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FEEDBACK STABILIZATION OF A TWO-DIMENSIONAL FLUID-STRUCTURE INTERACTION SYSTEM WITH MIXED BOUNDARY CONDITIONS

FOURNIÉ MICHEL*, NDIAYE MOCTAR*, AND RAYMOND JEAN-PIERRE*

Abstract. We study the stabilization of a fluid-structure interaction system around an unstable stationary solution. The system consists of coupling the incompressible Navier-Stokes equations, in a two dimensional polygonal domain with mixed boundary conditions, and a damped Euler-Bernoulli beam equations located at the boundary of the fluid domain. The control acts only in the beam equations. The feedback is determined by stabilizing the projection of the linearized model onto a finite dimensional invariant subspace. Here we have resolved two important challenges for applications in this field. One is the fact that we prove a stabilization result around a non zero stationary solution, which is new for such fluid-structure interaction systems. The other one is that the feedback laws that we determine do not depend on the Leray projector used to get rid of the algebraic constraints of partial differential equations. This is essential for numerical aspects.

Key words. Fluid-structure interaction, feedback control, stabilization, Navier-Stokes equations, beam equation.

AMS subject classifications. 93B52, 93C20, 93D15, 76D55, 76D05, 74F10.

1. Introduction. We are interested in stabilizing, around a non-zero stationary solution, a fluid-structure interaction system coupling beam equations and the Navier-Stokes equations. The initial configuration for the fluid domain is denoted by Ω , and is referred as the reference configuration. Its boundary Γ is split into different parts $\Gamma = \Gamma_s \cup \Gamma_i \cup \Gamma_e \cup \Gamma_n$, where Γ_s is a flat part of the boundary occupied by elastic structures satisfying Euler-Bernoulli damped beam equations. For the fluid, Neumann boundary conditions are prescribed on Γ_n , Navier type boundary conditions are prescribed on Γ_e , and Dirichlet boundary conditions are prescribed on Γ_i . The assumptions that are essential in our analysis are listed in Section 2. A particular configuration satisfying these assumptions corresponds to the right hand side of a wind tunnel, see Figure 1.1, in which a fluid flows around a thick plate, and two beams are located in the upper and lower boundaries of the plate. In Figure 1.1, the domain Ω is polygonal but the results of the paper can be easily extended to other geometrical configurations, see Remark 2.1. The stabilization problem for the associated semi-discrete model is studied in [8].

We assume that the inflow boundary condition in the computational domain Ω is a perturbed Blasius type profile, which is used to determine the stationary solution around which we want to stabilize the fluid-structure system. The goal is to use a force term as control in the beam equations in order to stabilize the full fluid-structure system. Since the structure is deformed under the action of the fluid, the domain occupied by the fluid at time t depends on the displacement $\eta(t)$ of the structure, see Figure 1.2. The fluid domain at time t is denoted by $\Omega_{\eta(t)}$ and the fluid-structure interface by $\Gamma_{\eta(t)}$. We use the notations

$$\begin{aligned} Q_\eta^\infty &= \bigcup_{t \in (0, \infty)} (\{t\} \times \Omega_{\eta(t)}), & \Sigma_\eta^\infty &= \bigcup_{t \in (0, \infty)} (\{t\} \times \Gamma_{\eta(t)}), & Q^\infty &= (0, \infty) \times \Omega, \\ \Sigma_s^\infty &= (0, \infty) \times \Gamma_s, & \Sigma_i^\infty &= (0, \infty) \times \Gamma_i, & \Sigma_e^\infty &= (0, \infty) \times \Gamma_e, & \Sigma_n^\infty &= (0, \infty) \times \Gamma_n. \end{aligned}$$

The Eulerian-Lagrangian system describing the evolution of the fluid-structure system is

$$\begin{aligned} u_t - \operatorname{div} \sigma(u, p) + (u \cdot \nabla)u &= 0, & \operatorname{div} u &= 0 \text{ in } Q_\eta^\infty, \\ u &= \eta_t n \text{ on } \Sigma_\eta^\infty, & u &= g_s + g_p \text{ on } \Sigma_i^\infty, & u \cdot n &= 0 \text{ and } \varepsilon(u)n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \\ \sigma(u, p)n &= 0 \text{ on } \Sigma_n^\infty, & u(0) &= u^0 \text{ on } \Omega, \\ \eta_{tt} - \beta \Delta_s \eta - \gamma \Delta_s \eta_t + \alpha \Delta_s^2 \eta &= -\sigma(u, p)|_{\Gamma_{\eta(t)}} n_{\eta(t)} \sqrt{1 + \eta_x^2} \cdot n + f_s + f \text{ on } \Sigma_s^\infty, \\ \eta &= 0 \text{ on } (0, \infty) \times \partial\Gamma_s, & \eta_x &= 0 \text{ on } (0, \infty) \times \partial\Gamma_s, \\ \eta(0) &= 0 \text{ on } \Gamma_s, & \eta_t(0) &= \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \tag{1.1}$$

where u and p stand for the fluid velocity and pressure, $\sigma(u, p)$ is the Cauchy stress tensor

$$\sigma(u, p) = 2\nu \varepsilon(u) - pI, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

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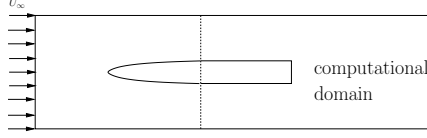


Fig. 1.1: Configuration of the wind tunnel.

ν is the fluid viscosity, $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are parameters of the structure, f_s is a stationary force precisely defined below after (1.2), and $n_{\eta(t)}$ (resp. n) is the unit normal to $\Gamma_{\eta(t)}$ (resp. Γ_s) exterior to $\Omega_{\eta(t)}$ (resp. Ω). Here $\Delta_s = \partial_{xx}$ stands for the Laplace-Beltrami operator on Γ_s . Clamped boundary conditions are imposed at the extremities of the beams, and η is a vector valued function representing the displacements of the upper and lower beams. The inflow boundary condition $g_s \in H^2(\Gamma_i; \mathbb{R}^2)$ is independent of time and g_p is a time dependent perturbation of g_s , taking into account the fact that the inflow boundary condition of the computational domain Ω is not precisely known. The precise assumptions on g_s are stated in Section 2.2. The function f_s is also assumed to be time independent, and the control function f is taken in the form

$$f(t, x, y) = \sum_{i=1}^{n_c} f_i(t) w_i(x, y),$$

where the functions $w_i \in L^2(\Gamma_s)$ are chosen in (8.4), so that some stabilizability condition stated in (8.7) is satisfied.

Let (u_s, p_s) be a solution of the stationary Navier-Stokes equations

$$\begin{aligned} -\operatorname{div} \sigma(u_s, p_s) + (u_s \cdot \nabla) u_s &= 0, \quad \operatorname{div} u_s = 0 \text{ in } \Omega, \\ u_s &= 0 \text{ on } \Gamma_s, \quad u_s = g_s \text{ on } \Gamma_i, \quad u_s \cdot n = 0 \text{ and } \varepsilon(u_s) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(u_s, p_s) n = 0 \text{ on } \Gamma_n. \end{aligned} \quad (1.2)$$

We choose $f_s = -p_s|_{\Gamma_s}$ in (1.1)₅. Thus, $(u, p, \eta) = (u_s, p_s, 0)$ is a stationary solution of system (1.1). We assume that it is an unstable stationary solution of that system. The goal of the paper is to find a control variable $\mathbf{f} = (f_1, \dots, f_{n_c})$, in feedback form, able to stabilize system (1.1) around the stationary solution $(u_s, p_s, 0)$, with any prescribed exponential decay rate $-\omega < 0$, provided that g_p , $u^0 - u_s$ and η_2^0 are small enough in appropriate functional spaces.

We follow a classical approach consisting of finding a feedback control law, stabilizing a linearized model, that is next applied to the nonlinear system. Let us explain why implementing this approach is not obvious and leads to new difficulties. In (1.1) the main nonlinearities come from the fact that the Navier-Stokes system is written in a time-dependent geometrical domain depending on the displacement of elastic structures. Therefore, before linearizing the system, we have to rewrite it in the reference configuration $(0, \infty) \times \Omega$ (see Section 3). For the stability analysis of our linearized fluid-structure system and for stabilization issues, we have to study the direct and adjoint eigenvalue problems. These problems are not standard because the algebraic constraints of the direct eigenvalue problem are

$$\operatorname{div} v = A_3 \eta_1 \text{ in } \Omega, \quad v = \eta_2 n \text{ on } \Gamma_s,$$

(see system (5.1)), while the algebraic constraints of the adjoint eigenvalue problem are

$$\operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 n \text{ on } \Gamma_s,$$

(see system (6.2)). It is a consequence of the fact that the system is linearized around a non zero stationary solution. Because the algebraic constraints are different, studying the spectrum and performing a Jordan decomposition of the linearized operator is not standard. In order to study the spectrum of the linearized operator and to establish its link with the direct and adjoint eigenvalue problems in PDE formulation, we have to rewrite them into equivalent forms with eigenfunctions belonging to the same state space. This can be done by using the so-called Leray projector to transform the direct and adjoint eigenvalue problems. This approach leads to direct and adjoint eigenvalue problems in the form of operator equations. Let

us emphasize that operator formulation is needed to justify the eigenvalue analysis, and that the PDE formulation will be needed in [8] for numerical simulations. For that, we have to establish the equivalence between these two formulations. This is done in Section 7.

To the best of our knowledge, the results of the present paper are the first ones dealing with the stabilization of a fluid-structure system involving mixed boundary conditions in the fluid equations. The analysis of such a system seems also to be new in the literature.

Feedback stabilization of strong solutions of such a system around the null solution, by controls acting only in the equations of the structure, is treated in [24]. In [1], the authors obtained stabilization results by means of Dirichlet boundary controls acting in the fluid equations. Several existence results of strong and weak solutions, and some uniqueness results, have been proved in [6, 10, 2, 16, 11].

The plan of the paper is as follows. The functional setting and the assumptions are given in Section 2. The main result is stated in Section 3. The analysis of Oseen operator and the characterization of the pressure in the Oseen system are done in Section 4. In Section 5, we characterize the infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ of the semigroup corresponding to the linearized system. We prove that this semigroup is analytic with compact resolvent. The adjoint of $(\mathcal{A}, D(\mathcal{A}))$ is characterized in Section 6. The equivalences between eigenvalue problems are established in Section 7. In Section 8, we prove that the bi-orthogonality condition satisfied by the families of eigenfunctions of the direct and adjoint partial differential equations implies another bi-orthogonality condition satisfied by the families of eigenfunctions of the direct and adjoint operator equations. This new relationship is used to define a feedback control law independent of the Leray projector, and which can be easily calculated (see [8]).

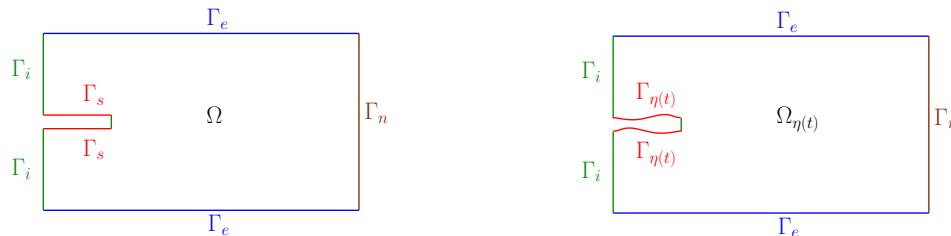


Fig. 1.2: Reference configuration (left) and deformed configuration (right).

