\mathrm{Der}(A)$, and so, induces a grading of $\mathrm{Der}(A)$ compatible with the M-grading, see, e.g., [29].

Example 4.15. Consider the standard action of the 2-torus $\mathbb{T}$ on $\mathbb{A}^2 = \mathrm{Spec}\,\mathbb{K}[x,y]$. This action induces an $\mathbb{N}^2$-grading on $\mathrm{Der}(\mathbb{K}[x,y])$ with graded pieces $\mathrm{Der}_e(\mathbb{K}[x,y])$ where $e = (i,j) \in \mathbb{Z}^2$ runs over the lattice vectors with $i,j \geq -1$ and $(i,j) \neq (-1,-1)$. The piece $\mathrm{Der}_e(\mathbb{K}[x,y])$ consists of all the homogeneous derivations of $\mathbb{K}[x,y]$ of degree e. For $i,j \geq 0$ one has

$$\mathrm{Der}_e(\mathbb{K}[x,y]) = \left\{ x^i y^j \left(ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \right) \,|\, a,b \in \mathbb{K} \right\}.$$

The pieces which correspond to the Demazure facets

$$\mathcal{S}_1 \cap M = \{(-1,j)\,|\,j \in \mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \mathcal{S}_2 \cap M = \{(i,-1)\,|\,i \in \mathbb{Z}_{\geq 0}\} \tag{10}$$

are

$$\mathrm{Der}_{(-1,j)}(\mathbb{K}[x,y]) = \left\{ ay^j\frac{\partial}{\partial x} \,|\, a \in \mathbb{K} \right\} \quad \text{resp.,} \quad \mathrm{Der}_{(i,-1)}(\mathbb{K}[x,y]) = \left\{ ax^i\frac{\partial}{\partial y} \,|\, a \in \mathbb{K} \right\}.$$

Any one-parameter subgroup $T \subset \mathbb{T}$ induces a $\mathbb{Z}$-grading on both $\mathbb{K}[x,y]$ and $\mathrm{Der}(\mathbb{K}[x,y])$. Letting l_T be the integral linear form on $\mathbb{Z}^2$ associated with T and $l_T = m$ be a supporting affine line for the Newton polytope $N(\partial)$, consider the corresponding T-principal part of ∂,

$$\partial_T = \sum_{e \in \{l_T = m\} \cap \mathbb{Z}^2} \partial_e \in \mathrm{Der}(\mathbb{K}[x,y])\,, \tag{11}$$

where ∂_e stand for the homogeneous component of ∂ of degree e in decomposition (3). According to Lemma 4.7, if ∂ is locally nilpotent then also ∂_T is. In particular, for any vertex e of the Newton polytope $N(\partial)$ the corresponding derivation ∂_e is locally nilpotent. Consequently, all the vertices of the Newton polytope $N(\partial)$ are situated on the Demazure facets (10). Hence the Newton polytope $N(\partial)$ is either a quadruple, a triangle, a line segment, or finally a point.

4.2. **Degeneration techniques.** We explore the M-grading on $\mathrm{Der}(A)$ in the following degeneration trick.

Proposition 4.16. *Consider a subgroup $G \subset \mathrm{Aut}(X)$ normalized by a one-parameter subgroup T of the torus $\mathbb{T}$. Let $H = \exp(\mathbb{K}\partial)$ be a $\mathbb{G}_a$-subgroup of G where $\partial \in \mathrm{LND}\,(A)$, and let $\partial_T \in \mathrm{LND}\,(A)$ be the T-principal part of ∂. Then $H_T = \exp(\mathbb{K}\partial_T)$ is a $\mathbb{G}_a$-subgroup of $\overline{G}$.*

Proof. Let $N(\partial)$ be the Newton polytope of ∂, let l_T be the linear form on M associated with T, and let

$$l_{\max} = \max\{l_T|_{N(\partial)}\} \quad \text{and} \quad l_{\min} = \min\{l_T|_{N(\partial)}\}\,.$$

Thus, one has $\partial_T = \partial_\tau$ for the face τ of $N(\partial)$ on which l_T achieves its maximal value.

The action of T on $\mathrm{Der}(A)$ defines a $\mathbb{Z}$-grading. Any $\partial \in \mathrm{Der}(A)$ admits a decomposition according with this grading:

$$\partial = \sum_{s=l_{\min}}^{l_{\max}} \partial_s \quad \text{where} \quad \partial_s = \sum_{e \in N(\partial) \cap \{l_T = s\}} \partial_e \in \mathrm{Der}(A) \quad \text{with} \quad \partial_e \in \mathcal{L}_e\,.$$

Given an isomorphism $T \cong \mathbb{G}_m$, an element $t_\lambda \in T$ with $\lambda \in \mathbb{G}_m$ acts on A via

$$t_\lambda . \chi^m = \lambda^{l_T(m)} \chi^m \quad \forall m \in \sigma^\vee\,.$$

It follows that

$$t_\lambda^{-1} \circ \partial_s \circ t_\lambda = \lambda^{-s} \partial_s\,.$$

Therefore, one has

$$t_\lambda^{-1} \circ \exp(\tau \partial_s) \circ t_\lambda = \exp(\tau \lambda^{-s} \partial_s)$$

and, furthermore,

$$t_\lambda^{-1} \circ \exp(\tau \partial) \circ t_\lambda = \exp\left(\sum_{s=l_{\min}}^{l_{\max}} \tau \lambda^{-s} \partial_s \right)\,.$$

Letting $\tau = h\lambda^{l_{\max}}$ one obtains:

$$t_\lambda^{-1} \circ \exp(\tau \partial) \circ t_\lambda = \exp\left(h \sum_{s=l_{\min}}^{l_{\max}} \lambda^{(l_{\max}-s)} \partial_s \right) \longrightarrow \exp(h \partial_T) \quad \text{as} \quad \lambda \to 0$$

on any monomial $\chi^m \in A$, $m \in \sigma^\vee$. This convergence guarantees the convergence with respect to the ind-group structure on $\mathrm{Aut}(X)$ associated to any given filtration $A = \bigcup_{r=1}^\infty A_r$ by finite dimensional graded subspaces of A such that $A_r \subset A_{r+1}$, see Lemma 3.2(b).

Since $t_\lambda^{-1} \circ \exp(\tau\partial) \circ t_\lambda \in G$ for any $\lambda \in \mathbb{K}^*$ and $\tau \in \mathbb{K}$ one concludes that $\exp(h\partial_T) \in \overline{G}$ for any $h \in \mathbb{K}$. □

Corollary 4.17. *Under the assumptions of Proposition 4.16 suppose that G is normalized by the torus $\mathbb{T}$. Then any vertex e of the Newton polytope $N(\partial)$ belongs to a Demazure facet $\mathcal{S}_i$ and the root subgroup H_e is contained in $\overline{G}$.*

Proof. It suffices to apply Proposition 4.16 to a one-parameter subgroup $T \subset \mathbb{T}$ such that $l_T|_{N(\partial)}$ achieves its maximum at e. □

Lemma 4.18. *Letting $n \geq 2$ consider two roots $e_i \in \mathcal{S}_i \cap M$, $i = 1, 2$. Let $\delta = \langle \rho_1, e_2 \rangle + 1$. Suppose that $\delta e_1 + e_2 \in \mathcal{S}_1$, that is, $\langle \rho_2, e_1 \rangle \geq 1$. Then one has $H_{\delta e_1 + e_2} \subset \overline{\langle H_{e_1}, H_{e_2} \rangle}$.*

Proof. This follows by Corollaries 4.14 and 4.17 applied to $U = \partial_{\rho_1, e_1}$, $V = \partial_{\rho_2, e_2}$, and $\partial = \exp(\mathrm{ad}_U)(V) = \mathrm{Ad}_{\exp(U)}(V)$, see (6). □

4.3. **Cox ring and total coordinates.** Let us recall some generalities on the Cox ring $R(X)$ of a toric affine variety X, see, e.g., [1, Ch. 2], [13, Sect. 1], [14, Ch. 5] for detailed expositions.

