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# Consensus-based Formation Control of Nonholonomic Robots using a Strict Lyapunov Function

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**Abstract:** This paper presents a novel decentralized consensus-based formation controller that considers both, the kinematic and the dynamic model, to uniformly and asymptotically drive a network composed of  $N$  nonholonomic mobile robots to a desired formation with a given orientation. The network is modeled as an undirected, static and connected graph. The controller is a smooth time-varying  $\delta$ -persistently exciting controller of the Proportional-Derivative type. The stability analysis is carried out using a novel strict Lyapunov function. Simulations, using a network with six agents, illustrate our theoretical contributions.

*Keywords:* Consensus, Nonholonomic systems, Persistence of Excitation, Lyapunov's direct method

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## 1. INTRODUCTION

In recent years, there has been an increasing interest in the distributed control for multiple mobile robots systems due, possibly, to the large number of applications that these systems can achieve, such as reconnaissance missions, detection of forest fires, search and rescue missions, object transportation and border patrol, among others (Kwon and Chwa, 2012; Li et al., 2014; Dimarogonas and Kyriakopoulos, 2007).

One of the objectives of distributed control in networks of multiple agents is consensus, where the control objective is to reach an agreement between certain coordinates of interest using only the information of the agent's neighbors. Consensus is a particular case of rendezvous, containment and formation control (Cortés et al., 2006; Moreau, 2004; Ren et al., 2005; Olfati-Saber and Murray, 2004; Cortés et al., 2006; Ferrari-Trecate et al., 2006; Lafferriere et al., 2005; Olfati-Saber, 2006). The consensus of multiple dynamical systems has been extensively studied for linear systems (Ren et al., 2005; Kranakis et al., 2006; Hui, 2011) and different classes of nonlinear systems (Nuño et al., 2011; Nuño et al., 2013b; Panteley et al., 2015). However, these results cannot be applied in multi-agent systems with nonholonomic restrictions.

One of the main difficulties appearing in the formation control of nonholonomic systems is that the designed controller has to be either discontinuous or time-varying (Brockett, 1983). Different approaches have been proposed to deal with consensus-like control objectives. Among them are: (Dimarogonas and Kyriakopoulos, 2007) where it is introduced a decentralized feedback control that

drives a system of multiple nonholonomic unicycles to a rendezvous point in terms of both position and orientation, the proposed control law is discontinuous and time-invariant; the work of (Lin et al., 2005) presents necessary and sufficient conditions for the feasibility of a class of formations; (Peng et al., 2015) propose a distributed formation control law using a consensus-based approach to drive a group of agents to a desired geometric pattern; (Yang et al., 2016) address the position/orientation formation control problem for multiple nonholonomic agents using a time-varying controller that leads the agents to a given formation using only their orientation; to solve the consensus problem, (Dong and Farrell, 2008) propose a cooperative control law that its robust to constant communication delays and its application to formation control. (Ajlou et al., 2015) present a distributed consensus control law for a network of nonholonomic agents in the presence of bounded disturbances with unknown dynamics in all inputs channels; for an undirected graph, (Peng et al., 2015) propose a smooth time varying controller that has been extended in (Bautista-Castillo et al., 2016) to a PD-like controller at the dynamical level. All these previous works, solve the consensus problem without uniformity on the initial time, and they only work on the kinematic model, except for (Bautista-Castillo et al., 2016).

In this paper we solve a consensus-based formation problem for a network of multiple second order nonholonomic robots with a given desired orientation. The solution is established using a novel smooth time varying  $\delta$ -Persistently Exciting (PE) controller (Loria et al., 1999); see also (Wang et al., 2015). The behavior of the agents takes into account both the kinematic and the dynamic models. Under the assumption that the inter-

connection graph is static, connected and undirected we propose a novel Strict Lyapunov Function (SLF) to establish uniform global asymptotic stability of the desired equilibrium point. The SLF is designed following (Malisoff and Mazenc, 2009) and (Mazenc, 2003). To the best of our knowledge this is the first work that provides a SLF in this scenario. It is well known that a SLF entails the system with robustness properties with regards to external perturbations and it also provides a tool for tuning the controller gains. Simulations are presented to provide evidence of our proposal.

**Notation.** For a vector  $z \in \mathbb{R}^n$ , we denote by  $\bar{z}$  its diagonal matrix representation, i.e.,  $\bar{z} = \text{diag}[z_i]$ .  $z^\perp$  is the orthogonal vector to  $z$  and  $\|z\|$  denotes the Euclidean norm of  $z$ . For a symmetric positive semi-definite matrix  $L \in \mathbb{R}^{n \times n}$ , we denote by  $\lambda_{max}(L)$  and  $\lambda_{min}(L)$ ,  $\lambda_i(L)$ , the maximum, the minimum and the  $i$ th eigenvalue of  $L$ , respectively.  $\|L\|$  is the induced Euclidian norm of  $L$ . For a time varying matrix  $M(t)$  we denote by  $\|M(t)\|_\infty = \sup_{t \geq 0} \{M(t)\}$ . The symbol  $\otimes$  stands for the Kronecker product and we define  $L_2 := L \otimes I_2$ .

