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TOWARDS THE RECOGNITION OF PGL_n VIA A HIGH DEGREE OF GENERIC TRANSITIVITY

TUNA ALTINEL AND JOSHUA WISCONS

ABSTRACT. In 2008, Borovik and Cherlin posed the problem of showing that the degree of *generic* transitivity of an infinite permutation group of finite Morley rank (X, G) is at most $n + 2$ where n is the Morley rank of X . Moreover, they conjectured that the bound is only achieved (assuming transitivity) by $\mathrm{PGL}_{n+1}(\mathbb{F})$ acting naturally on projective n -space. We solve the problem under the two additional hypotheses that (1) (X, G) is 2-transitive, and (2) $(X - \{x\}, G_x)$ has a definable quotient equivalent to $(\mathbb{P}^{n-1}(\mathbb{F}), \mathrm{PGL}_n(\mathbb{F}))$. The latter hypothesis drives the construction of the underlying projective geometry and is at the heart of an inductive approach to the main problem.

1. INTRODUCTION

Groups of finite Morley rank (fMr) are equipped with a model-theoretic notion of dimension generalizing the usual Zariski dimension for algebraic groups. Affine algebraic groups over algebraically closed fields are the primary examples, and as such, the study of groups of fMr is heavily inspired by the algebraic theory. However, as there is no topology, many of the algebraic tools have rudimentary analogs at best, and the methods of proof often follows lines from finite group theory. The guiding problem for the theory of groups of fMr is the classification of the simple ones, which is motivated by the *Algebraicity Conjecture* of Gregory Cherlin and Boris Zilber: the simple groups of fMr are simple algebraic groups over algebraically closed fields.

The theory of groups of fMr developed over the past forty years with a high point being the resolution of the Algebraicity Conjecture for those groups which contain an infinite elementary abelian 2-group [ABC08]. Much remains to be addressed around the Algebraicity Conjecture with a key issue being the lack of a Feit-Thompson Theorem—it is unknown if there exist simple groups of fMr without elements of order two. Nevertheless, the field has achieved a sufficient level of maturity that some attention is shifting to general questions about *permutation groups*.

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The current research on permutation groups is primarily organized around problems raised by Alexandre Borovik and Gregory Cherlin in [BC08]. The main result of [BC08] is that there exists a global bound on the rank of a *primitive* permutation group of fMr as a function of the rank of the set being acted upon, and the problems they pose are focused on tightening this bound. Their proof begins with the nontrivial observation that a bound on the rank of a primitive group can be derived from a bound on the *degree of generic transitivity* of the action (defined below), and it is an investigation of this latter bound that they propose as way to tighten the former one.

Definition 1.1. A permutation group of fMr (X, G) is called *generically n -transitive* if the induced action of G on X^n has an orbit \mathcal{O} for which $\text{rk}(X^n - \mathcal{O}) < \text{rk}(X^n)$. If (X, G) is generically n -transitive but not generically $(n+1)$ -transitive, we say that n is the *degree of generic transitivity*.

The notion of generic n -transitivity is quite natural—much more natural than ordinary n -transitivity in our context. The canonical example is that of $\text{GL}_n(\mathbb{C})$ acting on the vector space $V := \mathbb{C}^n$. The action is quite far from being 2-transitive (and not even transitive), but it is generically n -transitive. The large orbit \mathcal{O} of $\text{GL}_n(\mathbb{C})$ on V^n is precisely the set of bases for V . In [BC08], Borovik and Cherlin pose the following problem, which seeks to establish a *natural* bound on the degree of generic transitivity.

Problem A ([BC08, Problem 9]). Show that if (X, G) is an infinite, transitive, generically $(n+2)$ -transitive permutation group of fMr with $\text{rk } X = n$, then $(X, G) \cong (\mathbb{P}^n(\mathbb{F}), \text{PGL}_{n+1}(\mathbb{F}))$ for some algebraically closed field \mathbb{F} .

The early work of Ehud Hrushovski with actions on strongly minimal sets solves Problem A when X has rank 1 (see [BN94, Theorem 11.98]). The first investigation of the rank 2 case was by Ursula Gropp in 1992 [Gro92]; the full solution for rank 2 is much more recent [AW].

Generic n -transitivity has also been studied in the algebraic category where there is an open question about the degree of generic transitivity for actions of the simple algebraic groups on the various coset spaces of maximal parabolic subgroups. This question was raised by Vladimir Popov in [Pop07] where he also answered it in characteristic 0. Popov’s work may even have concrete implications for our context (via an analog of the O’Nan-Scott Theorem for primitive groups developed by Dugald Macpherson and Anand Pillay in [MP95]); however, there are several issues to overcome with such an approach, not the least of which is the restriction on the characteristic.

In this article, we take up Problem A in general, solving it under two additional hypotheses that appear to be the essential ingredients needed to recognize the underlying projective geometry. To streamline the discussion, we introduce some terminology.

Definition 1.2. An infinite permutation group (X, G) of fMr with $n := \text{rk } X$ is said to be *extremal* if it is transitive and generically $(n+2)$ -transitive.