4.19. As before, $\Xi = \{\rho_1, \ldots, \rho_k\}$ stands for the set of primitive ray generators of the cone $\sigma \subset N_\mathbb{Q}$. To any ray $\rho_i \in \Xi$ there corresponds a facet $\rho_i^\vee$ of the dual cone $\sigma^\vee$ and a $\mathbb{T}$-invariant prime Weil divisor $D_i = D(\rho_i)$ on X. The classes $[D_1], \ldots, [D_k]$ generate the class group $\mathrm{Cl}(X)$. The Cox ring $R(X)$ is the polynomial ring $\mathbb{K}[x_1, \ldots, x_k]$ graded by the class group $\mathrm{Cl}(X)$ in such a way that any variable x_i is a homogeneous element of degree $\deg(x_i) = [D_i] \in \mathrm{Cl}(X)$, $i = 1, \ldots, k$. This defines the grading uniquely.

Let $\mathbb{T}(k) \cong (\mathbb{G}_m)^k$ be the standard k-torus acting on $\mathbb{A}^k$, and let $F_{\mathrm{Cox}} = \mathrm{Hom}\,(\mathrm{Cl}(X), \mathbb{G}_m)$ be the dual group of the group $\mathrm{Cl}(X)$. This is a quasitorus, that is, the direct product of an algebraic torus and a finite Abelian group. By duality, $\mathrm{Cl}(X)$ is the group of characters of F_{Cox}. The $\mathrm{Cl}(X)$-grading on $\mathbb{K}[x_1, \ldots, x_k]$ defines an action on $\mathbb{A}^k$ of the quasitorus $F_{\mathrm{Cox}} \subset \mathbb{T}(k)$. The structure ring $\mathcal{O}_X(X)$ is canonically isomorphic to the ring of invariants $\mathbb{K}[x_1, \ldots, x_k]^{F_{\mathrm{Cox}}}$. This yields (canonical) isomorphisms $X \cong \mathbb{A}^k /\!/ F_{\mathrm{Cox}}$ and $\mathbb{T} \cong \mathbb{T}(k)/F_{\mathrm{Cox}}$.

The linear forms $\rho_1, \ldots, \rho_k$ on $M_\mathbb{Q}$ define a monomorphism of lattices $\varphi\colon M \hookrightarrow \mathbb{Z}^k$ which extends to the linear embedding

$$\Phi\colon M_\mathbb{Q} \hookrightarrow \mathbb{A}^k_\mathbb{Q}, \quad v \mapsto (\langle \rho_1, v \rangle, \ldots, \langle \rho_k, v \rangle)\,.$$

The coordinates of the image $\Phi(m)$ will be called the *total coordinates* of $m \in M$.

We let $\Delta^\vee_{\geq 0} \subset \mathbb{A}^k_\mathbb{Q}$ be the positive octant, and let $\hat{\mathcal{S}}_i = \mathcal{S}_i(\Delta^\vee_{\geq 0})$ be the ith Demazure facet of $\Delta^\vee_{\geq 0}$. The image $\Phi(e)$ of a Demazure root $e \in \mathcal{S}_i$ is a Demazure root, say, $\hat{e} \in \hat{\mathcal{S}}_i$. Any root vector $\hat{e} \in \Phi(M_\mathbb{Q}) \cap \mathbb{Z}^k$ appears in this way. The action of the root subgroup H_e on X induces the action of the root subgroup $H_{\Phi(e)}$ on $\mathbb{A}^k$, see [13, Sect. 4]. In more detail, one has the following Lemma 4.20 (cf. [13, Lem. 4.4]). Let us intruduce the necessary notation.

For an arbitrary lattice vector $e = (c_1, \ldots, c_k) \in \mathbb{Z}^k$ we let $x^e = x_1^{c_1} \cdots x_k^{c_k}$. For $e \in \mathcal{S}_1 \cap M$ one has $\hat{e} = (-1, c_2, \ldots, c_k) \in \mathbb{Z}^k$ where $c_i \in \mathbb{Z}_{\geq 0}$, $i = 2, \ldots, k$. The root subgroup $H_{\hat{e}}$ acts on $\mathbb{A}^k$ via

$$(x_1, \ldots, x_k) \mapsto (x_1 + t x^{\hat{e} + \varepsilon_1}, x_2, \ldots, x_k), \quad t \in \mathbb{K}\,, \tag{12}$$

where $(\varepsilon_1, \ldots, \varepsilon_k)$ is the standard basis in $\mathbb{A}^k$ and $x^{\hat{e}+\varepsilon_1} = x_2^{c_2} \cdots x_k^{c_k}$.

Lemma 4.20. (a) *A Demazure root $\hat{e} \in \hat{\mathcal{S}}_i \cap \mathbb{Z}^k$ belongs to the image $\Phi(\mathcal{S}_i \cap M)$ if and only if $\deg(x^{\hat{e}}) = 0$, that is, $x^{\hat{e}} \in \mathrm{Frac}\,(\mathbb{K}[x_1, \ldots, x_k])^{F_{\mathrm{Cox}}}$.*

(b) *The subgroups $H_{\hat{e}}$ and F_{Cox} of $\mathrm{Aut}(\mathbb{A}^k)$ commute if and only if $\hat{e} = \Phi(e)$ for a root e of $\sigma^\vee$. In the latter case the action of the root subgroup $H_{\hat{e}}$ on $\mathbb{A}^k$ descends to the action of the root subgroup H_e on X under the quotient morphism $\mathbb{A}^k \to X = \mathbb{A}^k /\!/ F_{\mathrm{Cox}}$.*

Proof. (a) Recall that the lattice of $\mathbb{T}$-invariant divisors on X is generated by $D_1, \ldots, D_k$. One may assume that $i = 1$. One has

$$\deg(\hat{e}) = 0 \Leftrightarrow [D_1] = c_2[D_2] + \ldots + c_k[D_k] \quad \text{in} \quad \mathrm{Cl}(X)\,.$$

The latter equality amounts to

$$c_2 D_2 + \ldots + c_k D_k - D_1 = \mathrm{div}\,(\chi^m) \in \mathrm{Princ}(X)^{\mathbb{T}} \quad \text{for some} \quad m \in M \tag{13}$$

where

$$\mathrm{div}\,(\chi^m) = \sum_{i=1}^{k} \langle \rho_i, m \rangle D_i\,.$$

Thus, (13) admits a solution $m \in M$ if and only if

$$\langle \rho_1, m \rangle = -1 \quad \text{and} \quad \langle \rho_i, m \rangle = c_i \geq 0 \quad \forall i = 2, \ldots, k\,,$$

that is, if $m = e \in \mathcal{S}_1 \cap M$ is a Demazure root and $\hat{e} = \Phi(e)$.

(b) The action (12) on $\mathbb{A}^k$ commutes with the F_{Cox}-action on $\mathbb{A}^k$ if and only if the LND $\partial_{\varepsilon_1^\vee, \hat{e}} = x_2^{c_2} \cdots x_k^{c_k} \partial/\partial x_1$ has zero F_{Cox}-degree, that is, if $\deg(x_1) = \deg(x^{\hat{e}+\varepsilon_1})$ or, which is equivalent,

$$[D_1] = c_2[D_2] + \ldots + c_k[D_k]\,.$$

So, the first assertion of (b) follows by an argument used in the proof of (a). The second one is a simple consequence of the first. Indeed, the LND $\partial_{\varepsilon_1^\vee, \hat{e}} \in \mathrm{LND}\,(\mathbb{K}[x_1, \ldots, x_k])$ of F_{Cox}-degree zero restricts to the ring of invariants $\mathbb{K}[x_1, \ldots, x_k]^{F_{\mathrm{Cox}}} = \mathcal{O}_X(X)$ yielding $\partial_{\rho_1, e} \in \mathrm{LND}\,(\mathcal{O}_X(X))$. Hence the $H_{\hat{e}}$-action on $\mathbb{A}^k$ descends to the H_e-action on X. □

Remark 4.21. The connected group $H_{\hat{e}}$ normalizes F_{Cox} in $\mathrm{Aut}(\mathbb{A}^k)$ if and only if these groups commute. Indeed, $\mathrm{Aut}(F_{\mathrm{Cox}})$ is a finite extension of $\mathrm{GL}(l, \mathbb{Z})$, hence a discrete group.