2. Functional setting.

2.1. Notations. The geometrical domain Ω and Γ_s , see Figure 1.2, are defined by

$$\Omega = ([0, L] \times [-\ell, \ell]) \setminus ([0, \ell_s] \times [-e, e]), \quad \Gamma_s = (0, \ell_s) \times \{e\} \cup (0, \ell_s) \times \{-e\},$$

where $L > 0$ is the length of the computational domain, 2ℓ is its height, ℓ_s is the length of the thick plate, and $2e$ is the thickness of the plate. Thus equations (1.1)₅₋₇ describe the displacement of the upper beam located at $(0, \ell_s) \times \{e\}$ and the lower beam located at $(0, \ell_s) \times \{-e\}$. We also have

$$\begin{aligned} \Gamma_i &= \Gamma_{i,1} \cup \Gamma_{i,2} \cup \Gamma_{i,3} \quad \text{with} \quad \Gamma_{i,1} = \{0\} \times (-\ell, -e), \quad \Gamma_{i,2} = \{0\} \times (e, \ell), \quad \Gamma_{i,3} = \{\ell_s\} \times (-e, e), \\ \Gamma_e &= \left((0, L) \times \{-\ell\} \right) \cup \left((0, L) \times \{\ell\} \right), \quad \Gamma_{i,e} = \Gamma_i \cup \Gamma_e, \quad \Gamma_n = \{L\} \times (-\ell, \ell), \quad \Gamma_0 = \Gamma_s \cup \Gamma_{i,e}. \end{aligned}$$

We introduce the functional spaces

$$\begin{aligned} \mathbf{L}^2(\Omega) &= L^2(\Omega; \mathbb{R}^2), \quad \mathbf{H}^s(\Omega) = H^s(\Omega; \mathbb{R}^2), \quad \forall s > 0, \\ \mathbf{H}_{\Gamma_0}^s(\Omega) &= \{v \in \mathbf{H}^s(\Omega) \mid v = 0 \text{ on } \Gamma_s \cup \Gamma_i, v \cdot n = 0 \text{ on } \Gamma_e\}, \quad \forall s > \frac{1}{2}, \\ \mathbf{V}_{n, \Gamma_0}^0(\Omega) &= \{v \in \mathbf{L}^2(\Omega) \mid \operatorname{div} v = 0 \text{ on } \Omega, v \cdot n = 0 \text{ on } \Gamma_0\}, \\ \mathbf{V}_{n, \Gamma_0}^s(\Omega) &= \mathbf{H}^s(\Omega) \cap \mathbf{V}_{n, \Gamma_0}^0(\Omega), \quad \forall s > 0, \\ \mathbf{V}_{\Gamma_{i,e}}^s(\Omega) &= \{v \in \mathbf{H}^s(\Omega) \mid \operatorname{div} v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_i, v \cdot n = 0 \text{ on } \Gamma_e\}, \quad \forall s > \frac{1}{2}. \end{aligned}$$

We also introduce the product spaces

$$H = \mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s) \quad \text{and} \quad Z = \mathbf{V}_{n, \Gamma_0}^0(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s).$$

The space $\mathbf{V}_{n,\Gamma_0}^0(\Omega)$ is equipped with the \mathbf{L}^2 -norm, the spaces $\mathbf{H}_{\Gamma_0}^s(\Omega)$, $\mathbf{V}_{\Gamma_{i,e}}^s(\Omega)$ and $\mathbf{V}_{n,\Gamma_0}^s(\Omega)$ are equipped with the \mathbf{H}^s -norm. The inner product in $H_0^2(\Gamma_s)$ is chosen at the beginning of Section 5.2, and the inner product in H and Z is defined in (5.2).

The space of boundary displacements. We consider displacements of the beams located on Γ_s belonging to the space denoted by $H_0^{\frac{3}{2}}(\Gamma_s)$, which is identified with $(H_0^{\frac{3}{2}}(0, \ell_s))^2 = (H^{\frac{3}{2}}(0, \ell_s) \cap H_0^1(0, \ell_s))^2$. By writing $H_0^{\frac{3}{2}}(\Gamma_s)$, we emphasize the fact that η is a vector function with one component defined on $(0, \ell_s) \times \{e\}$ and the other one defined on $(0, \ell_s) \times \{-e\}$.

The space of inflow boundary conditions. On the boundary $\Gamma_i = \Gamma_{i,1} \cup \Gamma_{i,2} \cup \Gamma_{i,3}$, we consider inflow conditions \mathbf{g} belonging to the space

$$\mathbf{H}(\Gamma_i) = \left\{ \mathbf{g} = (g_1, g_2) \mid g_2 = 0, g_1|_{\Gamma_{i,3}} = 0, g_1|_{\Gamma_{i,1}} \in H^2(-\ell, -e), g_1|_{\Gamma_{i,2}} \in H^2(e, \ell), \right. \\ \left. g_1'(-\ell) = g_1'(\ell) = 0, g_1(-e) = g_1(e) = 0 \right\}.$$

The space $\mathbf{H}(\Gamma_i)$ is equipped with the norm $(g_1, g_2) \mapsto \|g_1\|_{H^2((-\ell, -e) \cup (e, \ell))}$.

We denote by $\mathcal{C} = (\mathcal{C}_j)_{1 \leq j \leq 8}$ the set of the corners of Ω . For all $-1 < \delta < 1$, $s \in \mathbb{N}$, and for $n = 1$ or $n = 2$, we introduce the norms

$$\|v\|_{H_\delta^s(\Omega; \mathbb{R}^n)}^2 = \sum_{|\alpha| \leq s} \int_{\Omega} r^{2\delta} |\partial_\alpha v|^2,$$

where r stands for the distance to \mathcal{C} , $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ denotes a two-index, $|\alpha| = \alpha_1 + \alpha_2$ is its length, ∂_α denotes the corresponding partial differential operator. We denote by $H_\delta^s(\Omega; \mathbb{R}^n)$ the closure of $C^\infty(\bar{\Omega})$ in the norm $\|\cdot\|_{H_\delta^s(\Omega; \mathbb{R}^n)}$. We set

$$\mathbf{L}_\delta^2(\Omega) = H_\delta^0(\Omega; \mathbb{R}^2), \quad \mathbf{H}_\delta^s(\Omega) = H_\delta^s(\Omega; \mathbb{R}^2) \text{ for all } s > 0, \\ H_\delta = \mathbf{H}_\delta^2(\Omega) \times H_\delta^1(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s).$$

We also introduce the following spaces of time dependent functions

$$H^{2,1}(Q^\infty) = L^2(0, \infty; \mathbf{H}^2(\Omega)) \cap H^1(0, \infty; \mathbf{L}^2(\Omega)), \quad H_\delta^{2,1}(Q^\infty) = L^2(0, \infty; \mathbf{H}_\delta^2(\Omega)) \cap H^1(0, \infty; \mathbf{L}^2(\Omega)), \\ H_0^{2,1}(\Sigma_s^\infty) = L^2(0, \infty; H_0^2(\Gamma_s)) \cap H^1(0, \infty; L^2(\Gamma_s)), \quad H_0^{4,2}(\Sigma_s^\infty) = H_0^{2,1}(\Sigma_s^\infty) \cap H^{4,2}(\Sigma_s^\infty), \\ Y = L^2(0, \infty; \mathbf{L}^2(\Omega)) \times H^{2,1}(Q^\infty) \times L^2(0, \infty; L^2(\Gamma_s)), \\ X_\delta = H_\delta^{2,1}(Q^\infty) \times L^2(0, \infty; H_\delta^1(\Omega)) \times H_0^{4,2}(\Sigma_s^\infty) \times H_0^{2,1}(\Sigma_s^\infty).$$

2.2. Assumptions.

ASSUMPTION 1. We assume that $g_s \in \mathbf{H}(\Gamma_i)$, and that the system (1.2) admits a solution $(u_s, p_s) \in \mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ for some $0 < \delta_0 < \frac{1}{2}$ specified in (4.4).

ASSUMPTIONS 2 AND 3. These assumptions are stated in Section 8.2, see (8.5) and (8.7).

In Assumption 2, we state that the parts of spectrum of the Oseen operator and that of the elastic structure contained in the half plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq -\omega\}$ are disjoint. In Assumption 3, we state a unique continuation property sufficient to verify the Hautus stabilization test for the linearized system.

REMARK 2.1. The main results of the paper, Theorems 3.2 and 3.3, are proved for the geometrical configuration described in Section 2.1 and under Assumptions 1, 2 and 3. As mentioned in the introduction, Theorems 3.2 and 3.3 remain valid for geometrical configurations for which the regularity result stated in Theorem 4.1 is true. (See e.g. [21] where other geometrical configuration are considered.)

3. System in the reference configuration and main results. We denote by \mathcal{E} a continuous linear operator from $H_0^{4,2}(\Sigma_s^\infty)$ into $H_0^{4,2}((0, \infty) \times (0, L) \times \{e, -e\})$ such that $\mathcal{E}\eta(t, x) = 0$ for $(t, x) \in (0, \infty) \times ((\ell_s + L)/2, L)$. For beam displacements $\eta \in H_0^{4,2}(\Sigma_s^\infty)$ small enough, we define a \mathcal{C}^1 -diffeomorphism $\mathcal{T}_{\eta(t)}$, from $\Omega_{\eta(t)}$ into Ω , by setting

$$\mathcal{T}_{\eta(t)}(x, y) = (x, z) = \left(x, \frac{(\ell - e)y - \ell \mathcal{E}\eta(t, x, s(y)e)}{\ell - e - s(y)\mathcal{E}\eta(t, x, s(y)e)} \right), \quad (3.1)$$

where the sign function s is defined by $s(y) = -1$ if $y < 0$ and $s(y) = 1$ if $y \geq 0$.

REMARK 3.1. To simplify the notation, in all the transformations, the nonlinear terms and the operators involving $\mathcal{E}\eta$ for some $\eta \in H_0^{4,2}(\Sigma_s^\infty)$, we shall simply write η in place of $\mathcal{E}\eta$. We shall do the same abuse of notation when the partial derivatives of $\mathcal{E}\eta$ appear in some expression.

For a given $\eta \in H_0^{4,2}(\Sigma_s^\infty)$ such that the mapping $\mathcal{T}_{\eta(t)}$ is a C^1 -diffeomorphism from $\Omega_{\eta(t)}$ into Ω for all $t > 0$, we say that u belongs to $H_\delta^{2,1}(Q_\eta^\infty)$ (resp. p belongs to $L^2(0, \infty; H_\delta^1(\Omega_\eta))$) if and only if

$$u(\cdot, \mathcal{T}_{\eta(\cdot)}^{-1}) \text{ belongs to } H_\delta^{2,1}(Q^\infty) \text{ (resp. } p(\cdot, \mathcal{T}_{\eta(\cdot)}^{-1}) \text{ belongs to } L^2(0, \infty; H_\delta^1(\Omega))).$$

The main result of the paper is the following theorem.

THEOREM 3.2. Let ω be positive. We assume that Assumptions 1, 2 and 3 are satisfied. There exists $r > 0$ such that, for all $u^0 \in \mathbf{H}^1(\Omega)$, $\eta_2^0 \in H_0^1(\Gamma_s)$, and $e^{\omega t} g_p \in H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$ satisfying $u^0 - u_s \in \mathbf{V}_{\Gamma_i, e}^1(\Omega)$, $(u^0 - u_s)|_{\Gamma_s} = \eta_2^0 n$ and

$$\|u^0 - u_s\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} + \|e^{\omega t} g_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))} \leq r,$$

we can find $\mathbf{f} = (f_1, \dots, f_{n_c}) \in H^1(0, \infty; \mathbb{R}^{n_c})$, in feedback form, for which the nonlinear system (1.1) admits a solution (u, p, η) in $H_{\delta_0}^{2,1}(Q_\eta^\infty) \times L^2(0, \infty; H_{\delta_0}^1(\Omega_{\eta(t)})) \times H^{4,2}(\Sigma_s^\infty)$, for δ_0 defined in (4.4), satisfying

$$\begin{aligned} & \left\| \left(u \left(t, \mathcal{T}_{\eta(t)}^{-1} \right) - u_s, \eta(t), \eta_t(t) \right) \right\|_{\mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega) \times H^3(\Gamma_s) \times H^1(\Gamma_s)} \leq C e^{-\omega t}, \quad \forall t > 0, \quad \text{and} \\ & \|e^{-\omega \cdot} \eta_1\|_{L^\infty(\Sigma_s^\infty)} < e, \end{aligned}$$

where $C > 0$ depends on r and $0 < \varepsilon_0 < \frac{1}{2}$.