If $(X, G) \cong (\mathbb{P}^m(\mathbb{F}), \text{PGL}_{m+1}(\mathbb{F}))$ for some algebraically closed field \mathbb{F} of rank r , then we say that (X, G) is *projective over a field of rank r* .

It is not hard to see that projective permutation groups over a field of rank 1 are extremal—in this new language, Problem **A** seeks the converse.

Problem A'. Show that every extremal permutation group of fMr is projective over a field of rank 1.

To state our main result, we first recall that whenever (X, G) is a permutation group and \sim is a G -invariant equivalence relation on X , then G acts naturally on X/\sim , and if \overline{G} is the image of G in $\text{Sym}(X/\sim)$, the permutation group $(X/\sim, \overline{G})$ is called a *quotient* of (X, G) .

Theorem A. *Let (X, G) be an extremal permutation group of fMr. Further, assume that*

- (1) (X, G) is 2-transitive, and
- (2) for some $x \in X$, $(X - \{x\}, G_x)$ has a definable quotient, with classes of infinite size, that is projective over a field of rank 1.

Then (X, G) is projective over a field of rank 1.

The second hypothesis is key, and it is at the heart of an inductive approach to Problem **A**. Indeed, taking such an approach, the existence of a definable quotient of $(X - \{x\}, G_x)$ with infinitely many classes of infinite size immediately implies that the quotient is projective over a field of rank 1. We formalize this with a corollary. Recall that a permutation group (X, G) is *virtually (definably) primitive* if every (definable) G -invariant equivalence relation on X either has finite classes or finitely many classes.

Corollary A. *Let (X, G) be an extremal permutation group of fMr, and assume Problem **A** is solved for sets of rank less than $\text{rk } X$. Then (X, G) is virtually definably primitive, and we have the following case division:*

- (1) (X, G) is projective over a field of rank 1;
- (2) (X, G) is not 2-transitive;
- (3) (X, G) is 2-transitive with $(X - \{x\}, G_x)$ virtually definably primitive for all $x \in X$.

The idea behind our proof of Theorem **A** is that if the elements of X are the points of the expected geometry, then the classes of the quotient of $(X - \{x\}, G_x)$ ought to define the lines through x . The 2-transitivity hypothesis keeps things uniform and allows us to fairly easily define the line through two points. Theorem **A** is a general version of [AW, Proposition 5.1], but, notably, the so-called Fixed-point Assumption has now been removed. (However, our condition on the point stabilizer is now slightly stronger.)

Finally, we should emphasize that our proof of Theorem **A** is highly geometric and requires very little analysis of the internal structure of G . As in [AW], this will likely stand in sharp contrast to the treatment of the third case in Corollary **A** (and to some degree the second one as well), which we find to be a good reason for isolating the present theorem.

2. BACKGROUND

We assume familiarity with the theory of groups of fMr, and refer the reader to [Poi87], [BN94], and [ABC08] for the necessary background. Those unfamiliar with Morley rank will do well to translate “rank” to “dimension” and “definable” to “constructible.”

Regarding *permutation groups* of fMr, we provide the relevant definitions as needed and recommend [BC08] and [AW] to fill in the details. The solutions to Problem A in rank 1 and 2 are included below as they, among other things, allow us to ignore small geometries and focus on the generic case.

Fact 2.1 (Hrushovski, see [BN94, Theorem 11.98]). *If (X, G) is a transitive permutation group of fMr with X of rank and (Morley) degree 1, then either*

- (1) G° is abelian, acting regularly on X ,
- (2) $(X, G) \cong (\mathbb{F}, \text{AGL}_1(\mathbb{F}))$ for some algebraically closed field \mathbb{F} , or
- (3) $(X, G) \cong (\mathbb{P}^1(\mathbb{F}), \text{PSL}_2(\mathbb{F}))$ for some algebraically closed field \mathbb{F} .

Fact 2.2 ([AW, Theorem A]). *If (X, G) is a transitive and generically 4-transitive permutation group of fMr with X of rank 2, then $(X, G) \cong (\mathbb{P}^2(\mathbb{F}), \text{PGL}_3(\mathbb{F}))$ for some algebraically closed field \mathbb{F} .*

We now end this (very brief) section with some terminology that is frequently used in the sequel.

Definition 2.3. Let (X, G) be a generically n -transitive permutation group of fMr. We say that $(x_1, \dots, x_n) \in X^n$ is in *general position* if the orbit of G on X^n containing (x_1, \dots, x_n) is of maximal rank. The stabilizers of n -tuples in general position will be called *generic n -point stabilizers*.

Remark 2.4. It is easy to see that if $(x_1, \dots, x_n) \in X^n$ are in general position, then any permutation of the coordinates is as well ([AW, Lemma 4.8]). Thus, we often just write $x_1, \dots, x_n \in X$ are in general position.

3. DEFINING THE GEOMETRY

From the hypotheses of Theorem A, we define a notion of collinearity for the elements of X with respect to which we obtain a projective space. The goal of this section is simply the definition of collinearity and its connection to the given equivalence relation on $X - \{x\}$.

