Effects of network topology on the synchronized behaviour of coupled nonlinear oscillators: a case study

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1. EXTENDED ABSTRACT

The concept of network is ubiquitous to various disciplines and includes different phenomena that emerge in a collection of interacting systems. These may be technology-based, as in energy-distribution and transportation systems as well as in telecommunications, or they may appear naturally, as in the case of ecosystems, social or neuronal networks and biological systems at a cellular level.

Even though the nature of each constituting system and the interconnections among them differ drastically from one discipline to another, at the level of mathematical abstraction, they all possess common features and may be analysed via common approaches.

In particular, on the level of modelling, networks can be viewed as a set of nodes and links that represent individual dynamical systems and the interactions among them, respectively and the behaviour of isolated systems is described via nonlinear models. Even taken separately, both, the complexity of network topology and the nonlinear nature of individual dynamics can lead to the appearence of non-trivial network behaviour. However, the interplay of these two characteristics entails large scale *collective* behaviour, in which some form of global coordination arises out of the local interactions among initially disordered systems.

From such a perspective, network behaviour is often regarded as the *dichotomy* of two related processes: the emergence of collective behaivour and the reorganisation of the individual systems relative to the latter; these two processes can be described as a two-levels system in which a macroscopic level corresponds to the large-scale network behaviour and the microscopic level considers the network from the point of view of the individual systems that compose it.

Indirectly, through the notion of synchronisation, behaviour at the microscopic level is well-studied in dynamic control theory –see eg, Blekhman et al. (1997); Boccaletti et al. (2006); Brown and Kocarev (2000), using a variety of tools that stem both from the dynamical systems and automatic control domains for the synchronisation analysis of complex (networked) systems. Nonetheless, the analysis of emergent behaviour is hardly explored in the control-theory community.

In contrast to this, macroscopic-level coherent network behaviour is studied within numerous scientific disciplines, such as chemistry, biology, physics, sociology, physiology, complexity theory, systems sciences, philosophy of sciences, to name a few. Depending on the specific area of research this "new" behaviour is known under various aliases: collective behaviour, self-organised motion, emergence, synergy, cooperativeness, symbiosis, epistasis, threshold effects, phase transitions, co-evolution, heterosis, dynamical attractor ...

Our approach for analysis of heterogeneous networks, which is presented in Panteley et al. (2015), allows to decompose the network dynamics in two parts: on one hand the dynamics of the "averaged" motion generated by the so-called mean-field node and, on the other, the dynamics of each individual unit of the network relative to the dynamics of the mean-field's. The dynamics of the mean-field node is determined by the individual dynamics of the nodes and by the connection graph; it pertains to the emergent dynamics of the network. Then, the dynamics of the nodes relative to the mean-field node corresponds to the coordination of the nodes among them hence, to (some type of) synchronisation. In accordance to the duality of collective behaviour previously described, we broach the analysis problem by decomposing it in the study of two properties: the stability of the emergent dynamics and that of synchronisation manifold.

In particular we consider a network composed of N diffusively coupled Stuart-Landau oscillators that is, N dynamical systems

$$\dot{z}_i = f(z_i, \mu_i) + u_i, \quad i \in \mathcal{I} := \{1, \dots, N\} \quad (1)$$
$$f(z_i, \mu_i) := -|z_i|^2 z_i + \mu_i z_i$$

where $z_i,\ u_i \in \mathbb{C}$ are, respectively, the state and the input of ith oscillator, $\mu_i = \mu_{\mathrm{R}i} + \mathrm{i}\mu_{\mathrm{I}i} \in \mathbb{C}$ is a complex parameter which defines the asymptotic behaviour of the ith oscillator.

We assume that the oscillators are interconnected via diffusive coupling, *i.e.*, for the i-th oscillator the input is given by

$$u_i = -\gamma\Big[d_{i1}(z_i - z_1) + d_{i2}(z_i - z_2) \dots + d_{iN}(z_i - z_N)\Big], \quad (2)$$

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where all $d_{ij} \geq 0$ and the scalar parameter $\gamma > 0$ corresponds to the coupling strength.

We assume that the interconnections weights are real and the network graph is connected and undirected. Then, the corresponding Laplacian matrix L has exactly one eigenvalue (say, λ_1) equal to zero, while others are positive, *i.e.*, $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$. Therefore, denoting by $\boldsymbol{z} \in \mathbb{C}^N$ the overall network's state, that is $\boldsymbol{z} = [z_1, \ldots, z_N]^\top$, using (1) and the expression for the diffusive coupling, (2), the overall network dynamics can be rewritten in the following form

$$\dot{\boldsymbol{z}} = F(\boldsymbol{z}) - \gamma L \boldsymbol{z}, \tag{3}$$

where the function $F : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$F(\boldsymbol{z}) = [f(z_i, \mu_i)]_{i \in \mathcal{I}}. \tag{4}$$

Following Panteley et al. (2015) we rewrite this model as

$$\dot{\boldsymbol{z}} = A_\gamma \boldsymbol{z} - C(\boldsymbol{z})\boldsymbol{z}, \tag{5a}$$

$$A_\gamma := \mathcal{M} - \gamma L. \tag{5b}$$

where diagonal matrices $C(\boldsymbol{z})$ and $\mathcal{M}$ are defined as follows

$$C(\boldsymbol{z}) := \mathrm{diag}(|\boldsymbol{z}_1|^2, \ldots, |\boldsymbol{z}_N|^2), \quad \mathcal{M} := \mathrm{diag}(\mu_1, \ldots, \mu_N).$$

In case of homogeneous networks (*i.e.* $\mu_i = \mu_j$ for all i, $j \in \mathcal{I}$) the oscillators *completely* synchronise, that is asymptotically they oscillate at the same frequency and with zero phase differences if the coupling parameter γsatisfies the property

$$Re\lambda_2(A_\gamma) < 0. \tag{6}$$

Complete synchronization of homogeneos networks can be analysed using different analysis tools developed for semi-passive, incrementally passive or incrementally input-output stable systems Pogromsky et al. (1999); Pogromsky and Nijmeijer. (2001); Pham and Slotine (2007); Scardovi et al. (2009), among others.