In order to prove that theorem, we make the change of unknowns

$$\begin{aligned} \hat{u}(t, x, z) &:= e^{\omega t} [u(t, \mathcal{T}_{\eta(t)}^{-1}(x, z)) - u_s(x, z)], & \hat{p}(t, x, z) &:= e^{\omega t} [p(t, \mathcal{T}_{\eta(t)}^{-1}(x, z)) - p_s(x, z)], \\ \hat{\eta}_1(t, x) &:= e^{\omega t} \eta(t, x), & \hat{\eta}_2(t, x) &:= e^{\omega t} \eta_t(t, x), & \hat{\mathbf{f}}(t) &= (\hat{f}_i(t))_{1 \leq i \leq n_c} := e^{\omega t} \mathbf{f}(t), \\ \hat{g}_p(t, x, z) &:= e^{\omega t} g_p(t, x, z) \quad \text{and} \quad \hat{u}^0 = u^0 - u_s. \end{aligned} \tag{3.2}$$

The quadruplet $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2)$ satisfies the system

$$\begin{aligned} \hat{u}_t - \operatorname{div} \sigma(\hat{u}, \hat{p}) + (u_s \cdot \nabla) \hat{u} + (\hat{u} \cdot \nabla) u_s - A_1 \hat{\eta}_1 - A_2 \hat{\eta}_2 - \omega \hat{u} &= e^{-\omega t} \mathcal{F}_f[\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2] \text{ in } Q^\infty, \\ \operatorname{div} \hat{u} &= A_3 \hat{\eta}_1 + e^{-\omega t} \operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1] \text{ in } Q^\infty, \quad \hat{u} = \hat{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad \hat{u} = \hat{g}_p \text{ on } \Sigma_i^\infty, \\ \hat{u} \cdot n &= 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat{u}) n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat{u}, \hat{p}) n = 0 \text{ on } \Sigma_n^\infty, \\ \hat{\eta}_{1,t} - \hat{\eta}_2 - \omega \hat{\eta}_1 &= 0 \text{ on } \Sigma_s^\infty, \\ \hat{\eta}_{2,t} - \beta \Delta_s \hat{\eta}_1 - \gamma \Delta_s \hat{\eta}_2 + \alpha \Delta_s^2 \hat{\eta}_1 - A_4 \hat{\eta}_1 - \omega \hat{\eta}_2 &= \gamma_s \hat{p} + e^{-\omega t} \mathcal{F}_s[\hat{u}, \hat{\eta}_1] + \hat{f} \text{ on } \Sigma_s^\infty, \\ \hat{\eta}_1 &= 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \quad \hat{\eta}_{1,x} = 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \\ \hat{u}(0) &= \hat{u}^0 \text{ in } \Omega, \quad \hat{\eta}_1(0) = 0 \text{ on } \Gamma_s, \quad \hat{\eta}_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \tag{3.3}$$

where γ_s is the trace operator on Γ_s , the nonlinear terms \mathcal{F}_f and \mathcal{F}_s are given in Appendix A, $\mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1]$, and the linear differential operators A_1, A_2, A_3 and A_4 are defined by

$$\begin{aligned} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1] &= \frac{1}{\ell - e} (s(z) \hat{\eta}_1 \hat{u}_1 e_1 + (\ell - s(z)z) \hat{\eta}_{1,x} \hat{u}_1 e_2), \quad e_1 = (1, 0)^T, \quad e_2 = (0, 1)^T, \\ A_1 \hat{\eta}_1 &= \frac{s(z) \hat{\eta}_1}{\ell - e} (p_{s,z} e_2 + u_{s,2} u_{s,z} - 2\nu u_{s,zz} + \nu u_{s,1,xx} e_1 + \nu u_{s,1,xz} e_2) \\ &\quad + \frac{(\ell - s(z)z) \hat{\eta}_{1,x}}{\ell - e} (p_{s,z} e_1 - 2\nu u_{s,xz} + u_{s,1} u_{s,z} - \nu u_{s,1,zz} e_2 - \nu u_{s,1,xz} e_1) \\ &\quad + \frac{\nu \hat{\eta}_{1,x}}{\ell - e} (u_{s,1,z} e_2 - u_{s,1,x} e_1) - \frac{\nu(\ell - s(z)z) \hat{\eta}_{1,xx}}{\ell - e} (u_{s,z} + u_{s,1,x} e_1), \\ A_2 \hat{\eta}_2 &= \frac{(\ell - s(z)z) \hat{\eta}_2}{\ell - e} u_{s,z}, \quad A_3 \hat{\eta}_1 = \frac{1}{\ell - e} (s(z) \hat{\eta}_1 u_{s,1,x} + (\ell - s(z)z) \hat{\eta}_{1,x} u_{s,1,z}), \\ A_4 \hat{\eta}_1 &= \nu \left(\frac{s(z) \hat{\eta}_1}{\ell - e} u_{s,2,z} - \hat{\eta}_{1,x} u_{s,1,z} - 2\gamma_s(A_3 \hat{\eta}_1) \right). \end{aligned} \tag{3.4}$$

We notice that we have used the condition $\operatorname{div} \hat{u} = A_3 \hat{\eta}_1 + e^{-\omega t} \operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1]$, and the boundary condition $\hat{u}_1|_{\Sigma_s^\infty} = 0$ to replace $2\nu \hat{u}_{2,z}$, which appears in equation (3.3)₅, by

$$2\nu \gamma_s (A_3 \hat{\eta}_1 + e^{-\omega t} \operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1]).$$

The term $2\nu \gamma_s (A_3 \hat{\eta}_1)$ appears in $A_4 \hat{\eta}_1$, while the term $2\nu \gamma_s (\operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1])$ is involved in the nonlinear term \mathcal{F}_s of the equation.

In the nonlinear terms \mathcal{F}_f , \mathcal{F}_s and $\mathcal{F}_{\operatorname{div}}$, the functions $\hat{\eta}_1$ and $\hat{\eta}_2$, which are a priori defined over $[0, \ell_s]$, are extended by using the operator \mathcal{E} , see Remark 3.1. We can easily check that, if $\hat{u} \cdot n = 0$ and $\varepsilon(\hat{u})n \cdot \tau = 0$ on Σ_e^∞ , then $\mathcal{F}_{\operatorname{div}} = \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1]$ satisfies the following boundary conditions

$$\begin{aligned} \mathcal{F}_{\operatorname{div}} &= 0 \text{ on } \Sigma_s^\infty, & \mathcal{F}_{\operatorname{div}} &= 0 \text{ on } \Sigma_i^\infty, & \mathcal{F}_{\operatorname{div}} \cdot n &= 0 \text{ on } \Sigma_e^\infty, \\ \varepsilon(\mathcal{F}_{\operatorname{div}})n \cdot \tau &= 0 \text{ on } \Sigma_e^\infty, & \varepsilon(\mathcal{F}_{\operatorname{div}})n &= 0 \text{ on } \Sigma_n^\infty. \end{aligned}$$

Moreover $\mathcal{F}_{\operatorname{div}}|_{t=0} = 0$ since $\hat{\eta}_1|_{t=0} = 0$. Theorem 3.2 is a direct consequence of the following result.

THEOREM 3.3. *Let η_{\max} be in $(0, e)$ and ω be positive. We assume that Assumptions 1, 2 and 3 are satisfied. There exists $r > 0$ such that, for all $\hat{u}^0 \in \mathbf{V}_{\Gamma_{i,e}}^1(\Omega)$, $\eta_2^0 \in H_0^1(\Gamma_s)$, and $\hat{g}_p \in H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$ satisfying $\hat{u}^0|_{\Gamma_s} = \eta_2^0 n$ and*

$$\|\hat{u}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} + \|\hat{g}_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))} \leq r,$$

we can find a control $\hat{\mathbf{f}} \in H^1(0, \infty; \mathbb{R}^{n_c})$, in feedback form, for which the nonlinear system (3.3) admits a solution $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2) \in X_{\delta_0}$, where δ_0 is introduced in (4.4), obeying

$$\|e^{-\omega t} \hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \leq \eta_{\max} \quad \text{and} \quad \|(\hat{u}(t), \hat{\eta}(t), \hat{\eta}_t(t))\|_{\mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega) \times H^3(\Gamma_s) \times H^1(\Gamma_s)} \leq C, \quad \forall t > 0,$$

where $C > 0$ depends on r and $0 < \varepsilon_0 < \frac{1}{2}$.

REMARK 3.4. *The estimate $\|e^{-\omega t} \hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \leq \eta_{\max}$ guarantees that, for all $t > 0$, the change of variables $\mathcal{T}_{e^{-\omega t} \hat{\eta}(t)}$ is a \mathcal{C}^1 -diffeomorphism from $\Omega_{e^{-\omega t} \hat{\eta}(t)}$ into Ω .*

REMARK 3.5. *In Theorems 3.2 and 3.3, we assume that $\eta_1(0)$, the initial displacement of the beam, is zero, see (3.3). As in [24], the initial displacement can be considered to be non zero. In that case, the compatibility conditions are more complicated (see [24] for compatibility conditions stated when $\eta_1(0) \neq 0$).*

4. The Oseen system. The goal of this section is to study the non-homogeneous Oseen system

$$\begin{aligned} \lambda_0 w - \operatorname{div} \sigma(w, \pi) + (u_s \cdot \nabla)w + (w \cdot \nabla)u_s &= F, & \operatorname{div} w &= h \text{ in } \Omega, \\ w &= \eta n \text{ on } \Gamma_s, & w &= 0 \text{ on } \Gamma_i, & w \cdot n &= 0 \text{ and } \varepsilon(w)n \cdot \tau = 0 \text{ on } \Gamma_e, & \sigma(w, \pi)n &= 0 \text{ on } \Gamma_n, \end{aligned} \quad (4.1)$$

where $F \in \mathbf{L}^2(\Omega)$, $\eta \in H_0^{\frac{3}{2}}(\Gamma_s)$, $h \in H^1(\Omega)$. We choose $\lambda_0 > 0$ such that

$$\lambda_0 \int_{\Omega} |v|^2 + 2\nu \int_{\Omega} |\varepsilon(v)|^2 + \int_{\Omega} [(u_s \cdot \nabla)v + (v \cdot \nabla)u_s] \cdot v \geq \frac{\nu}{2} \|v\|_{\mathbf{H}^1(\Omega)}, \quad \forall v \in \mathbf{H}^1(\Omega). \quad (4.2)$$

