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► **To cite this version:**

Yash Lodha, Nicolás Matte Bon, Michele Triestino. Property FW, differentiable structures and smoothability of singular actions. *Journal of topology*, 2020, 13 (3), pp.1119-1138. 10.1112/topo.12151 . hal-01743993

HAL Id: hal-01743993

<https://hal.science/hal-01743993>

Submitted on 26 Mar 2018

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Property FW, differentiable structures, and smoothability of singular actions

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Abstract

We provide a smoothening criterion for group actions on manifolds by singular diffeomorphisms. We prove that if a countable group Γ has the fixed point property FW for walls (e.g. if it has property (T)), every aperiodic action of Γ by diffeomorphisms that are of class C^r with countably many singularities is conjugate to an action by diffeomorphisms of class C^r on a homeomorphic (possibly non-diffeomorphic) manifold. As applications, we show that Navas’s rigidity result for actions of Kazhdan groups on the circle, as well as the recent solutions to Zimmer’s conjecture, generalise to aperiodic actions by diffeomorphisms with countably many singularities. ¹

1 Introduction

Let Γ be a countable group acting by homeomorphisms on a compact differentiable manifold M . Assume that every $g \in \Gamma$ acts as a diffeomorphism on the complement of some finite subset of points, called the *singularities* of g . We are interested in the following question: when is such a singular action *smoothable*, i.e. conjugate to an action by honest diffeomorphisms? The purpose of this paper is to show that this is always the case provided Γ has Kazhdan’s property (T) and the action has no finite orbit. More generally, the same holds true for actions with a countable set of singularities (in a sense made precise below).

Recall that a countable group Γ is a *Kazhdan group* (or has *Kazhdan’s property (T)*) if every isometric affine action of Γ on a Hilbert space has a fixed point. Even if this is not the original definition, it is equivalent after the Delorme–Guichardet theorem. (See [3] as a general reference on property (T).) In fact, our results are based on a property weaker than property (T), namely the *fixed point property for walls* FW, named after Barnhill–Chatterji [2] and extensively studied by Cornuier [11, 12] (see also [10] for a recent application to birational dynamics). To recall its definition, let Γ be a countable group acting on a set X . A subset $A \subset X$ is said to be *commensurated* if the symmetric difference $g(A) \Delta A$ is finite for every $g \in \Gamma$. It is said to be *transfixed* if there exists a Γ -invariant subset $B \subset X$ such that $A \Delta B$ is finite. A group Γ has *property FW* if for every Γ -action, every commensurated subset is transfixed. Property FW is equivalent to a fixed point property for actions on CAT(0) cube complexes, and admits various other equivalent definitions, for which we refer to [12]. Property (T) implies property FW.

For any real $r \in [1, \infty]$ we will denote by $\text{Diff}^r(M)$ the group of C^r diffeomorphisms of a manifold M , with the convention that when r is not integer, this means functions whose $\lfloor r \rfloor$ -th

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¹**MSC2010:** Primary 37C85, 57M60. Secondary 37E10.

Key-words: Group actions, Groupoid of germs, Property (T), Property FW, Geometric structures, Piecewise linear homeomorphisms, Zimmer’s program, Singularities.

derivative exists and is $(r - \lfloor r \rfloor)$ -Hölder continuous. We further let $\Omega\text{Diff}^r(M)$ be the group of *countably singular* diffeomorphisms of class C^r , defined as the group of all homeomorphisms g of M such that there exists a dense open subset $U_g \subset M$ whose complement is countable and such that g is of class C^r in restriction to U_g .

It can happen that two manifolds M and N are homeomorphic through a countably singular diffeomorphism, yet they are not diffeomorphic. This famous discovery is due to Milnor [30], who showed that there are manifolds homeomorphic to the 7-sphere, but not diffeomorphic to it. Manifolds homeomorphic to spheres are now known as *exotic spheres*. In fact, every exotic sphere of dimension $d \geq 7$ admits a singular diffeomorphism to \mathbb{S}^d with just one singular point. On the other hand if M and N are as above, then the groups $\Omega\text{Diff}^r(M)$ and $\Omega\text{Diff}^r(N)$ are isomorphic. For this reason, exotic differentiable structures arise naturally in the context studied in this paper.

We prove the following theorem.

Theorem 1.1. *Let M be a closed differentiable manifold, and fix $r \in [1, \infty]$. Let Γ be a finitely generated group with property FW. For every homomorphism $\rho: \Gamma \rightarrow \Omega\text{Diff}^r(M)$, one of the following holds:*

- (i) *the action of $\rho(\Gamma)$ on M has a finite orbit;*
- (ii) *there exist a closed differentiable manifold N and a countably singular diffeomorphism $\varphi: M \rightarrow N$ such that $\varphi\rho(\Gamma)\varphi^{-1} \subset \text{Diff}^r(N)$.*

This is a special case of Theorem 3.1, which deals with more general geometric structures on manifolds.

In the second case, the manifold N delivered by the proof is a connected sum of M with finitely many exotic spheres. In particular, it is homeomorphic to M , but not necessarily diffeomorphic. If the dimension d of M is such that the d -sphere admits a unique differentiable structure, then it is possible to choose $N = M$ (see Theorem 3.1). It is known that the differentiable structure on \mathbb{S}^d is unique for $d = 1, 2, 3, 5, 6, 12, 56, 61$ [23, 24, 38], and never unique for odd $d \neq 1, 3, 5, 61$ [24, 38], while a general answer is unknown in even dimension. This is a widely open problem for $d = 4$, known as the *4-dimensional smooth Poincaré conjecture*.

In the particular case of representations of Γ into groups of singular diffeomorphisms with *finitely many* singularities, we obtain a more precise statement in the finite-orbit case (i), namely that the action is conjugate to an action that is of class C^r away from a finite invariant subset (Theorem 3.6).

Let us discuss applications of Theorem 1.1 in dimension $n = 1$, namely when $M = \mathbb{S}^1$. It is an open problem whether every action of a Kazhdan group on the circle by homeomorphisms factors through a finite quotient. Andrés Navas proved that the answer is affirmative for actions by diffeomorphisms of class $C^{3/2}$ [32]. (More precisely, this result is proven in [32] for C^r -actions with $r > 3/2$, but according to [33, Rmk. 5.2.24] it can be extended to $C^{3/2}$ -actions, relying on work by Bader–Furman–Gelder–Monod [1], by means of an L^p -analogue of property (T) due to Fisher–Margulis.)

Theorem 1.2 (Navas). *If Γ is a Kazhdan group, every homomorphism $\rho: \Gamma \rightarrow \text{Diff}^{3/2}(\mathbb{S}^1)$ has finite image.*

Motivated by a question asked by him (cf. [4, Question 4.8], [33, Rmk. 5.2.24] and [34, §1]), it is natural to study this problem for actions by countably singular diffeomorphisms. A relevant special case is the one of actions by *piecewise diffeomorphisms* (or *piecewise differentiable homeomorphisms*), i.e. those for which the singular set is finite, and moreover at every singular point the right and

left derivatives exist up to the required order of regularity. We denote by $\text{PDiff}^r(\mathbb{S}^1)$ the group of all piecewise differentiable homeomorphisms of the circle of class C^r . Well-studied examples of groups of piecewise differentiable homeomorphisms of the circle include groups of *piecewise linear* and *piecewise projective* homeomorphisms, and the existence of infinite Kazhdan groups was an open question even in this case. Since Kazhdan’s property (T) implies property FW, the combination of Theorem 1.1 with Navas’ theorem allows to answer to this question for actions by piecewise differentiable homeomorphisms. In the case of actions by countably singular diffeomorphisms, it allows to reduce the question to actions on the real line.

Corollary 1.3. *Let Γ be a Kazhdan group. Then the following hold.*

(i) *Every homomorphism $\rho: \Gamma \rightarrow \text{PDiff}^{3/2}(\mathbb{S}^1)$ has finite image.*

(ii) *For every homomorphism $\rho: \Gamma \rightarrow \Omega\text{Diff}^{3/2}(\mathbb{S}^1)$, the action of $\rho(\Gamma)$ on \mathbb{S}^1 has a finite orbit.*

In an independent preprint [13], Yves de Cornulier has obtained part (i) of Corollary 1.3 also allowing the presence of finitely many points of discontinuity for actions by piecewise linear or projective homeomorphisms (see Question 1.16 below and the remark following it).

Remark 1.4. The relevant difference between parts (i) and (ii) in Corollary 1.3 is that Thurston’s stability theorem holds in piecewise C^1 regularity.

Remark 1.5. Corollary 1.3 cannot be strengthened to obtain that every subgroup of $\text{PDiff}^r(\mathbb{S}^1)$ with property FW is finite. For example the group $\text{PSL}(2, \mathbb{Z}[\sqrt{2}])$ acts on the circle by Möbius transformations. This group has property FW, as proved by Cornulier relying on work of Haglund and Carter–Keller [12, Ex. 6.A.8].

Remark 1.6. There exist finitely generated subgroups of the group $\text{PDiff}^\infty(\mathbb{S}^1)$ that are not smoothable, and even not isomorphic to groups of C^1 -diffeomorphisms [4, 29].

Remark 1.7. If we allow an uncountable closed set of singularities (of empty interior), then every group of circle homeomorphisms is isomorphic to a group of singular diffeomorphisms. Blowing up an orbit, it is possible to obtain any desired level of regularity on the complement of a Cantor set.