## 2. SYSTEM DYNAMICS

As customary in multi-agent consensus (Olfati-Saber and Murray, 2004; Nuño et al., 2011), the complete dynamics of the systems is composed of a twofold: i) the dynamics of nodes, which are described by a second order nonholonomic differential equation; and ii) the interconnection topology, modeled using the Laplacian matrix from graph theory.

### 2.1 Node Dynamics

Without loss of generality, we consider the following model of  $N$  second order nonholonomic robots (Tzafestas, 2013),

$$\begin{cases} \dot{z} = \Phi(\theta)v \\ \dot{v} = u_v \\ \dot{\theta} = \omega \\ \dot{\omega} = u_\omega \end{cases} \quad (1)$$

where  $z = [z_1^\top, \dots, z_N^\top]^\top \in \mathbb{R}^{2N}$ ;  $z_i = [x_i - \delta_{xi}, y_i - \delta_{yi}]^\top \in \mathbb{R}^2$  is the translational error of the  $i$ th-robot;  $\delta_i := [\delta_{xi}, \delta_{yi}]^\top \in \mathbb{R}^2$  is the given desired position of the  $i$ th-robot relative to the barycentre of the formation;  $v = [v_1, \dots, v_N]^\top \in \mathbb{R}^N$ ;  $v_i$  is the linear velocity;  $\Phi(\theta) = \text{diag}[\phi(\theta_i)] \in \mathbb{R}^{2N \times N}$ ;  $\phi(\theta_i) = [\cos(\theta_i), \sin(\theta_i)]^\top \in \mathbb{R}^2$ ;  $\tilde{\theta} = \theta - \theta_d = [\theta_1 - \theta_{d1}, \dots, \theta_N - \theta_{dN}]^\top \in \mathbb{R}^N$  is the orientation error of each robot, with  $\theta_d$  a constant desired orientation;  $\omega = [\omega_1, \dots, \omega_N]^\top \in \mathbb{R}^N$ ;  $\omega_i$  is the angular velocity; and finally  $u_v = [u_{v1}, \dots, u_{vN}]^\top \in \mathbb{R}^N$  and  $u_\omega = [u_{\omega1}, \dots, u_{\omega N}]^\top \in \mathbb{R}^N$  are the control inputs.

Since  $\theta_d$  is constant, the two following equations hold

$$\dot{\Phi}(\theta) = -\Phi(\theta)^\perp \tilde{\omega}, \quad \dot{\Phi}(\theta)^\perp = \Phi(\theta)^\perp \tilde{\omega}, \quad (2)$$

where  $\tilde{\omega} = \text{diag}[\omega_i] \in \mathbb{R}^{N \times N}$ ,  $\Phi(\theta)^\perp = \text{diag}[\phi(\theta_i)^\perp] \in \mathbb{R}^{2N \times N}$  and  $\phi(\theta_i)^\perp = [\sin(\theta_i), -\cos(\theta_i)]^\top$ .

### 2.2 Interconnection Topology

The interconnection of the  $N$  agents is modeled using the Laplacian matrix  $L := [\ell_{ij}] \in \mathbb{R}^{N \times N}$ , whose elements are defined as

$$\ell_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} a_{ij} & i = j \\ -a_{ij} & i \neq j \end{cases} \quad (3)$$

where  $\mathcal{N}_i$  is the set of agents transmitting information to the  $i$ th robot,  $a_{ij} > 0$  if  $j \in \mathcal{N}_i$  and  $a_{ij} = 0$  otherwise.

Similar to passivity-based (energy-shaping) synchronization (Aldana et al., 2015; Nuño et al., 2013a) and in order to ensure that the interconnection forces are generated by the gradient of a potential function, the following assumption is used in this paper:

**A1.** The interconnection graph is *undirected, static and connected*.

*Remark 1.* By construction,  $L$  has a zero row sum, i.e.,  $L1_N = 0$ , where  $1_N$  is a vector of  $N$  ones. Moreover, Assumption **A1**, ensures that  $L$  is symmetric, has a single zero-eigenvalue and the rest of the spectrum of  $L$  is positive. Thus,  $\text{rank}(L) = N - 1$ .  $\triangle$