By Facts 2.1 and 2.2, Theorem A holds when X has rank 1 or 2, so we will assume throughout that $\text{rk } X \geq 3$. We adopt the following setup for the remainder of this section.

Setup. Let (X, G) satisfy all hypotheses of Theorem A. Set $n := \text{rk } X$, and assume $n \geq 3$. Fix $1 \in X$. Define $X_1 := X - \{1\}$, and let \sim_1 be a fixed, definable G_1 -invariant equivalence relation on X_1 such that

- $\overline{X_1} := X_1 / \sim_1$ contains classes of infinite size, and
- $(\overline{X_1}, \overline{G_1})$ is projective over a field of rank 1, where $\overline{G_1}$ is the image of G_1 in $\text{Sym}(\overline{X_1})$.

Finally, let K_1 be the kernel of the map $G_1 \rightarrow \overline{G_1}$.

As we are assuming that (X, G) is 2-transitive, (X_1, G_1) is transitive, so the \sim_1 -classes have constant rank at least 1. Thus, $1 \leq \mathrm{rk} \overline{X_1} \leq n-1$. We are also assuming that $(\overline{X_1}, \overline{G_1}) \cong (\mathbb{P}^m(\mathbb{F}), \mathrm{PGL}_{m+1}(\mathbb{F}))$ for some algebraically closed field \mathbb{F} of rank 1, and it is not hard to see that in fact $m = n-1$. Indeed, as (X, G) is generically $(n+2)$ -transitive, (X_1, G_1) is generically $(n+1)$ -transitive by general principles, so $(\overline{X_1}, \overline{G_1})$ is also generically $(n+1)$ -transitive, see [BC08, Lemma 6.1] or [AW, Lemma 4.17]. This forces $m \geq n-1$, so as $m = \mathrm{rk} \overline{X_1} \leq n-1$, we have $m = n-1$. We record this observation in the following remark.

Remark 3.1. We have that $(\overline{X_1}, \overline{G_1}) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \mathrm{PGL}_n(\mathbb{F}))$ for some algebraically closed field \mathbb{F} of rank 1. Consequently, $\overline{X_1}$ has rank $n-1$, and every \sim_1 -class has rank 1.

We begin with a lemma describing the action of point-stabilizers in the quotient; it will be followed by a more concrete description of \sim_1 . However, we first require a definition.

Definition 3.2. A generically m -transitive permutation group of fMr is said to be generically *sharply* m -transitive if the stabilizer of any m points in general position is trivial.

Note that $(\mathbb{P}^{n-1}(\mathbb{F}), \mathrm{PGL}_n(\mathbb{F}))$ is generically sharply $(n+1)$ -transitive. Also, if (Y, H) is any generically sharply m -transitive permutation group of fMr, then there exists a definable bijection between H and the large orbit $\mathcal{O} \subset Y^m$, implying that $\mathrm{rk} H = m \mathrm{rk} Y$.

Regarding notation, if $x \in X$, then \overline{x} is the image of x in $\overline{X_1}$, and by $G_{\overline{x}}$, we mean the setwise stabilizer of \overline{x} in G . Then, $G_{1, \overline{x}} = G_1 \cap G_{\overline{x}}$, so $\overline{G_{1, \overline{x}}} = (\overline{G_1})_{\overline{x}}$, where the latter is the stabilizer in $\overline{G_1}$ of the ‘‘point’’ \overline{x} .

Lemma 3.3. *If $2 \leq k \leq n+1$ and $1, 2, \dots, k \in X$ are in general position, then $\overline{G_{1, 2, \dots, k}} = \overline{G_{1, \overline{2}, \dots, \overline{k}}}$.*

Proof. Let $A := G_{1, 2, \dots, k}$, and $B := G_{1, \overline{2}, \dots, \overline{k}}$. Since (X, G) is generically $(n+2)$ -transitive, we find that A acts generically $(n+2-k)$ -transitively on X , so \overline{A} acts generically $(n+2-k)$ -transitively on $\overline{X_1}$ as well (see [BC08, Lemma 6.1] or [AW, Lemma 4.17]). Now, we know the action of \overline{B} on $\overline{X_1}$; it is acting as the stabilizer in PGL_n of $(k-1)$ points of projective $(n-1)$ -space. Thus, \overline{B} acts generically *sharply* $(n+2-k)$ -transitively on $\overline{X_1}$. Since $\overline{A} \leq \overline{B}$, we find that $\overline{A} = \overline{B}$. \square

Lemma 3.4. *Let $2 \in X_1$. Then every $G_{1, 2}$ -orbit on X of rank at most 1 belongs to $\overline{2} \cup \{1\}$.*

Proof. Set $A := \overline{2} \cup \{1\}$. Since A is $G_{1, 2}$ -invariant, the fact that A has rank 1 (by Remark 3.1) implies that every element of A is contained in a $G_{1, 2}$ -orbit of rank at most 1. Now assume $x \notin A$. We want to show that the orbit of

$G_{1,2}$ on x has rank at least 2. By 2-transitivity of (X, G) , 1 and 2 are in general position. By Lemma 3.3, $\overline{G_{1,2}} = \overline{G_{1,\bar{2}}}$, so as $(\overline{X_1}, \overline{G_1})$ is 2-transitive, the orbit of $\overline{G_{1,2}}$ on \bar{x} has rank $n - 1$. Since we are assuming that $n \geq 3$, the orbit of $\overline{G_{1,2}}$ on \bar{x} is at least 2, so the same must be true of $G_{1,2}$ on x . \square

We now define the lines of our geometry. The connection with \sim_1 will be made explicit below in Proposition 3.7.

