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Abstract

We solve the leader-follower tracking-agreement control problem for nonholonomic mobile robots with uncertainties; this consists in controlling a group of robots to take a pre-specified configuration (formation) and to move about following a vanishing reference trajectory. We assume that each robot has one unique leader and it is controlled by a local tracking controller that uses relative position and velocity measurements, but each robot may have one or several followers. We also assume that the reference trajectory is available to only one robot (the swarm leader). The control design is based on a δ-persistently exciting controller (for the kinematics model) that is robust to decaying perturbations and an outer control loop at the force-inputs level. Our proofs are constructive as they are based on Lyapunov's direct method; moreover, we establish strong integral input-to-state stability. To the best of our knowledge this is the first result of this nature in the literature of nonholonomic mobile robots.

Keywords: Tracking control, agreement, leader-follower, nonholonomic systems

1. Introduction

Over the turn of the last century there was a considerable bulk of literature on tracking and stabilization of non-holonomic mobile robots. Remarkable examples include, *e.g.*, the landmark paper [20], where the authors introduced a follow-the-leader control approach which consists in defining a virtual robot that generates a reference trajectory followed by the controlled robot. Depending on the velocities of the virtual robot, we distinguish, on one hand, the tracking problem, in which the leader velocities are defined by functions of time –see *e.g.*, [39, 37, 6] and, on the other, the set-point stabilization problem, in which

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case the non-holonomy restriction prevents the use of smooth autonomous feedback hence, one must employ either discontinuous controllers –*e.g.*, [3, 38, 8] or smooth *time-varying* controllers –[40, 30, 27].

For tracking purposes, a sufficient condition recurrently found in the literature is that the reference velocities do not vanish (*i.e.*, the virtual reference robot must not stop). For instance, in [39] it is assumed that at least one of the leader velocities is bounded from below in the limit, in [37] the rotational velocity of the leader is assumed to be persistently exciting, in [6] the translational leader velocity is assumed to be greater than zero. In the tracking-stabilization problem [13, 42], however, the leader velocities are functions of time that converge asymptotically to zero. Such problem may be regarded as a robust stabilization problem with respect to the leader velocities –see, for example, [22] where it is assumed that either the forward leader velocities is integrable, or the angular velocity is bounded from below.

The tracking-stabilization problem extends to the case of multi-agents control in which, in addition, it is required that the vehicles reach a predefined formation. Depending on the context, the formation can be defined on a case-by-case basis, by determining the positions of the vehicles relative to each other. In [14] the leader follower formation-tracking control problem is solved using a combination of the virtual structure and path-tracking approaches to generate the reference for each agent under the assumption that the translational leader velocity is persistently exciting; in [25] the tracking control of multiple mobile robots advancing in formation along straight-line paths, was addressed. Solutions to the agreement control problem for multiple mobile robots are provided, *e.g.*, in [10], for the case of undirected graphs, and in [15], for a swarm of vehicles interconnected via a direct graph. In the former, a distributed discontinuous feedback is used while in the latter a smooth time-varying controller is proposed. In [9] sliding-mode controllers are proposed to solve a leader-follower formation-tracking control problem, based on the measurement of the relative orientation and Euclidean distance separating the leader and its follower. In [41] the authors propose a distributed formation control method in which only a handful of robots have information of the reference trajectories, it is assumed that the system's parameters are unknown, and a result of ultimate boundedness on the tracking errors is established. See also [43].

In this paper we address the leader-follower tracking-agreement control problem for a group of mobile robots. Our control approach is decentralized; for each robot, we design a local δ-persistently exciting controller [30, 27] that uses relative velocity and position measurements of the actual robot, with respect to a leader robot. In contrast to most results in the literature, we provide a strict Lyapunov function, the construction of which is based on [32, 33, 34]. In addition, we establish, as far as we know for the first time in the literature, *strong integral input-to-state stability* (strong iISS) –see [5] and the Appendix. The importance of strong iISS is that not only this property guarantees robustness with respect to measurement noise, but it renders the solution to the tracking-agreement problem straightforward under a spanning tree communication topology.

The rest of the paper is organized as follows. In Section 2 we describe the control problem; in Section 3 we present our control approach and the main result on leader-follower tracking-stabilization control problem. In Section 4 we establish robustness results of the proposed controller with respect to the leader's velocities; we extend this result to the case of formation tracking-agreement control in Section 5. We provide some illustrative simulation results in Section 6, before concluding with some remarks.

2. Problem formulation

As for instance in [11], we consider mobile robots modeled by

$$\dot{z} = J(z)\nu \tag{1a}$$

$$M\dot{\nu} + C(\dot{z})\nu = \tau \tag{1b}$$

where $z := [x,\ y,\ \theta]$ contains the Cartesian coordinates (x, y) and the orientation θ of the robot, $\tau \in \mathbb{R}^2$ corresponds to the (torque) control input; $\nu := [\nu_1\ \nu_2]$ stands for the angular velocities corresponding to the two robot's wheels, M is the inertia matrix, which is constant, symmetric, and positive definite, and $C(\dot{z})$ is the matrix of Coriolis forces, which is skew-symmetric:

$$C(\dot{z}) = \begin{bmatrix} 0 & c\dot{\theta} \\ -c\dot{\theta} & 0 \end{bmatrix}, \qquad c > 0$$

—see [11]. for further detail on the model. In addition,

$$J(z) = \frac{r}{2}\begin{bmatrix} \cos(\theta) & \cos(\theta) \\ \sin(\theta) & \sin(\theta) \\ 1/b & -1/b \end{bmatrix}$$

where r is the radius of either steering wheel and b is the distance from the center of either wheel to the Cartesian point (x, y). The relation between the wheels' velocities, denoted by ν, and the robot's velocities in the fixed frame, denoted by $\dot{z}$, is

$$\begin{bmatrix} v \\ \omega \end{bmatrix} := \frac{r}{2b}\begin{bmatrix} b & b \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \ \Leftrightarrow\ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \frac{1}{r}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{bmatrix} v \\ \omega \end{bmatrix}. \tag{2}$$

These transformations may be used in (1a) to obtain the familiar model

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega, \end{aligned}$$

where $v = {}^1\!/_2|\dot{x}\ \dot{y}|$ denotes the forward velocity and ω corresponds to the angular velocity. See [11] for more details on the model (1).

The generic tracking problem consists in designing a control law τ^* such that z in (1) converges to $z_r := [x_r,\ y_r,\ \theta_r]$, which corresponds to the trajectories of a reference vehicle with kinematics

$$\dot{x}_r = v_r \cos\theta_r \tag{3a}$$
$$\dot{y}_r = v_r \sin\theta_r \tag{3b}$$
$$\dot{\theta}_r = \omega_r. \tag{3c}$$

According to the tracking-*stabilization* control goal [13, 42], it is assumed that the reference vehicle follows a vanishing trajectory, *i.e.*,

$$\lim_{t\to\infty} |v_r(t)| + |\omega_r(t)| = 0. \tag{4}$$

Remark 1 *The latter restriction naturally excludes control methods based on conditions of persistency of excitation hence, it also excludes the more restrictive case in which it is assumed that the references are always separated from zero –cf. [30, 7, 12, 22].*

From a *control* viewpoint, the goal is to steer to zero the differences between the Cartesian coordinates of the two robots, as well as orientation angles, that is, the quantities

$$\begin{aligned} p_x &= x_r - x - d_x \\ p_y &= y_r - y - d_y \\ p_\theta &= \theta_r - \theta \end{aligned}$$

where d_x, d_y are design parameters that determine the relative position of the follower vehicle with respect to that of the leader. Then, according to the approach of [20], we transform the error coordinates $[p_x,\ p_y,\ p_\theta]$ from the global coordinate frame to local coordinates fixed on the robot, to obtain

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_\theta \end{bmatrix}. \tag{5}$$

Hence, in the new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \tag{6a}$$
$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta) \tag{6b}$$
$$\dot{e}_\theta = \omega_r(t) - \omega \tag{6c}$$

and the leader-follower tracking-stabilization control problem comes to designing virtual control laws v^* and ω^* for the kinematics equations (6) and, using the latter as given references, a "tracking" control law τ^* for the force dynamics equation (1b), such that

$$\lim_{t\to\infty} |e(t)| = 0, \qquad e := [e_x\ e_y\ e_\theta]^\top. \tag{7}$$

3. The control approach

We address the tracking-stabilization control problem for (1) from a general perspective: we focus on the control design for the kinematics equations (6) and we establish the convergence of the tracking errors under the action of *any* controller τ^* at the dynamics level, *i.e.*, Eq. (1b). For the sake of completeness, however, we describe first a *particular* tracking controller τ^* guaranteeing that

$$\lim_{t\to\infty} |\tilde{v}(t)| = 0, \qquad \lim_{t\to\infty} |\tilde{\omega}(t)| = 0, \tag{8}$$

where $\tilde{v} = v - v^*$ and $\tilde{\omega} = \omega - \omega^*$.