We shall say that $(w, \pi) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ is a weak solution of (4.1) if and only if it satisfies the following mixed variational formulation

$$\begin{aligned} \text{Find } (w, \pi) &\in \mathbf{H}^1(\Omega) \times L^2(\Omega) \text{ such that} \\ a(w, \phi) - b(\phi, \pi) &= \int_{\Omega} F \cdot \phi \quad \text{for all } \phi \in \mathbf{H}_{\Gamma_0}^1(\Omega), & b(w, \psi) &= \int_{\Omega} h \psi \quad \text{for all } \psi \in L^2(\Omega), \\ w &= \eta n \text{ on } \Gamma_s, & w &= 0 \text{ on } \Gamma_i, & w \cdot n &= 0 \text{ on } \Gamma_e, \end{aligned} \quad (4.3)$$

where

$$a(w, \phi) = \int_{\Omega} (\lambda_0 w \phi + 2\nu \varepsilon(w) : \varepsilon(\phi)) dx + \int_{\Omega} ((u_s \cdot \nabla)w + (w \cdot \nabla)u_s) \phi dx, \quad b(w, \psi) = \int_{\Omega} \operatorname{div}(w) \psi dx.$$

In the case when $u_s = 0$, that is for the Stokes system, if $(F, h, \eta) \in \mathbf{L}^2(\Omega) \times H^1(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s)$, far from the set \mathcal{C} , that is in $\Omega \setminus \cup_{j=1,8} B(C_j, \varepsilon)$, the solution (w, π) to system (4.1) belongs to $H^2(\Omega \setminus \cup_{j=1,8} B(C_j, \varepsilon)) \times H^1(\Omega \setminus \cup_{j=1,8} B(C_j, \varepsilon))$ for all $\varepsilon > 0$ ($B(C_j, \varepsilon)$ is the ball centered at C_j of radius ε). In the neighborhood of \mathcal{C} , we can study the regularity of the solution to system (4.1) in weighted Sobolev spaces. According to [20, Theorem 9.4.5], there exists $\delta_0 < 1/2$ such that, if $(F, h, \eta) \in \mathbf{L}^2(\Omega) \times H^1(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s)$ and $u_s = 0$, the solution (w, π) to system (4.1) belongs to $\mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$. In particular $\delta_0 < 1/2$ can be chosen so that

$$\frac{1}{3} < \delta_0 < \frac{1}{2}. \quad (4.4)$$

The regularity result for the Oseen system is stated in the following theorem.

THEOREM 4.1. *For all $(F, h, \eta) \in \mathbf{L}^2(\Omega) \times H^1(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s)$, system (4.1) admits a unique solution $(w, \pi) \in \mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ satisfying the estimate*

$$\|w\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|\pi\|_{H_{\delta_0}^1(\Omega)} \leq C (\|F\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma_s)}).$$

Proof. As mentioned above, if $u_s = 0$, the existence of a unique solution to system (4.1) belonging to $\mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ is already proved in [20, Theorem 9.4.5]. However, we have to notice that the variational formulation (4.3) is different from that in [20, (9.1.7)-(9.1.9)]. It is different because equation (4.1)₁ is written in terms of $-\operatorname{div} \sigma(w, \pi)$, while in [20] equation (9.1.1) is written in terms of $-\nu \Delta w + \nabla p$. But the results stated in [20, Theorem 9.4.5] can be applied here to the Oseen system stated in (4.1).

The adaptation from the Stokes system to the Oseen system can be performed as in [21, Proof of Theorem 2.17, Step 2]. \square

REMARK 4.2. *If in system (4.1) the boundary condition $w = 0$ on Γ_i is replaced by $w = g_p$ on Γ_i , we can prove that, for $(F, h, \eta, g_p) \in \mathbf{L}^2(\Omega) \times H^1(\Omega) \times H_0^{\frac{3}{2}}(\Gamma_s) \times \mathbf{H}(\Gamma_i)$, the solution to system (4.1) satisfies*

$$\|w\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|\pi\|_{H_{\delta_0}^1(\Omega)} \leq C (\|F\|_{\mathbf{L}^2(\Omega)} + \|h\|_{H^1(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma_s)} + \|g_p\|_{\mathbf{H}(\Gamma_i)}).$$

LEMMA 4.3. *We set $\varepsilon_0 = \frac{1}{2} - \delta_0$. Then $\varepsilon_0 > 0$, $\mathbf{H}_{\delta_0}^2(\Omega) \subset \mathbf{H}^{\frac{3}{2} + \varepsilon_0}(\Omega)$ and $H_{\delta_0}^1(\Omega) \subset H^{\frac{1}{2} + \varepsilon_0}(\Omega)$, with continuous embeddings.*

Proof. It follows from [20, Lemma 6.2.1]. \square

4.1. The Leray projector. From [21, Lemma 2.2], we know that $\mathbf{L}^2(\Omega) = \mathbf{V}_{n, \Gamma_0}^0(\Omega) \oplus \nabla H_{\Gamma_n}^1(\Omega)$, where $H_{\Gamma_n}^1(\Omega) = \{p \in H^1(\Omega) \mid p|_{\Gamma_n} = 0\}$. We introduce the orthogonal projection P from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}_{n, \Gamma_0}^0(\Omega)$, called the Leray projector. For all v in $\mathbf{L}^2(\Omega)$, we have

$$Pv = v - \nabla q_v^1 - \nabla q_v^2,$$

where q_v^1 and q_v^2 are the solutions to the equations

$$\begin{aligned} q_v^1 &\in H_0^1(\Omega), \quad \Delta q_v^1 = \operatorname{div} v \text{ in } \Omega, \quad q_v^1 = 0 \text{ on } \partial\Omega, \\ q_v^2 &\in H^1(\Omega), \quad \Delta q_v^2 = 0 \text{ in } \Omega, \quad \frac{\partial q_v^2}{\partial n} = (v - \nabla q_v^1) \cdot n \text{ on } \Gamma_0, \quad q_v^2 = 0 \text{ on } \Gamma_n. \end{aligned} \quad (4.5)$$

PROPOSITION 4.4. *If v belongs to $\mathbf{H}^1(\Omega)$, then Pv and $(I - P)v$ belong to $\mathbf{H}^{\frac{1}{2} + \varepsilon_0}(\Omega)$. Moreover, we have $(I - P)v = \nabla q$, where q is the solution to the equation*

$$q \in H^1(\Omega), \quad \Delta q = \operatorname{div} v \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = v \cdot n \text{ on } \Gamma_0, \quad q = 0 \text{ on } \Gamma_n. \quad (4.6)$$

Proof. We have to prove that the solution q to equation (4.6) belongs to $H^{\frac{3}{2} + \varepsilon_0}(\Omega)$. Since v belongs to $\mathbf{H}^1(\Omega)$, we know that $v \cdot n|_{\Gamma_0}$ belongs to $H^{\frac{1}{2}}(\Gamma_0)$. Thus, far from the corners of the domain Ω , the solution q is of class H^2 . We have to analyze the regularity at the corners corresponding either to a

junction between two Neumann boundary conditions or to a junction between a Neumann and Dirichlet boundary condition. At the two corners corresponding to a Neumann-Dirichlet junction, the angle is equal to $\frac{\pi}{2}$, the condition on Γ_n is $q = 0$ and the condition on Γ_0 is $\frac{\partial q}{\partial n} \in H^{\frac{1}{2}}(\Gamma_0)$. Using an odd symmetry with respect to Γ_0 , we have to deal with a pure Neumann boundary condition on a flat boundary with the extension of $\frac{\partial q}{\partial n}$ belonging to $H^{\frac{1}{2}-\varepsilon}$ for all $\varepsilon > 0$, on that flat boundary. Thus q is of class $H^{2-\varepsilon}$, for all $\varepsilon > 0$, close to a Neumann-Dirichlet junction.

We may use the results in [12] to analyze the regularity at the corners corresponding to a Neumann-Neumann junction. From [12, Corollary 2.4.4, Remark 2.4.6, and Theorem 2.3.7 (iii)], it follows that $q \in H^s(\Omega)$ for all $s < 1 + \frac{2}{3}$. In particular $q \in H^{\frac{3}{2}+\varepsilon_0}(\Omega)$. This completes the proof. \square

4.2. Expression of the pressure. The goal of this section is to express the pressure π in equation (4.1) in terms of w , F , η and h . The method used in [24] consists in calculating the divergence of equation (4.1)₁ in order to get an elliptic equation for π . This method does not work here. Indeed, formally π is the solution to the equation

$$\begin{aligned} \Delta \pi &= -\lambda_0 h - \operatorname{div}((u_s \cdot \nabla)w + (w \cdot \nabla)u_s) + \operatorname{div} F \text{ in } \Omega, & \pi &= 2\nu \varepsilon(w)n \cdot n \text{ on } \Gamma_n, \\ \frac{\partial \pi}{\partial n} &= 2\nu \operatorname{div} \varepsilon(w) \cdot n - \lambda_0 w \cdot n - ((u_s \cdot \nabla)w + (w \cdot \nabla)u_s) \cdot n + F \cdot n \text{ on } \Gamma_0. \end{aligned} \quad (4.7)$$

Since $\operatorname{div} \varepsilon(w) \notin \mathbf{L}^2(\Omega)$, the boundary condition (4.7)₂ is not well-posed. Thus, we cannot use the classical transposition method because the Green formula needed for that is not valid (see [12, Theorem 1.5.3]). We are going to use a variant of the classical transposition method to define an equation for the pressure π . First, we consider the equation

$$\Delta \chi = \zeta \text{ in } \Omega, \quad \frac{\partial \chi}{\partial n} = 0 \text{ on } \Gamma_0, \quad \chi = 0 \text{ on } \Gamma_n. \quad (4.8)$$

LEMMA 4.5. *For all $\zeta \in L^2(\Omega)$, equation (4.8) admits a unique solution χ belonging to $H_\delta^2(\Omega)$, for all $\frac{1}{3} < \delta < 1$, and satisfying*

$$\|\chi\|_{H_\delta^2(\Omega)} \leq C_\delta \|\zeta\|_{L^2(\Omega)},$$

where C_δ depends on δ . In particular, this estimate is valid for $\delta = \delta_0$.

Proof. It follows from [20, Theorem 6.5.4], [20, Lemma 6.2.1] and [15, Chapter 2]. \square

Let us notice that

$$H_\delta^1(\Omega) \hookrightarrow L_{-\delta}^2(\Omega) \quad \text{for all } 0 < \delta < \frac{1}{2}. \quad (4.9)$$

Indeed, from [19, page 399], it follows that $H_\delta^1(\mathcal{D}) \hookrightarrow H_{-\delta}^\varepsilon(\mathcal{D})$ for a dihedral domain \mathcal{D} , when $1 - \delta = \varepsilon + \delta$ and $\varepsilon > 0$. Thus we have $H_\delta^1(\Omega) \hookrightarrow H_{-\delta}^\varepsilon(\Omega)$ for all $0 < \delta < \frac{1}{2}$ and $\varepsilon = 1 - 2\delta$. That proves (4.9), since $H_{-\delta}^\varepsilon(\Omega) \hookrightarrow L_{-\delta}^2(\Omega)$.

Using (4.9), we can define $k \in \mathcal{L}(L^2(\Omega), \mathbb{R})$ by

$$\begin{aligned} k(\zeta) &= \lambda_0 \int_{\Gamma_s} \eta \chi - \lambda_0 \int_{\Omega} h \chi - \int_{\Omega} F \cdot \nabla \chi + 2\nu \langle \varepsilon(w), \nabla^2 \chi \rangle_{\mathbf{L}_{-\delta_0}^2(\Omega), \mathbf{L}_{\delta_0}^2(\Omega)} \\ &\quad - 2\nu \int_{\Gamma_0} \varepsilon(w)n \cdot \nabla \chi + \int_{\Omega} [(u_s \cdot \nabla)w + (w \cdot \nabla)u_s] \cdot \nabla \chi, \quad \text{for all } \zeta \in L^2(\Omega), \end{aligned} \quad (4.10)$$

where $\chi \in H_{\delta_0}^2(\Omega)$ is the solution to equation (4.8).