Definition 3.5. Define a relation λ on X^3 by $\lambda(a, b, c)$ if and only if the orbit of G on the triple (a, b, c) has rank at most $2n + 1$.

- We read $\lambda(a, b, c)$ as “ a, b, c are *collinear*.”
- If a, b, c are not collinear, we say they form a *triangle*.
- For $a \neq b \in X$, define $\ell_{ab} := \lambda(a, b, X)$.

A *line* will be any subset of X of the form ℓ_{ab} with $a \neq b \in X$. The set of lines is denoted \mathcal{L} , and the set of lines through a is $\mathcal{L}(a) := \{\ell_{ab} : b \in X - \{a\}\}$.

Remark 3.6.

- (1) The orbits of G on X^3 are uniformly definable, as are their ranks, so λ is definable.
- (2) Note that triples with repeated entries are always collinear since the orbit of G on such a triple will have rank at most $2n$. Thus, triples forming a triangle must not have repeated entries.
- (3) As (X, G) is generically 3-transitive, any three points in general position form a triangle, but, a priori, there is no reason to believe that the points forming a triangle are necessarily in general position. That is, at this point, there may be different types of triangles; however, Lemma 5.1 will rule this out.
- (4) Note that $\ell_{ab} = \ell_{ba}$.

Proposition 3.7. *If $2 \in X_1$, then $\ell_{12} = \bar{2} \cup \{1\}$; thus $(\mathcal{L}(1), \overline{G_1}) \cong (\overline{X_1}, \overline{G_1})$.*

Proof. By 2-transitivity of (X, G) , the orbit of G on the pair $(1, 2)$ has rank $2n$. Thus, the points collinear with 1 and 2 are precisely those c for which the orbit $cG_{1,2}$ has rank at most 1, so by Lemma 3.4, $\ell_{12} = \bar{2} \cup \{1\}$. \square

Of course, the element $1 \in X$ is arbitrary; this leads to the following corollary, which also incorporates Remark 3.1.

Corollary 3.8. *For all $x \in X$, the relation on $X_x := X - \{x\}$ given by $y \sim z$ if and only if $\ell_{xy} = \ell_{xz}$ is an equivalence relation, and if K is the kernel of the action of G_x on $\mathcal{L}(x)$, then $(\mathcal{L}(x), G_x/K) \cong (\mathbb{F}^{n-1}, \text{PGL}_n(\mathbb{F}))$.*

4. ANALYZING THE GEOMETRY

We now begin to work out the details of our geometry. We start by recalling the axioms for a projective space.

Definition 4.1. A geometry consisting of points, lines, and an incidence relation is called a *projective space* provided

- (1) any two distinct points line on a unique line;
- (2) (Veblen's Axiom) if a, b, c, d are distinct points and the line through a and b meets the line through c and d , then the line through b and c meets the line through a and d ; and
- (3) there exist four points, no three of which are collinear.

Setup. Let (X, G) satisfy the hypotheses of Theorem A with $n := \mathrm{rk} X$. Set $\mathcal{P} := X$ and $\mathcal{L} := \{\ell_{ab} : a \neq b \in X\}$ (as defined in the previous section).

The goals of this section are to show that two distinct points lie on a unique line (Axiom 1 above) and to prepare for the proof of Veblen's Axiom.

Proposition 4.2 (Two points determine a unique line). *Let $a, b \in \mathcal{P}$ be distinct. Then ℓ_{ab} is the unique line containing a and b . Hence, any two lines intersect in at most 1 point.*

Proof. First, let $u, v, w \in \mathcal{P}$ with $u \neq v$ and $u \neq w$. Since, by Corollary 3.8 the sets $\ell_{uv} - \{u\}$ and $\ell_{uw} - \{u\}$ are classes of an equivalence relation, we have that if $w \in \ell_{uv}$ and $w \neq u$, then $\ell_{uv} = \ell_{uw}$.

Now assume that $a, b \in \ell_{pq}$ with p and q distinct elements of \mathcal{P} . We may assume that $a \neq p$, so by the previous observation, $\ell_{pq} = \ell_{pa} = \ell_{ap}$. Now $b \in \ell_{ap}$ (since $\ell_{ap} = \ell_{pq}$) and $b \neq a$, so again using the above observation, $\ell_{ap} = \ell_{ab}$. Thus, $\ell_{pq} = \ell_{ab}$. \square

We now begin to investigate the structure of G . Perhaps surprisingly, we already have enough information to compute its rank.