3.1. Control at the force level

Let v^* and ω^* be given smooth functions and consider the dynamics equation (1b) under the assumption that the inertia parameters and the constants contained in $C(\dot{z})$ are unknown while r and b are considered to be known. Let $\hat{M}$ and $\hat{C}$ denote, respectively, the estimates of M and C. Furthermore, let $\nu^* := [\nu_1^* \ \nu_2^*]^\top$,

$$\begin{bmatrix}\nu_1^*\\ \nu_2^*\end{bmatrix} = \frac{1}{r}\begin{bmatrix}1 & b\\ 1 & -b\end{bmatrix}\begin{bmatrix}v^*\\ \omega^*\end{bmatrix}, \tag{9}$$

and let us introduce the certainty-equivalence control law

$$\tau^* := \hat{M}\dot{\nu}^* + \hat{C}(\dot{z})\nu^* - k_d\tilde{\nu}, \quad k_d > 0 \tag{10}$$

where $\tilde{\nu} := \nu - \nu^*$. Then, let us define $\tilde{M} := \hat{M} - M$ and $\tilde{C} := \hat{C} - C$, so

$$\tau^* := M\dot{\nu}^* + C(\dot{z})\nu^* - k_d\tilde{\nu} + \tilde{M}\dot{\nu}^* + \tilde{C}\nu^* \tag{11}$$

and, setting $\tau = \tau^*$ in (1b), we obtain the closed-loop equation

$$M\dot{\tilde{\nu}} + [C(\dot{z}) + k_d I]\tilde{\nu} = \Psi(\dot{z}, \dot{\nu}^*, \nu^*)^\top\tilde{\Theta} \tag{12}$$

where $\Theta \in \mathbb{R}^m$ is a vector of constant (unknown) lumped parameters in M and C, $\hat{\Theta}$ denotes the estimate of Θ, $\tilde{\Theta} := \hat{\Theta} - \Theta$ is the vector of estimation errors, and $\Psi : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{m\times 2}$ is a continuous known function. For this, we used the property that (1b) is linear in the constant lumped parameters. In addition, we use the passivity-based adaptation law –*cf.* [36],

$$\dot{\hat{\Theta}} = -\gamma\Psi(\dot{z}, \dot{\nu}^*, \nu^*)\tilde{\nu}, \qquad \gamma > 0. \tag{13}$$

Then, a direct computation shows that the total derivative of

$$V(\tilde{\nu}, \tilde{\Theta}) := \frac{1}{2}\big[|\tilde{\nu}|^2 + \frac{1}{\gamma}|\tilde{\Theta}|^2\big]$$

along the trajectories of (12), (13), yields

$$\dot{V}(\tilde{\nu}(t), \tilde{\Theta}(t)) \le -k_d|\tilde{\nu}(t)|^2.$$

Integrating the latter from 0 to infinity we obtain that $\tilde{\nu} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\tilde{\Theta} \in \mathcal{L}_\infty$. It follows, *e.g.*, from [18, Lemma 3.2.5], that $\tilde{\nu} \to 0$ and, in view of (2), also (8) holds.

Remark 2 *In view of* (4) *it may not be expected that the estimation error parameters* $\tilde{\Theta}$ *converge to zero since the necessary and sufficient condition of persistency of excitation on* $\Psi(\dot{z}^*, \dot{\nu}^*, \nu^*)$ *is not met —cf. [26]. Consequently, in the context of the problem studied here, it is not possible to guarantee uniform global asymptotic stability but only that* $\tilde{\nu} \to 0$*, which is a rather weak property. It is all the more significant that this weak condition suffices to establish the control goal* (7)*.*

3.2. Control at kinematics level

The controller for (6) that we use is

$$v^* = k_x e_x + v_r(t) \cos e_\theta \tag{14a}$$

$$\omega^* = \omega_r(t) + k_\theta e_\theta + k_y \left[e_y^2 + e_x^2\right] p(t). \tag{14b}$$

Our standing assumption is that $\dot{p}$ is persistently exciting that is, we assume that there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \dot{p}(s)^2 ds \geq \mu \quad \forall t \geq 0. \tag{15}$$

The controller (14) is reminiscent of others in the literature. The control approach goes back to the tracking controller in [37] where controllers with persistency of excitation were introduced to solve the tracking control problem. As a matter of fact, if we set $p \equiv 0$, we recover the controller proposed in [37] which requires that ω_r is persistently exciting. Following [37], other controllers based on persistency of excitation of the reference velocities are reported —see, for instance, [23, 17, 4, 31]. In [31] it is showed, via Lyapunov's direct method, that the control law

$$\omega^* = \omega_r(t) + k_\theta e_\theta + k_y v_r(t) \frac{\sin(e_\theta)}{e_\theta}$$

—*cf.* [7], ensures uniform global asymptotic stability of the closed-loop kinematics under the assumption that either v_r or ω_r is persistently exciting. In all these cases, the restriction (4) is violated.

In the stabilization scenario, that is, if $v_r = \omega_r \equiv 0$ we recover the so-called *δ-persistently exciting (δ-PE) controllers* introduced in [30, 27], works which are strongly inspired by [40]. Other articles where δ-PE controllers are used include [42].

We recall that a function $(t, x) \mapsto \phi$ is called δ-persistently exciting (with respect to x) if and only if[2], for any $\delta > 0$, there exist $\mu > 0$ and $T > 0$ such that

$$|x| \geq \delta \quad \Longrightarrow \quad \int_t^{t+T} |\phi(s, x)| ds \geq \mu \quad \forall t \geq 0. \tag{16}$$

[2]The expression (16) is an equivalent characterization (for functions that are uniformly continuous in t) of the property of δ-persistency of excitation, defined in [28], that fits the purpose of this paper.

For instance, the term $\phi(t,x) := [e_y^2 + e_x^2]p(t)$, that appears in (14b), satisfies (16) with $x = [e_x,\ e_y]^\top$ and p being persistently exciting. Note that this is a weaker property than persistency of excitation along the state trajectories (as it is sometimes abusively used in adaptive control) since, notably, ϕ ceases to be exciting as the state trajectories converge, which is the control objective.

To understand the stabilisation mechanism of persistently-exciting controllers it is convenient to inspect the error equations that result after replacing v with $\tilde{v} + v^*$ and ω with $\tilde{\omega} + \omega^*$, and using (14) in (6). We obtain

$$\dot{e} = A(t,e)e + B(e)\eta, \qquad \eta := [v_r\ \omega_r\ \tilde{v}\ \tilde{\omega}]^\top \tag{17}$$

where

$$\begin{aligned} A(t,e) &:= \begin{bmatrix} -k_x & \psi(t,e) & 0 \\ -\psi(t,e) & 0 & 0 \\ -k_y p(t) e_x & -k_y p(t) e_y & -k_\theta \end{bmatrix}, \\ B(e) &= \begin{bmatrix} 0 & e_y & -1 & e_y \\ \sin(e_\theta) & -e_x & 0 & -e_x \\ 0 & 0 & 0 & -1 \end{bmatrix} \\ \psi(t,e) &:= k_\theta e_\theta + k_y p(t) \left[e_y^2 + e_x^2\right]. \end{aligned}$$

Therefore, the system (17) may be analysed as a perturbed system with nominal dynamics

$$\dot{e} = A(t,e)e \tag{18}$$

and a perturbation $B(e)\eta$ that vanishes in view of (4) and provided that the tracking-control problem is resolved (as for instance done in Section 3.1).