5. Infinite transitivity: the case of toric varieties

In this section we apply Theorem 2.2 to toric affine varieties X with no toric factor. It is known ([4, Thm. 2.1]) that the action of $\mathrm{SAut}(X)$ on the smooth locus $\mathrm{reg}(X)$ is infinitely transitive. However, $\mathrm{SAut}(X)$ is a huge group. Under a mild additional assumption we construct in Theorem 5.19 a subgroup $G \subset \mathrm{SAut}(X)$ which still acts infinitely transitively in $\mathrm{reg}(X)$ and is generated by a finite number of root subgroups, as it is predicted by Conjecture 1.1. We start with the case where X is an affine space.

5.1. Infinite transitivity on the affine spaces: an example.

5.1. The affine space $\mathbb{A}^n = \mathrm{Spec}\,\mathbb{K}[x_1, \ldots, x_n]$ can be regarded as a toric variety. The mutually dual lattices N and M are the standard lattices of integer vectors $N = \mathbb{Z}^n \subset \mathbb{A}^n_{\mathbb{Q}}$ and $M = \mathbb{Z}^n \subset (\mathbb{A}^n_{\mathbb{Q}})^*$. The cones $\sigma \subset \mathbb{A}^n_{\mathbb{Q}}$ and $\sigma^\vee \subset (\mathbb{A}^n_{\mathbb{Q}})^*$ are the positive octants. The ray generators $\rho_1, \ldots, \rho_n \in N$ form the standard basis of $\mathbb{A}^n_{\mathbb{Q}}$. The dual basis $(\varepsilon_1, \ldots, \varepsilon_n)$ is the standard base

of the lattice M. The LNDs associated with the Demazure roots $e_i = -\varepsilon_i \in \mathcal{S}_i$ are the partial derivatives

$$\partial_i = \partial/\partial x_i = \partial_{\rho_i, e_i} \in \mathrm{LND}\,(A), \quad i = 1, \ldots, n\,.$$

For a lattice vector $m = (m_1, \ldots, m_n) \in M$ we write $x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$. Given a root vector $e \in \mathcal{S}_i \cap M$ the associated root subgroup

$$H_e = \exp(\mathbb{K} x^{e+\varepsilon_i} \partial_i) \subset \mathrm{SAut}(\mathbb{A}^n)$$

acts on $\mathbb{A}^n$ via elementary transformations

$$x = (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_i + t x^{e+\varepsilon_i}, x_{i+1}, \ldots, x_n) \quad \text{where} \quad t \in \mathbb{K}\,.$$

For instance, letting $H_{i,j} = \exp(\mathbb{K} x_j^2 \partial_i)$ where $j \neq i$ the root subgroups $H_{1,2}$ and $H_{2,3}$ act on $\mathbb{A}^n$ via

$$(x_1, \ldots, x_n) \mapsto (x_1 + t x_2^2, x_2, \ldots, x_n) \text{ resp. } (x_1, \ldots, x_n) \mapsto (x_1, x_2 + t x_3^2, x_3, \ldots, x_n) \tag{14}$$

where $t \in \mathbb{K}$. To simplify the notation we write just the coordinates of the image for such an action. The following result confirms Conjecture 1.1 in the case $X = \mathbb{A}^n$, $n \geq 2$.

Theorem 5.2. *Consider the action of the symmetric group $\mathbb{S}(n)$ on $\mathbb{A}^n$ by permutations. Then for any $n \geq 3$ the subgroup*

$$G = \langle H_{1,2}, \mathbb{S}(n) \rangle \subset \mathrm{Aut}(\mathbb{A}^n)$$

acts infinitely transitively in $\mathscr{O}_G = \mathbb{A}^n \setminus \{0\}$.

The following corollary is straightforward.

Corollary 5.3. *For $n \geq 3$ the subgroup $\langle H_{1,2}, \mathrm{SL}\,(n, \mathbb{K}) \rangle \subset \mathrm{Aut}(\mathbb{A}^n)$ acts infinitely transitively in $\mathbb{A}^n \setminus \{0\}$.*

The proof of Theorem 5.2 is preceded by the following lemmas.

Lemma 5.4. *Assume that $H_u \subset \overline{G}$ where $u = (-1, c_2, \ldots, c_n) \in \mathcal{S}_1 \cap M$ with $c_2 \geq 1$. Letting $v = (0, -1, 2, 0, \ldots, 0) \in \mathcal{S}_2 \cap M$ consider the root vector*

$$e = u + v = (-1,\ c_2 - 1,\ c_3 + 2,\ c_4, \ldots, c_n) \in \mathcal{S}_1 \cap M\,.$$

Then $H_e \subset \overline{G}$.

Proof. This follows immediately from Lemma 4.18. Indeed, the pair (u, v) satisfies the assumptions of this lemma with $\delta = 1$. □

Lemma 5.5. *For three indices $s, i, j \in \{2, \ldots, n\}$ where $i \neq j$ consider a root vector of the form*

$$w = (-1, 1, \ldots, 1, 2) + 3k_s \varepsilon_s + k_{i,j}(\varepsilon_i + \varepsilon_j) \in \mathcal{S}_1 \cap M\,. \tag{15}$$

Then $H_w = \exp(\mathbb{K} x^{w+\varepsilon_1} \partial_1) \subset \overline{G}$ for any k_s, $k_{i,j} \in \mathbb{Z}_{\geq 0}$.

Proof. Let $v_i = -\varepsilon_i + 2\varepsilon_{i+1}$, $i = 1, \ldots, n-1$. The Demazure root $u = v_1 = (-1, 2, 0, \ldots, 0) \in \mathcal{S}_1 \cap M$ generates the root subgroup $H_u = H_{1,2} \subset G$. Starting with u and adding $v_2, \ldots, v_{n-1}$ one gets the root vector $w_0 = (-1, 1, \ldots, 1, 2) \in \mathcal{S}_1 \cap M$. By Lemma 5.4 the associated root subgroup H_{w_0} is contained in $\overline{G}$. The same conclusion holds if one adds to w_0 the lattice vectors

$$(-\varepsilon_i + 2\varepsilon_j) + (2\varepsilon_i - \varepsilon_j) = \varepsilon_i + \varepsilon_j \quad \text{and} \quad (\varepsilon_i + \varepsilon_j) + (2\varepsilon_i - \varepsilon_j) = 3\varepsilon_i\,.$$

Iterating one arrives at the desired conclusion. □

We need also the following elementary lemma.

Lemma 5.6. *For $n \geq 4$ the vectors*

$$3\varepsilon_i \quad \textit{and} \quad \varepsilon_i + \varepsilon_j, \quad i \neq j,\ i,j \geq 2$$

span the sublattice $L = \langle \varepsilon_2, \ldots, \varepsilon_n \rangle \subset M$ of rank $n-1$.

Proof. One has

$$\varepsilon_2 = (\varepsilon_3 + \varepsilon_4) + 2(\varepsilon_2 + \varepsilon_4) - (\varepsilon_2 + \varepsilon_3) - 3\varepsilon_4\,.$$

Similar expressions hold for ε_i, $i = 3, \ldots, n$. □

Proof of Theorem 5.2. By our assumption one has $G \supset H_{i,j}$, $i \neq j$, $i,j \in \{1,\ldots,n\}$. By Lemma 5.5, $\overline{G} \supset H_w$ for any root vector w in (15). Letting $w_0 = (-1,1,\ldots,1,2)$ consider

$$\hat{\partial}_1 = x^{w_0}\partial/\partial x_1 \quad \text{and} \quad A_1 = \mathbb{K}[x_i^3, x_i x_j \,|\, i \neq j,\ i,j \geq 2] \subset \ker \hat{\partial}_1\,.$$

By Lemma 5.5, $\overline{G} \supset \exp(\mathbb{K} f \hat{\partial}_1)$ for any $f \in A_1$. The conjugation by the $\mathbb{S}(n)$-action yields a collection $\{(\hat{\partial}_i, A_i)\}_{i=1,\ldots,n}$ and $\overline{G} \supset \exp(\mathbb{K} f \hat{\partial}_i)$ for any $f \in A_i$.