Corollary 4.3. *The action of G on X is generically sharply $(n+2)$ -transitive, and consequently, G is connected of rank $n(n+2)$. Moreover, all generic k -point-stabilizers are connected for $k \leq n+2$.*

Proof. Let $1, \dots, n+2 \in \mathcal{P}$ be in general position. Let K_1 be the kernel of the action of G_1 on $\mathcal{L}(1)$, and similarly define K_2 . (So K_1 fixes each line through 1 setwise, and similarly for K_2 .) The key observation is that any point $z \in \mathcal{P} - \{1, 2\}$ is the *unique* point of intersection of ℓ_{1z} and ℓ_{2z} , so z is fixed by $K_1 \cap K_2$. As the action of G on \mathcal{P} is faithful, $K_1 \cap K_2 = \{1\}$.

By [AW, Lemma 4.17], $(\ell_{12}, \ell_{13}, \dots, \ell_{1(n+2)})$ is a generic tuple in $\mathcal{L}(1)^{n+1}$, so as $(\mathcal{L}(1), \overline{G}_1) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \mathrm{PGL}_n(\mathbb{F}))$, the stabilizer in \overline{G}_1 of the tuple is trivial. Of course, $G_{1,2,\dots,n+2}$ fixes each of the lines $\ell_{12}, \ell_{13}, \dots, \ell_{1(n+2)}$, so $G_{1,2,\dots,n+2} \leq K_1$. Similarly, we find that $G_{1,2,\dots,n+2} \leq K_2$, so $G_{1,2,\dots,n+2} = 1$.

We now have that the action of G on X is generically sharply $(n+2)$ -transitive, so there is a definable *bijection* between G and the orbit $\mathcal{O} := (1, \dots, n+2) \cdot G$ in X^n . This shows that $\mathrm{rk} G = n(n+2)$. Also, since (X, G) is generically 2-transitive, G° has a single orbit on X , so X has degree 1. This implies that X^n , hence \mathcal{O} , also has degree 1, so as G is in definable bijection with \mathcal{O} , G is connected. The connectedness of the generic k -point-stabilizers follows by a similar argument. \square

The next lemma, in conjunction with Fact 2.1, says that the group induced by G on any line is of the form AGL_1 or PSL_2 .

Lemma 4.4. *If $\ell \in \mathcal{L}$, then G_ℓ acts 2-transitively on ℓ , and consequently, ℓ has degree 1.*

Proof. Set $N := G_\ell$. Fix $x \neq y \in \ell$. Let $z \in \ell = \ell_{xy}$ with z different from x and y . Since G_x is transitive on $\mathcal{P} - \{x\}$, there is a $g \in G_x$ taking y to z . Of course, this implies that $g \in N$, so N_x is transitive on $\ell - \{x\}$. Similarly, we find that N_y is transitive on $\ell - \{y\}$, so N is 2-transitive on ℓ . This also implies that N° must have a single orbit on ℓ , so ℓ has degree 1. \square

We are now in a position to identify the torus. Recall that a *good torus* of G is a definable divisible abelian subgroup T such that every definable subgroup of T is equal to the definable closure of its torsion subgroup. We will make use of the following fundamental fact about actions of tori.

Fact 4.5 ([BC08, Lemma 3.11]). *Let (Y, H) be a transitive permutation group of fMr , T a definable divisible abelian subgroup of H , and $O(T)$ the largest definable torsion free subgroup of T . Then $\text{rk}(T/O(T)) \leq \text{rk} Y$.*

Proposition 4.6. *If $1, \dots, n+2 \in \mathcal{P}$ are in general position, then $G_{1, \dots, n+1}$ is a self-centralizing, maximal good torus of G .*

Proof. Set $H := G_{1, \dots, n+1}$; let K be the kernel of $(\mathcal{L}(1), G_1)$. By Lemma 3.3 and the fact that $(\mathcal{L}(1), G_1) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \text{PGL}_n(\mathbb{F}))$, \overline{H} , hence $H/(H \cap K)$ is a good torus of rank $n-1$. As a consequence of Corollary 4.3, K has rank $n+1$, and $A := (H \cap K)^\circ$ has rank 1. We must show that A is a good torus.

Set $\ell_i := \ell_{1i}$, the line through 1 and i , and consider the canonical map from K to $G^{\ell_2} \times \dots \times G^{\ell_{n+2}}$ where G^{ℓ_i} is the group induced by G on ℓ_i . The kernel of the map, N , fixes $1, \dots, n+2$, so by Corollary 4.3, $N = 1$. Since $(\mathcal{L}(1), \overline{G_1}) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \text{PGL}_n(\mathbb{F}))$, G_1 permutes the lines $\ell_2, \dots, \ell_{n+2}$ transitively while normalizing K , so the images of K in each G^{ℓ_i} are isomorphic.