## 3. PROBLEM FORMULATION AND ITS SOLUTION

**Consensus Problem.** Consider a network of  $N$  non-holonomic robots satisfying (1). Design a decentralized controller verifying Assumption **A1** such that all robots positions converge, globally, uniformly and asymptotically, to a given formation pattern with a desired orientation, i.e., there exists  $z_c \in \mathbb{R}^2$  such that

$$\lim_{t \rightarrow \infty} z(t) = 1_N \otimes z_c; \quad \lim_{t \rightarrow \infty} \theta_i(t) = \theta_{di}, \quad (4)$$

where  $\theta_{di} \in \mathbb{R}$  is a given desired constant orientation for each robot, and  $z_c$  is the barycentre of the formation pattern.  $\nabla$

*Remark 2.* The consensus problem defined above is typically referred to as *leaderless consensus*, since the barycentre of the formation  $z_c$  is not *a priori* known.  $\triangle$

We solve the consensus problem by recasting it into a classical stabilisation problem (of the origin). To that end, we first need to introduce suitable error coordinates. Let

$$\begin{aligned} e &= \Phi(\theta)^\top L_2 z, \\ s &= \Phi(\theta)^\perp L_2 z \end{aligned} \quad (5)$$

and we recall that  $L_2 = L \otimes I_2$ . Then, the control objective (4) is achieved if we prove that  $(e, s) \rightarrow (0, 0)$ . This is due to the following fundamental fact.

*Lemma 1.* Consider  $(e, s)$ , given by (5), and assume that  $L$  satisfies **A1**, then  $L_2 z = 0 \Leftrightarrow (e, s) = (0, 0)$ . Moreover,

$$\lambda_2(L) z^\top L_2 z \leq |e|^2 + |s|^2 \leq \lambda_N(L) z^\top L_2 z, \quad (6)$$

where  $\lambda_2(L)$  and  $\lambda_N(L)$  are the second smallest and the largest eigenvalue of  $L$ , respectively.  $\square$

*Proof of Lemma 1.* Since the matrix  $\begin{bmatrix} \Phi(\theta)^\top \\ \Phi(\theta)^\perp \end{bmatrix}$  is non singular. The first fact follows directly. For the second fact, we remark that  $|e|^2 + |s|^2 = z^\top L_2^2 z = z^\top L_2^\frac{1}{2} L_2 L_2^\frac{1}{2} z$ . Since  $L_2^\frac{1}{2} z$  is orthogonal to the eigenspace associated to the zero eigenvalue of  $L_2$ , it holds that

$$\lambda_2(L) z^\top L_2^\frac{1}{2} L_2^\frac{1}{2} z \leq z^\top L_2^\frac{1}{2} L_2 L_2^\frac{1}{2} z \leq \lambda_N(L) z^\top L_2^\frac{1}{2} L_2^\frac{1}{2} z,$$

and so (6) follows.  $\square$

In the new error coordinates, for the translation, we employ a simple PD-like controller originally proposed in (Bautista-Castillo et al., 2016). That is,

$$u_v = -K_{dt}v - K_{pt}e, \quad (7)$$

while for the rotational dynamics we use

$$u_\omega = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s, e). \quad (8)$$

By design,  $K_{dt}, K_{pt}, K_{d\theta}, K_{p\theta}$  are diagonal positive definite matrices and  $\kappa(s, e)$  is defined as

$$\kappa(s, e) = \frac{1}{2}[s_1^2 + e_1^2, \dots, s_N^2 + e_N^2]^\top \in \mathbb{R}^N \quad (9)$$

and the function  $p: \mathbb{R}^+ \rightarrow \mathbb{R}$  satisfies the following.

**A2.**  $p(t)$ , and up to its third derivative, is bounded and  $\dot{p}(t)$  is persistently exciting (see the Appendix) with excitation parameters  $(T, \mu)$ . Thus, there exists  $b_p > 0$  such that

$$\max \left\{ \|p\|_\infty, \|\dot{p}\|_\infty, \|\ddot{p}\|_\infty, \|p^{(3)}\|_\infty \right\} \leq b_p.$$

*Remark 3.* Two remarks are in order: i) for simplicity, and without losing generality, the function  $p$  is taken equal for all the agents; ii) the function  $\kappa$  in (9) may correspond to any class- $\mathcal{K}$  function with the following form  $\kappa(s, e) = \frac{1}{2}[G(s_1^2 + e_1^2), \dots, G(s_n^2 + e_n^2)]^\top$ . The only condition on  $\kappa$  is that there exist two positive polynomials  $P_1(\cdot)$  and  $P_2(\cdot)$  such that:

$$G(\cdot) \leq P_1(\cdot), \quad \text{and}, \quad \left\| \frac{\partial G(\cdot)}{\partial(\cdot)} \right\| \leq P_2(\cdot).$$

Now, the closed-loop system, which results from Equations (1), (5), (7), and (8), is

$$\begin{cases} \dot{z} = \Phi(\theta)v \\ \dot{v} = -K_{dt}v - K_{pt}e \\ \dot{e} = -\bar{\omega}s + \Phi(\theta)^\top L_2 \Phi(\theta)v \\ \dot{s} = \bar{\omega}e + \Phi(\theta)^{\perp\top} L_2 \Phi(\theta)v \\ \dot{\theta} = \omega \\ \dot{\omega} = -K_{d\theta}\omega - K_{p\theta}\tilde{\theta} - p(t)\kappa(s, e). \end{cases} \quad (10)$$