Hence, the stability of (17) relies on that of (18). Then, in order to analyse the latter it is useful to note that it has the feedback interconnection form (illustrated in Figure 1) of two systems that, without inputs, are stable. On one hand, it is clear that Σ_2 is exponentially stable at the origin if $u_2 \equiv 0$. On the other hand, for $u_1 \equiv 0$, Σ_1 has the familiar form of model-reference adaptive control systems [35], the function φ playing the role of a regressor. For such systems, after [29], uniform global asymptotic stability follows, roughly speaking, if φ is persistently exciting provided that $e_x,\ e_y \neq 0$. Formally, this is captured by the property of δ-persistency of excitation; see, for instance, [25] where such a result is established for the particular case in which $\omega_r \equiv 0$. See also Proposition 2 in next section, whose original proof relies on Lyapunov's direct method.

Following this rationale we present our main statement.

Proposition 1 *(Main result) Let k_x, k_θ, and $k_y > 0$ and let p and $\dot{p}$ be bounded and persistently exciting functions. Consider the system* (1) *in closed loop with <u>any</u> control law τ^* guaranteeing that* (8) *holds. Then,* (7) *also holds.*

Sketch of proof. The detailed proof of Proposition 1 is lengthy hence, for the sake of clarity, it is organized in the following three steps, which are developed in Section 4:

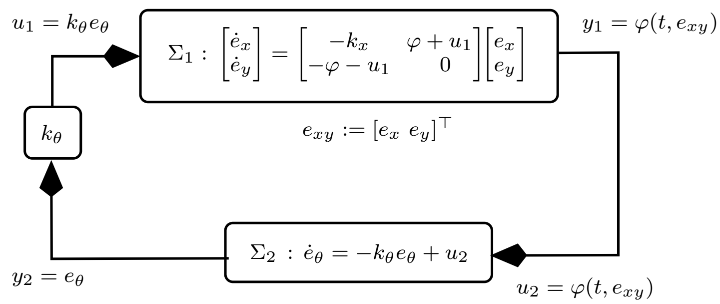

Figure 1: "Small-gain" feedback representation of Eq. (18) in which we use $\varphi(t, e_{xy}) := k_y p(t) \left[e_y^2 + e_x^2\right]$.

(1) First, a strict Lyapunov function for (18) is constructed —see Proposition 2 below.

(2) With such Lyapunov function it is established small-input-to-state stability (small ISS) with respect to η —see Lemma 1.

(3) Finally, integral input-to-state stability (iISS) of (17) with respect to η is proved.

The two properties together, small-ISS and iISS, imply the so-called property of strong iISS [5], in view of which, under the assumption that (8) and (4) hold, the result follows. ■

4. Robust leader-follower tracking and stabilization control

4.1. Stability of the nominal system (18)

We establish uniform global asymptotic stability for (18) via Lyapunov's direct method. Even though this is only a preliminary step in the construction of the proof of our main result we stress that, to the best of our knowledge, the following statement has no precedent in the literature.

Proposition 2 (set-point stabilization) *Let $\omega_r \equiv 0$ and $v_r \equiv 0$, let k_x, k_y, and $k_\theta > 0$ be positive constants, and let $t \mapsto p$ be once continuously differentiable and satisfy* (15)*. Then, the origin $\{e = 0\}$ for the closed-loop system* (6), (14) *is uniformly globally asymptotically stable. Moreover, it admits the Lyapunov function*

$$V_3(t, e) := \gamma_1\big(V_1(e)\big)V_1(e) + V_2(t, e) + \gamma_2\big(V_1(e)\big)e_z^2 \tag{19}$$

where

$$\begin{aligned} V_1(e) &:= e_x^2 + e_y^2, \\ V_2(t, e) &:= \gamma_3\big(V_1(e)\big)V_1(e) + \Phi(t)V_1(e)^3 - \dot\phi(t)V_1(e)e_x e_y \end{aligned} \tag{20}$$

$$\begin{aligned} e_z &:= e_\theta + \phi(t)\left[e_y^2 + e_x^2\right] && (21)\\ \dot{\phi} &= -k_\theta \phi + k_y p(t), && (22)\\ \Phi(t) &:= 1 + \bar{\phi}^2 T - \frac{1}{T}\int_t^{t+T}\int_t^m \dot{\phi}(s)^2 ds dm, &&\\ \bar{\phi} &:= \sup_{t\geq 0}|\phi(t)|, && (23)\end{aligned}$$

and $\gamma_i(V_1)$ are polynomials of V_1 with positive coefficients. Moreover, V_3 satisfies

$$\dot{V}_3(t,e) \leq -\frac{\mu}{8T}e_y^6 - k_\theta \gamma_2(V_1)e_z^2 - k_x e_x^2 - \frac{\mu}{4T}V_1^3. \quad (24)$$

Proof. Consider the function ϕ defined in (22); this function satisfies

$$\ddot{\phi} = -k_\theta \dot{\phi} + k_y \dot{p}. \quad (25)$$

Hence, since both p and $\dot{p}$ are persistently exciting and bounded, it follows, in view of (22), (25), and [18, Lemma 4.8.3], that ϕ and $\dot{\phi}$ are also persistently exciting and bounded. Hence, there exist μ, T, and $\bar{\phi} > 0$ such that[3]

$$\int_t^{t+T}\dot{\phi}(s)^2 ds \geq \mu \quad \forall t \geq 0. \quad (26)$$

Note also that

$$1 \leq |\Phi(t)| \leq 1 + \bar{\phi}^2 T < \infty.$$

Next, we use the error coordinate e_z defined in (21), which satisfies

$$\dot{e}_z = -k_\theta e_z - 2\phi k_x e_x^2. \quad (27)$$

Then, in the new coordinates (e_x, e_y, e_z), the nominal system $\dot{e} = A(t,e)e$ becomes

$$\begin{aligned} \begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} &= \begin{bmatrix}-k_x & \dot{\phi}V_1\\ -\dot{\phi}V_1 & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} + e_z\begin{bmatrix}0 & k_\theta\\ -k_\theta & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} && (28a)\\ \dot{e}_z &= -k_\theta e_z - 2\phi k_x e_x^2. && (28b)\end{aligned}$$

We proceed to show that the statement of Proposition 2 holds with

$$\begin{aligned} \gamma_1(V_1) &:= \frac{\mu}{16Tk_x}V_1^2 + \frac{1}{2}\bar{\phi}V_1 + \frac{4k_x\bar{\phi}^2}{k_\theta}\gamma_2(V_1)V_1 + \frac{1}{2}, && (29)\\ \gamma_2(V_1) &:= \frac{8T\bar{\phi}^2 k_\theta}{\mu}V_1 + 1 && (30)\\ \gamma_3(V_1) &:= \frac{\bar{\phi}}{k_x}\left[2\bar{\phi}V_1^2 + \frac{1}{4}\left[3k_x + 1\right]V_1 + \frac{T\bar{\phi}}{\mu}\left[k_x^2 + 1\right]\right]. && (31)\end{aligned}$$

[3]Without loss of generality, we assume that (15) and (26) hold with the same μ and T.

First, we remark that V_2, hence V_3, is positive definite and radially unbounded; this follows since $\gamma_1(V_1)V_1 > (\bar{\phi}/2)V_1^2$ and

$$-\dot{\phi}(t)V_1(e)e_x e_y + \frac{\bar{\phi}}{2}V_1(e)^2 = \frac{V_1}{2}\begin{bmatrix} e_x \\ e_y \end{bmatrix}^\top \begin{bmatrix} \bar{\phi} & -\dot{\phi} \\ -\dot{\phi} & \bar{\phi} \end{bmatrix}\begin{bmatrix} e_x \\ e_y \end{bmatrix} \geq 0.$$

Next, we proceed to compute the total derivative of V_3. By the fundamental theorem of calculus, we have

$$\dot{\Phi}(t) = -\frac{1}{T}\int_t^{t+T} \dot{\phi}(s)^2 ds + \dot{\phi}(t)^2.$$