By Lemma 5.6 for $n \geq 4$ this collection satisfies condition (β) of 2.1. For $n = 3$ it satisfies condition (α) of 2.1, that is, the function field $\mathrm{Frac}\,(\mathbb{K}[x_1,x_2,x_3])$ is generated by $\{\mathrm{Frac}\,(A_i)\}_{i=1,2,3}$. By Theorem 2.2, $\overline{G}$ acts infinitely transitively on the open orbit $\mathscr{O}_{\overline{G}}$. By virtue of Proposition 3.4(c) the same is true for G and the open orbit $\mathscr{O}_G = \mathbb{A}^n \setminus \{0\}$. □

Remarks 5.7. 1. Theorem 5.2 does not hold any longer if one replaces $x_1 + x_2^2$ in (14) by $x_1 + x_2^k$ with $k \geq 3$. Indeed, under such a replacement any $g \in G$ sends the pair of points $(Q, \omega Q)$ with $Q \in \mathbb{A}^n \setminus \{0\}$ and $\omega^{k-1} = 1$ to a pair $(g(Q), \omega g(Q))$. Thus, the 2-transitivity of G fails.

2. Theorem 5.2 does not hold in the case $n = 2$. More generally, fixing $a, b \in \mathbb{Z}_{\geq 0}$ consider the root subgroups

$$H_1 :\ (x,y) \mapsto (x + t_1 y^a, y) \quad \text{and} \quad H_2 :\ (x,y) \mapsto (x, y + t_2 x^b), \quad t_1, t_2 \in \mathbb{K}\,.$$

Claim. *If the group $G = \langle H_1, H_2 \rangle$ acts 2-transitively on its open orbit then one has $ab = 2$.*

Proof. Assume first that $ab = 0$; let, say, $a = 0$. Then H_1 acts on $\mathbb{A}^2$ by translations, and G acts on the first coordinate also by translations. Hence one has $x(g.P) - x(g.Q) = x(P) - x(Q)$ for any $P, Q \in \mathbb{A}^2$. The latter is an obstacle to the 2-transitivity.

Suppose that $a = b = 1$. Then $G = \mathrm{SL}(2,\mathbb{K})$. However, a linear group preserves the collinearity, hence it does not act 2-transitively on $\mathbb{A}^n$ for $n \geq 2$.

Let further $ab > 2$. Fixing a primitive root of unity ω of degree $ab - 1 > 1$ consider the set

$$S = \{(P,Q) \in \mathbb{A}^2 \times \mathbb{A}^2 \,|\, P = (x,y),\ Q = (\omega x, \omega^b y)\}\,.$$

It is easily seen that S is invariant under the diagonal action of G on $\mathbb{A}^2 \times \mathbb{A}^2$. Once again, this makes an obstacle to the 2-transitivity. □

It would be interesting to determine the degree of transitivity of the G-action on $\mathbb{A}^2 \setminus \{0\}$ in the remaining case $ab = 2$. We can show that this action is 2-transitive. However, we do not know whether a higher transitivity holds.

5.2. **Infinite transitivity on $\mathbb{A}^n$ and cotameness.**

5.8. Let Aff_n stand for the group of affine transformations of the affine space $\mathbb{A}^n$, and let

$$\mathrm{SAff}_n = \{f \in \mathrm{Aff}_n \,|\, \mathrm{Jac}\,(f) = 1\} = \langle \mathrm{Transl}_n,\, \mathrm{SL}\,(n, \mathbb{K})\rangle$$

be the subgroup of volume preserving affine transformations, where Jac stands for the Jacobian. Notice that the subgroup of translations Transl_n has generators

$$H_i = \exp(\mathbb{K}\partial_i),\ \ i = 1, \ldots, n\,.$$

Letting

$$H(i,j) = \exp(\mathbb{K}x_j\partial_i), \quad i,j \in \{1,\ldots,n\},\ i \neq j$$

the group $\mathrm{SL}\,(n,\mathbb{K})$ is generated by $H(n,1)$ and the root subgroups $\{H(i,i+1)\}_{i=1,\ldots,n-1}$ corresponding to the root system of type A_{n-1}. Therefore,

$$\mathrm{SAff}_n = \langle H_1,\, H(1,2), \ldots, H(n-1,n), H(n,1)\rangle \tag{16}$$

is generated by $n+1$ root subgroups. (In fact, there exists a smaller generating set.)

Definition 5.9 (cf. [18])**.** Let Tame_n stand for the tame subgroup of $\mathrm{Aut}(\mathbb{A}^n)$. Consider the subgroup

$$\mathrm{STame}_n = \{g \in \mathrm{Tame}_n \,|\, \mathrm{Jac}(g) = 1\} \subset \mathrm{Tame}_n\,.$$

One says that $h \in \mathrm{Aut}(\mathbb{A}^n) \smallsetminus \mathrm{Aff}_n$ is *cotame* if $\langle \mathrm{Aff}_n, h\rangle \supset \mathrm{Tame}_n$ and *topologically cotame* if $\overline{\langle \mathrm{Aff}_n, h\rangle} \supset \mathrm{Tame}_n$.

The following result due to Edo ([17, Thm. 1.2]) extends and refines the earlier results of Bodnarchuk ([8, Thm. 3]) and Furter ([24, Thm. D]).

Theorem 5.10. *For $n \geq 2$ any element $h \in \mathrm{Aut}(\mathbb{A}^n) \smallsetminus \mathrm{Aff}_n$ is topologically cotame.*

Remark 5.11. Recall that the triangular (de Jonquères) subgroup $\mathcal{B}_n \subset \mathrm{Aut}(\mathbb{A}^n)$ is the subgroup generated by the torus $\mathbb{T}$ and the triangular root subgroups $\exp(x^m\partial_i)$ where $x^m = x_{i+1}^{m_{i+1}} \cdots x_n^{m_n}$, $i = 1, \ldots, n$. It is known ([9], [10, 1.4 and Thm. 1.8], [18]) that for $n \geq 3$ any triangular $h \in \mathrm{Aut}(\mathbb{A}^n) \smallsetminus \mathrm{Aff}_n$ is cotame, while there is no triangular cotame $h \in \mathrm{Aut}(\mathbb{A}^2) \smallsetminus \mathrm{Aff}_2$.

Using Theorem 5.10 it is not difficult to deduce the following result on infinite transitivity (see [7], [8], [10, Thm. 1.2]).

Theorem 5.12. *For any $n \geq 2$ and any $h \in \mathrm{Aut}(\mathbb{A}^n) \smallsetminus \mathrm{Aff}_n$ the group $\langle \mathrm{Aff}_n, h\rangle$ acts infinitely transitively on $\mathbb{A}^n$.*

Proof. According to Theorem 5.10, Aff_n is a maximal closed subgroup in $\mathrm{Aut}(\mathbb{A}^n)$. Its normalizer $\mathcal{N}_n$ is a closed subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ containing Aff_n. Since Aff_n is not a normal subgroup of $\mathrm{Aut}(\mathbb{A}^n)$, that is, $\mathcal{N}_n \neq \mathrm{Aut}(\mathbb{A}^n)$, one has $\mathcal{N}_n = \mathrm{Aff}_n$.