It is a classical leitmotiv that Kazhdan groups should behave like lattices in higher-rank simple Lie groups, which are the primary source of examples of Kazhdan groups. This is certainly far from true in full generality (e.g. there are Kazhdan groups that are hyperbolic), and it is an interesting problem to see how far one can go with this comparison. That is, what are the rigidity properties of Kazhdan groups? For higher-rank lattices, beyond the classical rigidity theorems by Mostow, Margulis and Zimmer, one of the most interesting rigidity properties is described by the so-called Zimmer’s program. An important conjecture in this program states that a lattice in a higher rank simple Lie group has only (virtually) trivial actions on closed manifolds of dimension $< d$, where d is an explicit constant depending on the ambient Lie group (bounded below by its real rank). This conjecture has been (partially) solved recently with the breakthrough work of Brown, Fisher and Hurtado [6, 7, 15]. For what concerns actions on one-dimensional manifolds, earlier and more precise results were obtained by Ghys, Burger–Monod, and Witte Morris [8, 17, 40]. In relation to our work, the main result of [8, 17] implies that any morphism from a lattice in a simple Lie group of rank ≥ 2 to the group $\text{PDiff}^1(\mathbb{S}^1)$ has finite image.

It is a well known open problem whether Zimmer’s conjecture holds for action by homeomorphisms that are not diffeomorphisms. In combination with the aforementioned results of Brown–Fisher–Hurtado [6, 7], Theorem 1.1 yields the following.

Corollary 1.8. *Let M^d be a closed manifold of dimension d . Let G be a connected Lie group, whose Lie algebra is simple and with finite centre. Assume that the real rank of G is $r > d$ and let $\Gamma \subset G$ be a cocompact lattice, or $\Gamma = \mathrm{SL}(r+1, \mathbb{Z})$. For any morphism $\rho : \Gamma \rightarrow \Omega\mathrm{Diff}^2(M)$, the action of $\rho(\Gamma)$ on M has a finite orbit.*

Using a more general version of Theorem 1.1 (namely Theorem 3.1), we also establish an analogous statement in the case of actions preserving a volume form, see Corollary 3.7.

Remark 1.9. Here we have simplified the statement for avoiding introducing unnecessary notations. The interested reader will be able to improve the above statement from the results appearing in [6, 7, 9, 15]. Note also that Theorem 3.6 gives a stronger result for actions by elements with finitely many singularities.

Remark 1.10. For the particular classes of *groups of elementary matrices*, and *automorphism groups of free groups* (and some relatives) questions related to Zimmer’s program have been largely investigated by Ye, with no restrictions on regularity (see e.g. [41, 42, 44]).

Let us give an outline of the paper and of the proof of Theorem 1.1. The core of our argument has purely topological dynamical nature (no manifolds are involved, only group actions on locally compact spaces), and is contained in Section 2. We consider an action of a countable group Γ on a locally compact space X whose *germs* at every point belong to a given *groupoid of germs* \mathcal{H} , except for a finite set of singular points for every group element (all the necessary terminology will be defined in Section 2). We provide a criterion on \mathcal{H} that implies, when Γ has property FW, that the action is conjugate to an action whose germ at every point belongs to \mathcal{H} . To do so we construct a commensurating action of Γ inspired from work of Juschenko–Nekrashevych–de la Salle on amenability of groups of homeomorphisms [21]. We then extend this to the case of countably many singularities by transfinite induction on the *Cantor–Bendixson rank* of the set of singular points of the generators of Γ .

In Section 3 we apply this criterion to singular actions on manifolds. In fact, the set of diffeomorphisms of class C^r do not play any special role in the proof of Theorem 1.1. We shall work in a more general setting (see Theorem 3.1), by allowing actions that are countably singular with respect to an arbitrary given groupoid of germs \mathcal{G} of local homeomorphisms of \mathbb{R}^d , provided M has the structure of a *\mathcal{G} -manifold*, i.e. admits an atlas whose changes of charts have germs in \mathcal{G} (see Thurston’s book [37, Ch. 3] as a standard reference on \mathcal{G} -manifolds). In the course of the proof, we are faced to the problem of extending a finite collection of germs of singular diffeomorphisms to the whole ambient manifold, without introducing new singularities. Such an extension does not always exist. In the case of actions by diffeomorphisms the obstruction to its existence is measured by the group of twisted n -spheres $\Gamma_n = \pi_0(\mathrm{Diff}_+(\mathbb{S}^{n-1}))/\pi_0(\mathrm{Diff}_+(\mathbb{D}^n))$, which is in one-to-one correspondence with the differentiable structures of \mathbb{S}^n (for $n \neq 4$). Taking a connected sum with exotic spheres allows to eliminate this obstruction by producing another manifold, homeomorphic to the original one, on which the action can be regularised.

In Section 4 we discuss the case of piecewise differentiable circle diffeomorphisms, and then focus on the example of groups of *piecewise linear* (PL) homeomorphisms of the circle. In this case, we provide a more elementary proof of the fact that they do not have property (T), which only relies on Hölder’s theorem. Namely, we will exhibit an isometric action of $\mathrm{PL}_+(\mathbb{S}^1)$ on the Hilbert space $\ell^2(\mathbb{S}^1)$, with linear part defined by the action of the groups on \mathbb{S}^1 : we twist the linear action with a cocycle that measures the failure of elements to be affine. This cocycle has been widely used (implicitly or explicitly) for understanding many properties of groups of piecewise linear homeomorphisms [18, 19, 27, 31] (just to cite a few). This is to be compared to the result by Farley and Hughes [16, 26] that Thompson’s group V has the Haagerup property, which is also proved by exhibiting an explicit

proper action of V on a Hilbert space (Farley’s proof of this result in [16] had a gap and has been fixed by Hughes [26]).

Let us end this introduction with a list of problems.

Question 1.11. Which subgroups of $\text{PDiff}^r(\mathbb{S}^1)$ have the Haagerup property?

Question 1.12. Does the group $\Omega\text{Diff}^r(\mathbb{R})$ of countably singular diffeomorphisms of the real line contain infinite property (T) subgroups?

Question 1.13. Study Zimmer’s conjecture for piecewise differentiable (or PL) actions of higher rank lattices on manifolds of small dimension. (cf. the work of Ye [43].)

Question 1.14. Consider the renormalised linear action of $\text{SL}(n+1, \mathbb{Z})$ on the sphere \mathbb{S}^n , with $n \geq 7$. For which differentiable structures on \mathbb{S}^n is this action topologically conjugate to a smooth action?

Question 1.15. Extend Corollary 1.3 to groups of discontinuous transformations, such as AIET, the group of affine interval exchange transformations. (The group of isometric interval exchange transformations does not contain infinite Kazhdan subgroups [14].)

Question 1.16 (cf. [35]). Classify the subgroups of the group of piecewise projective circle homeomorphisms $\text{PP}_+(\mathbb{S}^1)$ that are topologically conjugate to $\text{PSL}(2, \mathbb{R})$, up to PP conjugacy.

Questions 1.15 and 1.16 (that were written in a preliminary version of this paper) have been answered by Yves de Cornulier in [13]. His answer to Question 1.16 makes use of the classification of projective one-manifolds and their automorphism groups: using the fact that $\text{PSL}(2, \mathbb{Z}[\sqrt{2}])$ has property FW (Remark 1.5) and is dense in $\text{PSL}(2, \mathbb{R})$, he concludes that the action of a topological $\text{PSL}(2, \mathbb{R})$ must preserve a projective structure on \mathbb{S}^1 .

2 Property FW and singular actions on locally compact spaces

2.1 Groupoids of germs

Throughout the section, we let X be a locally compact Hausdorff space (not necessarily a manifold). We will be primarily interested in the case where the space X is compact. However, in the course of the proofs, it will be useful to consider non-compact spaces as well.

Recall that the *support* of a homeomorphism h of X is the subset $\text{supp}(h) = \overline{\{x \in X : h(x) \neq x\}}$ of X , which is the closure of the set of points which are moved by h . A *germ* (on the space X) is the equivalence class of a pair (h, x) , where $x \in X$ and h is a homeomorphism defined from a neighbourhood of x to a neighbourhood of $h(x) \in X$, where (h_1, x_1) and (h_2, x_2) are equivalent if $x_1 = x_2$ and h_1 and h_2 coincide on a neighbourhood of x_1 . We denote by $[h]_x$ the equivalence class of (h, x) . If $\gamma = [h]_x$ is a germ, where (h, x) is a representative pair of γ , the points $s(\gamma) = x$ and $t(\gamma) = h(x)$ are called the *source* and the *target* of γ . Two germs γ_1, γ_2 can be multiplied provided $t(\gamma_2) = s(\gamma_1)$, and in this case $\gamma_1\gamma_2 = [h_1h_2]_{s(\gamma_2)}$, where $(h_1, s(\gamma_1))$ and $(h_2, s(\gamma_2))$ are any choice of representatives of γ_1, γ_2 . They are inverted according to the rule $[h]_x^{-1} = [h^{-1}]_{h(x)}$. Given a point $x \in X$, we will denote by 1_x the germ of the identity homeomorphism at x .

Definition 2.1. A *groupoid of germs* (over X) is a set \mathcal{G} of germs on X which verifies the following properties:

- (i) if $\gamma_1, \gamma_2 \in \mathcal{G}$ are such that $s(\gamma_1) = t(\gamma_2)$, then $\gamma_1\gamma_2 \in \mathcal{G}$, and we have $\gamma^{-1} \in \mathcal{G}$ for every $\gamma \in \mathcal{G}$;

- (ii) we have $1_x \in \mathcal{G}$ for every $x \in X$;
- (iii) every $\gamma \in \mathcal{G}$ admits a representative pair $(h, s(\gamma))$ with the property that $[h]_y \in \mathcal{G}$ for every y in the domain of definition of h .