Now, let μ, $T > 0$ be generated by the assumption that $\dot{\phi}$ is persistently exciting. Then,

$$\dot{\Phi}(t) \leq -\frac{\mu}{T} + \dot{\phi}(t)^2.$$

Therefore, the time derivative of V_2 along the trajectories of the system

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} -k_x & \dot{\phi}[e_y^2 + e_x^2] \\ -\dot{\phi}[e_y^2 + e_x^2] & 0 \end{bmatrix}\begin{bmatrix} e_x \\ e_y \end{bmatrix} \tag{32}$$

satisfies

$$\begin{aligned}
\dot{V}_2 \;\; &\leq \;\; -\frac{\mu}{T}V_1^3 + \dot{\phi}^2 V_1^3 - \dot{\phi}^2 e_y^2 V_1^2 + k_x\dot{\phi}e_x e_y V_1 \\
&\quad\; -2k_x\gamma_3(V_1)e_x^2 - \ddot{\phi}e_x e_y V_1 + 2\dot{\phi}e_y k_x e_x^3 + \dot{\phi}^2 V_1^2 e_x^2 \\
&\leq \;\; -\frac{\mu}{2T}V_1^3 - \frac{\mu}{2T}e_y^6 + \dot{\phi}^2\left(e_x^4 + 3e_x^2e_y^2 + 3e_y^4\right)e_x^2 \\
&\quad\; +\dot{\phi}^2 e_y^6 - \dot{\phi}^2 e_y^2 V_1^2 + k_x\dot{\phi}e_x e_y[e_x^2 + e_y^2] - 2k_x\gamma_3(V_1)e_x^2 \\
&\quad\; -\ddot{\phi}e_x e_y V_1 + 2\dot{\phi}e_x e_y k_x e_x^2 + \dot{\phi}^2 V_1^2 e_x^2.
\end{aligned}$$

Now, we use $V_1 = [e_x^2 + e_y^2]$ and the inequalities

$$\begin{aligned}
\dot{\phi}^2\left(e_x^4 + 3e_x^2e_y^2 + 3e_y^4\right)e_x^2 \;\; &\leq \;\; 3\bar{\phi}^2 V_1^2 e_x^2, \\
\dot{\phi}^2 e_y^6 - \dot{\phi}^2 e_y^2 V_1^2 \;\; &\leq \;\; 0, \\
3\dot{\phi}e_x e_y k_x e_x^2 \;\; &\leq \;\; \frac{3}{2}V_1\bar{\phi}k_x e_x^2, \\
\dot{\phi}e_x e_y k_x e_y^2 \;\; &\leq \;\; \frac{\bar{\phi}}{2}[\frac{1}{\epsilon}e_y^6 + \epsilon k_x^2 e_x^2], \\
-\ddot{\phi}e_x e_y[e_y^2 + e_x^2] \;\; &\leq \;\; \frac{\bar{\phi}}{2}[\frac{1}{\epsilon}e_y^6 + \epsilon e_x^2 + e_x^2 V_1]
\end{aligned}$$

to obtain

$$\dot{V}_2 \;\; \leq \;\; -\frac{\mu}{2T}V_1^3 - \left[\frac{\mu}{2T} - \frac{\bar{\phi}}{\epsilon}\right]e_y^6$$

$$-\left[2k_x\gamma_3(V_1) - 4\bar{\phi}^2 V_1^2 - \frac{3}{2}\bar{\phi}k_x V_1 - \frac{\epsilon\bar{\phi}}{2}[k_x^2+1] - \frac{\bar{\phi}}{2}V_1\right]e_x^2$$

so, setting $\epsilon = \frac{4T\bar{\phi}}{\mu}$ and $\gamma_3(V_1)$ as in (31), we obtain

$$\dot{V}_2 \le -\frac{\mu}{2T}V_1^3 - \frac{\mu}{4T}e_y^6. \tag{33}$$

Next, we compute the total derivative of V_3 in (19) along the trajectories of (28). Using (33), we obtain

$$\begin{aligned}
\dot{V}_3 \quad \le \quad & -2\gamma_1(V_1)k_x e_x^2 - \frac{\mu}{2T}V_1^3 - \frac{\mu}{4T}e_y^6 \\
& +\frac{\partial V_2}{\partial V_1}\frac{\partial V_1}{\partial [e_x \ e_y]^\top}\begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix}\begin{bmatrix} e_x \\ e_y \end{bmatrix} e_z \\
& -\frac{\partial(\dot{\phi}V_1 e_x e_y)}{\partial[e_x \ e_y]^\top}\begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix}\begin{bmatrix} e_x \\ e_y \end{bmatrix} e_z \\
& +2\gamma_2(V_1)e_z[-k_\theta e_z - 2\phi k_x e_x^2]
\end{aligned}$$

and we use the fact that

$$\frac{\partial V_1}{\partial [e_x \ e_y]^\top}\begin{bmatrix} 0 & k_\theta \\ -k_\theta & 0 \end{bmatrix}\begin{bmatrix} e_x \\ e_y \end{bmatrix} = 0$$

to obtain

$$\begin{aligned}
\dot{V}_3 \quad \le \quad & -\frac{\mu}{4T}e_y^6 - k_\theta\dot{\phi}e_z[e_y^4 - e_x^4] - 2k_\theta\gamma_2(V_1)e_z^2 \\
& -4\phi k_x\gamma_2(V_1)e_z e_x^2 - 2\gamma_1(V_1)k_x e_x^2 - \frac{\mu}{2T}V_1^3.
\end{aligned}$$

Now, for any ϵ_1, $\epsilon_2 > 0$ we have

$$\begin{aligned}
-k_\theta\dot{\phi}e_z e_y^4 \quad & \le \quad \frac{1}{2\epsilon_1}\bar{\phi}k_\theta e_z^2 e_y^2 + \frac{\epsilon_1}{2}\bar{\phi}k_\theta e_y^6 \\
k_\theta\dot{\phi}e_z e_x^4 \quad & \le \quad \frac{1}{2\epsilon_1}\bar{\phi}k_\theta e_z^2 V_1 + \frac{\epsilon_1}{2}\bar{\phi}k_\theta e_x^2 V_1^2 \\
-4\phi k_x\gamma_2(V_1)e_z e_x^2 \quad & \le \quad \frac{2\bar{\phi}}{\epsilon_2}k_x\gamma_2(V_1)e_z^2 + 2\epsilon_2\bar{\phi}k_x\gamma_2(V_1)V_1 e_x^2,
\end{aligned}$$

therefore

$$\begin{aligned}
\dot{V}_3(t,e) \quad \le \quad & -\frac{\mu}{4T}V_1^3 - \left[\frac{\mu}{4T} - \frac{\epsilon_1}{2}\bar{\phi}k_\theta\right]e_y^6 \\
& -\left[2\gamma_1(V_1)k_x - \frac{\epsilon_1}{2}\bar{\phi}k_\theta V_1^2 - 2\epsilon_2 k_x\bar{\phi}\gamma_2(V_1)V_1\right]e_x^2 \\
& -\left[2k_\theta\gamma_2(V_1) - [\frac{k_\theta}{\epsilon_1}V_1 + \frac{2}{\epsilon_2}k_x\gamma_2(V_1)]\bar{\phi}\right]e_z^2.
\end{aligned}$$

Thus, (24) follows from (29)–(31) and setting

$$\epsilon_1 := \frac{\mu}{4T\bar{\phi}k_\theta}, \quad \epsilon_2 := \frac{4k_x\bar{\phi}}{k_\theta}.$$

■

4.2. Small-ISS property of Eq. (18)

We use the Lyapunov function V_3 from Proposition 2 to establish that the nominal system (18) is small-input-to-state stable.

Lemma 1 *Under the conditions of Proposition 2, the system* (17) *is small-input-to-state stable with respect to the input* η*.*

Proof. The proof relies on the function V_3 defined in (19); specifically, on its order of growth in V_1. For the purpose of analysis we remark that V_3 satisfies

$$V_3(t,e) \equiv \mathcal{V}_3(t,e,V_1) \tag{34}$$

where

$$\begin{aligned} \mathcal{V}_3(t,e,V_1) &:= \rho(t,V_1)V_1 - \dot{\phi}(t)V_1 e_x e_y + \gamma_2(V_1)e_z^2 & (35)\\ \rho(t,V_1) &:= \gamma_1(V_1) + \gamma_3(V_1) + \Phi(t)V_1^2 & (36)\end{aligned}$$

that is, $\rho : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function, uniformly bounded in t and, for each t, $\rho(t,\cdot)$ is a polynomial of degree 2 with strictly positive coefficients. Consequently, since $\Phi(t) \geq 1$,

$$\frac{\partial \rho}{\partial V_1} \geq 0 \quad \forall\, (t, V_1) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \tag{37}$$