Our main contribution is to establish uniform global asymptotic stability for the origin of this system. Moreover, our proof is constructive as it relies on the construction of a strict Lyapunov function (globally positive definite and with negative definite derivative). To that end, let us define the following change of coordinates:

$$\begin{aligned} e_\theta &= \tilde{\theta} + \bar{f}(t)\kappa(s, e) \\ e_\omega &= \omega + \bar{f}(t)\kappa(s, e), \end{aligned} \quad (11)$$

where  $\bar{f}(t) := \text{diag}[f_i(t)] \in \mathbb{R}^{N \times N}$ ,  $\dot{\bar{f}}(t) := \text{diag}[\dot{f}_i(t)] \in \mathbb{R}^{N \times N}$  and  $f_i(t)$  satisfies the differential equation

$$\dot{f}_i + k_{dti}\dot{f}_i + k_{pti}f_i = p(t), \quad (12)$$

where  $k_{dti}, k_{pti}$  are the elements of the diagonal matrices  $K_{dt}$  and  $K_{pt}$ , respectively. If  $\dot{p}(t)$  satisfies **A2** then, after Lemma 2 in the Appendix, it follows that  $\dot{f}_i$  is also persistently exciting and so is the matrix  $\dot{\bar{f}}(t)$ . Furthermore, there exists  $b_{\bar{f}} > 0$  such that

$$\max \left\{ \|\bar{f}\|_\infty, \|\dot{\bar{f}}\|_\infty, \|\ddot{\bar{f}}\|_\infty, \|\bar{f}^{(3)}\|_\infty \right\} \leq b_{\bar{f}}.$$

Lemma 2 also provides an explicit estimation of the excitation parameters  $(T_f, \mu_f)$  of  $\dot{f}$  and the constant  $b_{\bar{f}}$  that are used in the construction of the Lyapunov function.

Next, let us define  $X_t = [v^\top, e^\top, s^\top]^\top \in \mathbb{R}^{3N}$  and  $X_r = [e_\theta^\top, e_\omega^\top]^\top \in \mathbb{R}^{2N}$ , as the translational and rotational parts

of the state, respectively. Additionally, let  $\bar{e} = \text{diag}[e_i]$ ,  $\bar{s} = \text{diag}[s_i]$ ,  $\bar{e}_\omega = \text{diag}[e_{\omega i}]$  and  $\bar{\kappa} = \text{diag}[\kappa_i]$ . Then,

$$\begin{aligned} \dot{X}_t &= \begin{bmatrix} -K_{dt} & -K_{pt} & 0 \\ 0 & 0 & \bar{f}\bar{\kappa} - \bar{e}_\omega \\ 0 & -\bar{f}\bar{\kappa} + \bar{e}_\omega & 0 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ \Phi^\top L_2 \\ \Phi^{\perp\top} L_2 \end{bmatrix} \Phi v \\ \dot{X}_r &= \begin{bmatrix} 0 & I_N \\ -K_{p\theta} & -K_{d\theta} \end{bmatrix} X_r + \begin{bmatrix} \bar{f} \\ \dot{\bar{f}} \end{bmatrix} (\bar{e}\Phi^\top L_2 + \bar{s}\Phi^{\perp\top} L_2) \Phi v. \end{aligned} \quad (13)$$

Note that in view of Lemma 1,  $(X_t, X_r) = (0, 0) \Leftrightarrow (v, z, \theta, \omega) = (0, 1_N \otimes z_c, \theta_a, 0)$  and the dynamics (10) are embedded in (13). Therefore, our analysis problem comes to study the stability of the origin for (13). This is the subject of our main result, which is stated next.

*Theorem 1.* Controller (7) and (8) solves the Consensus Problem provided that  $p(t)$  satisfies **A2**. Moreover, the closed-loop system (13) admits a strict Lyapunov function.  $\triangle$

*Proof:* (Sketch) Due to space constraints we do not include here a complete proof, but the main steps are given.