The behaviour of networks with non-identical oscillators is more complex, synchronization properties of such networks were addressed e.g. in Panteley et al. (2015) where it was shown that in the case of undirected graphs, the behaviour of the network (3), may be studied via two separate properties: the first relates to the dynamic behaviour of the mean-field solutions $z_m(t) \in \mathbb{C}$ defined as a projection of the network dynamics on the left eigenvector $v_{1\gamma}$ corresponding to the largest left eigenvalue of the matrix A_γ, i. e. $z_m = v_{1\gamma}^\top \mathbf{z}$.

The second part relates to the synchronisation errors, that is the differences between each unit's trajectories, $z_i(t)$, and $z_m(t)$. In particular, the synchronisation errors manifold was defined as

$$\mathcal{S} = \{\boldsymbol{e} \in \mathbb{C}^N : \boldsymbol{e}_1 = \boldsymbol{e}_2 = \ldots = \boldsymbol{e}_N = 0\} \tag{7}$$

where $\boldsymbol{e}_i = z_i - z_m$ and it was shown that this manifold is practically globally asymptotically stable under the assumption that the coupling gain γ satisfies inequality (6).

In this paper we link different possible behaviours of the network with the properties of the matrix A_γ. Namely, we consider the system (5) with $\mu_1 = \mu_2 \ldots = \mu_N = \mu$, that is the network defined as

$$\dot{\boldsymbol{z}} = A_\gamma \boldsymbol{z} - C(\boldsymbol{z})\boldsymbol{z}, \tag{8}$$

where the matrix A_γ takes now the form $A_\gamma = (\mu I - \gamma L)$.

When the network graph is directed and the network is strongly connected, the Laplacian matrix L can be presented in the form

$$L = V \Lambda V^{-1}, \tag{9}$$

where $\Lambda \in \mathbb{C}^{N \times N}$ is a diagonal matrix whose elements correspond to the eigenvalues of L and columns of the matrices V, $V^{-1} \in \mathbb{C}^{N \times N}$) correspond to the right and left eigenvectors of the Laplacian. It is easy to see that the matrix A_γ has the same eigenvectors as L while the eigenvalues of the two matrices are relates as

$$\lambda_i(A_\gamma) = \mu - \gamma \lambda_i(L), \quad i = 1, \ldots, N. \tag{10}$$

The eigenvalues of A_γ can be always ordered in decreasing order, that is, $\lambda_1(A_\gamma)$ has the largest real part and $\Re e[\lambda_1] \geq \Re e[\lambda_2] \geq \ldots \geq \Re e[\lambda_N]$.

We consider the three different types of networks depending on the eigenvalues of the matrix A_γ.

- all eigenvalues have nonpositive real parts,
- only one eigenvalue has a positive real part,
- two eigenvalues have a positive real part, i.e. $\Re e[\lambda_1] = \Re e[\lambda_2] > 0$.

As before, we use the matrix V, defined in (10) to decompose the network dynamics in two parts: the dynamics of a "mean-field" network and the dynamics of each individual unit of the network relative to the dynamics of the mean-field's. In the first two cases we obtain the mean field dynamics described by a single oscillator (cases considered in Panteley et al. (2015)).

In the last case matrix A_γ has two nonnegative eigenvalues $\lambda_1(A_\gamma)$ and $\lambda_2(A_\gamma)$ and therefore condition (6) is not satisfied.

In this case we project the network dynamics on the subspace defined by 2 eigenvectors and, as a result, obtain the mean-field dynamics described by a network of two oscillators which we define as

$$z_m = V_{l1}^* \mathbf{z}, \tag{11}$$

where $z_m \in \mathbb{C}^2$ and $V_{l1} \in \mathbb{C}^{2 \times N}$ is a matrix composed from the two left eigenvectors corresponding to $\lambda_1(A_\gamma)$ and $\lambda_2(A_\gamma)$. As a result the reduced network behaviour is defined not by a single oscillator but by a network of 2 coupled oscillators. In this case we define syncronization errors with respect to the dynamics of the reduced order network as

$$\boldsymbol{e} = \boldsymbol{z} - V_{r1} z_m,$$

where V_{r1} is a matrix composed from the two right eigenvectors of the matrix A_γ.

Under an additional assumption on the eigenvectors of A_γ which is satisfied for example, if the Laplacian matrix is circulant, we ensure asymptotic stability of the synchronization errors $\boldsymbol{e}$ and show that the behaviour of the overall network is defined by the behaviour of the reduced one.

Next we use results of Aroson et al. (1990), where detailed analysis of possible behaviours is done for a network composed of two oscillators with linear coupling, to characterize possible behaviours of the mean-field model and show how oscillations death, phase locking, phase drift and bistability can appear in a network depending on the properties of eigenvalues and eigenvectors of A_γ.

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