By the previous lemma, G^{ℓ_i} is of the form AGL_1 or PSL_2 , so as the image of K in each G^{ℓ_i} fixes a point (namely 1), the image is either a good torus or a Borel subgroup of PSL_2 of the form AGL_1 . In the former case, K embeds into a product of good tori. However, this implies that K° is a good torus of rank $n+1$, which contradicts Fact 4.5. Thus, the image of K in each G^{ℓ_i} is of the form AGL_1 , so K° is solvable and nonnilpotent. In particular, if $F := F^\circ(K)$ is the connected Fitting subgroup of K , then the image of F in G^{ℓ_i} is the unipotent radical of AGL_1 .

We now claim that $A \not\leq F$. Assume $A \leq F$. For $2 \leq i \leq n+2$, define

$$A_{\hat{i}} := (G_{1, \dots, \hat{i}, \dots, n+2} \cap K)^\circ$$

where \hat{i} denotes that i has been removed; thus, $A = \widehat{A_{n+2}}$. The $A_{\hat{i}}$ are G_1 -conjugate, so our assumption implies that $A_{\hat{i}} \leq F$ for all $2 \leq i \leq n+2$. Now consider the sequence of points stabilizers $F_1, F_{1,2}, \dots, F_{1, \dots, n+2}$; of course $F = F_1$. Note that, for $2 \leq i \leq n+2$, $A_{\hat{i}} \subseteq F_{1, \dots, i-1}$ with $A_{\hat{i}} \not\subseteq (F_{1, \dots, i})^\circ$; this implies that F has rank at least $n+1$. However, we have observed that K° is nonnilpotent of rank $n+1$, so it must be that $A \not\leq F$.

Now, A must have a nontrivial image in some G_i^ℓ . The image of F in G_i^ℓ is the unipotent radical of AGL_1 , so as $A \not\leq F$, we find that A is a good torus. We conclude that H is a good torus, which by Fact 4.5, is maximal. \square

The previous proposition has an extremely important consequence for the action of G , namely that the so-called Fixed-point Assumption holds (cf. [AW, Proposition 5.1(3)]). The proof utilizes (very lightly) the Σ -groups from [AW, Section 4.4], which we now define.

Definition 4.7. If $x_1, \dots, x_m \in \mathcal{P}$ are in general position, we define the group $\Sigma(x_1, \dots, x_{k-1}; x_k, \dots, x_m)$ to be the subgroup of G that normalizes the set $\{x_1, \dots, x_{k-1}\}$ while fixing x_k, \dots, x_m pointwise.

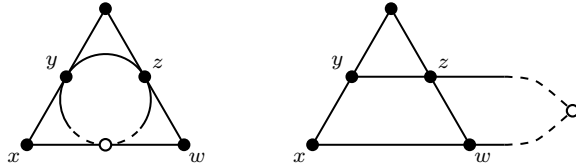
Corollary 4.8 (Fixed-Point Assumption). *If $1, \dots, n+1 \in \mathcal{P}$ are in general position, then $\mathrm{Fix}(G_{1, \dots, n+1}) = \{1, \dots, n+1\}$.*

Proof. Set $H := G_{1, 2, \dots, n+1}$. Let $\ell_i := \ell_{1i}$, and set $\widehat{H} := G_{1, \ell_2, \dots, \ell_{n+1}}$. Assume that H fixes a point $x \notin \{1, 2, \dots, n+1\}$. Then H certainly fixes the line $\ell_{1,x}$ (setwise), so by Lemma 3.3, \widehat{H} does as well. However, since $(\mathcal{L}(1), G_1) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \mathrm{PGL}_n(\mathbb{F}))$, the image of \widehat{H} in $(\mathcal{L}(1), G_1)$ fixes no more lines than those in $\{\ell_2, \dots, \ell_{n+1}\}$, so without loss of generality, we may assume that $\ell_{1,x} = \ell_{1,2} = \ell_2$.

Now, the image of H in G^{ℓ_2} fixes the points 1, 2, and x , so by Fact 2.1, H acts trivially on ℓ_2 . Moreover, the normalizer of H in G_1 contains $\Sigma(2, 3, \dots, n+1; 1)$, which permutes the set $\ell_2, \dots, \ell_{n+1}$ transitively, so H acts trivially on ℓ_i for every $2 \leq i \leq n+1$. This implies that H coincides with the subgroup of G_1 centralizing (the points of) each of the lines $\ell_2, \dots, \ell_{n+1}$. By definition, the subgroup of G_1 fixing each of the lines (setwise) is \widehat{H} , so we find that \widehat{H} normalizes H . However, by Proposition 4.6, H is a self-centralizing good torus, which implies that H is almost self-normalizing (see [ABC08, Lemma 4.23]). Thus, $\mathrm{rk} H = \mathrm{rk} \widehat{H}$. But, it is easily seen (for example using Corollary 4.3) that $\mathrm{rk}(\widehat{H}/H) = n$, a contradiction. \square

5. VEBLÉN'S AXIOM

We now prove Veblen's axiom, which we restate here for convenience. The claim is that if x, y, z, w are distinct points for which ℓ_{xy} and ℓ_{zw} intersect, then the lines ℓ_{yz} and ℓ_{xw} intersect as well, see below for a couple of different pictures. Our idea is to show that for some $p \in \ell_{yz}$ the group $G_{x,y,z}$ contains an element g taking ℓ_{xp} to ℓ_{xw} ; in this case, $p \cdot g$ will be the desired point of intersection.