LEMMA 4.6. *Let (w, F, η, h) belong to $\mathbf{H}_{\delta_0}^2(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Gamma_s) \times H^1(\Omega)$. The operator k belongs to $\mathcal{L}(L^2(\Omega), \mathbb{R})$ and we have*

$$|k(\zeta)| \leq C \left(\|\eta\|_{L^2(\Gamma_s)} + \|h\|_{L^2(\Omega)} + \|F\|_{\mathbf{L}^2(\Omega)} + \|w\|_{\mathbf{H}_{\delta_0}^2(\Omega)} \right) \|\zeta\|_{L^2(\Omega)} \quad \text{for all } \zeta \in L^2(\Omega).$$

Proof. The lemma follows from the following estimates

$$\begin{aligned}
\left| \int_{\Gamma_s} \eta \chi \right| &\leq C \|\eta\|_{L^2(\Gamma_s)} \|\chi\|_{L^2(\Gamma_s)} \leq C \|\eta\|_{L^2(\Gamma_s)} \|\zeta\|_{L^2(\Omega)}, \\
\left| \int_{\Omega} h \chi \right| &\leq C \|h\|_{L^2(\Omega)} \|\chi\|_{L^2(\Omega)} \leq C \|h\|_{L^2(\Omega)} \|\zeta\|_{L^2(\Omega)}, \\
\left| \int_{\Omega} F \cdot \nabla \chi \right| &\leq C \|F\|_{\mathbf{L}^2(\Omega)} \|\nabla \chi\|_{L^2(\Omega)} \leq C \|F\|_{\mathbf{L}^2(\Omega)} \|\zeta\|_{L^2(\Omega)}, \\
\left| \int_{\Gamma_0} \varepsilon(w) n \cdot \nabla \chi \right| &\leq C \|\varepsilon(w)\|_{\mathbf{L}^2(\Gamma_0)} \|\nabla \chi\|_{L^2(\Gamma_0)} \leq C \|w\|_{\mathbf{H}^{\frac{3}{2}+\varepsilon_0}(\Omega)} \|\zeta\|_{L^2(\Omega)}, \\
\left| \int_{\Omega} \left[(u_s \cdot \nabla) w + (w \cdot \nabla) u_s \right] \cdot \nabla \chi \right| &\leq C \|w\|_{\mathbf{H}^1(\Omega)} \|\nabla \chi\|_{L^2(\Omega)} \leq C \|w\|_{\mathbf{H}^1(\Omega)} \|\zeta\|_{L^2(\Omega)}, \\
\left| \langle \varepsilon(w), \nabla^2 \chi \rangle_{\mathbf{L}^2_{-\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} \right| &\leq C \|\varepsilon(w)\|_{\mathbf{L}^2_{-\delta_0}(\Omega)} \|\nabla^2 \chi\|_{\mathbf{L}^2_{\delta_0}(\Omega)} \leq C \|w\|_{\mathbf{H}^{1+\delta_0}(\Omega)} \|\zeta\|_{H^2_{\delta_0}(\Omega)}.
\end{aligned}$$

The last inequality comes from [20, Lemma 6.2.1]. \square

We introduce the following operators:

- $N_s \in \mathcal{L}(L^2(\Gamma_s), H^1(\Omega))$ is defined by $N_s \eta = q$ where q is the solution to the equation

$$\Delta q = 0 \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = \eta \text{ on } \Gamma_s, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0 \setminus \Gamma_s, \quad q = 0 \text{ on } \Gamma_n. \quad (4.11)$$

- $N_{\text{div}} \in \mathcal{L}(L^2(\Omega), H^1(\Omega))$ is defined by $N_{\text{div}} h = q$ where q is the solution to the equation

$$\Delta q = h \text{ in } \Omega, \quad \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_0, \quad q = 0 \text{ on } \Gamma_n. \quad (4.12)$$

- $N_p \in \mathcal{L}(\mathbf{L}^2(\Omega), H^1(\Omega))$ is defined by $N_p F = q_F^1 + q_F^2$ where q_F^1 and q_F^2 are the solutions to equations (4.5)₁ and (4.5)₂ respectively, with $v = F$.

- $N_v \in \mathcal{L}(\mathbf{H}^2_{\delta_0}(\Omega), L^2(\Omega))$ is defined by $N_v w = q$ where q is the solution to the variational problem

Find $q \in L^2(\Omega)$ such that

$$\begin{aligned}
\int_{\Omega} q \zeta &= 2\nu \langle \varepsilon(w), \nabla^2 \chi \rangle_{\mathbf{L}^2_{-\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} - 2\nu \int_{\Gamma_0} \varepsilon(w) n \cdot \nabla \chi + \int_{\Omega} \left[(u_s \cdot \nabla) w + (w \cdot \nabla) u_s \right] \cdot \nabla \chi, \\
\text{for all } \zeta \in L^2(\Omega), \quad \text{where } \chi \in H^2_{\delta_0}(\Omega) &\text{ is the solution to equation (4.8).}
\end{aligned}$$

Thanks to the Lax-Milgram Theorem, we can easily prove that the operators N_s , N_{div} , N_p and N_v are well-defined.

REMARK 4.7. We notice that $\nabla N_p = I - P$.

THEOREM 4.8. If $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is a solution to system (4.1), where δ_0 obeys (4.4), then π is the unique solution to the problem

$$\text{Find } \pi \in L^2(\Omega) \text{ such that } \int_{\Omega} \pi \zeta dx = k(\zeta), \quad \forall \zeta \in L^2(\Omega). \quad (4.13)$$

As a consequence, π is determined by

$$\pi = -\lambda_0 N_s \eta - \lambda_0 N_{\text{div}} h + N_p F + N_v w.$$

Proof. Let $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ be the solution to system (4.1) and let ζ belong to $L^2(\Omega)$. Multiplying the first equation of the system by $\nabla \chi$, where χ is the solution to equation (4.8), we obtain

$$\lambda_0 \int_{\Omega} w \cdot \nabla \chi - \int_{\Omega} \text{div } \sigma(w, \pi) \cdot \nabla \chi + \int_{\Omega} \left[(u_s \cdot \nabla) w + (w \cdot \nabla) u_s \right] \cdot \nabla \chi - \int_{\Omega} F \cdot \nabla \chi = 0. \quad (4.14)$$

By using integration by parts, we get

$$\lambda_0 \int_{\Omega} w \cdot \nabla \chi = -\lambda_0 \int_{\Omega} \text{div } w \cdot \chi + \int_{\partial \Omega} w \cdot n \chi = -\lambda_0 \int_{\Omega} h \chi + \lambda_0 \int_{\Gamma_s} \eta \chi, \quad (4.15)$$

and

$$\begin{aligned}
\int_{\Omega} \operatorname{div} \sigma(w, \pi) \cdot \nabla \chi &= -\langle \sigma(w, \pi), \nabla^2 \chi \rangle_{\mathbf{L}^2_{-\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} + \int_{\partial\Omega} \sigma(w, \pi) n \cdot \nabla \chi \\
&= -2\nu \langle \varepsilon(w), \nabla^2 \chi \rangle_{\mathbf{L}^2_{-\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} + \int_{\Omega} \pi \cdot \zeta + \int_{\Gamma_0} \sigma(w, \pi) n \cdot \nabla \chi.
\end{aligned} \tag{4.16}$$

Let us notice that (4.16)₁ can be first obtained for $\chi \in C^\infty(\overline{\Omega})$ and next by density for $\chi \in \mathbf{H}^2_{\delta_0}(\Omega)$. From equations (4.14)-(4.16), it follows that π is a solution of equation (4.13). Thanks to Lemma 4.6 and to the Lax-Milgram Lemma in $L^2(\Omega)$, we prove that (4.13) admits a unique solution.

To prove the last part of the theorem, it is sufficient to verify that

$$\begin{aligned}
\int_{\Gamma_s} \eta \chi &= \int_{\Gamma_s} \frac{\partial N_s \eta}{\partial n} \chi = - \int_{\Omega} \zeta N_s \eta, \\
\int_{\Omega} h \chi &= \int_{\Omega} \Delta N_{\operatorname{div} h} \chi = \int_{\Omega} \zeta N_{\operatorname{div} h} \quad \text{and} \quad \int_{\Omega} F \cdot \nabla \chi = \int_{\Omega} \zeta N_p F.
\end{aligned} \tag{4.17}$$

□

LEMMA 4.9. *Let $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ be the solution of system (4.1), where δ_0 obeys (4.4). Then $N_v w$ belongs to $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$.*

Proof. We start from the identity

$$\pi = -\lambda_0 N_s \eta - \lambda_0 N_{\operatorname{div} h} + N_p F + N_v w.$$

We know that $N_p F$, $N_{\operatorname{div} h}$ and $N_s \eta$ belong to $H^1(\Omega)$ and that π belongs to $H^1_{\delta_0}(\Omega) \subset H^{\frac{1}{2}+\varepsilon_0}(\Omega)$ (see Lemma 4.3). Hence $N_v w$ belongs to $H^{\frac{1}{2}+\varepsilon_0}(\Omega)$. □

4.3. The Oseen operator. We introduce the Oseen operator $(A, D(A))$ in $\mathbf{V}^0_{n, \Gamma_0}(\Omega)$ defined by

$$\begin{aligned}
D(A) &= \left\{ v \in \mathbf{V}^{\frac{3}{2}+\varepsilon_0}_{n, \Gamma_0}(\Omega) \mid \exists q \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ such that } \operatorname{div} \sigma(v, q) \in \mathbf{L}^2(\Omega), \right. \\
&\quad \left. \varepsilon(v) n \cdot \tau = 0 \text{ on } \Gamma_e, \sigma(v, q) n = 0 \text{ on } \Gamma_n \right\} \quad \text{and} \\
Av &= P \operatorname{div} \sigma(v, q) - P[(u_s \cdot \nabla)v + (v \cdot \nabla)u_s].
\end{aligned} \tag{4.18}$$

THEOREM 4.10. *The operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on $\mathbf{V}^0_{n, \Gamma_0}(\Omega)$ and its resolvent is compact.*

Proof. The proof is similar to that of [21, Theorem 2.8]. □

PROPOSITION 4.11. *The adjoint of $(A, D(A))$ in $\mathbf{V}^0_{n, \Gamma_0}(\Omega)$ is defined by*

$$\begin{aligned}
D(A^*) &= \left\{ v \in \mathbf{V}^{\frac{3}{2}+\varepsilon_0}_{n, \Gamma_0}(\Omega) \mid \exists q \in H^{\frac{1}{2}+\varepsilon_0}(\Omega) \text{ such that } \operatorname{div} \sigma(v, q) \in \mathbf{L}^2(\Omega), \right. \\
&\quad \left. \varepsilon(v) n \cdot \tau = 0 \text{ on } \Gamma_e, \sigma(v, q) n + u_s \cdot n v = 0 \text{ on } \Gamma_n \right\} \quad \text{and} \\
A^* v &= P \operatorname{div} \sigma(v, p) + P(u_s \cdot \nabla)v - P(\nabla u_s)^T v.
\end{aligned}$$

Proof. See [21, Theorem 2.11]. □

Finally, we introduce the lifting operators $L \in \mathcal{L}(H^{\frac{3}{2}}_0(\Gamma_s) \times H^1(\Omega), \mathbf{H}^2_{\delta_0}(\Omega))$ and $L_p \in \mathcal{L}(H^{\frac{3}{2}}_0(\Gamma_s) \times H^1(\Omega), H^1_{\delta_0}(\Omega))$ defined by

$$L(\eta, h) = w \quad \text{and} \quad L_p(\eta, h) = \pi, \tag{4.19}$$

where (w, π) is the solution to system (4.1) for $F = 0$.