Now, by Proposition 2 the time derivative of V_3 along the nominal system (28) satisfies (24) hence, the time derivative of V_3 along the trajectories of (17) satisfies

$$\dot{V}_3 \leq -\frac{\mu}{4T}V_1^3 - k_\theta \gamma_2(V_1)e_z^2 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 + \frac{\partial V_3}{\partial e}B(e)\eta. \tag{38}$$

Now, note that $B(e)\eta = K_1(\eta)e + K_2(\eta,e)$ where

$$K_1(\eta) := \begin{bmatrix} 0 & \omega_r + \tilde{\omega} & 0 \\ -(\omega_r + \tilde{\omega}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_2(\eta,e) = \begin{bmatrix} -\tilde{v} \\ v_r \sin(e_\theta) \\ -\tilde{\omega} \end{bmatrix},$$

so, using (34)—(36), $\dot{V}_1 \leq 0$, and the fact that

$$\frac{\partial V_1}{\partial e}K_1(\eta)e = 0,$$

we obtain

$$\begin{aligned} \dot{V}_3 \;\leq\; & -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 - k_x e_x^2 - \frac{\mu}{8T}e_y^6 \\ & - \dot{\phi}[\omega_r + \tilde{\omega}]V_1\left[e_y^2 - e_x^2\right] + \frac{\partial V_3}{\partial e}K_2(\eta,e) \\ \leq & -\frac{\mu}{4T}V_1^3 - k_\theta\gamma_2(V_1)e_z^2 + \bar{\phi}\big[\,|\omega_r| + |\tilde{\omega}|\,\big]V_1^2 + \left|\frac{\partial V_3}{\partial e}\right||K_2| \end{aligned}$$

$$- k_x e_x^2 - \frac{\mu}{8T} e_y^6. \tag{39}$$

On the other hand, from (34) and (35), we obtain

$$\begin{aligned}
\left|\frac{\partial V_3}{\partial e}\right| \;\leq 2&\left[\frac{\partial \rho}{\partial V_1} V_1 + \rho(t, V_1) + \bar{\phi} V_1\right]\big[\,|e_y| + |e_x|\,\big] \\
&+ 2\frac{\partial \gamma_2}{\partial V_1}\big[\,|e_y| + |e_x|\,\big] e_z^2 \\
&+ 4\gamma_2(V_1)\bar{\phi}\,|e_z|\,\big[\,|e_y| + |e_x|\,\big] + 2\gamma_2(V_1)|e_z|.
\end{aligned} \tag{40}$$

Next, having $\rho(t, V_1)$ uniformly bounded in t, let us introduce the positive polynomial of second degree $\gamma_4 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\gamma_4(V_1) \geq \frac{\partial \rho}{\partial V_1} V_1 + \rho(t, V_1) + \bar{\phi} V_1$$

and let the positive constant

$$\alpha := \frac{\partial \gamma_2}{\partial V_1}$$

–see (30), so that, using them in (40) and observing that $|K_2| \leq |\eta|$, we obtain

$$\begin{aligned}
\dot{V}_3 \leq &- \frac{\mu}{4T} V_1^3 - k_\theta \gamma_2(V_1) e_z^2 + 2\bar{\phi}|\eta| V_1^2 - k_x e_x^2 - \frac{\mu}{8T} e_y^6 \\
&+ 2\gamma_4(V_1)|\eta|\big[|e_y| + |e_x|\big] + 2\alpha|\eta|\big[|e_y| + |e_x|\big] e_z^2 \\
&+ 4\gamma_2(V_1)\bar{\phi}|\eta||e_z|\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)|\eta||e_z|.
\end{aligned} \tag{41}$$

Then, using the inequality $|e_z|\big[\,|e_y| + |e_x|\big] \leq e_z^2 + V_1/2$ in (41) we obtain

$$\begin{aligned}
\dot{V}_3 \leq &- \frac{\mu}{4T} V_1^3 - k_x e_x^2 - \frac{\mu}{8T} e_y^6 \\
&- \Big[\big[k_\theta - 4\bar{\phi}|\eta|\big]\gamma_2(V_1) - 2\alpha|\eta|\Big[|e_y| + |e_x|\Big] - |\eta|\Big]\, e_z^2 \\
&+ 2\bar{\phi}|\eta| V_1^2 + 2\gamma_4(V_1)|\eta|\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)\bar{\phi}|\eta| V_1 + \gamma_2(V_1)^2|\eta| \\
\leq \; &- \Big[\frac{\mu}{4T} V_1^3 - \Phi_1|\eta|\Big] - \Big[\frac{k_\theta}{2}\gamma_2(V_1) - \Phi_2|\eta|\Big] e_z^2 \\
&- k_x e_x^2 - \frac{\mu}{8T} e_y^6 - \frac{k_\theta}{2}\gamma_2(V_1) e_z^2
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
\Phi_1 \;&:=\; 2\bar{\phi} V_1^2 + 2\gamma_4(V_1)\big[|e_y| + |e_x|\big] + 2\gamma_2(V_1)\bar{\phi} V_1 + \gamma_2(V_1)^2, \\
\Phi_2 \;&:=\; 4\bar{\phi}\gamma_2(V_1) + 2\alpha\big[|e_y| + |e_x|\big] + 1.
\end{aligned}$$

Then, since $|e_y|+|e_x| \leq \sqrt{2V_1}$, $\gamma_2(V_1) = \mathcal{O}(V_1)$, and $\gamma_4(V_1) = \mathcal{O}(V_1^2)$ there exist positive constants a_i, with $i \in [0,4]$, of innocuous values[4], such that

$$\begin{aligned} \Phi_1 &\leq [a_2 V_1^2 + a_1 V_1 + a_0][1 + a_4 V_1^{1/2}] && (43) \\ \Phi_2 &\leq a_1 V_1 + a_4 V_1^{1/2} + a_0. && (44) \end{aligned}$$

Furthermore, since $V_1^{1/2} \leq a_0 + a_1 V_1$ for all $a_0 \geq 1$, $a_1 \geq 1$, and $V_1 \geq 0$,

$$\begin{aligned} \Phi_1 &\leq a_3 V_1^3 + a_2 V_1^2 + a_1 V_1 + a_0 && (45) \\ \Phi_2 &\leq a_1 V_1 + a_0. && (46) \end{aligned}$$

Now, let $R > 0$ and

$$|\eta| \leq R \min\left\{V_1(e)^3 + e_z^2,\ 1\right\} \tag{47}$$

which, in particular, implies that $|\eta| \leq R$. We see that the first factor of e_z^2 in (42) is non-positive for sufficiently small R. In regards to the term involving Φ_1 in (42), note that in case that $V_1 \geq 1$, since $|\eta| \leq R$, we have $\Phi_1|\eta| \leq c_1 R V_1^3$ for some constant $c_1 > 0$. Otherwise, if $V_1 \leq 1$, there exists $c_2 > 0$ such that $\Phi_1 \leq c_2$ and, in view of (47),

$$\Phi_1|\eta| \leq c_2 R\,[V_1^3 + e_z^2]. \tag{48}$$

We conclude that, for sufficiently small R, (42) and (47) imply that

$$\dot{V}_3(t,e) \leq -\frac{k_\theta}{4} e_z^2 - k_x e_x^2 - \frac{\mu}{8T} e_y^6,$$

so the system is small-input-to-state stable. ∎

4.3. The iISS property

Lemma 2 *Under the conditions of Proposition 2, the nominal system* (17) *is integral input-to-state stable with respect to the input η.*

Proof. Consider the proper positive-definite Lyapunov function

$$W(t,e) = \ln\big(1 + V_3(t,e)\big) \tag{49}$$

and a positive definite function $\alpha : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha(e) \geq \frac{1}{1 + V_3(t,e)}\left[k_x e_x^2 + \frac{\mu}{8T} e_y^6 + \frac{k_\theta^2}{2} e_z^2\right] \tag{50}$$

[4]Let, by convention, a_i (for any integer $i \geq 0$) denote positive coefficients of polynomials so that, without loss of generality, we may implicitly assume that they are redefined as needed, *e.g.*, $a_i := a_i a_j + a_i^2 - a_i \ldots$

Then, in view of (42), the time derivative of W along the trajectories of (17) satisfies

$$\dot{W}(t,e) \quad \leq \quad -\alpha(e) + \frac{\Phi_1 + \Phi_2 e_z^2}{1 + V_3(t,e)}|\eta|. \tag{51}$$