The *topological full group* of \mathcal{G} is the group $F(\mathcal{G})$ of all compactly supported homeomorphisms g of X with the property that $[g]_x \in \mathcal{G}$ for every $x \in X$.

2.2 Finitely many singularities

In this subsection we study group actions on locally compact spaces whose germs at every point belong to a given groupoid of germs, except for finitely many isolated “singularities”. To formalise this idea, we will work in the following setting.

Definition 2.2. Let $\mathcal{H} \subset \mathcal{G}$ be groupoids of germs over X . We say that the pair $\mathcal{H} \subset \mathcal{G}$ is *co-discrete* if every $\gamma \in \mathcal{G}$ admits a representative $(h, s(\gamma))$ with the property that $[h]_y \in \mathcal{H}$ for every $y \neq s(\gamma)$ in the domain of definition of h . We will say that $\gamma \in \mathcal{G}$ is *singular* if it does not belong to \mathcal{H} , and that it is *regular* otherwise.

Similarly given $g \in F(\mathcal{G})$, we will say that a point $x \in X$ is *singular* for g if $[g]_x \notin \mathcal{H}$, and *regular* otherwise.

The terminology “co-discrete” comes from the fact that a groupoid of germs is naturally endowed with a topology (the so-called *sheaf topology*), and the condition above is equivalent to the fact that the complement $\mathcal{G} \setminus \mathcal{H}$ is discrete with the induced topology (however we will not take this point of view). A simple compactness argument yields the following lemma:

Lemma 2.3. *Assume that $\mathcal{H} \subset \mathcal{G}$ is a co-discrete pair. Then every $g \in F(\mathcal{G})$ has finitely many singular points.*

Proof. This follows from the fact that elements of $F(\mathcal{G})$ are compactly supported, and from a simple compactness argument. \square

We also give the following definition.

Definition 2.4. Let $\mathcal{H} \subset \mathcal{G}$ be a co-discrete pair. We say that it has *resolvable singularities* if the following holds. For every choice of finitely many germs $\gamma_1, \dots, \gamma_\ell \in \mathcal{G}$ such that $t(\gamma_1), \dots, t(\gamma_\ell)$ are distinct points, and every compact subset $K \subset X$ there exists an element $\varphi \in F(\mathcal{G})$ with the following properties.

- (i) We have $[\varphi]_{t(\gamma_i)} \gamma_i \in \mathcal{H}$ for every $i = 1, \dots, \ell$.
- (ii) We have $[\varphi]_y \in \mathcal{H}$ for every $y \in K \setminus \{t(\gamma_1), \dots, t(\gamma_\ell)\}$.

Intuitively, this means that any finite family of singular germs $\gamma_1, \dots, \gamma_\ell \in \mathcal{G}$ can be “resolved” (i.e. brought back to \mathcal{H}) by post-composing them with an element $\varphi \in F(\mathcal{G})$ that can be chosen without any additional singularities in any arbitrarily large compact subset. Note that if X is compact, the compact subset K is redundant in the definition as we can choose $K = X$.

The reader can have in mind the following example: \mathcal{H} is the groupoid of all germs of partially defined diffeomorphisms of the circle, and \mathcal{G} is defined similarly by allowing isolated singularities for the derivatives. In this situation, the fact that the pair $\mathcal{H} \subset \mathcal{G}$ has resolvable singularities is easy to verify (and will follow from Proposition 3.5).

We consider now subgroups of the topological full group $F(\mathcal{G})$.

Remark 2.5. If a group Γ is countable, then property FW implies automatically that it is finitely generated (this is already a consequence of property FA of Serre, which is implied by FW, see [12]).

Given a subgroup $\Gamma \subset F(\mathcal{G})$, its *support* $\text{supp}(\Gamma) = \overline{\bigcup_{h \in \Gamma} \text{supp}(h)}$ is the closure in X of the set of points of X that are moved by some element of Γ . When the subgroup is finitely generated, as it is the case for countable subgroups with property FW, the support of Γ is just the union of the supports of elements in a symmetric generating system. We also say that a point $x \in X$ is *singular* for Γ if there exists $g \in \Gamma$ such that x is singular for g .

Proposition 2.6. *Let $\mathcal{H} \subset \mathcal{G}$ be a co-discrete pair of groupoids of germs over X with resolvable singularities. Let $\Gamma \subset F(\mathcal{G})$ be a countable subgroup, with property FW. Then there exists $\varphi \in F(\mathcal{G})$ such that the set of singular points of the group $\varphi\Gamma\varphi^{-1}$ is finite and consists of points with a finite $\varphi\Gamma\varphi^{-1}$ -orbit.*

In particular, if the action of Γ on its support has no finite orbits, then Γ is conjugate in $F(\mathcal{G})$ to a subgroup of $F(\mathcal{H})$.

In the proof, we will use the following terminology. Given a pair of groupoids of germs $\mathcal{H} \subset \mathcal{G}$, there is a well-defined *coset space* \mathcal{G}/\mathcal{H} . The latter is defined as the set of equivalence classes of the equivalence relation on \mathcal{G} that identifies $\gamma_1, \gamma_2 \in \mathcal{G}$ if $t(\gamma_1) = t(\gamma_2)$ and $\gamma_1^{-1}\gamma_2 \in \mathcal{H}$. The equivalence class of γ is denoted by $\gamma\mathcal{H}$. Note that the target map $t: \mathcal{G} \rightarrow X$ descends to a well-defined map $t: \mathcal{G}/\mathcal{H} \rightarrow X$, $t(\gamma\mathcal{H}) = t(\gamma)$.

Proof. We let the group Γ act on the coset space \mathcal{G}/\mathcal{H} by the rule

$$g \cdot \gamma\mathcal{H} = [g]_{t(\gamma)}\gamma\mathcal{H}, \quad g \in \Gamma, \gamma\mathcal{H} \in \mathcal{G}/\mathcal{H}.$$

Observe that the target map $t: \mathcal{G}/\mathcal{H} \rightarrow X$ is equivariant for this action. Let $Y = \text{supp}(\Gamma) \subset X$ be the support of Γ . The action of Γ is trivial in restriction to fibres $t^{-1}(x)$, $x \notin Y$. Therefore, the subset $t^{-1}(Y) = \{\gamma\mathcal{H} \in \mathcal{G}/\mathcal{H} : t(\gamma) \in Y\}$ is a Γ -invariant subset of \mathcal{G}/\mathcal{H} . Consider the trivial section $A = \{1_x\mathcal{H} : x \in Y\} \subset t^{-1}(Y)$ in \mathcal{G}/\mathcal{H} .

Claim 1. *The subset A is commensurated.*

Proof of claim. Take $g \in \Gamma$. For every $1_x\mathcal{H} \in A$, the condition that $g \cdot 1_x\mathcal{H} = [g]_x\mathcal{H} \notin A$ is equivalent to the fact that $[g]_x \notin \mathcal{H}$, and there are only finitely many points $x \in Y$ with this property by Lemma 2.3. This shows that $g(A) \setminus A$ is finite. The same reasoning applied to g^{-1} shows that $A \setminus g(A)$ is also finite. Hence A is commensurated. \square

Since we are assuming that Γ has FW, it follows that A is transfixed. Let $B \subset t^{-1}(Y)$ in \mathcal{G}/\mathcal{H} , be a Γ -invariant subset such that $A\Delta B$ is finite.

Claim 2. *There exists a finite subset $E \subset Y$ which is Γ -invariant and such that for every $x \in Y \setminus E$, we have $|t^{-1}(x) \cap B| = 1$.*

Proof of claim. We set $E = \{x \in Y : |t^{-1}(x) \cap B| \neq 1\}$. Then, by invariance of B and equivariance of t , it follows that E is Γ -invariant. Since $B\Delta A$ is finite, and $A \cap t^{-1}(x) = \{1_x\mathcal{H}\}$ for every $x \in Y$, we deduce that E must be finite. \square

For every $x \in Y \setminus E$, denote by $\gamma_x\mathcal{H}$ the unique element of $t^{-1}(x) \cap B$ (where we fix arbitrarily a representative γ_x of the coset for every $x \in Y \setminus E$). Note that, since $B\Delta A$ is finite, we have $\gamma_x\mathcal{H} = 1_x\mathcal{H}$ for all but finitely many x . Note also that $t(\gamma_x) = x$ for every $x \in Y$. Invariance of B reads as follows.

$$\gamma_{g(x)}\mathcal{H} = [g]_x\gamma_x\mathcal{H}, \quad \forall x \in Y \setminus E, \forall g \in \Gamma. \quad (1)$$

Claim 3. *There exists $\varphi \in F(\mathcal{G})$ with the property that*

$$[\varphi^{-1}]_{\varphi(x)}\mathcal{H} = \gamma_x\mathcal{H} \quad \forall x \in Y \setminus E.$$

Proof of claim. Let $\Sigma \subset Y \setminus E$ be the finite subset of points $x \in Y \setminus E$ such that $\gamma_x\mathcal{H} \neq 1_x\mathcal{H}$. We use the assumption that the pair $\mathcal{H} \subset \mathcal{G}$ has resolvable singularities, applied to the finite collection of germs $\{\gamma_x : x \in \Sigma\}$ and to the compact subset $K = Y$. We obtain that there exists $\varphi \in F(\mathcal{G})$ such that

- (a) for every $x \in \Sigma$ we have $[\varphi]_x\gamma_x \in \mathcal{H}$,
- (b) for every $x \in Y \setminus \Sigma$ we have $[\varphi]_x \in \mathcal{H}$.