First, we observe that the translational part of the system admits the following non-strict Lyapunov function

$$V(\theta, X_t) = v^\top K_{pt}^{-1}v + z^\top L_2 z. \quad (14)$$

Indeed, in view of (6), it is concluded that  $V(\theta, X_t)$  is positive definite and radially unbounded with regards to  $X_t$ . Moreover, the time derivative of  $V$  along the trajectories of (10) yields

$$\dot{V}(\theta, X_t) = -2v^\top K_{pt}^{-1}K_{dt}v. \quad (15)$$

Now, the Lyapunov function for the closed-loop system (13) is

$$\Gamma(t, X_t, X_r) = W(t, X_t, X_r) + \rho_1(V)Z(X_r) + \rho_2(V)V \quad (16)$$

where

$$\begin{aligned} W &= \gamma(V)V + V\kappa^\top \bar{Q}_{j_2}(t)\kappa + \alpha(V)e^\top v - c_1 V e^\top \bar{f}s \\ &\quad + c_1 b_f \lambda_N(L)V^2 + (\lambda_N(L) + \|K_{pt}\|)\alpha(V)V, \\ Z &= c_2 (e_\omega^\top e_\omega + e_\theta^\top K_{p\theta} e_\theta) + e_\theta^\top e_\omega, \\ \rho_1(V) &= \frac{2\sigma(V)}{c_2 \lambda_{\min}(K_{d\theta})} (\alpha(V) + c_1 b_f V) + 1, \\ \sigma(V) &= \max \left\{ \frac{16Tc_1 b_f}{\mu}, \frac{4\lambda_N(L) \|K_{dt}^{-1}K_{pt}\| \alpha(V)V}{\gamma(V)} \right\}, \\ \alpha(V) &= 4b_{\bar{f}}^2 \lambda_N(L)V^2 \|K_{pt}^{-1}\| + 4c_1 b_{\bar{f}}^2 \lambda_N(L) \|K_{pt}^{-1}\| V^2 \\ &\quad + \frac{4c_1}{c_4} \left\| \bar{f}^2 (\Phi^{\perp\top} L_2 \Phi)^2 \right\|_\infty \|K_{dt}^{-1}\| V + c_1^2 c_4 b_{\bar{f}}^2 \|K_{pt}^{-1}\|, \\ \gamma(V) &= 2c_4 V^2 \lambda_N(L) \|K_{dt}^{-1}K_{pt}\| \left\| \bar{Q}_{j_2} \Phi^\top L_2 \Phi \right\|_\infty^2 \\ &\quad + 2c_4 V^2 \lambda_N(L) \|K_{dt}^{-1}K_{pt}\| \left\| \bar{Q}_{j_2} \Phi^{\perp\top} L_2 \Phi \right\|_\infty^2 \\ &\quad + \frac{\partial \alpha}{\partial V} V (\|K_{pt}\| + \lambda_N(L)) + \frac{c_4}{2} c_1 V \\ &\quad + 2\alpha(V) \|\Phi^\top L_2 \Phi\|_\infty \|K_{dt}^{-1}K_{pt}\|_\infty \\ &\quad + \frac{c_4}{2} \|K_{pt}K_{dt}^{-1}\| \alpha^2(V) + \frac{c_4}{2} \alpha(V) \|K_{dt}\| \\ &\quad + 2c_1 b_f \lambda_N(L)V + \frac{4}{c_4} V^2 \lambda_N(L) \|K_{dt}^{-1}\| \end{aligned}$$

$$\begin{aligned}
& + \frac{c_4}{2} c_1^2 \|K_{dt}^{-1} K_{pt}\| \left\| \bar{f}^2 (\Phi^\top L_2 \Phi)^2 \right\|_\infty, \\
\rho_2(V) &= \rho_1(V) \rho_3(V) V \\
\rho_3(V) &= \frac{c_3 \lambda_N(L) \|K_{dt}^{-1} K_{pt}\|}{2} \left( \|\Phi^\top L_2 \Phi\|_\infty^2 + \|\Phi^{\perp\top} L_2 \Phi\|_\infty^2 \right),
\end{aligned}$$

and the constants  $c_1, c_2, c_3$  and  $c_4$  are:

$$\begin{aligned}
c_1 &= 1 + \frac{\lambda_N(L)}{\max \left\{ 2, \frac{2T}{\mu} \left( 1 + \frac{2N}{\lambda_2(L)} \right) \right\}} \\
c_2 &= \frac{2}{\lambda_{\min}(K_{d\theta})} + \frac{\lambda_{\max}(K_{d\theta}) + 1}{\lambda_{\min}(K_{p\theta})} + 1 \\
c_3 &= \max \left\{ \frac{8(2c_2 b_f + b_f)^2}{c_2 \lambda_{\min}(K_{d\theta})}, \frac{8(2c_2 b_f \lambda_{\max}(K_{p\theta}) + b_f)^2}{\lambda_{\min}(K_{p\theta})} \right\}, \\
c_4 &= \max \left\{ 2, \frac{2T}{\mu} \left( 2 + \frac{8N}{\lambda_2(L)} \right) \right\}.
\end{aligned}$$

Additionally, we have defined  $\bar{Q}_{\dot{f}_i^2}(t) := \text{diag} [Q_{\dot{f}_i^2}(t)]$ , with

$$Q_{\dot{f}_i^2}(t) := 1 + 2b_f^2 T - \frac{2}{T} \int_t^{t+T} \int_t^m \dot{f}_i(s)^2 ds dm. \quad (17)$$