From (29) and the fact that $V_2 \geq 0$, there exist a_1, a_2, and $a_3 > 0$, such that

$$V_3(t,e) \geq a_3 V_1^3 + a_2 V_1^2 + a_1 V_1 + \gamma_2(V_1) e_z^2 \tag{52}$$

so, in view of (45), (46), and (30), the factor of $|\eta|$ in (51) is bounded that is, there exists $c > 0$ such that $\dot{W}(t,e) \leq -\alpha(e) + c|\eta|$, so the system (17) is integral input-to-state stable. ■

5. Leader-follower tracking-agreement control

We use the previous results to solve the problem of multi-agent tracking-agreement control for a group of N robots modelled by (1), *i.e.*,

$$\dot{z}_i = J(z_i)\nu_i \tag{53a}$$

$$M\dot{\nu}_i + C(\dot{z}_i)\nu_i = \tau_i, \qquad i \leq N. \tag{53b}$$

The control objective is to make the N robots take specific postures and to make the swarm follow a path determined by a virtual reference vehicle; as before, the reference velocities are assumed to converge to zero. Any physically feasible geometrical configuration may be achieved and one can choose any point in the Cartesian plane to follow the virtual reference vehicle. We solve this problem using a slightly modified recursive implementation of the tracking-stabilization leader-follower controller of the previous section. For each vehicle the local control law depends on the reference trajectory generated by the virtual leader. From a configuration viewpoint, the robots are interconnected in a spanning-tree topology, that is, the minimal configuration to achieve consensus. Accordingly, each robot has only one leader and may have one or several followers.

The fictitious vehicle, which serves as reference to the swarm, describes a reference trajectory defined by the desired linear and angular velocities v_r and w_r which are communicated to the swarm leader robot only. According to this communication topology, consensus is achieved if it holds for each and all possible open chains, subsets of the spanning tree. Hence, without loss of generality, in what follows we assume that there exists an open chain composed of all agents, each having one follower and one leader.

Following the setting for tracking control, the formation-agreement control problem reduces to stabilizing the origin of the error systems,

$$\dot{e}_{x_i} \quad = \quad \omega_i e_{y_i} - v_i + v_{i-1}\cos(e_{\theta_i}) \tag{54a}$$

$$\dot{e}_{y_i} \quad = \quad -\omega_i e_{x_i} + v_{i-1}\sin(e_{\theta_i}) \tag{54b}$$

$$\dot{e}_{\theta_i} \quad = \quad \omega_{i-1} - \omega_i \tag{54c}$$

at $e_i = 0$ and (53b) at $\nu_i = \nu_i^*$. Although it should be clear from the context, we stress that

$$\nu_i := \frac{1}{r}\begin{bmatrix}1 & b\\ 1 & -b\end{bmatrix}\begin{bmatrix}v_i\\ \omega_i\end{bmatrix}, \qquad e_i := \begin{bmatrix}e_{x_i} \ e_{y_i} \ e_{\theta_i}\end{bmatrix}^\top .$$

We use the index $i-1$ for the leader of the ith robot. The swarm leader is indexed $i = 1$ and the reference robot is indexed $i = 0$ hence, by definition, $\omega_0 := \omega_r$ and $v_0 := v_r$.

Similarly to the controller (14) we define

$$\begin{aligned} v_i^* &= v_{i-1}\cos(e_{\theta_i}) + k_{xi}e_{x_i} && (55a)\\ \omega_i^* &= \omega_{i-1} + k_{\theta_i}e_{\theta_i} + k_{y_i}p_i(t)\left[e_{y_i}^2 + e_{x_i}^2\right] && (55b)\end{aligned}$$

where $p_i : \mathbb{R}_+ \to [p_{mi}, p_{Mi}]$, are bounded and smooth for all $i \leq N$ with bounded derivatives up to the second. Moreover, we assume that each p_i and its first derivative, $\dot{p}_i$, are persistently exciting.

Remark 3 *For the ith robot, the local controller depends on the measurement of its leader velocities, v_{i-1} and ω_{i-1}, only. The controller for the swarm leader (for which $i = 1$) is the only one to employ v_r and ω_r (by convention $v_0 = v_r$ and $\omega_0 = \omega_r$).*

Remark 4 *Our control approach may also be used under a general time-varying uni-directional connected communication graph and using distributed exponentially-convergent estimators for the leader positions and velocities, as in [24] and [1]. In this case, the controllers (55) reduce to (14) in which we replace (v_r, ω_r) by the estimated leader's velocities and we compute the errors e_i with respect to the estimated leader's trajectories.*

Proposition 3 *For each $i \in [1, N]$, consider the systems (53) in closed loop with the controller defined by*

$$\begin{aligned} \tau_i^* &:= \hat{M}_i\dot{\nu}_i^* + \hat{C}_i(\dot{z}_i)\nu_i^* - k_{d_i}\tilde{\nu}, \quad k_{d_i} > 0 && (56a)\\ \dot{\hat{\Theta}}_i &= -\gamma_i\Psi(\dot{z}_i, \dot{\nu}_i^*, \nu_i^*)\tilde{\nu}_i, \quad \gamma_i > 0 && (56b)\\ \nu_i^* : &= \frac{1}{r}\begin{bmatrix}1 & b\\ 1 & -b\end{bmatrix}\begin{bmatrix}v_i^*\\ \omega_i^*\end{bmatrix}, && (56c)\end{aligned}$$

and (55). Let k_{xi}, k_{yi}, $k_{\theta i} > 0$ and let p_i and $\dot{p}_i$ be bounded and persistently exciting. Then, under (4), the property (7) holds for $e_i := [e_{xi},\ e_{yi},\ e_{\theta i}]$ for all $i \leq N$.

Proof. We only analyze the closed-loop equations corresponding to the kinematics since the rest of the equations correspond to

$$M_i\dot{\tilde{\nu}}_i + [C_i(\dot{z}_i) + k_{d_i}I]\tilde{\nu}_i = \Psi(\dot{z}_i, \dot{\nu}_i^*, \nu_i^*)^\top\tilde{\Theta}_i$$

which is a repetition of (12). Therefore, we have $\tilde{\nu}_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\tilde{\nu}_i \to 0$ for all $i \leq N$.

Now, to compact the notation, let us define

$$V_{1i}(e_i) := e_{x_i}^2 + e_{y_i}^2 \tag{57}$$

$$\psi_i(t, e_i) := k_{\theta i} e_{\theta_i} + k_{y_i} p_i(t) V_{1i}$$

so that, replacing

$$v_i = v_i^* + \tilde{v}_i, \quad \omega_i = \tilde{\omega}_i + \omega_i^*, \tag{58}$$

and (55) in (54) we obtain

$$\begin{aligned} \dot{e}_{x_i} &= [\tilde{\omega}_i + \omega_{i-1} + \psi_i] e_{y_i} - \tilde{v}_i - k_{x_i} e_{x_i} && \text{(59a)} \\ \dot{e}_{y_i} &= -[\tilde{\omega}_i + \omega_{i-1} + \psi_i] e_{x_i} + v_{i-1} \sin(e_{\theta_i}) && \text{(59b)} \\ \dot{e}_{\theta_i} &= -\psi_i - \tilde{\omega}_i && \text{(59c)} \end{aligned}$$

which has exactly the same structure as (17). Indeed, the equations (59) may be re-written in the compact form

$$\dot{e}_i = A_i(t, e_i) e_i + B(e_i) \eta_i \tag{60}$$

where $e_i := [e_{x_i} \ e_{y_i} \ e_{\theta_i}]^\top$,

$$\begin{aligned} A_i(t, e_i) &:= \begin{bmatrix} -k_{x_i} & \psi_i(t, e_i) & 0 \\ -\psi_i(t, e_i) & 0 & 0 \\ -k_{y_i} p_i(t) e_{x_i} & -k_{y_i} p_i(t) e_{y_i} & -k_{\theta_i} \end{bmatrix} \\ \eta_i &:= [v_{i-1} \ \omega_{i-1} \ \tilde{v}_i \ \tilde{\omega}_i]^\top. \end{aligned}$$

For $i = 1$, the system (60) corresponds to (17) hence, by Proposition 1, $e_1 \to 0$. For $i = 2$, $\eta_2 := [v_1, \ \omega_1, \ \tilde{v}_1, \ \tilde{\omega}_1]$ where

$$\begin{aligned} v_1 &= \tilde{v}_1 + k_{x_1} e_{x_1} + v_r \cos(e_{\theta_1}) \\ \omega_1 &= \omega_r + k_{\theta 1} e_{\theta_1} + k_{y_1} p_1(t) V_{11} \end{aligned}$$

hence, $\eta_2 \to 0$ and, by Proposition 1 we obtain that $e_2 \to 0$. The statement follows by induction.