Property (a) can be rewritten as $[\varphi^{-1}]_{\varphi(x)}\mathcal{H} = \gamma_x\mathcal{H}$ for $x \in \Sigma$. Moreover if $x \in Y \setminus (E \cup \Sigma)$ property (b) implies that $[\varphi^{-1}]_{\varphi(x)}\mathcal{H} = 1_x\mathcal{H}$, which is equal to $\gamma_x\mathcal{H}$ since $x \notin \Sigma$. This concludes the proof of the claim. \square

Claim 4. *All singular points of $\varphi\Gamma\varphi^{-1}$ are contained in $\varphi(E)$.*

Proof of claim. Let $y \notin \varphi(E)$ and let us show that for every $g \in \Gamma$ the germ of $\varphi g \varphi^{-1}$ at y belongs to \mathcal{H} . Since $\varphi\Gamma\varphi^{-1}$ is supported on $\varphi(Y)$, we can assume that $y = \varphi(x)$ for some $x \in Y \setminus E$ and the claim is equivalent to:

$$[\varphi]_{g(x)}[g]_x[\varphi^{-1}]_{\varphi(x)} \in \mathcal{H} \quad \forall x \in Y \setminus E,$$

which is equivalent to

$$[\varphi^{-1}]_{\varphi(g(x))}\mathcal{H} = [g]_x[\varphi^{-1}]_{\varphi(x)}\mathcal{H},$$

which by Claim 3 is equivalent to

$$\gamma_{g(x)}\mathcal{H} = [g]_x\gamma_x\mathcal{H}$$

which is exactly (1). \square

It follows that all the singular points of $\varphi\Gamma\varphi^{-1}$ are contained in the finite $\varphi\Gamma\varphi^{-1}$ -invariant subset $\varphi(E)$. This concludes the proof of Proposition 2.6. \square

2.3 Countably many singularities

The purpose of the remainder of this section is to study a more general situation where we allow group actions with a *countable* set of singularities with respect to a given groupoid of germs \mathcal{H} . We work in the following setting.

Definition 2.7. Let \mathcal{H} be a groupoid of germs over X . We let $\Omega\mathcal{H}$ be the groupoid of all germs γ that admit a representative $(h, s(\gamma))$ with the property that $[h]_y \in \mathcal{H}$ for all but at most countably many points y in the domain of definition of h .

It is easy to check that $\Omega\mathcal{H}$ is a well-defined groupoid of germs. Given an element $g \in F(\Omega\mathcal{H})$, we will say that a point $x \in X$ is *singular* for g if $[g]_x \notin \mathcal{H}$.

Lemma 2.8. *For every $g \in F(\Omega\mathcal{H})$, the set of singular points of g is countable and closed.*

Proof. Recall that every element $g \in F(\Omega\mathcal{H})$ has compact support, by the definition we use of the full group. Countability follows from a compactness argument. The fact that the complement of the set of singular points of g is open is a consequence of condition (iii) in Definition 2.1. \square

Our aim is to prove a result analogous to Proposition 2.6 that applies to the pair $\mathcal{H} \subset \Omega\mathcal{H}$. In order to state it we first need to give an alternative point of view on the groupoid $\Omega\mathcal{H}$ via the notion of *Cantor–Bendixson rank* of a compact space (see [22, Sec. 6]). Let us recall here its definition in the special case of *countable* compact spaces, that will be enough for our purposes. Let C be a countable compact space. If such a space is non-empty, then it must contain isolated points by Baire’s theorem. Its *Cantor–Bendixson derivative*, denoted C' , is defined as the complement of the subset of isolated points in C . Given a countable ordinal α define a compact countable subset $C_\alpha \subset C$ by transfinite induction by setting $C_0 = C$, and

$$C_\alpha = \begin{cases} C'_\beta & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta < \alpha} C_\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

The family C_α is a decreasing ordered family of closed subsets of C . Using the fact that C is countable, it follows that there exists a smallest countable ordinal ρ such that $C_\rho = \emptyset$. The ordinal ρ will be called the *Cantor–Bendixson rank* of C . Note also that, as a consequence of compactness of C , the ordinal ρ is of the form $\rho = \beta + 1$ and C_β is a finite subset (the reader should be warned that sometimes its predecessor β is called the Cantor–Bendixson rank of C).

Now let \mathcal{H} be a groupoid of germs over X . We denote by $S\mathcal{H}$ the groupoid of germs consisting of all germs γ that admit a representative $(h, s(\gamma))$ such that $[h]_y \in \mathcal{H}$ for all $y \neq s(\gamma)$ in the domain of definition of h . Note that the pair $\mathcal{H} \subset S\mathcal{H}$ is co-discrete, by definition. For every countable ordinal α we define a groupoid \mathcal{H}_α by setting:

$$\mathcal{H}_\alpha = \begin{cases} S\mathcal{H}_\beta & \text{if } \alpha = \beta + 1, \\ S(\bigcup_{\beta < \alpha} \mathcal{H}_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Lemma 2.9. *Assume X is compact. Let $g \in F(\Omega\mathcal{H})$, and let $C \subset X$ be the set of singular points of g . Let ρ be the Cantor–Bendixson rank of C . Then $g \in F(\mathcal{H}_\rho)$.*

Note that C is countable and compact by Lemma 2.8.

Proof. Let C_α be the transfinite sequence as in the definition of Cantor–Bendixson rank. We show by induction on $\alpha \leq \rho$ that if $x \in C \setminus C_\alpha$ then $[g]_x \in \mathcal{H}_\alpha$. First assume that $\alpha = 1$. Then $x \in C \setminus C_1$ is an isolated point in C , and therefore x has a neighbourhood U in X that does not intersect C , i.e. such that $[g]_y \in \mathcal{H}$ for every $y \in U$, $y \neq x$. It follows that $[g]_x \in S\mathcal{H} = \mathcal{H}_1$. For the induction step, assume first that α is a limit ordinal. This implies that $x \in C \setminus C_\beta$ for some $\beta < \alpha$ and by the inductive hypothesis we have $[g]_x \in \mathcal{H}_\beta \subset \mathcal{H}_\alpha$. Assume now that $\alpha = \beta + 1$ is a successor. We can assume that $x \in C_\beta \setminus C_\alpha$, or we are done again by the inductive hypothesis. Note that $C_\beta \setminus C_\alpha$ consists of isolated points in C_β , and therefore we can find a neighbourhood U of x such that $(U \setminus \{x\}) \cap C \subset (C \setminus C_\beta)$ and it follows by inductive hypothesis that for every $y \in U \cap C$, $y \neq x$, we have $[g]_y \in \mathcal{H}_\beta$, from which we conclude that $[g]_x \in \mathcal{H}_\alpha = S\mathcal{H}_\beta$. \square

Definition 2.10. We say that \mathcal{H} has *countably resolvable singularities* if for every countable ordinal α the following holds:

- (i) if $\alpha = \beta + 1$ is a successor, then the pair $\mathcal{H}_\beta \subset \mathcal{H}_\alpha$ has resolvable singularities;
- (ii) if α is a limit ordinal, then the pair $\bigcup_{\beta < \alpha} \mathcal{H}_\beta \subset \mathcal{H}_\alpha$ has resolvable singularities.

Once again, the reader can have in mind the example of the groupoid of germs of all partially defined diffeomorphisms of the circle (Proposition 3.5). We are now ready to state the final result of this section:

Proposition 2.11. *Let \mathcal{H} be a groupoid of germs over X with countably resolvable singularities. Let $\Gamma \subset F(\Omega\mathcal{H})$ be a finitely generated subgroup with property FW. Then one of the following possibilities holds.*

- (i) *The action of the group Γ on its support has a finite orbit.*
- (ii) *The group Γ is conjugate in $F(\Omega\mathcal{H})$ to a subgroup of $F(\mathcal{H})$.*

Proof of Proposition 2.11. We assume that (i) does not hold and show that (ii) holds. Note that the support of the action of Γ is compact (contained in the union of the supports of its generators). Using Lemma 2.9 we obtain that $\Gamma \subset F(\mathcal{H}_\rho)$ for some countable ordinal ρ (it is enough to take the largest such ordinal over a finite generating subset of Γ). We show by induction on ρ that Γ is conjugate in $F(\mathcal{H}_\rho)$ to a subgroup of $F(\mathcal{H})$. Assume first that $\rho = \beta + 1$ is a successor, so that $\mathcal{H}_\rho = S\mathcal{H}_\beta$. Then the pair $\mathcal{H}_\beta \subset \mathcal{H}_\rho$ is co-discrete by definition, and it has resolvable singularities by Definition 2.10. As we are assuming that the action of Γ has no finite orbit in its support, Proposition 2.6 yields that Γ is conjugate in $F(\mathcal{H}_\rho)$ to a subgroup of $F(\mathcal{H}_\beta)$, and since the assumption that Γ does not have finite orbits in its support is invariant under conjugacy, we are reduced to the inductive hypothesis. Next, assume that ρ is a limit ordinal, so that $\mathcal{H}_\rho = S(\bigcup_{\beta < \rho} \mathcal{H}_\beta)$. The pair $\bigcup_{\beta < \rho} \mathcal{H}_\beta \subset \mathcal{H}_\rho$ is co-discrete and has resolvable singularities, and using again Proposition 2.6 we obtain that there exists $\varphi \in F(\mathcal{H}_\rho)$ such that $\varphi\Gamma\varphi^{-1} \subset F(\bigcup_{\beta < \rho} \mathcal{H}_\beta) = \bigcup_{\beta < \rho} F(\mathcal{H}_\beta)$. Using again that $\varphi\Gamma\varphi^{-1}$ is finitely generated, it cannot be written as an increasing union of proper subgroups. Therefore it must be contained in $F(\mathcal{H}_\beta)$ for some $\beta < \rho$. By the inductive hypothesis, this concludes the proof of the proposition. \square