Setup. As in the previous section, (X, G) satisfies the hypotheses of Theorem A with $n := \text{rk } X$, $\mathcal{P} := X$, and $\mathcal{L} := \{\ell_{ab} : a \neq b \in X\}$.

Lemma 5.1 (Triangles are in general position). *The points x, y, z form a triangle if and only if they are in general position.*

Proof. The reverse direction is clear. Now, suppose that x, y, z are not in general position; we show that they are collinear. Assume not, so $\ell_{xy} \neq \ell_{xz}$. Choose a such that x, y, a are in general position. Combining Lemma 3.3 with the fact that $(\mathcal{L}(x), G_x)$ is 2-transitive, we see that $G_{x,y}$ is transitive on $\mathcal{L}(x) - \ell_{xy}$, so there exists $g \in G_{x,y}$ taking ℓ_{xa} to ℓ_{xz} . Thus, for $w = ag$, we have that x, y, w are in general position, and $\ell_{xw} = \ell_{xz}$. We now work to show that $G_{x,y,w}$ fixes z , contradicting Corollary 4.8.

Since x, y, z are not in general position, the corank of $G_{x,y,z}$ in $G_{x,y}$ is at most $n - 1$. Also, by Lemma 3.3, the orbit of $G_{x,y}$ on ℓ_{xz} is generic in $\mathcal{L}(x)$, so the corank of $G_{x,y,\ell_{xz}}$ in $G_{x,y}$ is equal to $n - 1$. Thus, it must be that $(G_{x,y,z})^\circ = (G_{x,y,\ell_{xz}})^\circ$. Now, as $\ell_{xw} = \ell_{xz}$, we find that $(G_{x,y,w})^\circ \leq (G_{x,y,\ell_{xz}})^\circ = (G_{x,y,z})^\circ$, so $(G_{x,y,w})^\circ$, which is equal to $G_{x,y,w}$ by Corollary 4.3, fixes z . This contradicts Corollary 4.8. \square

Proposition 5.2. *Veblen's axiom holds.*

Proof. Let x, y, z, w be distinct points for which ℓ_{xy} and ℓ_{zw} intersect. We may assume that $\ell_{xy} \neq \ell_{zw}$, so x, y, z are in general position (by Lemma 5.1). Set $H := G_{x,y,z}$. We know that $(\mathcal{L}(x), \overline{G_x}) \cong (\mathbb{P}^{n-1}(\mathbb{F}), \text{PGL}_n(\mathbb{F}))$ where $\overline{G_x} = G_x/K$, with K the kernel of the action of G_x on $\mathcal{L}(x)$. Additionally, by Lemma 3.3, we know that $\overline{H} = \overline{G_{x,\ell_{xy},\ell_{xz}}}$.

The key observation to make is that, by inspection of the action of $\text{PGL}_n(\mathbb{F})$ on $\mathbb{P}^{n-1}(\mathbb{F})$, the stabilizer of two lines $G_{x,\ell_{xy},\ell_{xz}}$ has precisely 4 orbits on $\mathcal{L}(x)$: $\{\ell_{xy}\}$, $\{\ell_{xz}\}$, one of rank 1 that we call Y , and one of (full) rank $n - 1$. And importantly, the same statement holds for H (by Lemma 3.3). As such, our goal is to show that for some $p \in \ell_{yz}$ with p different from y and z , ℓ_{xp} and ℓ_{xw} are both in Y . This will imply that there exists an $h \in H$ taking ℓ_{xp} to ℓ_{xw} , and as h fixes ℓ_{yz} setwise, ℓ_{yz} and ℓ_{xw} must intersect at the point $p \cdot h$.

Let $p \in \mathcal{P}(\ell_{yz}) - \{y, z\}$ be arbitrary. Since H fixes ℓ_{yz} setwise, the orbit pH is confined to the rank 1 set of points on ℓ_{yz} . Since each line in the orbit $\ell_{xp}H$, is determined by $(x$ and) its point of intersection with ℓ_{yz} , the orbit $\ell_{xp}H$ has rank one; thus $\ell_{xp} \in Y$.

It remains to show that $\ell_{xw} \in Y$. Using the Veblen configuration, we first show that wH has rank at most 2 by identifying an H -invariant rank 2 set containing w . Define $A := \{q : \ell_{qz} \cap \ell_{xy} \neq \emptyset\}$. The map $A \rightarrow \mathcal{P}(\ell_{xy}) : q \mapsto p \in \ell_{qz} \cap \ell_{xy}$ is a definable surjection with fibers of rank 1. Thus, $\text{rk } A = 2$. As $w \in A$ and A is H -invariant, $\text{rk}(wH) \leq 2$.