6. Simulation results

In order to illustrate our results we have performed some simulation tests under Simulink ™ of Matlab™.

We consider several scenarios: first, the leader-follower tracking-stabilization problem with one leader and only one follower only; then, we consider a group of five mobile robots following each one leader, as explained in Section 5. The physical parameters are the same for all the robots in both simulation case-studies and are taken from [16]:

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_1 \end{bmatrix}, \quad C(\dot{z}) = \begin{bmatrix} 0 & c\omega \\ -c\omega & 0 \end{bmatrix},$$

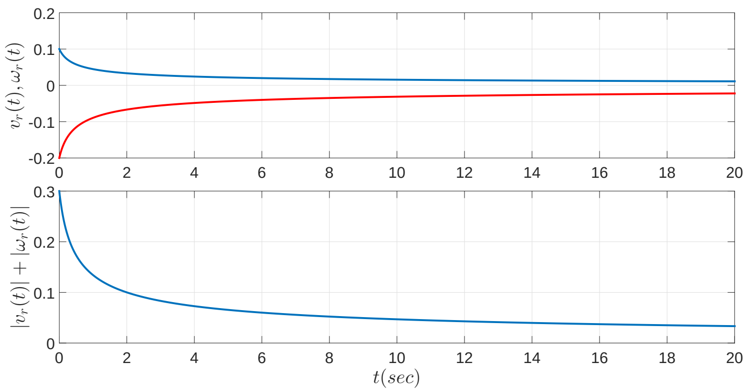

Figure 2: Unperturbed leader's velocities v_r and ω_r

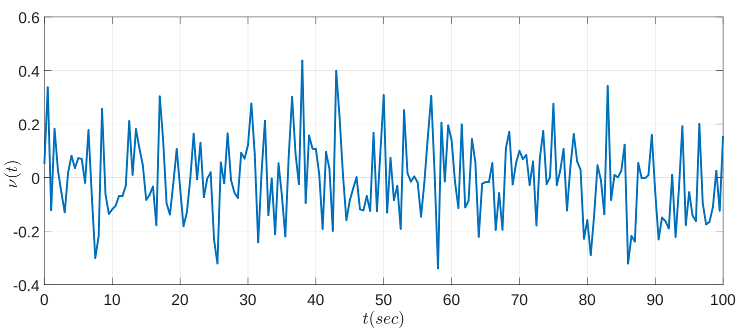

Figure 3: Measurement noise affecting the information transfer along the network

with $m_1 = 0.6227$, $m_2 = -0.2577$, $c = 0.2025$, $r = 0.15$, and $b = 0.5$. The control gains were set to $k_x = k_{x_i} = k_y = k_{y_i} = 2.5$ and $k_\theta = k_{\theta_i} = 1$ and the function $p(t) = 20\sin(t/6) + 0.5$, which has a persistently exciting time derivative. The parameters (γ, k_d) are taken equal to $(10^{-6}, 12)$, and $\hat{\Theta}(0) = (\hat{m}_1, \hat{m}_2, \hat{c}) = (0, 0, 0)$.

The virtual robot's reference velocities v_r and ω_r are defined in a way that they converge to zero asymptotically but relatively slowly, *i.e.*,

$$v_r = \frac{1}{\sqrt{400t + 100}}, \quad \omega_r = \frac{-1}{\sqrt{100t + 25}}$$

–see Figure 2.

In addition, in some of the simulation tests, we added noise to the leader's measured variables. The random noise signal that we use is depicted in Figure 3 and the reference velocities, severly affected by the measurement noise, are illustrated in Figure 4. For this first scenario, with only one robot and one virtual leader, the initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$ for

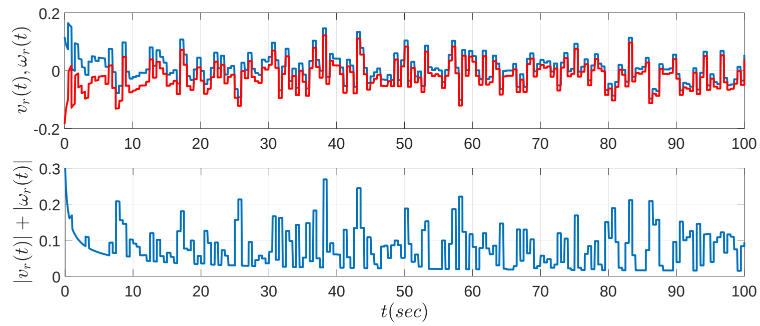

Figure 4: Perturbed leader's velocities v_r and ω_r

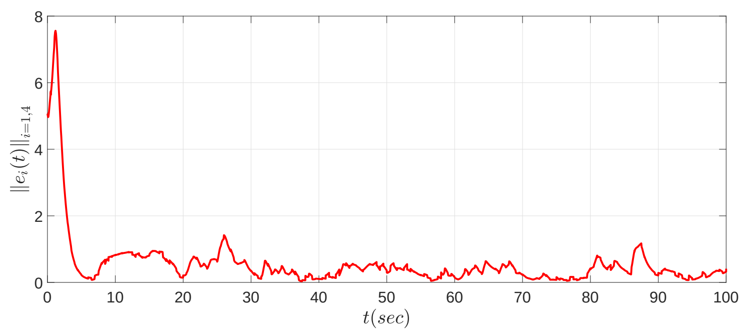

Figure 5: Tracking errors in the presence of measurement noise

the leader positions and $[x(0), y(0), \theta(0)] = [1, 3, 4]$ for the follower; the offset between the positions of the two robots is $[d_{x_{r,1}}, d_{y_{r,1}}] = [0, 0]$. Furthermore, Figure 5 illustrates the robustness of our controller to measurement noise; the tracking errors remains close to the origin in spite of the noisy measurements. Figure 6 shows the torque response for the follower robot in the presence of measurement noise.

In a second scenario we consider a group of five mobile robots following one leader. The desired formation shape of the six mobile robots is a hexagonal configuration that tracks the trajectory of the leader under slowly converging leader's velocities. The desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0.5, -0.5]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [1, 0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, 1/2]$, $[d_{x_{3,4}}, d_{y_{3,4}}] = [0.5, -0.5]$ and $[d_{x_{4,5}}, d_{y_{4,5}}] = [1, 0]$. See Figure 9.

The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1, 3, 4]$, $[x_2(0), y_2(0), \theta_2(0)] = [0, 2, 2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0, 4, 1]$, $[x_4(0), y_4(0), \theta_4(0)] = [2, 2, 1]$ and $[x_5(0), y_5(0), \theta_5(0)] = [-2, 2, 1]$.

In the absence of measurement noise, simulation results are shown in Figures 7–10. In Figures 7 and 8, it is showed the convergence of the tracking errors

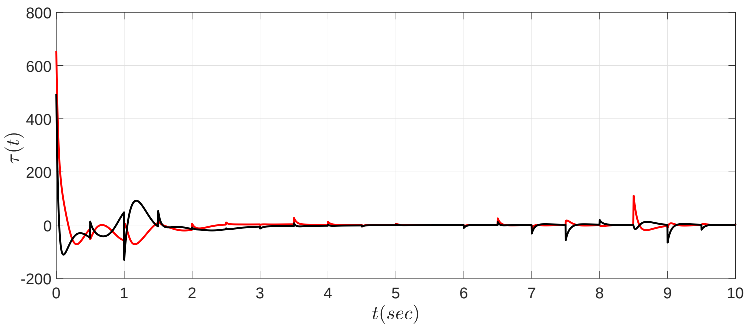

Figure 6: Illustration of the torque inputs for the follower robot

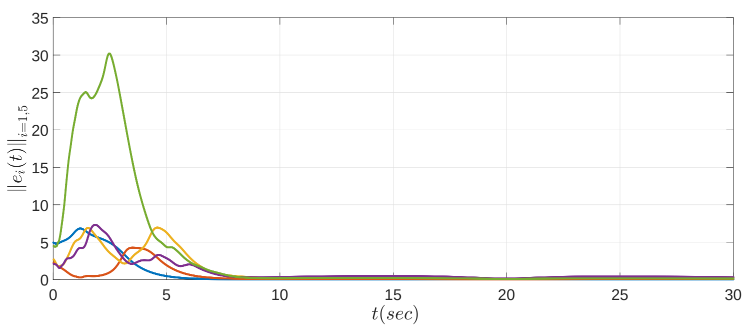

Figure 7: Convergence of the relative errors (in norm) for each pair leader-follower

between the agent and its neighborhood and the torque inputs of each agent.