THEOREM 4.12. *A pair $(w, \pi) \in \mathbf{H}^2_{\delta_0}(\Omega) \times H^1_{\delta_0}(\Omega)$ is solution to system (4.1) if and only if*

$$\begin{aligned}
(\lambda_0 I - A)Pw + (A - \lambda_0 I)PL(\eta, h) &= PF, \quad (I - P)w = \nabla N_s \eta + \nabla N_{\operatorname{div} h}, \\
\pi &= -\lambda_0 N_s \eta - \lambda_0 N_{\operatorname{div} h} + N_p F + N_v w.
\end{aligned} \tag{4.20}$$

Proof. Let $(w, \pi) \in \mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ be a solution of system (4.1). We set

$$\widehat{w} = w - L(\eta, h) \quad \text{and} \quad \widehat{\pi} = \pi - L_p(\eta, h).$$

The couple $(\widehat{w}, \widehat{\pi})$ is solution to system (4.1) with $(\eta, h) = (0, 0)$. Thus, \widehat{w} belongs to $D(A)$ and $\lambda_0 P\widehat{w} - AP\widehat{w} = PF$. Hence $(\lambda_0 I - A)Pw + (A - \lambda_0 I)PL(\eta, h) = PF$. The last two equations in (4.20) come from Proposition 4.4 and Theorem 4.8.

Now, we suppose that $(w, \pi) \in \mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ is solution to system (4.20). Since $(I - P)w = \nabla N_s \eta + \nabla N_{\text{div}} h = (I - P)L(\eta, h)$, we have

$$\widehat{w} = w - L(\eta, h) \in D(A). \quad (4.21)$$

Thus, there exists $\pi_1 \in H^{\frac{1}{2}+\varepsilon_0}(\Omega)$ such that

$$A\widehat{w} = P \operatorname{div} \sigma(\widehat{w}, \pi_1) - P(u_s \cdot \nabla)\widehat{w} - P(\widehat{w} \cdot \nabla)u_s \quad \text{and} \quad \sigma(\widehat{w}, \pi_1)n = 0 \text{ on } \Gamma_n. \quad (4.22)$$

From the first equation of system (4.20), it follows that

$$P[\lambda_0 \widehat{w} - \operatorname{div} \sigma(\widehat{w}, \pi_1) + (u_s \cdot \nabla)\widehat{w} + (\widehat{w} \cdot \nabla)u_s - F] = 0.$$

Thus, there exists $\pi_2 \in H_{\Gamma_n}^1(\Omega)$ such that

$$\lambda_0 \widehat{w} - \operatorname{div} \sigma(\widehat{w}, \pi_1 + \pi_2) + (u_s \cdot \nabla)\widehat{w} + (\widehat{w} \cdot \nabla)u_s = F. \quad (4.23)$$

With equations (4.21), (4.22)₂ and (4.23), we obtain the system

$$\begin{aligned} \lambda_0 \widehat{w} - \operatorname{div} \sigma(\widehat{w}, \pi_1 + \pi_2) + (u_s \cdot \nabla)\widehat{w} + (\widehat{w} \cdot \nabla)u_s &= F, \quad \operatorname{div} \widehat{w} = 0 \text{ in } \Omega, \\ \widehat{w} &= 0 \text{ on } \Gamma_s, \quad \widehat{w} = 0 \text{ on } \Gamma_i, \quad \widehat{w} \cdot n = 0 \text{ and } \varepsilon(\widehat{w})n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\widehat{w}, \pi_1 + \pi_2)n = 0 \text{ on } \Gamma_n. \end{aligned}$$

We deduce that (w, π) , with

$$\pi = \pi_1 + \pi_2 + L_p(\eta, h) = -\lambda_0 N_s \eta - \lambda_0 N_{\text{div}} h + N_p F + N_v w,$$

is the solution to system (4.1). \square

5. Reformulation of the linearized system. For all (F_f, F_s^1, F_s^2) belonging to $H = \mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s)$, we consider the system

$$\begin{aligned} \lambda v - \operatorname{div} \sigma(v, q) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1 \eta_1 - A_2 \eta_2 &= F_f, \quad \operatorname{div} v = A_3 \eta_1 \text{ in } \Omega, \\ v &= \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \quad v \cdot n = 0 \text{ and } \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v, q)n = 0 \text{ on } \Gamma_n, \\ \lambda \eta_1 - \eta_2 &= F_s^1 \text{ on } \Gamma_s, \\ \lambda \eta_2 - \beta \Delta_s \eta_1 - \gamma \Delta_s \eta_2 + \alpha \Delta_s^2 \eta_1 - A_4 \eta_1 &= \gamma_s q + F_s^2 \text{ on } \Gamma_s, \\ \eta_1 &= 0 \text{ on } \partial \Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial \Gamma_s. \end{aligned} \quad (5.1)$$

5.1. Properties of the operators A_1, A_2, A_3 and A_4 .

PROPOSITION 5.1. *The differential operators A_1, A_2, A_3 and A_4 obey*

$$\begin{aligned} A_1 &\in \mathcal{L}(H_0^2(\Gamma_s), \mathbf{L}^2(\Omega)), \quad A_2 \in \mathcal{L}(L^2(\Gamma_s), \mathbf{L}^2(\Omega)), \\ A_3 &\in \mathcal{L}(H_0^2(\Gamma_s), H^1(\Omega)) \quad \text{and} \quad A_4 \in \mathcal{L}(H_0^2(\Gamma_s), H^{\varepsilon_0}(\Gamma_s)). \end{aligned}$$

Proof. For A_1, A_2 and A_3 , the proposition follows from Lemmas 5.2 and 5.3 below, and for A_4 , it follows from [13, Proposition B.1]. \square

LEMMA 5.2. *There exists $C > 0$ such that, for all (η, w) belonging to $L^2(\Gamma_s) \times \mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)$, ηw belongs to $\mathbf{L}^2(\Omega)$ and*

$$\|\eta w\|_{\mathbf{L}^2(\Omega)} \leq C \|\eta\|_{L^2(\Gamma_s)} \|w\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)}.$$

Proof. Thanks to the continuous embeddings $\mathbf{H}^{\frac{1}{2}+\varepsilon_0}((\pm e, \pm \ell) \times (0, \ell_s)) \hookrightarrow L^2(\pm e, \pm \ell; L^\infty(0, \ell_s))$, and using the continuity of the extension operator \mathcal{E} (see Remark 3.1), it follows that

$$\begin{aligned} \int_{\Omega} |\eta w|^2 &\leq C \int_{-\ell}^{-e} \int_0^{\ell_s} |\eta(x, -e)w(x, z)|^2 dx dz + C \int_e^\ell \int_0^{\ell_s} |\eta(x, e)w(x, z)|^2 dx dz \\ &\leq C \|\eta\|_{L^2(\Gamma_s)}^2 \|w\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)}^2. \end{aligned}$$

□

LEMMA 5.3. *There exists $C > 0$ such that, for all (η, w) belonging to $H_0^1(\Gamma_s) \times \mathbf{L}_{\delta_0}^2(\Omega)$, ηw belongs to $\mathbf{L}^2(\Omega)$ and*

$$\|\eta w\|_{\mathbf{L}^2(\Omega)} \leq C \|\eta\|_{H_0^1(\Gamma_s)} \|w\|_{\mathbf{L}_{\delta_0}^2(\Omega)}.$$

Proof. Let us denote by $(C_j^\pm)_{1 \leq j \leq 2}$, the four corners corresponding to the boundary of Γ_s , and by (x_j^\pm, z_j^\pm) the coordinates of C_j . Here the exponent \pm refers to the upper and lower beams. Since η^\pm belongs to $H_0^1(\Gamma_s)$, using an appropriate corner, we have $|\eta^\pm(x)| = \left| \int_{x_j^\pm}^x \eta_x(s) ds \right| \leq \|\eta_x\|_{L^2(\Gamma_s)} |x - x_j^\pm|^{\frac{1}{2}}$. Hence

$$\begin{aligned} \int_{\Omega} |\eta^\pm w|^2 &\leq \|\eta_x\|_{L^2(\Gamma_s)}^2 \int_{\Omega} \frac{|x - x_j^\pm|}{(|x - x_j^\pm|^2 + |z - z_j^\pm|^2)^{2\delta_0}} (|x - x_j^\pm|^2 + |z - z_j^\pm|^2)^{\delta_0} |w(x, z)|^2 dx dz \\ &\leq \|\eta_x\|_{L^2(\Gamma_s)}^2 \int_{\Omega} (|x - x_j^\pm|^2 + |z - z_j^\pm|^2)^{1-2\delta_0} (|x - x_j^\pm|^2 + |z - z_j^\pm|^2)^{\delta_0} |w(x, z)|^2 dx dz. \end{aligned}$$

Since $0 < \delta_0 < \frac{1}{2}$, it follows that $\|\eta w\|_{\mathbf{L}^2(\Omega)} \leq C \|\eta\|_{H_0^1(\Gamma_s)} \|w\|_{\mathbf{L}_{\delta_0}^2(\Omega)}$. □

We denote by $A_1^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$, $A_2^* \in \mathcal{L}(\mathbf{L}^2(\Omega), L^2(\Gamma_s))$, $A_3^* \in \mathcal{L}(\mathbf{L}^2(\Omega), H^{-2}(\Gamma_s))$ and $A_4^* \in \mathcal{L}(L^2(\Gamma_s), H^{-2}(\Gamma_s))$ the adjoints of A_1, A_2, A_3 and A_4 respectively, where $H^{-2}(\Gamma_s)$ is the dual of $H_0^2(\Gamma_s)$.

5.2. Definition of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$. For the beam, we introduce the unbounded operator $(A_{\alpha, \beta}, D(A_{\alpha, \beta}))$ in $H_0^2(\Gamma_s)$ defined by

$$D(A_{\alpha, \beta}) = H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \quad \text{and} \quad A_{\alpha, \beta} = \beta \Delta_s - \alpha \Delta_s^2.$$

We equip the space $H_0^2(\Gamma_s)$ with the norm

$$(\eta, \xi)_{H_0^2(\Gamma_s)} = ((-A_{\alpha, \beta})^{\frac{1}{2}} \eta, (-A_{\alpha, \beta})^{\frac{1}{2}} \xi)_{L^2(\Gamma_s)} = \int_{\Gamma_s} (\beta \nabla_s \eta \cdot \nabla_s \xi + \alpha \Delta_s \eta \Delta_s \xi),$$

and the spaces $H = \mathbf{L}^2(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ and $Z = \mathbf{V}_{n, \Gamma_0}^0(\Omega) \times H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ with the inner product

$$((v, \eta_1, \eta_2), (\phi, \xi_1, \xi_2))_H = (v, \phi)_{\mathbf{L}^2(\Omega)} + (\eta_1, \xi_1)_{H_0^2(\Gamma_s)} + (\eta_2, \xi_2)_{L^2(\Gamma_s)}. \quad (5.2)$$

As in [24, Lemma 3.2], we can show that $I + \gamma_s N_s$ is an automorphism in $L^2(\Gamma_s)$, and we can introduce $M_s \in \mathcal{L}(Z)$ and $M_s^{-1} \in \mathcal{L}(Z)$ two automorphisms in Z defined by

$$M_s = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma_s N_{\text{div} A_3} & I + \gamma_s N_s \end{pmatrix} \quad \text{and} \quad M_s^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -(I + \gamma_s N_s)^{-1} \gamma_s N_{\text{div} A_3} & (I + \gamma_s N_s)^{-1} \end{pmatrix}.$$

REMARK 5.4. *We notice that, in M_s , $\gamma_s N_s$ corresponds to the so-called added mass effect (see [5]). Here, the added mass operator M_s is not symmetric.*