Since $h \notin \mathrm{Aff}_n$ it does not normalize Aff_n, that is, $h\,\mathrm{Aff}_n\,h^{-1} \neq \mathrm{Aff}_n$. Pick up $g \in h\,\mathrm{Aff}_n\,h^{-1} \smallsetminus \mathrm{Aff}_n$. By Theorem 5.10 one has $\overline{\langle \mathrm{Aff}_n, g\rangle} \supset \mathrm{Tame}_n$. Letting

$$G = \langle \mathrm{Aff}_n, h\mathrm{Aff}_n h^{-1}\rangle \supset \langle \mathrm{Aff}_n, g\rangle$$

one obtains $\overline{G} \supset \mathrm{Tame}_n$. Since Tame_n acts infinitely transitively in $\mathbb{A}^n$ then also $\overline{G}$ does. The group G is algebraically generated. By Proposition 3.4(c), G acts infinitely transitively in $\mathbb{A}^n$. Since $G \subset \langle \mathrm{Aff}_n, h\rangle$ the latter group does as well. □

Remark 5.13. Let $g = \alpha_1 g_1 \cdots \alpha_l g_l \alpha_{l+1} \in \mathrm{Aut}(A^n)$ where $\mathrm{Jac}(g_i) = 1$ and $\alpha_i \in \mathrm{Aff}_n$. If $\mathrm{Jac}(g) = 1$ then there exists another decomposition

$$g = \beta_1 h_1 \cdots \beta_l h_l \beta_{l+1} \quad \text{with} \quad \mathrm{Jac}(h_i) = 1 \quad \text{and} \quad \beta_i \in \mathrm{SAff}_n,\ i = 1, \ldots, l$$

where

$$\beta_i = c_{i-1} \alpha_i c_i^{-1},\ i = 1, \ldots, l+1, \text{ and } h_i = c_i g_i c_i^{-1},\ i = 1, \ldots, l$$

with

$$c_0 = 1,\ c_i = \prod_{j=1}^{i} d_j,\ d_i = \mathrm{Jac}(\alpha_i), \quad \text{and so,} \quad c_{l+1} = \prod_{j=1}^{l+1} d_j = 1\,.$$

In the setting of Theorem 5.10 this observation yields the following corollary.

Corollary 5.14. *For any $h \in \mathrm{SAut}(\mathbb{A}^n) \smallsetminus \mathrm{SAff}_n$ one has*

$$\overline{\langle \mathrm{SAff}_n, h \rangle} \supset \mathrm{STame}_n\,.$$

The group STame_n acts infinitely transitively on $\mathbb{A}^n$. This leads to the following result.

Corollary 5.15. *For any $n \geq 2$ and any $h \in \mathrm{SAut}(\mathbb{A}^n) \smallsetminus \mathrm{SAff}_n$ the subgroup $\langle \mathrm{SAff}_n, h \rangle \subset \mathrm{SAut}(\mathbb{A}^n)$ acts infinitely transitively on $\mathbb{A}^n$.*

Let us provide an alternative direct proof of a similar result which does not apply the notion of cotameness.

Theorem 5.16. *For any $n \geq 2$ and any non-affine root subgroup $H_u \subset \mathrm{Aut}(\mathbb{A}^n)$ the subgroup*

$$\langle \mathrm{SAff}_n,\ H_u \rangle \subset \mathrm{STame}_n$$

generated by $n+2$ root subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ acts infinitely transitively on $\mathbb{A}^n$. Furthermore, for $n = 2$ there exists a collection of three root subgroups with the latter property.

Proof. Suppose first that $n \geq 3$ and $u = (-1, c_2, \ldots, c_n) \in \mathcal{S}_1 \cap \mathbb{Z}^n$. Since H_u is not affine one has $c_2 + \ldots + c_n \geq 2$. Letting $e_i = -\varepsilon_i$, $i = 1, \ldots, n$ and assuming that $c_i \geq 1$ one can deduce from Lemma 4.18 the relations

(i) $H_{u+e_i} \subset \overline{\langle H_u, H_{e_i} \rangle}$;
(ii) $H_{u+e_i-e_j} \subset \overline{\langle H_u, H_{e_i-e_j} \rangle}$ $\forall\, j \geq 2,\ j \neq i$;
(iii) $H_{2u+e_i-e_1} \subset \overline{\langle H_u, H_{e_i-e_1} \rangle}$.

Claim. *One has $H_v \subset \overline{G} := \overline{\langle \mathrm{SAff}_n,\ H_u \rangle}$ for any root subgroup H_v with $v \in \mathcal{S}_1 \cap \mathbb{Z}^n$.*

Proof of the Claim. Applying (ii) and (iii) successefully to u and the vectors obtained from u on each step one can get a root $u' \in \mathcal{S}_1 \cap \mathbb{Z}^n$ whose coordinates dominate the corresponding coordinates of v and such that $H_{u'} \subset \overline{G}$. Applying now (i) one can conclude. □

Applying the cyclic permutations of coordinates one can see that the Claim holds as well for any Demazure root v. According to Theorem 2.2 the group $\overline{G}$ acts infinitely transitivily on its open orbit $\mathscr{O}_{\overline{G}}$. Being algebraically generated the group $G = \langle \mathrm{SAff}_n,\ H_u \rangle$ does as well, see Proposition 3.4(c). Since $G \supset \mathrm{Transl}_n$ one has $\mathscr{O}_G = \mathbb{A}^n$. This gives the first assertion for $n \geq 3$.

Let further $n = 2$. Consider two affine and one non-affine roots

$$e_2 = (0, -1) \in \mathcal{S}_2, \quad v = e_2 - e_1 = (1, -1) \in \mathcal{S}_2, \quad \text{and} \quad u = (-1, 2) \in \mathcal{S}_1\,.$$

We claim that the group $G = \langle H_{e_2}, H_u, H_v\rangle$ acts infinitely transitively on $\mathbb{A}^2$. Indeed, by (i) one has $H_{u+e_2} \subset \overline{\langle H_u, H_{e_2}\rangle}$ where $u + e_2 = (-1, 1) \in \mathcal{S}_1$. Since

$$\mathrm{SL}(2,\mathbb{K}) = \langle H_{u+e_2}, H_v\rangle \quad \text{and} \quad \mathrm{SAff}_2 = \langle \mathrm{SL}(2,\mathbb{K}), H_{e_2}\rangle$$

it follows that $\langle \mathrm{SAff}_2, H_u\rangle \subset \overline{G}$ for the non-affine group H_u.

The rest of the proof proceeds likewise in the case $n \geq 3$. Notice first that $H_{e_1} \subset \mathrm{SAff}_2 \subset \overline{G}$. Let $u_i = (-1, i) \in \mathcal{S}_1$. We know already that $H_{u_i} \subset \overline{G}$ for $u_0 = e_1$, $u_1 = u + e_2$, and $u_2 = u$. Assume by induction that $H_{u_i} \subset \overline{G}$ for $i = 1, \ldots, d$ where $d \geq 2$. Due to (iii) one has $H_{u_{2d-1}} \subset \overline{G}$ where $u_{2d-1} = 2u_d + v$. Then by (i) one has $H_{u_{2d-2}} \subset \overline{G}$ where $u_{2d-2} = u_{2d-1} + e_2$, etc., $H_{u_{d+1}} \subset \overline{G}$. By induction, $H_e \subset \overline{G}$ for any $e = u_i \in \mathcal{S}_1 \cap M$. Since $\mathrm{SL}(2,\mathbb{K}) \subset \overline{G}$ as well one has $H_f \subset \overline{G}$ for any $f = (j, -1) \in \mathcal{S}_2 \cap M$. It follows by the Jung-van der Kulk Theorem that $\overline{G} = \mathrm{SAut}(\mathbb{A}^2)$. The latter group acts infinitely transitively in $\mathbb{A}^2$. Then also G does in view of Proposition 3.4(c). □

The following result completes the picture.

Theorem 5.17. *For any $n \geq 3$ one can find four $\mathbb{G}_a$-subgroups $U_1, \ldots, U_4 \subset \mathrm{SAut}(\mathbb{A}^n)$ such that*

$$G = \langle U_1, \ldots, U_4\rangle \subset \mathrm{SAut}(\mathbb{A}^n)$$

acts infinitely transitively on $\mathbb{A}^n$.