In Figure 9 we illustrate the phase portrait of the six agents, we can see that the group of robots reach the desired formation shape in steady state and track the leader trajectory. The phase portrait is not necessarily smooth even if the trajectories are sufficiently smooth with respect to time, such a behavior occurs since the problem is formulated as a trajectory tracking problem and not as path-following problem.

The estimation parameter errors for each agent are presented in Figure 10, the estimation errors are not converging since the adaptive controller in (10) guarantees only the convergence of the error velocities and the global boundedness of the estimation errors.

Finally, we present simulation results for a swarm of unicycles and in the presence of measurement noise. As before, we consider that *all* the exchanged variables, *i.e.*, positions, velocities, and accelerations, are affected by measurement noise —see Figure 3.

In Figures 11 and 12 we show the tracking errors between the agent and its neighborhood and the torque inputs of each agent in the presence of measurement noise. This illustrates the robustness of the controller from Proposition 3.

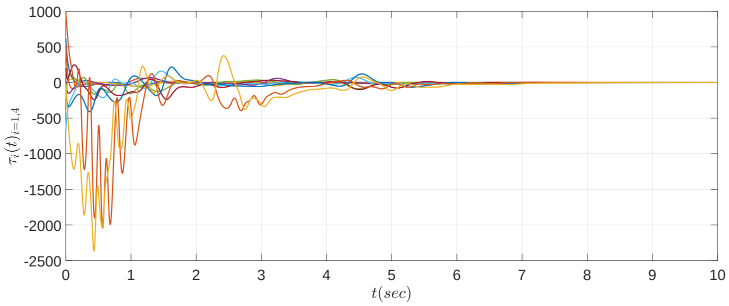

Figure 8: Illustration of the torque inputs for each agent

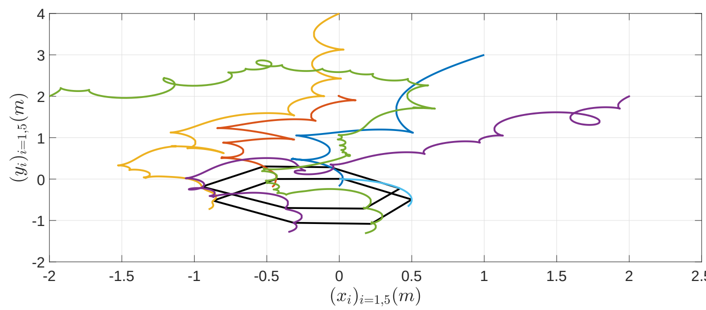

Figure 9: Illustration of the path-tracking in formation

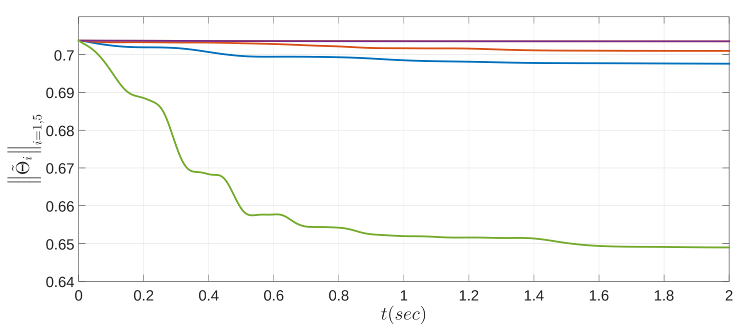

Figure 10: Illustration of the estimation parameter errors for each agent

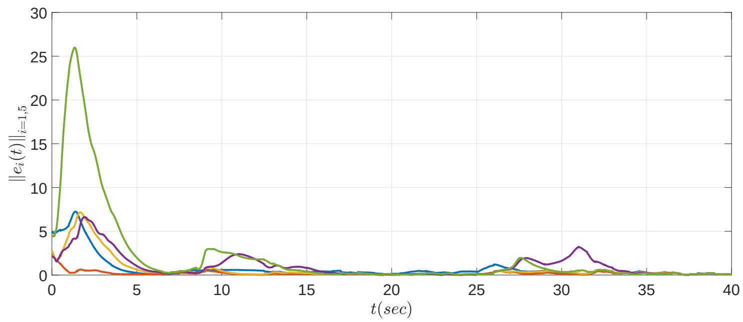

Figure 11: The relative errors (in norm) for each pair leader-follower in the presence of measurement noise

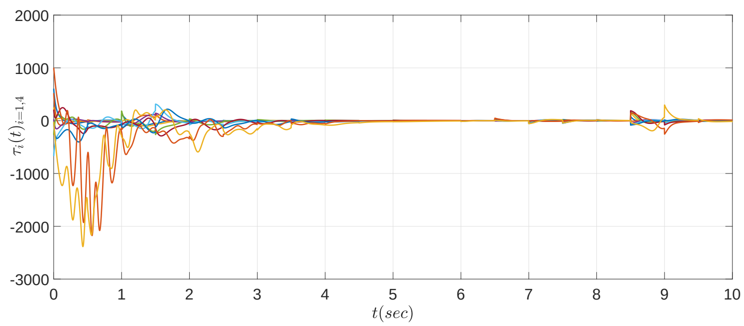

Figure 12: The torque inputs for each agent in the presence of measurement noise

7. Conclusions

We presented a simple decentralized controller for leader-follower tracking-agreement problem that is, we consider that the leader velocities converge and we assume that each robot has only one leader and may have one or more followers. Further research is being carried out to incorporate the general directed graph case.

Our control approach is decentralized hence, each robot uses only limited information (from one neighbor) and, in contrast to multiple-tracking control schemes, it has no knowledge of the reference robot. Even though the bulk of needed transmitted information through the communications network is significantly reduced, the price paid for this is that the effect of perturbations to a leader robot may propagate. This might be coped with using switching topologies, in which case, a robot might "decide" to switch to an alternative leader *cf.* [2]. This is an interesting open problem.

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Appendix A.

On input-to-state stability

We recall here some definitions and facts on input-to-state stability.

Definition 1 (Strong iISS [5]) *The dynamical system $\dot{x} = f(t, x, \eta)$ is said to be strongly integral input-to-state stable (strongly iISS) with respect to η, if it is integral input-to-state stable (iISS) with respect to η, and input to state stable (ISS) with respect to sufficiently small values of η. That is, there exist $R > 0$, $\beta \in \mathcal{KL}$, as well as μ_1, μ_1, and μ, such that*

$$|x(t)| \leq \beta(|x_\circ|, t - t_\circ) + \mu_1 \left(\int_{t_\circ}^{t} \mu_2(|\eta(s)|) ds \right)$$
$$|\eta| < R \quad \Longrightarrow \quad |x(t)| \leq \beta(|x_\circ|, t - t_\circ) + \mu(|\eta|)$$

Lemma 3 (Lyapunov characterization of ISS [21]) *Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that:*

$$\underline{\alpha}\,(|x|) \leq V(t, x) \leq \overline{\alpha}\,(|x|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, \eta) \leq -W(x), \quad \forall |x| \geq \rho\,(|\eta|) > 0$$

where $\underline{\alpha}$, $\overline{\alpha}$ are $\mathcal{K}_\infty$ functions, ρ a class K function, and W is a continuous positive definite function. Then, the system $\dot{x} = f(t, x, \eta)$ is ISS with respect to the input η.

Lemma 4 (Lyapunov characterization of iISS [19]) *Let $V : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that*

$$\underline{\alpha}\,(|x|) \leq V(t,x) \leq \overline{\alpha}\,(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,\eta) \leq -\alpha_1(|x|) + \rho(|\eta|)$$

where $\underline{\alpha}$, $\overline{\alpha}$, and ρ are class $\mathcal{K}_\infty$ functions and α_1 is positive definite. Then, the system $\dot{x} = f(t,x,\eta)$ is integral ISS with respect to u.