3 Singular actions on manifolds

Throughout the section we fix $d \geq 1$ and let \mathcal{G} be a groupoid of germs over \mathbb{R}^d . Let M be a topological manifold of dimension d (all topological manifolds will be assumed to be Hausdorff and second countable). We will say that M is a \mathcal{G} -manifold if it is endowed with an atlas such that the corresponding changes of charts have all germs in the groupoid \mathcal{G} . See Thurston's book [37, Ch. 3] for more details. Whenever M, N are \mathcal{G} -manifolds, a homeomorphism $\varphi: M \rightarrow N$ between \mathcal{G} -manifolds will be called a \mathcal{G} -homeomorphism if in coordinate charts, all the germs of φ belong to \mathcal{G} . An $\Omega\mathcal{G}$ -homeomorphism (or a countably singular \mathcal{G} -homeomorphism) $\varphi: M \rightarrow N$ between \mathcal{G} -manifolds is a homeomorphism of topological manifolds which is moreover a \mathcal{G} -homeomorphism in restriction to an open subset of M whose complement is countable. We will denote by $\text{Homeo}_{\mathcal{G}}(M)$ the group of self \mathcal{G} -homeomorphisms of a \mathcal{G} -manifold M , and by $\Omega\text{Homeo}_{\mathcal{G}}(M)$ the group of countably singular ones.

As the main example, fix $r \geq 1$ and let \mathcal{D}^r be the groupoid of germs of all local diffeomorphisms of class C^r of \mathbb{R}^d . Then a \mathcal{D}^r -manifold M is simply a C^r -differentiable manifold, and a \mathcal{D}^r -homeomorphism is a diffeomorphism of class C^r . By a classical theorem of Whitney [39], every C^r -differentiable manifold admits a unique compatible C^∞ -differentiable structure up to diffeomorphism, therefore in this case we simply say that M is a differentiable manifold. We will say that a differentiable manifold is an *exotic sphere* if it is homeomorphic to a euclidean sphere \mathbb{S}^d (in particular, we use the convention that the standard euclidean sphere is itself an exotic sphere).

As another example that will be relevant, let $\mathcal{D}_{\text{Leb}}^r$ be the set of C^r -diffeomorphisms of \mathbb{R}^d whose Jacobian at every point has determinant 1 (that is, those that preserve the Lebesgue volume form $dx_1 \wedge \cdots \wedge dx_d$). Then a \mathcal{G} -manifold M is an oriented C^r -manifold endowed with a volume form ω

(see [37, Ex. 3.1.12]), and \mathcal{G} -homeomorphisms are homeomorphisms that preserve the corresponding volume forms.

We are now ready to state the main theorem of this section, which implies Theorem 1.1.

Theorem 3.1. *Let \mathcal{G} be a groupoid of germs of \mathbb{R}^d , and let M be a closed \mathcal{G} -manifold. Let Γ be a finitely generated group with property FW. For every homomorphism $\rho: \Gamma \rightarrow \Omega\text{Homeo}_{\mathcal{G}}(M)$, one of the following holds:*

- (i) *the action of $\rho(\Gamma)$ on M has a finite orbit;*
- (ii) *there exist a closed \mathcal{G} -manifold N and an $\Omega\mathcal{G}$ -homeomorphism $\varphi: M \rightarrow N$ such that $\varphi\rho(\Gamma)\varphi^{-1} \subset \text{Homeo}_{\mathcal{G}}(N)$.*

If moreover $\mathcal{G} = \mathcal{D}^r$ (so that M is a differentiable manifold) and if the dimension d is such that the euclidean sphere \mathbb{S}^d is the unique exotic d -sphere up to diffeomorphism, then in part (ii) one can choose $N = M$.

Remark 3.2. In [13], Cornulier obtains results in similar spirit but in a different setting, namely by considering partially defined actions where the action of each element is defined on the complement of a finite set and preserves the \mathcal{G} -structure on its domain of definition. The main differences are that while [13] only treats actions with finitely many singularities, in dimension one the results of [13] can be applied to discontinuous actions with finitely many discontinuity points, while we only treat actions that are globally continuous. Another difference is that [13] works more generally with modelled spaces in the sense of Ehresmann (not necessarily manifolds), i.e. spaces with charts taking values in a given model space X (not necessarily homeomorphic to \mathbb{R}^d) with changes of charts prescribed by a given pseudogroup of transformations of X . Here we consider only manifolds, but we point out that this is not essential. Theorem 3.1 (except for the last sentence) extends with the same proof if \mathbb{R}^d is replaced by any model space X satisfying mild local conditions (more precisely, any Hausdorff space which does not contain open countable subsets).

In the proof we will use the following lemma, whose proof is an elementary cut-and-paste argument.

Lemma 3.3. *Let M be a \mathcal{G} -manifold of dimension d , and $x_1, \dots, x_\ell \in M$ be distinct points in M . Assume that U_1, \dots, U_ℓ are pairwise disjoint open neighbourhoods of x_1, \dots, x_ℓ , and that for every $i = 1, \dots, \ell$ we are given an $\Omega\mathcal{G}$ -homeomorphism $h_i: U_i \rightarrow V_i$, where V_i is a \mathcal{G} -manifold. Then there exists a \mathcal{G} -manifold N such that the following hold.*

1. *As topological manifolds, N and M are homeomorphic.*
2. *There exists an $\Omega\mathcal{G}$ -homeomorphism $\tau: M \rightarrow N$ such that for every $i = 1, \dots, \ell$ there exist an open neighbourhood $W_i \subset U_i$ of x_i , and \mathcal{G} -homeomorphic embeddings $\lambda_i: h_i(W_i) \hookrightarrow N$ such that*
 - (i) *the set of singular points of τ is compactly contained in $W_1 \cup \dots \cup W_\ell$;*
 - (ii) *for every $i = 1, \dots, \ell$ the restriction $\tau|_{W_i}$ coincides with $\lambda_i \circ h_i|_{W_i}$.*
3. *Assume further $\mathcal{G} \subset \mathcal{D}^r$ for some $r \geq 1$ (in particular M and N also inherit a structure of differentiable manifolds). Then, seen as a differentiable manifold, N is diffeomorphic to a connected sum of M with finitely many exotic spheres. In particular if the dimension d is such that the standard sphere \mathbb{S}^d is the unique exotic d -sphere up to diffeomorphism, then N is diffeomorphic to M .*

For a definition of the connected sum of differentiable manifolds and its basic properties, we refer the reader to [23, §1].

Proof. Since every h_i is a \mathcal{G} -homeomorphism on the complement of a countable closed subset of U_i , using compactness we can choose disjoint neighbourhoods $W_i \subset U_i$ of the x_i 's such that the following properties are satisfied for every $i = 1, \dots, \ell$:

- (a) the closure \overline{W}_i is a \mathcal{G} -manifold with boundary, topologically homeomorphic to a closed ball and is contained in U_i ;
- (b) h_i is a \mathcal{G} -homeomorphism in restriction to some open neighbourhood L_i of $\partial\overline{W}_i$ in U_i .

Set $Z_i = h_i(W_i)$, and let \overline{Z}_i be its closure in V_i , $i = 1, \dots, \ell$. Consider the open subset $M_0 = M \setminus (\overline{W}_1 \cup \dots \cup \overline{W}_\ell)$ of M , and let \overline{M}_0 be its closure in M . We have $\partial\overline{M}_0 = \partial\overline{W}_1 \cup \dots \cup \partial\overline{W}_\ell$. Let N be the space obtained by gluing \overline{M}_0 to $\overline{Z}_1 \sqcup \dots \sqcup \overline{Z}_\ell$, by identifying every $\partial\overline{W}_i$ with $\partial\overline{Z}_i$ via the map h_i , for every $i = 1, \dots, \ell$. We denote by $\lambda_0: \overline{M}_0 \hookrightarrow N$ and $\lambda_i: \overline{Z}_i \hookrightarrow N, i = 1, \dots, \ell$ the canonical inclusions. Define the (topological) homeomorphism $\tau: M \rightarrow N$ given by

$$\tau(x) = \begin{cases} \lambda_0(x) & x \in M_0, \\ \lambda_i \circ h_i(x) & x \in \overline{W}_i, i = 1, \dots, \ell. \end{cases}$$

In particular M and N are homeomorphic as topological manifolds. Let us endow N with the structure of a \mathcal{G} -manifold as follows. Write $L_i = L_i^+ \cup L_i^-$, where $L_i^+ = L_i \setminus W_i$ and $L_i^- = L_i \cap \overline{W}_i$, so that $L_i^+ \subset \overline{M}_0$, $L_i^- \subset \overline{W}_i$, and $L_i^+ \cap L_i^- = \partial\overline{W}_i$. The subset $K_i = \tau(L_i)$ is an open neighbourhood in N of $\partial\overline{W}_i \simeq_{h_i} \partial\overline{Z}_i$. We transport the structure of \mathcal{G} -manifold of L_i to K_i via the homeomorphism $\tau|_{L_i}: L_i \rightarrow K_i, i = 1, \dots, \ell$. Moreover we transport the \mathcal{G} -manifold structures of M_0, Z_1, \dots, Z_ℓ to $\lambda_0(M_0), \lambda_1(Z_1), \dots, \lambda_\ell(Z_\ell)$ via the identifications $\lambda_0, \dots, \lambda_\ell$. Note that the corresponding structures are compatible on the intersections K_i , because every h_i is a \mathcal{G} -homeomorphism on the interior of L_i^- . Therefore this defines uniquely a structure of \mathcal{G} -manifold on N . By construction, the homeomorphism τ defined above satisfies the conclusion of part 2.