Proof. Let $u = (-1, 2, 0, \ldots, 0) \in \mathcal{S}_2$. Using Lemma 4.18 one can deduce the relations

$$H_{u+e_2} = H_{e_1-e_2} = \exp(\mathbb{K}x) \subset \overline{\langle H_u, H_{e_2}\rangle} \cap \mathrm{SL}(n,\mathbb{K}) \tag{17}$$

where $x \in \mathrm{sl}(n,\mathbb{K})$ is the infinitesimal nilpotent generator of $H_{e_1-e_2}$. According to [27, Sect. 4.1] there exists a regular semisimple element $h \in \mathrm{sl}(n,\mathbb{K})$ such that $\mathrm{sl}(n,\mathbb{K}) = \mathrm{Lie}\langle x, h\rangle$. According to [47, Thm. 1] one can write $h = [y, z]$ where $y, z \in \mathrm{sl}(n,\mathbb{K})$ are nilpotent matrices. Hence $\mathrm{sl}(n,\mathbb{K}) = \mathrm{Lie}\langle x, y, z\rangle$. Letting

$$U_x = H_{e_1-e_2} = \exp(\mathbb{K}x),\ U_y = \exp(\mathbb{K}y), \quad \text{and} \quad U_z = \exp(\mathbb{K}z)$$

it follows that

$$\mathrm{SL}(n,\mathbb{K}) = \langle U_x, U_y, U_z\rangle, \quad \text{and so,} \quad \mathrm{SAff}_n = \langle U_x, U_y, U_z, H_{e_2}\rangle \subset \overline{\langle U_y, U_z, H_{e_2}, H_u\rangle}$$

where H_u is not affine. Let $G = \langle U_y, U_z, H_{e_2}, H_u\rangle$. By Theorem 5.16 the subgroup $\langle \mathrm{SAff}_n, H_u\rangle \subset \overline{G}$ acts infinitely transitively on $\mathbb{A}^n$. Hence $\overline{G}$ does. By Proposition 3.4(c) the same holds for G. □

5.3. **Infinite transitivity on toric varieties.**

5.18. All the toric varieties in this paper are supposed to be normal, in particular, smooth in codimension 1. We consider below the class of toric affine varieties X smooth in codimension 2. The latter condition is equivalent to the following one: any two-dimensional face τ of the cone $\sigma \subset N_{\mathbb{Q}}$ of X is regular, that is, the pair of ray generators (ρ_i, ρ_j) of τ can be included in a base of the lattice N. For instance, the cone $\sigma \subset \mathbb{A}^3_{\mathbb{Q}}$ with the primitive ray generators

$$\rho_i = \varepsilon_i,\ \ i = 1, 2, 3,\ \ \rho_4 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$$

defines a toric threefold X with a single singular point; this X is smooth in codimension 2.

The following theorem is the main result of this section.

Theorem 5.19. *Let X be a toric affine variety of dimension $n \geq 2$ with no toric factor. Suppose that X is smooth in codimension 2. Then one can find a finite collection of root subgroups $H_1, \ldots, H_N$ such that the group*

$$G = \langle H_1, \ldots, H_N \rangle$$

acts infinitely transitively in the regular locus $\mathrm{reg}(X)$.

Proof. If $n = 2$ then X is smooth, hence $X \cong \mathbb{A}^2$. In this case the result (with $N = 3$) follows from Theorem 5.16.

Assume in the sequel that $n \geq 3$. Due to Theorem 2.1 in [4], $\mathrm{reg}(X)$ coincides with the open orbit of the group $\mathrm{SAut}(X)$. By [2, Prop. 1.5] there exists a finite collection of root subgroups $H_1, \ldots, H_r$ such that the group generated by $H_1, \ldots, H_r$ acts transitively in $\mathrm{reg}(X)$ too. To get infinite transitivity we need to enlarge this collection.

Recall that Ξ stands for the set of the primitive ray generators $\rho_1, \ldots, \rho_k$ of the cone $\sigma \subset N_{\mathbb{Q}}$ associated with X. Given a ray generator, say, $\rho_1 \in \Xi$ there exists $\bar{m}_1 \in M$ such that the hyperplane $L_{\bar{m}_1} = \{v \in N_{\mathbb{Q}} \,|\, \langle v, \bar{m}_1 \rangle = 0\}$ is strictly supporting for the ray $\mathbb{Q}_{\geq 0}\rho_1$ of σ, that is,

$$\langle \rho_1, \bar{m}_1 \rangle = 0 \quad \text{and} \quad \langle \rho_j, \bar{m}_1 \rangle > 0 \quad \forall\, j = 2, \ldots, k\,.$$

Since ρ_1 is a primitive lattice vector its coordinates are coprime. So, $\langle \rho_1, \breve{m}_1 \rangle = -1$ for some $\breve{m}_1 \in M$. Fix $r \gg 1$ and a root vector

$$e_1 = r\bar{m}_1 + \breve{m}_1 \in \mathcal{S}_1 \quad \text{where} \quad \langle \rho_j, e_1 \rangle \geq 2 \qquad \forall\, j = 2, \ldots, k\,.$$

Up to renumbering one may suppose that the 2-cones $\tau_{1,2}$ and $\tau_{1,3}$ spanned by the pairs of ray generators (ρ_1, ρ_2) and (ρ_1, ρ_3), respectively, are two-dimensional faces of σ containing the common ray $\mathbb{Q}_{\geq 0}\rho_1$. By our assumption, $\tau_{1,2}$ is regular. Hence one can find $\breve{m}_{1,2} \in M$ such that

$$\langle \rho_1, \breve{m}_{1,2} \rangle = 0 \quad \text{and} \quad \langle \rho_2, \breve{m}_{1,2} \rangle = -1\,.$$

Choose a strictly supporting hyperplane $L_{1,2} = \{\langle v, \bar{m}_{1,2} \rangle = 0\}$ of the face $\tau_{1,2}$ of σ where $\bar{m}_{1,2} \in M$ satisfies

$$\langle \rho_1, \bar{m}_{1,2} \rangle = \langle \rho_2, \bar{m}_{1,2} \rangle = 0 \quad \text{and} \quad \langle \rho_j, \bar{m}_{1,2} \rangle > 0 \quad \forall\, j \geq 3\,.$$

Fixing $r \gg 1$ consider the root

$$e_2 = r\bar{m}_{1,2} + \breve{m}_{1,2} \in \mathcal{S}_2 \text{ with } \langle \rho_1, e_2 \rangle = 0, \quad \langle \rho_2, e_2 \rangle = -1, \quad \text{and} \quad \langle \rho_j, e_2 \rangle \geq 2 \quad \forall\, j \geq 3\,. \tag{18}$$

Choose a root $e_3 \in \mathcal{S}_3$ in a similar fashion. Then in the total coordinates one has

$$\hat{e}_1 = (-1, *, \ldots, *), \quad \hat{e}_2 = (0, -1, *, \ldots, *), \quad \text{and} \quad \hat{e}_3 = (0, *, -1, * \ldots, *)$$

where the stars are integers ≥ 2.

Let $\tau_{1,2}^\vee$ be the face of $\sigma^\vee$ of codimension 2 dual to $\tau_{1,2}$, that is,

$$\tau_{1,2}^\vee = \left\{ \bar{m}_{1,2} \in M \,\middle|\, \begin{cases} \langle \rho_i, \bar{m}_{1,2} \rangle = 0, & i = 1, 2 \\ \langle \rho_i, \bar{m}_{1,2} \rangle \geq 0, & i = 3, \ldots, k \end{cases} \right\}.$$

Choosing $n - 2$ linearly independent primitive ray generators $\{\eta_1, \ldots, \eta_{n-2}\}$ of $\tau_{1,2}^\vee$ consider the sequence of roots

$$u_1 = e_2,\ u_2 = e_2 + \eta_1, \ldots, u_{n-1} = e_2 + \eta_{n-2} \in \mathcal{S}_2 \cap M \tag{19}$$

with total coordinates $\hat{u}_i = (0, -1, *, \ldots, *)$ where " $*$ " ≥ 2. The lattice vectors

$$v_1 = u_1 + e_3 = e_2 + e_3, \;\; v_2 = u_2 + e_3, \;\; \ldots, \;\; v_{n-1} = u_{n-1} + e_3 \in \tau_1 \tag{20}$$

have total coordinates $\hat{v}_i = (0, *, \ldots, *)$ where " $*$ " ≥ 1. We claim that these vectors are linearly independent, that is, $v_1 = e_2 + e_3 \notin \mathrm{span}(\eta_1, \ldots, \eta_{n-2}) =: W$. Indeed, any $\eta \in W$ has total coordinates $(0, 0, *, \ldots, *)$ whereas $\langle \rho_2, v_1 \rangle > 0$.