It should be underscored that  $Q_{\dot{f}_i^2}(t)$  admits the following bounds  $1 \leq Q_{\dot{f}_i^2}(t) < b_{Q_i} := 1 + 2b_f^2 T$  and, furthermore,

$$\dot{Q}_{\dot{f}_i^2}(t) = -\frac{2}{T} \int_t^{t+T} \dot{f}_i(s)^2 ds + 2\dot{f}_i(t)^2. \quad (18)$$

Since  $\rho_1$  and  $\rho_2$  are positive functions and radially unbounded, positive definiteness of  $\Gamma$  is ensured with the fact that  $\Gamma(t, 0, 0) = 0$ , for all  $t \geq 0$ , and

$$\begin{aligned}
W &\geq \gamma(V)V, \\
W &\leq \gamma(V)V + V \kappa^\top(e, s) \bar{Q}_{\dot{f}_i^2}(t) \kappa(e, s) + 2c_1 b_f \lambda_N(L) V^2, \\
&\quad + 2(\lambda_N(L) + \|K_{pt}\|) \alpha(V)V, \\
Z &\geq \min \{1, \lambda_{\min}(K_{p\theta})\} (e_\theta^\top e_\theta + e_\omega^\top e_\omega), \\
Z &\leq \max \{1 + c_2, c_2 \lambda_{\max}(K_{p\theta}) + 1\} (e_\theta^\top e_\theta + e_\omega^\top e_\omega).
\end{aligned}$$

After some term chasing and some cumbersome manipulations we get

$$\begin{aligned}
\dot{\Gamma} &\leq -\frac{\mu}{4T} V^3 - \frac{\rho_1(V)}{8} [c_2 e_\omega^\top K_{d\theta} e_\omega + e_\theta^\top K_{p\theta} e_\theta] \\
&\quad - \frac{1}{4} \gamma(V) v^\top K_{dt} K_{pt}^{-1} v - \frac{1}{8} \alpha(V) e^\top K_{pt} e
\end{aligned} \quad (19)$$

Therefore  $\dot{\Gamma}$  is negative definite and  $\Gamma$  qualifies as a strict Lyapunov function for system (13). Global uniformly asymptotic stability of the equilibrium  $(X_t, X_r) = (0, 0)$  is ensured and thus the Consensus Problem is solved.

#### 4. SIMULATIONS

This section presents some numerical simulations using six differential wheeled mobile robots. The desired formation pattern is an hexagon and the communication topology, together with the required vectors  $\delta_i$  are depicted in Fig. 1. The initial states positions of the robots are:

For simplicity, all the robots are the same, with mass equal to 10kg and the moment of inertia equal to 3kg m<sup>2</sup>. The

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$	$q_6$
$x_i$	5	6	-3	-4	3	2
$y_i$	0	2	0	-2	-4	-6
$\theta_i$	90	15	80	0	-90	100

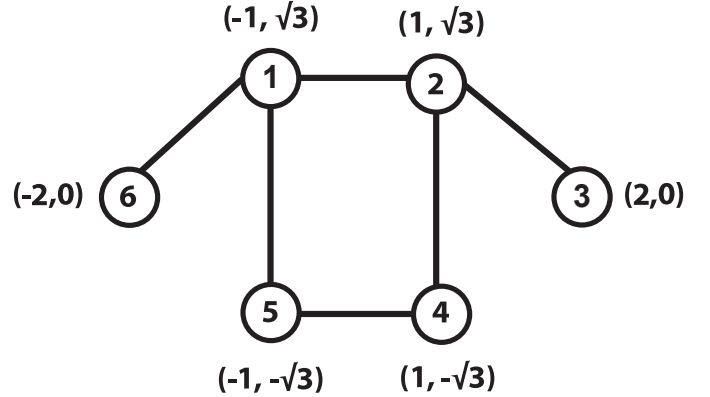


Fig. 1. Interconnection graph and bias of each mobile robot in the network.

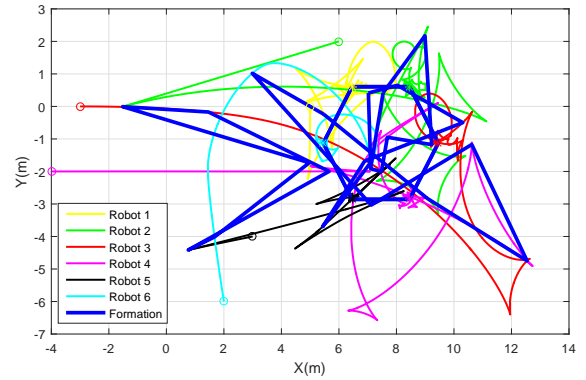


Fig. 2. Trajectories and formation of the simulation of the network of mobile robots.

distance parameters are:  $R = 0.1$  and  $r = 0.01$ . The gains have been set to  $d_i = k_{d_i} = 7$ ,  $k_{p_i} = 100$ ,  $p_i = 100$ ,  $k_{\alpha_i} = 300$ .