It remains to check part 3. To this end, recall that by (a) the sets W_i above are such that \overline{W}_i is a differentiable manifold with boundary homeomorphic to a closed ball. For $i = 1, \dots, \ell$, let R_i be the differentiable manifold obtained by gluing \overline{W}_i and \overline{Z}_i by identifying $\partial\overline{W}_i$ with $\partial\overline{Z}_i$ using the diffeomorphism $h_i|_{\partial\overline{W}_i}$. Since both \overline{W}_i and \overline{Z}_i are homeomorphic to disks, and R_i is an exotic sphere, it readily follows from the construction of N that it is diffeomorphic to a connected sum of M with R_1, \dots, R_ℓ . The last sentence in part 3 follows from the fact that taking a connected sum with standard spheres does not change the diffeomorphism class of a manifold. \square

Given a closed \mathcal{G} -manifold M we denote by \mathcal{G}_M the groupoid consisting of all germs of \mathcal{G} -homeomorphisms between open subsets of M . Observe that, keeping the notations introduced in the previous section, we have $F(\mathcal{G}_M) = \text{Homeo}_{\mathcal{G}}(M)$ and $F(\Omega\mathcal{G}_M) = \Omega\text{Homeo}_{\mathcal{G}}(M)$. The groupoid \mathcal{G}_M , however, needs not satisfy the requirements for Proposition 2.11. In order to get around this, we will instead view it as a subgroupoid of a larger groupoid, consisting of all germs of \mathcal{G} -homeomorphisms between \mathcal{G} -manifolds of dimension d . More precisely, let \mathcal{B}_d be a set of closed compact \mathcal{G} -manifolds of dimension d that contains infinitely many representatives for each \mathcal{G} -homeomorphism class (one representative would be enough, but taking infinitely many will slightly simplify the discussion).

We define a space \mathbb{X}_d as the disjoint union $\mathbb{X}_d = \bigsqcup_{M \in \mathcal{B}_d} M$. We let $\mathcal{G}_{\mathbb{X}_d}$ be the groupoid of germs over the space \mathbb{X}_d consisting of all germs of all \mathcal{G} -homeomorphisms defined from an open subset of some manifold in \mathcal{B}_d to an open subset of some (perhaps different) manifold in \mathcal{B}_d .

For a fixed \mathcal{G} -manifold $M \in \mathcal{B}_d$, the group $\Omega\text{Homeo}_{\mathcal{G}}(M)$ is naturally a subgroup of $F(\Omega\mathcal{G}_{\mathbb{X}_d})$ supported on M . In particular, it acts on the groupoid $\Omega\mathcal{G}_{\mathbb{X}_d}$ by post-composition whenever the target of the element of the groupoid lies in M , and as the identity otherwise.

Proposition 3.4. *The groupoid $\mathcal{G}_{\mathbb{X}_d}$ has countably resolvable singularities.*

Proof. To simplify the notations set $\mathcal{H} = \mathcal{G}_{\mathbb{X}_d}$ and let \mathcal{H}_α be the transfinite sequence as in the previous section. Assume that $\alpha = \beta + 1$ is a successor, and let us show that the pair $\mathcal{H}_\beta \subset \mathcal{H}_\alpha$ has resolvable singularities (the case of a limit ordinal is totally analogous and we omit it). To this end take $\gamma_1, \dots, \gamma_\ell \in \mathcal{H}_\alpha$ and let $K \subset \mathbb{X}_d$ be a compact subset as in Definition 2.4. Setting $x_i = t(\gamma_i)$, $i = 1, \dots, \ell$, we can assume that x_1, \dots, x_ℓ belong to a closed \mathcal{G} -manifold $M \subset \mathbb{X}_d$ (equal to the union of all the connected \mathcal{G} -manifolds that contain x_1, \dots, x_ℓ). Choose representatives $(h_1, x_1), \dots, (h_\ell, x_\ell)$ of $\gamma_1^{-1}, \dots, \gamma_\ell^{-1}$ whose domains of definitions are open neighbourhoods U_1, \dots, U_ℓ of x_1, \dots, x_ℓ respectively, that we can assume to be pairwise disjoint and contained in M . Moreover, by definition of $\mathcal{H}_\alpha = S\mathcal{H}_\beta$, upon reducing the U_i 's we can assume that $[h_i]_y \in \mathcal{H}_\beta$ for every $y \in U_i \setminus \{x_i\}$.

We apply Lemma 3.3 to this data, and let N be the \mathcal{G} -manifold and $\tau: M \rightarrow N$ and $W_i \subset U_i$ and $\lambda_i: h_i(W_i) \hookrightarrow N$ be as in the conclusion of the lemma. Without loss of generality, we can suppose that N belongs to \mathcal{B}_d , so that $N \subset \mathbb{X}_d$. Moreover, since we assume that \mathcal{B}_d contains infinitely many representatives of every \mathcal{G} -homeomorphism class, we can assume that $N \cap K = \emptyset$ and that $N \cap M = \emptyset$ in \mathbb{X}_d . Let $\varphi \in F(\mathcal{H}_\alpha)$ be the element given by:

$$\varphi(x) = \begin{cases} \tau(x) & x \in M, \\ \tau^{-1}(x) & x \in N, \\ x & x \notin M \cup N. \end{cases}$$

We claim that φ satisfies the conclusions (i,ii) of Definition 2.4. To see this, observe that for every $i = 1, \dots, \ell$ we have

$$[\varphi]_{x_i} \gamma_i = [\lambda_i]_{h_i(x_i)} [h_i]_{x_i} \gamma_i = [\lambda_i]_{h_i(x_i)} \gamma_i^{-1} \gamma_i = [\lambda_i]_{h_i(x_i)} \in \mathcal{H} \subset \mathcal{H}_\beta.$$

In a similar way one checks that for every $x \in M \setminus \{x_1, \dots, x_\ell\}$ we have $[\varphi]_x \in \mathcal{H}_\beta$. Since φ is supported on $M \cup N$ and $N \cap K = \emptyset$, this shows that $[\varphi]_x \in \mathcal{H}_\beta$ for every $x \in K \setminus \{x_1, \dots, x_\ell\}$. This concludes the proof. \square

Note that in the above proof, we have not used part 3 of Lemma 3.3. Taking it into account, the same proof yields the following when $\mathcal{G} = \mathcal{D}^r$, $r \geq 1$. (In the statement, \mathcal{D}_M^r denotes the groupoid consisting of all germs of C^r diffeomorphisms between open subsets of M .)

Proposition 3.5. *Let M be a closed differentiable manifold of dimension d , and assume that d is such that the standard sphere \mathbb{S}^d is the unique standard exotic d -sphere up to diffeomorphism. Then the groupoid \mathcal{D}_M^r has countably resolvable singularities.*

Proof of Theorem 3.1. Let M be a closed manifold and $\rho: \Gamma \rightarrow \Omega\text{Diff}^r(M)$ be as in the statement, and assume that Γ has no finite orbit. Since property FW passes to quotients, we can identify Γ with its image. We identify M with a subset of \mathbb{X}_d , where d is the dimension of M , and Γ with a subgroup of $F(\Omega\mathcal{D}_d^r)$ supported on M . By Proposition 3.4 we are in position to apply Proposition 2.11 and we obtain that either Γ has a finite orbit in M , or that there exists an element $\varphi \in F(\Omega\mathcal{D}_d^r)$ such that $\varphi\Gamma\varphi^{-1} \subset F(\mathcal{D}_d^r)$. Note that $\varphi\Gamma\varphi^{-1}$ is supported on the manifold $N = \varphi(M)$, which is the manifold in the statement. If the dimension d is as in the last sentence, then we may repeat the same reasoning using the groupoid \mathcal{D}_M^r by Proposition 3.5 and we obtain that $M = N$. \square

If we restrict the attention to actions with *finitely many* singularities, the conclusion can be made more precise in the finite-orbit case. Given a \mathcal{G} -manifold M we denote $S\text{Homeo}_{\mathcal{G}}(M) = F(S\mathcal{G}_M)$ the group of finitely singular \mathcal{G} -homeomorphisms of M , i.e. those that are \mathcal{G} -homeomorphisms in restriction to the complement of a finite set. It is naturally a subgroup of the topological full group $F(S\mathcal{G}_{\mathbb{X}_d})$ (with the notations of the previous section). It follows from Proposition 3.4 that the pair $S\mathcal{G}_d \supset \mathcal{G}_d$ satisfies the requirements for Proposition 2.6. Using this, one obtains:

Theorem 3.6. *Let M be a closed \mathcal{G} -manifold. Let Γ be a finitely generated group with property FW. For every homomorphism $\rho: \Gamma \rightarrow S\text{Homeo}_{\mathcal{G}}(M)$, there exists a closed \mathcal{G} -manifold N and a finitely singular \mathcal{G} -homeomorphism $\varphi: M \rightarrow N$ such that the set of singular points of $\varphi\rho(\Gamma)\varphi^{-1}$ is finite and consists of points with a finite $\varphi\rho(\Gamma)\varphi^{-1}$ -orbit.*

Again, if $\mathcal{G} = \mathcal{D}^r$ and if the dimension d is such that \mathbb{S}^d is the unique exotic d -sphere up to diffeomorphism, then we can choose $M = N$.