Claim. *Consider the cone $\omega \subset \tau_1$ of dimension $n-1$ with ray generators $v_1, \ldots, v_{n-1}$. Consider also the submonoid $\mathcal{M}_1 = \mathbb{Z}_{\geq 0} v_1 + \ldots + \mathbb{Z}_{\geq 0} v_{n-1}$ of ω of rank $n-1$ and the subgroup*

$$G_1 = \langle H_{e_1}, H_{u_1}, H_{u_2}, \ldots, H_{u_{n-1}}, H_{e_3} \rangle \subset \mathrm{SAut}(X) \, .$$

Then one has $H_w \subset \overline{G}_1$ for any root $w \in e_1 + \mathcal{M}_1 \subset \mathcal{S}_1 \cap M$.

Proof of the claim. The assertion is true for $w = e_1$. Assume by recursion that $H_w \subset \overline{G}_1$ for some root $w \in e_1 + \mathcal{M}_1$. It suffices to show that then the same holds as well for any root $w + v_i \in e_1 + \mathcal{M}_1$, $i = 1, \ldots, n-1$.

Notice that $\hat{w} = (-1, *, \ldots, *)$ where " $*$ " ≥ 2. Since $\langle \rho_2, w \rangle \geq 1$ the pair (w, u_i) satisfies the assumptions of Lemma 4.18 with $\langle \rho_1, u_i \rangle = 0$ and $\delta = 1$ for any $i = 1, \ldots, n-1$. Applying the recursive hypothesis and Lemma 4.18 one deduces that

$$H_{w+u_i} \subset \overline{\langle H_w, H_{u_i} \rangle} \subset \overline{G}_1 \quad \forall i = 1, \ldots, n-1 \, .$$

Likewise, since $\langle \rho_3, w + u_i \rangle \geq 1$ the pair $(w + u_i, e_3)$ satisfies the assumptions of Lemma 4.18 with $\langle \rho_1, e_3 \rangle = 0$ and $\delta = 1$ for any $i = 1, \ldots, n-1$. Applying Lemma 4.18 one deduces by (20) that

$$H_{w+v_i} \subset \overline{\langle H_{w+u_i}, H_{e_3} \rangle} \subset \overline{G}_1 \quad \forall i = 1, \ldots, n-1 \, .$$

This yields the inductive step and ends the recursion. □

Now one can constitute the data verifying the assumptions 2.1 of Theorem 2.2. Recall that we fixed already a collection of root subgroups $H_1, \ldots, H_r$ such that the open orbit of the group $\langle H_1, \ldots, H_r \rangle$ coincides with $\mathrm{reg}(X)$.

Letting $\partial_1 = \partial_{\rho_1, e_1} \in \mathrm{LND}(\mathcal{O}_X(X))$ consider the subalgebra

$$A_1 = \mathbb{K}[\chi^v \,|\, v \in \mathcal{M}_1] = \mathbb{K}[\chi^{v_1}, \ldots, \chi^{v_{n-1}}] \subset \ker(\partial_1) \, ,$$

see Remark 4.5. According to the Claim for any $f \in A_1$ the replica $\exp(\mathbb{K} f \partial_1)$ of H_{e_1} is a subgroup of $\overline{G}_1$. Since $\mathrm{rank}\,(\mathcal{M}_1) = n-1$ one has

$$[\mathrm{Frac}(\ker(\partial_1)) : \mathrm{Frac}(A_1)] < +\infty \, .$$

Hence there exists $b_1 \in \ker \partial_1$ such that $\mathrm{Frac}\,(\ker \partial_1)$ is generated by b_1 and $\mathrm{Frac}\,(A_1)$. According to Remark 4.5 one can write $b_1 = \sum_{j=1}^{s} c_j \chi^{m_j}$ where $m_j \in \tau_1 \cap M$. Then $H = \exp(\mathbb{K} b_1 \partial_1)$ is contained in the product of the root subgroups $H_{r+j} := \exp(\mathbb{K} \chi^{m_j} \partial_1)$, $j = 1, \ldots, s$.

Choose linearly independent ray generators $\rho_1, \ldots, \rho_n \in \Xi$. Repeating the same construction one obtains for any $i = 1, 2, \ldots, n$ a triple (G_i, ∂_i, A_i) with properties similar to the ones of (G_1, ∂_1, A_1). Let now

$$G = \langle H_1, \ldots, H_{r+s}, G_1, \ldots, G_n \rangle \subset \mathrm{SAut}(X) \, .$$

The group $\overline{G}$ satisfies the assumptions 2.1(γ) of Theorem 2.2. Due to this theorem, $\overline{G}$ acts infinitely transitively on its open orbit $\mathcal{O}_{\overline{G}} = \mathcal{O}_G = \mathrm{reg}(X)$. By virtue of Proposition 3.4(c) the same is true for G. □

5.3.1. *Final remarks.* Theorem 5.19 leads to the following questions.

5.20. Problem. *Let X be a toric affine variety with no toric factor. What is the smallest number of root subgroups ($\mathbb{G}_a$-subgroups, respectively) $H_1, \ldots, H_s$ of $\mathrm{Aut}(X)$ such that $G = \langle H_1, \ldots, H_s \rangle$ acts on X with an open orbit and is infinitely transitive on this orbit?*

We cannot exclude that this number equals 2, at least in the setup of arbitrary $\mathbb{G}_a$-subgroups; cf. Theorem 5.17. Let us remind the question (V. L. Popov [41, Problem 3.1]) as to when (the closure of) the subgroup $G = \langle H_1, H_2 \rangle$ generated by $\mathbb{G}_a$-subgroups $H_1, H_2 \subset \mathrm{Aut}(\mathbb{A}^n)$ is an algebraic group. The third author thanks Hanspeter Kraft for an inspiring example of two root subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ whose product is an (infinite dimensional) free product. The discussions with Hanspeter Kraft resulted in the following theorem ([32, Thm. 5.5.1]) which answers, in particular, the question above.

Theorem 5.21. *Given an affine variety X the subgroup $G \subset \mathrm{Aut}(X)$ generated by a family $\mathfrak{F}$ of connected algebraic subgroups of $\mathrm{Aut}(X)$ is a (closed) algebraic group if and only if the Lie algebras $\mathrm{Lie}(H)$ for $H \in \mathfrak{F}$ generate a finite dimensional Lie algebra.*

The following conjecture arises naturally (cf., e.g., Lemma 4.18).

Conjecture 5.22. *Let X be an affine variety, and let $A = \mathcal{O}_X(X)$ be its structure algebra. Consider the group $G = \langle H_1, \ldots, H_k \rangle$ generated by a finite collection of $\mathbb{G}_a$-subgroups $H_i = \exp(\mathbb{K}\partial_i) \subset \mathrm{SAut}(X)$ where $\partial_i \in \mathrm{LND}(A)$, $i = 1, \ldots, k$. Then the $\mathbb{G}_a$-subgroup $H = \exp(\mathbb{K}\partial) \subset \mathrm{SAut}(X)$ where $\partial \in \mathrm{LND}(A)$ is contained in $\overline{G}$ if and only if $\partial \in \mathrm{Lie}\langle \partial_1, \ldots, \partial_k \rangle$.*

Of course, the latter holds if G is an algebraic group. One more justification is provided by Lemmas 4.13 and 4.18. Indeed, letting $X = \mathrm{Spec}(A)$ be a nondegenerate toric affine variety of dimension $n \geq 2$, in the notation of Lemma 4.13 for two LNDs $U = \partial_1$ and $V = \partial_2$ of A and for $m = \delta$ the nonzero homogeneous derivation $W = \mathrm{ad}_U^{\delta}(V)$ is an LND. According to Lemma 4.18 the associated root subgroup $H_W = \exp(\mathbb{K}W)$ is contained in the closure $\overline{G}$ where $G = \langle H_U, H_V \rangle$.

References

[1] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface. *Cox rings*. Cambridge Studies in Advanced Mathematics 144. Cambridge University Press, Cambridge, 2015.

[2] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg, *Flexible varieties and automorphism groups*. Duke Math. J. 162 (2013), 767–823.

[3] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg, *Infinite transitivity on affine varieties*. In: Birational Geometry, Rational Curves, and Arithmetic. F. Bogomolov, B. Hassett and Yu. Tschinkel (eds.), 1–14. Springer-Verlag, New York e.a. 2013.

[4] I. Arzhantsev, K. Kuyumzhiyan, and M. Zaidenberg, *Flag varieties, toric varieties, and suspensions: three instances of infinite transitivity*. Sb. Math. 203 (2012), 3–30.