Fig. 2 show the trajectories of the nonholonomic mobile robots in order to form the desired pattern. Fig. 3 present the orientation behaviour of the network, where  $\theta_d = 70$ . From these simulations it can be concluded that the novel proposed controller asymptotically solves the desired control objective, as expected.

#### 5. CONCLUSION

This paper deals with the formation control of multiple nonholonomic robots. We report a novel decentralized consensus-based formation controller that considers both, the kinematic and the dynamic model, to uniformly and asymptotically drive a network composed of  $N$  agents to a desired formation with a given orientation. The network is modeled as an undirected, static and connected graph. The controller is a smooth time-varying PD-like scheme that is  $\delta$ -persistently exciting the nonholonomic robots. Up to the authors' knowledge this is the first work that provides a strict Lyapunov function, thereby guaranteeing uniform global asymptotic stability for the closed-loop

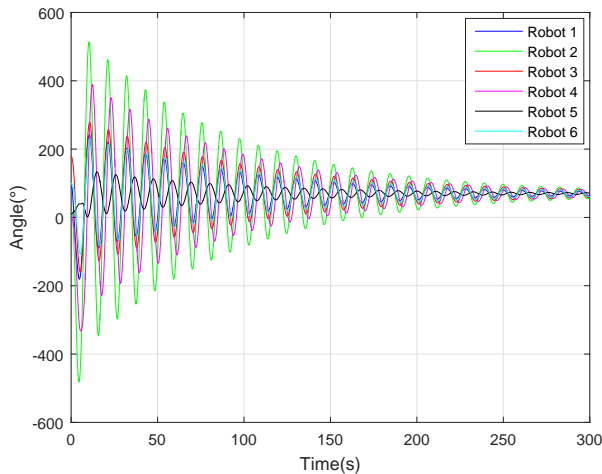


Fig. 3. Orientation of each mobile robot in the network.

system. Hence, the system is robust with respect to (small) external perturbations. Simulations, using a network with six agents, have been provided to illustrate our theoretical contributions.

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## Appendix A. ON PERSISTENCY OF EXCITATION

We recall that a function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m}$  is said to be persistently exciting, with excitation parameters  $(T, \mu)$ , if there exist  $T, \mu > 0$  such that

$$\int_t^{t+T} \psi(s)\psi(s)^\top ds \geq \mu I_n \quad \forall t \geq 0. \quad (\text{A.1})$$

The following lemma extends a well-known filtration property of persistently exciting functions (Ioannou and Sun, 1996).

*Lemma 2.* Consider the scalar second order system:

$$\ddot{f} + k_1 \dot{f} + k_2 f = p(t) \quad (\text{A.2})$$

where  $k_1, k_2 > 0$  and  $p(t)$  is a time varying input such that  $\dot{p}(t)$  is PE with excitation parameters  $(T, \mu)$  and there exists  $b_p > 0$  such that  $\max\{p, \dot{p}, \ddot{p}, p^{(3)}\} \leq b_p$ . Then  $\dot{f}(t)$  is PE with excitation parameters  $(T_f, \mu_f)$  given by  $T_f = kT$ ,

$$\mu_f = \frac{2(1+k_2^{-1})b_p r}{b_p^2(1+k_1 k_2^{-1} + k_2^{-1})^2 T_f}, \quad (\text{A.3})$$

and  $k = \left\lceil \frac{4(1+k_2^{-1})b_p r}{\mu k_2^{-1}} \right\rceil + 1$ , where

$$r^2 = \frac{(a+1)y(0)^2 + 4(ak_2+1)(f(0)^2 + k_2^{-1}b_p^2) + \frac{bb_p^2}{c}}{\min\{1, k_2\}}, \quad (\text{A.4})$$

$a = 2k_1^{-1} + k_1 k_2^{-1} + k_2^{-1} + 1$ ,  $b := 4k_2^{-1} + \frac{1}{ak_1 k_2^2}$  and  $c := \frac{1}{4} \frac{\min\{ak_1, k_2\}}{a+2+ak_2}$ .

Furthermore,  $\max\{\dot{f}, \ddot{f}, f^{(3)}\} \leq b_f$ , with:

$$b_f = r(k_1^2 + k_1 k_2 + k_2 + k_1 + 1) + b_p. \quad (\text{A.5})$$

□

*Proof of Lemma 2.* Consider the following linear change of coordinates  $x = f - k_2^{-1}p(t)$ ,  $y = \dot{f}$ . Then  $\dot{x} = y - k_2^{-1}\dot{p}(t)$  and  $\dot{y} = -k_2 y - k_1 x$ .