Let us now explain the applications related to Zimmer's conjecture.

Corollary 3.7. *Let G be a connected Lie group of real rank $r \geq 2$, whose Lie algebra is simple and with finite centre. Let $\Gamma \subset G$ be a cocompact lattice, or $\Gamma = \text{SL}(r+1, \mathbb{Z})$. Let M^d be a closed manifold of dimension d , and let $\rho: \Gamma \rightarrow \Omega\text{Diff}^2(M)$. Then*

- (i) *If $d < r$, then the action of $\rho(\Gamma)$ on M has a finite orbit.*
- (ii) *If there is a volume form ω on M such that $\rho(\Gamma)$ preserves the measure associated to ω , and if $d \leq r$, then $\rho(\Gamma)$ has a finite orbit.*

Remark 3.8. Note that the condition that $\rho(\Gamma)$ preserve the measure associated to ω is equivalent to say that for every $\gamma \in \Gamma$, the element $\rho(\gamma)$ preserves the form ω outside its singular set (as the singular set of $\rho(\gamma)$ is countable, it has measure zero).

Proof of Corollary 3.7. Both facts follow by combining the main results in [6, 7] with Theorem 3.1, applied to the groupoids $\mathcal{G} = \mathcal{D}^2$ and $\mathcal{G} = \mathcal{D}_{\text{Leb}}^2$ respectively. \square

4 Singular actions on the circle

This section is devoted to the more precise results that we can obtain in the case of actions on the circle.

4.1 Piecewise differentiable homeomorphisms

Definition 4.1. A homeomorphism $g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is said to be *piecewise differentiable* of class C^r if there exists a finite subset $\text{BP}(g) \subset \mathbb{S}^1$ such that g is a C^r -diffeomorphism in restriction to $\mathbb{S}^1 \setminus \text{BP}(g)$, and for every $x \in \text{BP}(g)$ the right and left derivatives of g exist up to order $\lfloor r \rfloor$ at x , and the derivative $D^{(\lfloor r \rfloor)}g$ is $(r - \lfloor r \rfloor)$ -Hölder continuous. The subset $\text{BP}(g)$ will be called the set of *breakpoints* of g . The group of all piecewise differentiable homeomorphisms of \mathbb{S}^1 of class C^r is denoted by $\text{PDiff}^r(\mathbb{S}^1)$.

In the following, we will simply denote by \mathcal{D}^r the groupoid of diffeomorphic germs $\mathcal{D}_{\mathbb{S}^1}^r$ over \mathbb{S}^1 , and by \mathcal{PD}^r the groupoid of piecewise- C^r germs over \mathbb{S}^1 . The difference between $S\mathcal{D}^r$ and \mathcal{PD}^r is that in the latter case we require that every germ is represented by a local homeomorphism having left and right first $\lfloor r \rfloor$ derivatives defined at every point of the domain of definition, and with the

$\lfloor r \rfloor$ -th left and right derivatives $(r - \lfloor r \rfloor)$ -Hölder continuous. Observe that the pair of groupoids $\mathcal{PD}^r \supset \mathcal{D}^r$ is co-discrete, and it is not difficult to show that it has resolvable singularities.

We let \mathcal{PD}_+^r be the subgroupoid consisting of *orientation preserving* germs. For every $x \in X$ we let $(\mathcal{PD}_+^r)_x$ be the *isotropy group* at x of this groupoid, which consists of all $\gamma \in \mathcal{PD}_+^r$ such that $s(\gamma) = t(\gamma) = x$. We first settle the following special case of Theorem 4.3.

Proposition 4.2. *Let Γ be a countable property FW subgroup of $\text{PDiff}^1(\mathbb{S}^1)$ that has a finite orbit. Then Γ is finite.*

Proof. Assume by contradiction that Γ is infinite, and note that it is finitely generated because of property FW. Since property FW passes to finite index subgroups, we can assume that Γ preserves orientation and fixes a point $x \in \mathbb{S}^1$, and using that Γ is finitely generated, without loss of generality we can assume that there exists an element $g \in \Gamma$ such that g is not the identity in restriction to a right neighbourhood of x . Hence we obtain a non-trivial group homomorphism $\Gamma \rightarrow (\mathcal{PD}_+^1)_x$. The image of this homomorphism is a non-trivial finitely generated subgroup of $(\mathcal{PD}_+^1)_x$, and therefore admits a non-trivial homomorphism to \mathbb{R} by the Thurston's stability theorem [36]. Hence so does Γ . This contradicts property FW, since FW passes to quotients, and an abelian infinite finitely generated group never has FW. \square

Theorem 4.3. *Fix $r \in [1, \infty]$. Every countable subgroup of $\text{PDiff}^r(\mathbb{S}^1)$ with property FW is conjugate in $\text{PDiff}^r(\mathbb{S}^1)$ to a subgroup of $\text{Diff}^r(\mathbb{S}^1)$.*

Proof. Let $\Gamma \subset \text{PDiff}^r(\mathbb{S}^1)$ be a subgroup with property FW. As the pair $\mathcal{PD}^r \subset \mathcal{D}^r$ is co-discrete, with resolvable singularities, we can invoke Proposition 2.6 and claim that the action of Γ on \mathbb{S}^1 has a finite orbit, or Γ is conjugate in $F(\mathcal{SD}^r) = \text{PDiff}^r(\mathbb{S}^1)$ to a subgroup of $F(\mathcal{D}^r) = \text{Diff}^r(\mathbb{S}^1)$. However, because of Proposition 4.2, the former is possible only if Γ is finite. \square

Proof of Corollary 1.3. Let Γ be a subgroup of $\text{PDiff}^{3/2}(\mathbb{S}^1)$, with property (T) and hence property FW. (We again identify Γ with its image.) After Theorem 4.3, there exists a homeomorphism $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\varphi\Gamma\varphi^{-1}$ is a group of C^r circle diffeomorphisms. Navas' theorem (Theorem 1.2) gives that Γ is finite. \square

4.2 PL circle homeomorphisms and isometric actions

In this subsection we will give an alternative and more elementary proof of Corollary 1.3 for groups of piecewise linear homeomorphisms, that we state below. For simplicity, we will always assume that homeomorphisms do preserve the orientation.

Definition 4.4. A homeomorphism $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is *piecewise linear* if, when seeing \mathbb{S}^1 as the flat torus \mathbb{R}/\mathbb{Z} (so with its quotient affine structure), for all but finitely many points $x \in \mathbb{S}^1$ there exists an open neighbourhood $I(x)$ such that the restriction $h|_{I(x)}$ is an affine map, that is of the form $y \mapsto ay + b$. A point $x \in \mathbb{S}^1$ where this condition is not verified is a *breakpoint* of h . We write $\text{BP}(h)$ for the set of breakpoints of h . We denote by $\text{PL}_+(\mathbb{S}^1)$ the group of piecewise linear homeomorphisms of \mathbb{S}^1 .

Theorem 4.5. *If Γ is a Kazhdan group, every homeomorphism $\rho : \Gamma \rightarrow \text{PL}_+(\mathbb{S}^1)$ has finite image.*

We consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{S}^1)$. If a group Γ acts on \mathbb{S}^1 , the action induces an isometric action on \mathcal{H} , just by permutation of the basis: $h \cdot (a_x)_{x \in \mathbb{S}^1} = (a_{h^{-1}(x)})_{x \in \mathbb{S}^1}$. We write

$\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$ for this (left) representation. Given an element $h \in \text{PL}_+(\mathbb{S}^1)$, we write D^-h and D^+h for its left and right derivatives (which are well-defined at every point). The map

$$\begin{aligned} b : \text{PL}_+(\mathbb{S}^1) \times \mathbb{S}^1 &\rightarrow \mathbb{R} \\ (h, x) &\mapsto \log \frac{D^+h(x)}{D^-h(x)} \end{aligned}$$

is a cocycle (i.e. $b(gh, x) = b(g, h(x)) + b(h, x)$), as one sees by applying the chain rule. Observe that we have $b(h^{-1}, x) = -b(h, h^{-1}(x))$. As h is a piecewise linear homeomorphism, for fixed h , the function $b(h, -)$ is zero at all but finitely many points (the breakpoints of h). This allows to define an isometric affine action $\rho : \Gamma \rightarrow \text{Isom}(\mathcal{H})$, with linear part π and translation part b :

Lemma 4.6. *Let Γ be a subgroup of $\text{PL}_+(\mathbb{S}^1)$. The map ρ , defined for $h \in \Gamma$ and $(a_x) \in \ell^2(\mathbb{S}^1)$ by*

$$\begin{aligned} \rho(h)(a_x)_{x \in \mathbb{S}^1} &:= \pi(h)(a_x)_{x \in \mathbb{S}^1} + (b(h^{-1}, x))_{x \in \mathbb{S}^1} \\ &= \left(a_{h^{-1}(x)} + b(h^{-1}, x) \right)_{x \in \mathbb{S}^1}, \end{aligned}$$

defines a homomorphism $\rho : \Gamma \rightarrow \text{Isom}(\mathcal{H})$.