[5] I. Arzhantsev, A. Perepechko, and H. Süss. *Infinite transitivity on universal torsors*. J. London Math. Soc. 89 (2014), 762–778.

[6] Yu. Berest, A. Eshmatov, and F. Eshmatov. *Multitransitivity of Calogero–Moser spaces*. Transform. Groups 21 (2016), 35–50.

[7] Yu. Bodnarchuk. *On the transitivity of the action of a group of biregular automorphisms containing the affine group*. (Ukrainian) Dop. NAN Ukraine (Comptes Rendues Nat. Acad. Sci. Ukraine) 1 (1995), 5–7.

[8] Yu. Bodnarchuk. *Some extreme properties of the affine group as an automorphisms group of the affine space*. Contribution to General Algebra 13 (2001), 15–29.

[9] Yu. Bodnarchuk. *Affine group as a subgroup of biregular transformation group of an affine space.* Nauk. Zap., Kyïv 20 (2002), 6–10.
[10] Yu. Bodnarchuk. *On generators of the tame invertible polynomial maps group.* Int. J. Algebra Comput. 15 (2005), 851–867.
[11] F. Bogomolov, I. Karzhemanov, and K. Kuyumzhiyan. *Unirationality and existence of infinitely transitive models.* In: Birational Geometry, Rational Curves, and Arithmetic. F. Bogomolov, B. Hassett, and Yu. Tschinkel (eds.), 77–92. Springer-Verlag, New York e.a. 2013.
[12] A. Bonfiglioli and R. Fulci. *Topics in noncommutative algebra. The theorem of Campbell, Baker, Hausdorff and Dynkin.* Lect. Notes Math. 2034, Springer, 2011.
[13] D. A. Cox. *The homogeneous coordinate ring of a toric variety.* J. Algebraic Geom. 4 (1995), 17–50. Erratum: J. Algebraic Geom. 23 (2014), 393–398.
[14] D. A. Cox, J. B. Little and H. K. Schenck. *Toric Varieties.* Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.
[15] M. Demazure. *Sous-groupes algébriques de rang maximum du groupe de Cremona.* Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588.
[16] A. Dubouloz. *Flexible bundles over rigid affine surfaces.* Comment. Math. Helv. 90 (2015), 121–137.
[17] E. Edo. *Closed subgroups of the polynomial automorphism group containing the affine subgroup.* arXiv:1609.02670 (2016), 5p.
[18] E. Edo and D. Lewis. *Co-tame polynomial automorphisms.* arXiv:1705.01120 (2017), 23p.
[19] H. Flenner, S. Kaliman, and M. Zaidenberg. *The Gromov-Winkelmann theorem for flexible varieties.* J. Eur. Math. Soc. 18 (2016), 2483–2510.
[20] H. Flenner, S. Kaliman, and M. Zaidenberg. *Cancellation for surfaces revisited.* I. arXiv:1610.01805v2 (2016), 67p; II. arXiv:1801.02274 (2017), 26p.
[21] F. Forstnerič. *Stein Manifolds and Holomorphic Mappings. The Homotopy Principle in Complex Analysis.* A Series of Modern Surveys in Mathematics, 3. Springer-Verlag, 2011.
[22] G. Freudenburg. *Algebraic theory of locally nilpotent derivations.* Encyclopaedia of Mathematical Sciences 136, Springer-Verlag, 2006.
[23] W. Fulton. *Introduction to Toric Varieties.* Princeton University Press, 1993.
[24] J.-P. Furter. *Polynomial composition rigidity and plane polynomial automorphisms.* J. London Math. Soc. (2) 91 (2015), 180–202.
[25] J.-P. Furter and H. Kraft. *On the geometry of the automorphism group of affine n-space.* In preparation (2016).
[26] M. H. Gizatullin. *Quasihomogeneous affine surfaces.* Math. USSR Izvestiya 5 (1971), 1057–1081.
[27] T. Ionescu. *On the generators of semisimple Lie algebras.* Linear Algebra App. 15 (1976), 271–292.
[28] S. Kaliman and F. Kutzschebauch. *On the present state of the Andersen-Lempert theory.* In: Affine Algebraic Geometry. The Rassell Festschrift. Centre de Recherches Mathématiques, Daniel Daigle e.a. (eds.), 85–122. CRM Proceedings and Lecture Notes 54, 2011.
[29] P. Kotenkova. *On restriction of roots on affine $\mathbb{T}$-varieties.* Beiträge zur Algebra und Geometrie 55 (2014), 621–634.
[30] S. Kovalenko. *Transitivity of automorphism groups of Gizatullin surfaces.* Int. Math. Res. Notices 21 (2015), 11433–11484.
[31] S. Kovalenko, A. Perepechko, and M. Zaidenberg. *On automorphism groups of affine surfaces.* In: Algebraic varieties and automorphism groups. Advanced Studies in Pure Mathematics 75, 207–286. K. Masuda, T. Kishimoto, H. Kojima, M. Miyanishi, M. Zaidenberg (eds.). Hackensack, NJ: World Scientific, 2017.
[32] H. Kraft and M. Zaidenberg. *Locally finite group actions and vector fields.* Preprint (2014), 20p. (in preparation).
[33] S. Kumar. *Kac-Moody groups, their flag varieties and representation theory.* Progress in Math. 204, Birkhauser Boston Inc., Boston, MA, 2002.
[34] A. Liendo. *Affine T-varieties of complexity one and locally nilpotent derivations.* Transform. Groups 15 (2010), 389–425.
[35] A. Liendo. *$\mathbb{G}_a$-actions of fiber type on affine T-varieties.* J. Algebra 324 (2010), 3653–3665.
[36] A. Liendo. *Roots of the affine Cremona group.* Transform. Groups 16 (2011), 1137–1142.

[37] M. Manetti. *The Baker-Campbell-Hausdorff formula*. Notes of a course on deformation theory 2011-12. http://www1.mat.uniroma1.it/people/manetti/DT2011/BCHformula.pdf
[38] M. Michałek, A. Perepechko, and H. Süss. *Flexible affine cones and flexible coverings*. arXiv:1612.01144 (2016), 19p.
[39] J. Park and J. Won. *Flexible affine cones over del Pezzo surfaces of degree* 4. Eur. J. Math. 2 (2016), 304–318.
[40] A. Perepechko. *Flexibility of affine cones over del Pezzo surfaces of degree* 4 *and* 5. Func. Anal. Appl. 47 (2013), 45–52.
[41] V. L. Popov. *Roots of the affine Cremona group*. in: Affine Algebraic Geometry, J. Gutierrez, V. Shpilrain, J.–T. Yu (eds.), 12–13. Contemp. Math. 369, Amer. Math. Soc. Providence, RI, 2005.
[42] V. L. Popov. *On infinite dimensional algebraic transformation groups*. Transform. Groups 19 (2014), 549–568.
[43] Yu. Prokhorov and M. Zaidenberg. *Fano-Mukai fourfolds of genus* 10 *as compactifications of* $\mathbb{C}^4$. arXiv:1706.04926 (2017), 57p.
[44] E. Romaskevich. *Sums and commutators of homogeneous locally nilpotent derivations of fiber type*. J. Pure Appl. Algebra 218 (2014), 448–455.
[45] A. A. Shafarevich. *Flexibility of affine horospherical varieties of semisimple groups*. Sbornik: Math. 208 (2017), 285–310.
[46] I. P. Shestakov and U. U. Umirbaev. *The Nagata automorphism is wild*. Proc. Nat. Acad. Sci. USA 100 (2003), 12561–12563.
[47] J. H. Smith. *Commutators of nilpotent matrices*. Linear and Multilinear Algebra 4 (1976), 17–19.

National Research University Higher School of Economics, Faculty of Computer Science, 3 Kochnovskiy Proezd, Moscow, 125319 Russia
E-mail address: arjantsev@hse.ru

National Research University Higher School of Economics, Faculty of Mathematics, 6 Usacheva str., Moscow, 119048 Russia
E-mail address: karina@mccme.ru

Université Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France
E-mail address: Mikhail.Zaidenberg@univ-grenoble-alpes.fr