First, note that the overall trajectories are bounded, i.e., there exists  $r > 0$  that is a function of  $(x(0), y(0), b_p)$ , such that  $\|(x, y)\| \leq r$ ,  $\forall t \geq 0$ .

Consider now the following time derivative

$$\begin{aligned} \frac{d}{dt} [-\dot{p}x - k_2^{-1}\ddot{p}y] &= [-\dot{p} + k_1 k_2^{-1}\ddot{p} - k_2^{-1}p^{(3)}]y + k_2^{-1}\dot{p}^2 \\ &\geq -b_p [1 + k_1 k_2^{-1} + k_2^{-1}] |y| + k_2^{-1}\dot{p}^2, \end{aligned}$$

then

$$\begin{aligned} b_p [1 + k_1 k_2^{-1} + k_2^{-1}] \int_t^{t+kT} |y(s)| ds &\geq \\ \int_t^{t+kT} \frac{d}{ds} [\dot{p}(s)x(s) + k_2^{-1}\ddot{p}(s)y(s)] ds &+ \\ k_2^{-1} \int_t^{t+kT} \dot{p}(s)^2 ds &\geq -2(1+k_2^{-1})b_p r + k_2^{-1}k\mu \end{aligned}$$

where  $k$  is a positive integer and, to obtain the last inequality, we used the fact that trajectories are bounded and that  $\dot{p}$  is PE with parameters  $(T, \mu)$ .

Invoking the Cauchy-Schwartz inequality on  $\int_t^{t+kT} |y(s)| ds$ , we obtain

$$\begin{aligned} b_p^2 (1 + k_1 k_2^{-1} + k_2^{-1})^2 kT \int_t^{t+kT} y(s)^2 ds &\geq \\ (k_2^{-1}k\mu - 2(1+k_2^{-1})b_p)^2 & \end{aligned}$$

Finally, we get

$$\int_t^{t+kT} y(s)^2 ds \geq \frac{(k_2^{-1}k\mu - 2(1+k_2^{-1})b_p r)^2}{b_p^2(1+k_1 k_2^{-1} + k_2^{-1})^2 kT} \quad (\text{A.6})$$

Taking  $k = \left\lceil \frac{4(1+k_2^{-1})b_p r}{\mu k_2^{-1}} \right\rceil + 1$ , we find  $T_f = kT$  and

$$\mu_f = \frac{2(1+k_2^{-1})b_p r}{b_p^2(1+k_1 k_2^{-1} + k_2^{-1})^2 T_f}, \text{ such that } \int_t^{t+T_f} y(s)^2 ds \geq \mu_f$$

In order to have an explicit estimation of  $(T_f, \mu_f)$  it only remains to estimate the upper bound of the trajectories. For, let us define the Lyapunov function candidate  $V(x, y) = a(y^2 + k_2 x^2) + xy$  with  $a = 2k_1^{-1} + k_1 k_2^{-1} + k_2^{-1} + 1$ .

$V(x, y)$  verifies the following bounds

$$\begin{aligned} \min\{1, k_2\}(y^2 + x^2) &\leq V(x, y) \leq \\ \max\{a+1, ak_2+1\}(x^2 + y^2). & \end{aligned}$$

$\dot{V}$ , along the trajectories of the system, satisfies

$$\begin{aligned} \dot{V}(\cdot) &\leq -ak_1 y^2 + y^2 - k_1 yx - k_2 x^2 + 2\dot{p}x + k_2^{-1}y\dot{p} \\ &\leq -\frac{a}{4}k_1 y^2 - \frac{1}{4}k_2 x^2 + \left[4k_2^{-1} + \frac{1}{ak_1 k_2^2}\right] \dot{p}^2 \\ &\leq -cV + b\dot{p}^2 \end{aligned}$$

where  $c := \frac{1}{4} \frac{\min\{ak_1, k_2\}}{a+2+ak_2}$  and  $b := 4k_2^{-1} + \frac{1}{ak_1 k_2^2}$ .

Since  $x^2 + y^2 \leq \frac{1}{\min\{1, k_2\}} V$ , we can calculate the upper bound of the trajectories as

$$\begin{aligned} \|(x, y)\|^2 &\leq \frac{1}{\min\{1, k_2\}} \max\left\{V(0), \frac{b\dot{p}^2}{c}\right\} \\ &\leq \frac{(a+1)y(0)^2 + 4(ak_2+1)(f_1(0)^2 + k_2^{-1}b_p^2) + \frac{bb_p^2}{c}}{\min\{1, k_2\}} = r^2. \end{aligned}$$

Finally, from the system dynamics and (A.7), we can find that  $\dot{y} \leq (k_1 + k_2)r$  and  $\ddot{y} \leq (k_1^2 + k_1 k_2 + k_2)r + b_p$  so (A.5) follows. This concludes the proof. □