Proof. One needs a few simple verifications. First, we check that the map ρ actually defines a homomorphism. This is guaranteed by the fact that b is a cocycle (over π):

$$\begin{aligned} \rho(g)\rho(h)(a_x)_{x \in \mathbb{S}^1} &= \rho(g) \left(a_{h^{-1}(x)} + b(h^{-1}, x) \right)_{x \in \mathbb{S}^1} \\ &= \left(a_{h^{-1}g^{-1}(x)} + b(h^{-1}, g^{-1}(x)) + b(g^{-1}, x) \right)_{x \in \mathbb{S}^1} \\ &= \left(a_{(gh)^{-1}(x)} + b(h^{-1}g^{-1}, x) \right)_{x \in \mathbb{S}^1} = \rho(gh)(a_x)_{x \in \mathbb{S}^1}. \end{aligned}$$

Secondly, we have to verify that if a sequence $(a_x)_{x \in \mathbb{S}^1}$ is in \mathcal{H} , then also its $\rho(h)$ -image is. This is because for a fixed element $h \in \text{PL}_+(\mathbb{S}^1)$, the cocycle $b(h^{-1}, -)$ is zero at all but finitely many points. Finally, we have to verify that for fixed $h \in \text{PL}_+(\mathbb{S}^1)$, the map $\rho(h)$ is an isometry. This is because $\rho(h)$ is the composition of the isometry $\pi(h)$ with the translation $b(h^{-1}, -)$. \square

Lemma 4.7. *Let $f \in \text{PL}_+(\mathbb{S}^1)$ be a non-trivial element acting with a fixed point on \mathbb{S}^1 , then $\rho(f) : \mathcal{H} \rightarrow \mathcal{H}$ has an unbounded orbit.*

Proof. We reproduce an argument appearing in [28, §C]. Let $I = (x_0, x_1)$ be a connected component of the open support $\{x \in \mathbb{S}^1 : f(x) \neq x\}$ (possibly $x_0 = x_1$). Without loss of generality, we can assume that f is contracting on I , i.e. $f(y) < y$ for every $y \in I$.

Claim 1. *The set of values $S = \{b(f^n, x) : n \in \mathbb{N}, x \in I\}$ is finite.*

Proof of claim. Choose $x_0 < z < x_1$ such that the restriction of f to $I_0 = (x_0, z)$ is linear. Given $x \in I$ and $n \in \mathbb{N}$, the logarithm of the jump of derivatives $b(f^n, x)$ can be written as the sum

$$b(f^n, x) = \sum_{k=0}^{n-1} b(f, f^k(x)).$$

Notice that only points $f^k(x)$ that are breakpoints of f contribute to the sum above. As f is a contraction, the sequence $\{f^k(x)\}_{k \in \mathbb{N}}$ is strictly decreasing to x_0 , so the forward orbit of x visits any breakpoint of f at most once. Therefore $b(f^n, x)$ can only take values in the finite set $\{\sum_{c \in E} b(f, c)\}_{E \subset \text{BP}(f)}$. \square

Claim 2. *Let M_n denote the number of breakpoints of f^n . Then M_n is unbounded as $n \rightarrow \infty$.*

Proof of claim. Suppose, by way of contradiction, that every f^n has at most M breakpoints. Fix $n \in \mathbb{N}$ and set $\lambda_n = \inf_{x \in I} D^+ f^n(x)$. By the previous claim, the set S is finite, hence $C := (\max_{\sigma \in S} e^\sigma)^M$ is finite. As the derivative $D^+ f^n$ can change only when passing through breakpoints, we have the upper bound $D^+ f^n(y) \leq C\lambda_n$ for every $y \in I$. By averaging over I , using the fact that $\int_I D^+ f^n(x) dx = |I|$, we obtain the inequality $1 \leq C\lambda_n$. As f is a contraction, $\lambda_n \rightarrow 0$ as $n \rightarrow +\infty$. This gives the desired contradiction. \square

Take $\beta = \min_{\sigma \in S \setminus \{0\}} |\sigma|$. The image $\rho(f^{-n})\vec{0}$ of the vector $\vec{0}$ is exactly the vector $(b(f^{-n}, x))_{x \in \mathbb{S}^1}$. Its ℓ^2 -norm is at least $M_n\beta$. As the sequence M_n is unbounded, also the sequence $\|\rho(f^n)(0)\|$ is unbounded. \square

Remark 4.8. In the proof of Claim 2, the fact that elements are piecewise linear is fundamental for the proof. Indeed, every non-trivial piecewise linear homeomorphism of an interval has at least one breakpoint. This is no longer true, even for Möbius transformations.

We next recall a classical result due to Hölder [33, Theorem 2.2.32]. For its statement, recall that a group action is *free* if every point has trivial stabiliser.

Theorem 4.9 (Hölder). *Let Γ be a subgroup of $\text{Homeo}_+(\mathbb{S}^1)$, whose action on \mathbb{S}^1 is free. Then Γ is isomorphic to a group of rotations. More precisely, if $\Phi : \Gamma \rightarrow \Phi(\Gamma) \subset \text{SO}(2)$ denotes the isomorphism, there exists a monotone continuous degree 1 map $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $hg = \Phi(g)h$ (i.e. Γ is semi-conjugate to a group of rotations).*

We can now prove the main result of the section:

Proof of Theorem 4.5. As the quotient of a Kazhdan group is also a Kazhdan group, it is enough to prove that every subgroup of $\text{PL}_+(\mathbb{S}^1)$ which has property (T) is actually finite. So let $\Gamma \subset \text{PL}_+(\mathbb{S}^1)$ be an infinite subgroup. We consider the representation $\rho : \Gamma \rightarrow \text{Isom}(\mathcal{H})$ of Lemma 4.6. Even if property (T) is not inherited by subgroups, we can however restrict to subgroups to show that a particular isometric affine action has unbounded orbits. If Γ contains a non-trivial element acting with a fixed point, then Lemma 4.7 implies that the ρ has an unbounded orbit, disproving property (T). Otherwise, Theorem 4.9 implies that Γ is isomorphic to a group of rotations (in fact, by Denjoy's theorem, which holds for piecewise linear homeomorphisms [20], it is topologically conjugate), which is abelian, and hence cannot have property (T). \square

A direct application of the work of Minakawa on *exotic circles* of $\text{PL}_+(\mathbb{S}^1)$ [31] (not to be confused with exotic spheres in the sense of differentiable topology) gives a more precise statement for Corollary 1.3 in the PL case. For this, we set some notations. A *topological circle* in $\text{PL}_+(\mathbb{S}^1)$ is a one-parameter subgroup $S = \{g_\alpha; \alpha \in \mathbb{S}^1\} \subset \text{PL}_+(\mathbb{S}^1)$ which is topologically conjugate to the group of rotations $\text{SO}(2)$. The topological circle is called *exotic*, if the conjugating map cannot be taken in $\text{PL}_+(\mathbb{S}^1)$. For $A > 1$ one sets $I_A = [1/(A-1), A/(A-1)]$, which is an interval of length 1, and defines

$$\begin{aligned} h : I_A &\rightarrow [0, 1] \\ x &\mapsto \frac{\log(A-1)x}{\log A} \end{aligned}$$

which naturally extends to a homeomorphism \widetilde{h}_A commuting with the translation by 1 and hence defines, by quotient, a homeomorphism h_A of \mathbb{S}^1 . Set $S_A = h_A^{-1}\text{SO}(2)h_A$. When $A < 1$, we set $S_A = \iota S_A \iota^{-1}$, where $\iota : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the order-reversing involution defined by $\iota(x) = -x$. These are exotic circles. For $A = 1$, S_A will simply stay for $\text{SO}(2)$. The circles S_A are contained in $\text{PL}_+(\mathbb{S}^1)$

(these examples had previously appeared in [5, 20]) and Minakawa shows that every topological circle in $\text{PL}_+(\mathbb{S}^1)$ is PL conjugate to one of the exotic circles S_A (and vice versa). More precisely, he proves that the only irrational rotations for which 1) the number of breakpoints of iterates is bounded and 2) the set of jumps is finite, are contained in a topological circle (see also [27]). Relying on this, one can prove:

Theorem 4.10. *Let Γ be an infinite discrete subgroup of $\text{PL}_+(\mathbb{S}^1)$. The following statements are equivalent:*

1. *the affine isometric action $\rho : \Gamma \rightarrow \text{Isom}(\mathcal{H})$ defined in Lemma 4.6 has bounded orbits;*
2. *the subgroup Γ is PL conjugate into a topological circle of rotations S_A .*

Remark 4.11. Exotic circles were one of the main reasons for suspecting that $\text{PL}_+(\mathbb{S}^1)$ might have contained subgroups with property (T) (cf. [4]). The statement in Theorem 4.10 clearly gives a negative answer, and at the same time explains the rôle of exotic circles in this problem.

Acknowledgements M.T. thanks Christian Bonatti for the helpful discussion about germs of singular diffeomorphisms. N.M.B. thanks Peter Feller for useful clarifications about exotic spheres. The authors show their gratitude to Yves de Cornulier for sharing his work [13] and for the several remarks.

Y.L. was supported by an SNSF *Ambizione* grant. M.T. was partially supported by PEPS – Jeunes Chercheur-e-s – 2017 (CNRS) and Projet “Jeunes Géomètres” of F. Labourie (financed by the Louis D. Foundation).

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