We are going to define the infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ in Z of the linearized system around $(0, 0, 0, 0)$ associated with (3.3). For that we first introduce

$$D(\mathcal{A}) := \left\{ (Pv, \eta_1, \eta_2) \in \mathbf{V}_{n, \Gamma_0}^{\frac{1}{2} + \varepsilon_0}(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \mid Pv - PL(\eta_2, A_3\eta_1) \in D(\mathcal{A}) \right\}.$$

LEMMA 5.5. *If (Pv, η_1, η_2) belongs to $D(\mathcal{A})$, then $N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1)$ belongs to $H^{\frac{1}{2} + \varepsilon_0}(\Omega)$.*

Proof. Let (Pv, η_1, η_2) belong to $D(\mathcal{A})$. Then, there exists $F \in \mathbf{V}_{n, \Gamma_0}^0(\Omega)$ such that

$$(\lambda_0 I - A)Pv + (A - \lambda_0 I)PL(\eta_2, A_3\eta_1) = F.$$

We set $v = Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1$ and $\rho = -\lambda_0 N_s \eta_2 - \lambda_0 N_{\text{div}} A_3 \eta_1 + N_v v$. Then, according to Theorem 4.12, we know that (v, ρ) is the unique solution of the equation

$$\begin{aligned} \lambda_0 v - \text{div } \sigma(v, \rho) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s &= F, \quad \text{div } v = A_3 \eta_1 \text{ in } \Omega, \\ v &= \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \quad v \cdot n = 0 \text{ and } \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v, \rho)n = 0 \text{ on } \Gamma_n. \end{aligned} \quad (5.3)$$

According to Lemma 4.9, we conclude that $N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1)$ belongs to $H^{\frac{1}{2} + \varepsilon_0}(\Omega)$. \square

We are now in position to introduce the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in Z where $D(\mathcal{A})$ is defined above and

$$\mathcal{A} = M_s^{-1} \begin{pmatrix} A & PA_1 + (\lambda_0 I - A)PL(0, A_3 \cdot) & PA_2 + (\lambda_0 I - A)PL(\cdot, 0) \\ 0 & 0 & I \\ 0 & A_{\alpha, \beta} + \gamma_s N_p A_1 + A_4 & \gamma \Delta + \gamma_s N_p A_2 \end{pmatrix} + M_s^{-1} \mathcal{A}_p, \quad (5.4)$$

where $(\mathcal{A}_p, D(\mathcal{A}_p))$ is the unbounded operator in Z defined by $D(\mathcal{A}_p) = D(\mathcal{A})$ and

$$\mathcal{A}_p(Pv, \eta_1, \eta_2)^T = (0, 0, \gamma_s N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1))^T, \quad \forall (Pv, \eta_1, \eta_2) \in D(\mathcal{A}). \quad (5.5)$$

5.3. Resolvent of \mathcal{A} in Z . In what follows, we are going to consider system (5.1) either when $\lambda \in \mathbb{C}$ and (F_f, F_s^1, F_s^2) are complex valued functions (in which case the solutions are also complex valued functions), or when $\lambda \in \mathbb{R}$ and (F_f, F_s^1, F_s^2) are real valued functions (in which case the solutions are also real valued functions). The context will clearly indicate in which case we shall be.

The goal is to prove that the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ has a compact resolvent.

PROPOSITION 5.6. *Let (F_f, F_s^1, F_s^2) belong to H and λ belong to \mathbb{C} . A quadruplet $(v, q, \eta_1, \eta_2) \in H_{\delta_0}$ is solution to system (5.1) if and only if*

$$\begin{aligned} \lambda(Pv, \eta_1, \eta_2)^T &= \mathcal{A}(Pv, \eta_1, \eta_2)^T + M_s^{-1}(PF_f, F_s^1, F_s^2 + \gamma_s N_p F_f)^T, \\ (I - P)v &= \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1, \\ q &= -\lambda N_s \eta_2 - \lambda N_{\text{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1) + N_p F_f. \end{aligned} \quad (5.6)$$

Proof. It follows from Theorem 4.12. \square

PROPOSITION 5.7. *Let (F_f, F_s^1, F_s^2) belong to Z . If $\lambda \in \mathbb{R}$ is large enough, then system (5.1) admits a unique solution $(v, q, \eta_1, \eta_2) \in H_{\delta_0}$ satisfying the estimate*

$$\|v\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|q\|_{H_{\delta_0}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

Proof. Step 1. Lifting of the divergence condition. We are going to rewrite system (5.1) in the form of a system with divergence free condition. For that, we make the change of unknowns

$$\widehat{v} = (\widehat{v}_1, \widehat{v}_2) = v - L(0, A_3 \eta_1) \quad \text{and} \quad \widehat{p} = p - L_p(0, A_3 \eta_1), \quad (5.7)$$

where L and L_p are introduced in (4.19). We have $\widehat{v} = \eta_2 n$ on Γ_s , which gives

$$\eta_2 = \widehat{v}_2 \text{ on } \Gamma_s \quad \text{and} \quad \eta_1 = \frac{1}{\lambda} \widehat{v}_2 + \frac{1}{\lambda} F_s^1 \text{ on } \Gamma_s.$$

We introduce the operator L_λ in $H_0^2(\Gamma_s)$ defined by

$$L_\lambda = \lambda^2 - (\beta + \gamma\lambda)\Delta_s + \alpha\Delta_s^2 - A_4 - \gamma_s L_p(0, A_3 \cdot).$$

We choose λ large enough so that L_λ satisfies a coercivity condition in $H_0^2(\Gamma_s)$. From Proposition 5.1, it follows that $L_\lambda \in \mathcal{L}(H_0^2(\Gamma_s), H^{-2}(\Gamma_s))$, and with the Lax-Milgram Theorem, we can prove that L_λ is an isomorphism from $H_0^2(\Gamma_s)$ into $H^{-2}(\Gamma_s)$. The operator L_λ is also an isomorphism from $H^4(\Gamma_s) \cap H_0^2(\Gamma_s)$ into $L^2(\Gamma_s)$. The system satisfied by $(\widehat{v}, \widehat{p}, \eta_1, \eta_2)$ can be rewritten in the form

$$\begin{aligned} \lambda\widehat{v} - \operatorname{div} \sigma(\widehat{v}, \widehat{p}) + (u_s \cdot \nabla)\widehat{v} + (\widehat{v} \cdot \nabla)u_s - A_5\widehat{v} &= G_f \text{ in } \Omega, \\ \operatorname{div} \widehat{v} &= 0 \text{ in } \Omega, \quad \widehat{v} = \lambda L_\lambda^{-1} \gamma_s \widehat{p} n + G_b n \text{ on } \Gamma_s, \quad \widehat{v} = 0 \text{ on } \Gamma_i, \\ \widehat{v} \cdot n &= 0 \text{ on } \Gamma_e, \quad \varepsilon(\widehat{v})n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\widehat{v}, \widehat{p})n = 0 \text{ on } \Gamma_n, \\ \lambda\eta_1 - \eta_2 &= F_s^1 \text{ on } \Gamma_s, \quad L_\lambda \eta_1 = \gamma_s \widehat{p} + G_s \text{ on } \Gamma_s, \\ \eta_1 &= 0 \text{ on } \partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial\Gamma_s, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} A_5\widehat{v} &= \frac{1}{\lambda} A_1 \gamma_s \widehat{v}_2 + A_2 \gamma_s \widehat{v}_2 + \frac{\lambda_0 - \lambda}{\lambda} L(0, A_3 \gamma_s \widehat{v}_2) - \frac{1}{\lambda} L_p(0, A_3 \gamma_s \widehat{v}_2), \\ G_f &= F + \frac{1}{\lambda} A_1 F_s^1 + \frac{\lambda_0 - \lambda}{\lambda} L(0, A_3 F_s^1) - \frac{1}{\lambda} L_p(0, A_3 F_s^1), \\ G_s &= F_s^2 + \lambda F_s^1 - \gamma \Delta F_s^1, \quad G_b = \lambda L_\lambda^{-1} G_s - F_s^1. \end{aligned}$$

From Theorem 4.1 and Proposition 5.1, it follows that there exists $C > 0$ such that

$$\begin{aligned} \|A_5\widehat{v}\|_{\mathbf{L}^2(\Omega)} &\leq C \|\gamma_s \widehat{v}_2\|_{H_0^2(\Gamma_s)} \quad \text{for all } \widehat{v}_2 \in H_0^2(\Gamma_s), \\ \text{and} \\ \|G_f\|_{\mathbf{L}^2(\Omega)} + \|G_s\|_{L^2(\Gamma_s)} + \|G_b\|_{H_0^2(\Gamma_s)} &\leq C \|(F_f, F_s^1, F_s^2)\|_{\mathbf{Z}}. \end{aligned}$$

Step 2. Existence of weak solutions for system (5.8). We introduce the space

$$\mathbf{E} = \left\{ \widehat{v} = (\widehat{v}_1, \widehat{v}_2) \in \mathbf{H}^1(\Omega) \mid \operatorname{div} \widehat{v} = 0 \text{ in } \Omega, \widehat{v} = 0 \text{ on } \Gamma_i, v \cdot n = 0 \text{ on } \Gamma_e, \widehat{v}_1 = 0 \text{ on } \Gamma_s, \widehat{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s) \right\},$$

equipped with the norm $\|\widehat{v}\|_{\mathbf{E}} = \left(\|\widehat{v}\|_{\mathbf{H}^1(\Omega)} + \|L_\lambda^{\frac{1}{2}} \widehat{v}_2|_{\Gamma_s}\|_{L^2(\Gamma_s)} \right)^{\frac{1}{2}}$. We set

$$\begin{aligned} a_\lambda(\widehat{v}, \phi) &= \lambda \int_\Omega \widehat{v} \cdot \phi + 2\nu \int_\Omega \varepsilon(\widehat{v}) : \varepsilon(\phi) + \int_\Omega ((u_s \cdot \nabla)\widehat{v} + (\widehat{v} \cdot \nabla)u_s) \cdot \phi - \int_\Omega A_5 \widehat{v} \cdot \phi + \frac{1}{\lambda} \int_{\Gamma_s} L_\lambda^{\frac{1}{2}} \widehat{v}_2 L_\lambda^{\frac{1}{2}} \phi_2, \\ \text{and} \quad \ell_\lambda(\phi) &= \int_\Omega G_f \cdot \phi + \frac{1}{\lambda} \int_{\Gamma_s} L_\lambda^{\frac{1}{2}} G_b L_\lambda^{\frac{1}{2}} \phi_2. \end{aligned}$$

If $(\widehat{v}, \widehat{p}) \in \mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ is a solution to system (5.8)₁₋₃, then \widehat{v} is solution to the variational problem

$$\text{Find } \widehat{v} \in \mathbf{E} \text{ such that } a_\lambda(\widehat{v}, \phi) = \ell_\lambda(\phi), \quad \forall \phi \in \mathbf{E}. \tag{5.9}$$

Thanks to the Lax-Milgram Theorem, problem (5.9) admits a unique solution. Moreover it can be shown that if $\widehat{v} \in \mathbf{E}$ is the solution to (5.9), then there exists a pressure $\widehat{p} \in L^2(\Omega)$ such that the pair $(\widehat{v}, \widehat{p})$ is a variational solution to the system

$$\begin{aligned} \lambda_0 \widehat{v} - \operatorname{div} \sigma(\widehat{v}, \widehat{p}) + (u_s \cdot \nabla)\widehat{v} + (\widehat{v} \cdot \nabla)u_s &= (\lambda_0 - \lambda)\widehat{v} + G_f - A_5\widehat{v}, \quad \operatorname{div} \widehat{v} = 0 \text{ in } \Omega, \\ \widehat{v} &= \widehat{v}_2 n \text{ on } \Gamma_s, \quad \widehat{v} = 0 \text{ on } \Gamma_i, \quad \widehat{v} \cdot n = 0 \text{ and } \varepsilon(\widehat{v})n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\widehat{v}, \widehat{p})n = 0 \text{ on } \Gamma_n. \end{aligned} \tag{5.10}$$