Now, towards a contradiction, assume that $\ell_{xw} \notin Y$, so $\text{rk}(\ell_{xw}H) = n - 1$. Since $\text{rk}(\ell_{xw}H) \leq \text{rk}(wH)$, we have $n - 1 = \text{rk}(\ell_{xw}H) \leq \text{rk}(wH) \leq 2$, so as $n \geq 3$, it must be that $\text{rk}(\ell_{xw}H) = \text{rk}(wH) = 2$ and $n = 3$.

Next, we find an $r \in \ell_{xw}$ such that x, y, z, r are in general position. Indeed, choose s so that x, y, z, s are in general position. Thus, $\ell_{xy}, \ell_{xz}, \ell_{xs}$ are in general position in $\mathcal{L}(x)$, and by our assumption that $\ell_{xw} \notin Y$, $\ell_{xy}, \ell_{xz}, \ell_{xw}$ are in general position as well. Then ℓ_{xw} and ℓ_{xs} are both in the generic orbit of $\overline{H} = \overline{G_{x, \ell_{xy}, \ell_{xz}}}$ on $\mathcal{L}(x)$, so ℓ_{xw} contains an H -conjugate of s , which we take to be r .

Finally, we obtain the contradiction. We know $\text{rk}(wH) = \text{rk}(\ell_{xw}H) = 2$, so as $H_w \leq H_{\ell_{xw}}$, we find that $(H_w)^\circ = (H_{\ell_{xw}})^\circ$. Also, $\ell_{xw} = \ell_{xr}$, so $H_r \leq H_{\ell_{xw}}$. Now, H_r is connected by Corollary 4.3, so $H_r \leq (H_{\ell_{xw}})^\circ = (H_w)^\circ$, which contradicts Corollary 4.8. Thus, $\ell_{xw} \in Y$, and we are done. \square

Corollary 5.3. *The geometry is that of projective n -space for $n \geq 3$.*

Proof. It only remains to verify that there exist four points, no three of which are collinear. This is clearly satisfied by four points in general position. \square

6. PROOF OF THE MAIN RESULTS

Proof of Theorem A. Let (X, G) satisfy the hypotheses of Theorem A with $n := \text{rk } X$. By our work above, the geometry with respect to $\mathcal{P} := X$ and $\mathcal{L} := \{\ell_{ab} : a \neq b \in X\}$ is that of a projective n -space for $n \geq 3$. Our geometry is definable, and as it is automatically arguesian (because $n \geq 3$), it can be definably coordinatized (see for example [Art57, Hil59]). As our lines have rank 1, we find that the geometry is that of $\mathbb{P}^n(\mathbb{F})$, for some algebraically closed field \mathbb{F} of rank 1.

We now have that $G \leq \text{PGL}_{n+1}(\mathbb{F}) \rtimes \text{Aut}(\mathbb{F})$. This copy of $\text{PGL}_{n+1}(\mathbb{F})$ is definable, and we consider $N := G \cap \text{PGL}_{n+1}(\mathbb{F})$. Since G/N is a definable group of field automorphisms, $G/N = 1$ (see [ABC08, Lemma 4.5]). Thus, $G \leq \text{PGL}_{n+1}(\mathbb{F})$. Further, $\text{PGL}_{n+1}(\mathbb{F})$ acts generically *sharply* $(n+2)$ -transitively on \mathcal{P} , so as G acts generically $(n+2)$ -transitively, we find that $G = \text{PGL}_{n+1}(\mathbb{F})$. \square

Proof of Corollary A. Let (X, G) be an extremal permutation group of fMr with $n := \text{rk } X$. Assume Problem A is solved for sets of rank less than n .

We first show that the action is virtually definably primitive. If not, there is a definable quotient $(\overline{X}, \overline{G})$ with infinitely many classes of infinite size. Thus, $m := \text{rk } \overline{X} < \text{rk } X$ and $m > 0$. Moreover, as we have noted before, $(\overline{X}, \overline{G})$ is transitive and generically $(n+2)$ -transitive by [BC08, Lemma 6.1] (or [AW, Lemma 4.17]), so $(\overline{X}, \overline{G})$ is transitive and generically $(m+3)$ -transitive on a set of rank $m < n$. This contradicts our assumption that Problem A is solved for sets of rank less than n .

If (X, G) is not 2-transitive, we are done (falling into case (2) of the statement of the corollary), so assume it is. Then, if $(X - \{x\}, G_x)$ is virtually definably primitive for all $x \in X$, we are also done (and in case (3)). Thus, (X, G) is 2-transitive and $(X - \{x\}, G_x)$ is virtually definably imprimitive for some $x \in X$. Set $X_x := X - \{x\}$. We now have that (X_x, G_x) is transitive, and by general principles, (X_x, G_x) is generically $(n+1)$ -transitive. Let

$(\overline{X_x}, \overline{G_x})$ be any definable quotient with infinitely many classes of infinite size, and set $r := \text{rk } \overline{X_x}$. Applying our inductive hypothesis to $(\overline{X_x}, \overline{G_x})$, we find that $(\overline{X_x}, \overline{G_x})$ is projective, and we are finished by Theorem A. \square

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