By variational solution, we understand a solution to the mixed variational problem associated with (5.10), as introduced in (4.3). Since \widehat{v} belongs to $\mathbf{V}^1(\Omega)$, $(\lambda_0 - \lambda)\widehat{v} + G_f - A_5\widehat{v}$ belongs to $\mathbf{L}^2(\Omega)$ and \widehat{v}_2 belongs to $H_0^{\frac{3}{2}}(\Gamma_s)$. Thus, from Theorem 4.1, it follows that $(\widehat{v}, \widehat{p})$ belongs to $\mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ and

$$\|\widehat{v}\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|\widehat{p}\|_{H_{\delta_0}^1(\Omega)} \leq C(\|(\lambda_0 - \lambda)\widehat{v} + G_f - A_5\widehat{v}\|_{\mathbf{L}^2(\Omega)} + \|\widehat{v}_2\|_{H_0^{\frac{3}{2}}(\Gamma_s)}) \leq C\|(F_f, F_s^1, F_s^2)\|_{\mathbf{Z}}.$$

Step 3. From equations (5.8)_{4–6}, we get

$$\|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

We deduce that system (5.8) admits a unique solution $(\widehat{v}, \widehat{p}, \eta_1, \eta_2) \in H_{\delta_0}$ satisfying

$$\|\widehat{v}\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|\widehat{p}\|_{H_{\delta_0}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z. \quad (5.11)$$

Step 4. *Existence of solutions for system (5.1).* According to Theorem 4.1 and Proposition 5.1, $(L(0, A_3\eta_1), L_p(0, A_3\eta_1))$ belongs to $\mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ and we have

$$\|L(0, A_3\eta_1)\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|L_p(0, A_3\eta_1)\|_{H_{\delta_0}^1(\Omega)} \leq C\|A_3\eta_1\|_{H^1(\Omega)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

Thus, thanks to (5.7) and (5.11), system (5.1) admits a unique solution (v, q, η_1, η_2) in H_{δ_0} satisfying

$$\|v\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|q\|_{H_{\delta_0}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

□

THEOREM 5.8. *The resolvent of the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is compact.*

Proof. Let $\lambda > 0$ large enough, and let (F_f, F_s^1, F_s^2) belong to Z . There exists $(Pv, \eta_1, \eta_2) \in Z$ such that

$$(\lambda I - \mathcal{A})(Pv, \eta_1, \eta_2)^T = M_s^{-1}(F_f, F_s^1, F_s^2)^T.$$

Thanks to Proposition 5.6, (v, q, η_1, η_2) with

$$\begin{aligned} v &= Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1, \\ q &= -\lambda N_s \eta_2 - \lambda N_{\text{div}} A_3 \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v (Pv + \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1), \end{aligned}$$

is solution of system (5.1). Due to Proposition 5.7, (v, q, η_1, η_2) belongs to H_{δ_0} and

$$\|v\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|q\|_{H_{\delta_0}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

From Lemma 4.3 and Proposition 4.4, it follows that (Pv, η_1, η_2) belongs to $\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s)$, and

$$\|Pv\|_{\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C\|(F_f, F_s^1, F_s^2)\|_Z.$$

Since the imbedding $\mathbf{H}^{\frac{1}{2}+\varepsilon_0}(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \hookrightarrow Z$ is compact, the proof is complete. □

5.4. Analyticity.

THEOREM 5.9. *The unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup of class \mathcal{C}^0 on Z .*

Proof. First, we introduce the unbounded operator $(\widehat{\mathcal{A}}, D(\widehat{\mathcal{A}}))$ in Z defined by $D(\widehat{\mathcal{A}}) = D(\mathcal{A})$ and

$$\widehat{\mathcal{A}} = \begin{pmatrix} A & (\lambda_0 I - A)PL(0, A_3 \cdot) & (\lambda_0 I - A)PL(\cdot, 0) \\ 0 & 0 & I \\ 0 & A_{\alpha, \beta} & \gamma \Delta \end{pmatrix}.$$

We set $K_s := (I + \gamma_s N_s)^{-1}$. We decompose \mathcal{A} in the form $\mathcal{A} = \widehat{\mathcal{A}} + \widehat{\mathcal{A}}_1 + \widehat{\mathcal{A}}_2 + \widehat{\mathcal{A}}_3 + \mathcal{A}_p$, where the operators $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{A}}_2$ and $\widehat{\mathcal{A}}_3$ are defined in $D(\widehat{\mathcal{A}})$ by

$$\begin{aligned} \widehat{\mathcal{A}}_1(Pv, \eta_1, \eta_2)^T &= (0, 0, (K_s - I)A_{\alpha, \beta}\eta_1)^T, & \widehat{\mathcal{A}}_2(Pv, \eta_1, \eta_2) &= (0, 0, \gamma(K_s - I)\Delta\eta_2)^T, \\ \widehat{\mathcal{A}}_3(Pv, \eta_1, \eta_2)^T &= (PA_1\eta_1 + PA_2\eta_2, 0, K_s\gamma_s N_p A_1\eta_1 + K_s A_4\eta_1 + K_s\gamma_s N_p A_2\eta_2 - K_s\gamma_s N_{\text{div}} A_3\eta_2)^T, \end{aligned}$$

for all $(Pv, \eta_1, \eta_2) \in D(\widehat{\mathcal{A}})$, and \mathcal{A}_p is given in (5.5). The theorem will be a direct consequence of Theorem 5.10, Propositions 5.11, 5.14 and 5.7. □

THEOREM 5.10. *The unbounded operator $(\widehat{\mathcal{A}}, D(\widehat{\mathcal{A}}))$ is the infinitesimal generator of an analytic semigroup of class \mathcal{C}^0 on Z .*

Proof. The proof is similar to that in [24, Theorem 3.6]. \square

PROPOSITION 5.11. *There exist $0 < \theta_1, \theta_2 < 1$ such that $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ belong to $\mathcal{L}(D((-\widehat{\mathcal{A}})^{\theta_1}), Z)$ and $\mathcal{L}(D((-\widehat{\mathcal{A}})^{\theta_2}), Z)$ respectively.*

Proof. See [24, Lemma 3.9]. \square

LEMMA 5.12. *There exists $C > 0$ such that, for all $(Pv, \eta_1, \eta_2) \in D(\widehat{\mathcal{A}})$, we have the estimate*

$$\|N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1)\|_{H^{\frac{1}{2} + \varepsilon_0}(\Omega)} \leq C \|(Pv, \eta_1, \eta_2)\|_{D(\widehat{\mathcal{A}})}.$$

Proof. Let λ be positive and (Pv, η_1, η_2) be in $D(\widehat{\mathcal{A}})$. We set

$$(F_f, F_s^1, F_s^2)^T := (\lambda M_s - M_s \widehat{\mathcal{A}})(Pv, \eta_1, \eta_2)^T.$$

As in Proposition 5.6, we can prove that (v, q, η_1, η_2) with

$$\begin{aligned} v &= Pv + \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1, \\ q &= -\lambda N_s \eta_2 - \lambda N_{\text{div} A_3} \eta_1 + N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1), \end{aligned}$$

is a solution of system (5.1). As in the proof of Proposition 5.7, we can show that, for λ large enough, system (5.1) admits a unique solution $(v, q, \eta_1, \eta_2) \in H_{\delta_0}$ satisfying the estimate

$$\|v\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|q\|_{H_{\delta_0}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H^2(\Gamma_s)} \leq C \|(F_f, F_s^1, F_s^2)\|_H.$$

With this estimate and Lemma 4.3, we obtain

$$\|q\|_{H^{\frac{1}{2} + \varepsilon_0}(\Omega)} \leq C \|q\|_{H_{\delta_0}^1(\Omega)} \leq C \|(F_f, F_s^1, F_s^2)\|_H \leq C \|(Pv, \eta_1, \eta_2)\|_{D(\widehat{\mathcal{A}})}.$$

Using Proposition 5.1, we deduce that

$$\|N_v(Pv + \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1)\|_{H^{\frac{1}{2} + \varepsilon_0}(\Omega)} \leq C \|(Pv, \eta_1, \eta_2)\|_{D(\widehat{\mathcal{A}})}.$$

\square

LEMMA 5.13. *The operator K_s belongs to $\mathcal{L}(H^{\varepsilon_0}(\Gamma_s))$.*

Proof. Let g belong to $H^{\varepsilon_0}(\Gamma_s)$. Since $I + \gamma_s N_s \in \text{isom}(L^2(\Gamma_s))$, there exists a unique $h \in L^2(\Gamma_s)$ such that $h + \gamma_s N_s h = g$. Since $\gamma_s N_s h$ belongs to $H^{\varepsilon_0}(\Gamma_s)$, we deduce that h belongs to $H^{\varepsilon_0}(\Gamma_s)$. Thus, the operator $I + \gamma_s N_s$ is an automorphism in $H^{\varepsilon_0}(\Gamma_s)$. \square

PROPOSITION 5.14. *The operator $\mathcal{A}_p \in \mathcal{L}(D(\widehat{\mathcal{A}}), L^2(\Gamma_s))$ is compact.*

Proof. Thanks to Proposition 5.1, Lemmas 5.12 and 5.13, we prove that the operator \mathcal{A}_p belongs to $\mathcal{L}(D(\widehat{\mathcal{A}}), H^{\varepsilon_0}(\Gamma_s))$. From the compact embedding $H^{\varepsilon_0}(\Gamma_s) \hookrightarrow L^2(\Gamma_s)$, it follows that \mathcal{A}_p is a compact operator from $D(\widehat{\mathcal{A}})$ into $L^2(\Gamma_s)$. \square

PROPOSITION 5.15. *The operators $\widehat{\mathcal{A}}_3$ is $\widehat{\mathcal{A}}$ -bounded with relative bound zero.*

Proof. It is sufficient to argue by contradiction as in [24, Lemma 3.8]. \square

PROPOSITION 5.16. *We have*

$$[Z, D(\mathcal{A})]_{\frac{1}{2}} = \left\{ (Pv, \eta_1, \eta_2) \in \mathbf{V}_{n, \Gamma_0}^{\frac{1}{2} + \varepsilon_0}(\Omega) \times (H^3 \cap H_0^2)(\Gamma_s) \times H_0^1(\Gamma_s) \mid Pv - PL(\eta_2, A_3 \eta_1) \in \mathbf{V}_{\Gamma_0}^1(\Omega) \right\}.$$

Proof. The result can be established by proving that the mapping

$$\Upsilon : (Pv, \eta_1, \eta_2) \longmapsto (Pv - PL(\eta_2, A_3 \eta_1), \eta_1, \eta_2),$$

is an isomorphism from $D(\mathcal{A})$ into $D(\mathcal{A}) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s)$, and from $[Z, D(\mathcal{A})]_{\frac{1}{2}}$ into $\mathbf{V}_{\Gamma_0}^1(\Omega) \times (H^3(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^1(\Gamma_s)$, and by using that $[Z, D(\mathcal{A})]_{\frac{1}{2}} \subset \mathbf{V}_{n, \Gamma_0}^{\frac{1}{2} + \varepsilon_0}(\Omega) \times (H^3(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^1(\Gamma_s)$. \square

6. Adjoint of $(\mathcal{A}, D(\mathcal{A}))$. Using (4.9), we introduce the operator $N_* \in \mathcal{L}(\mathbf{H}_{\delta_0}^2(\Omega), L^2(\Omega))$ defined by $N_*\phi = \psi$, where ψ is the solution of the variational problem

Find $\psi \in L^2(\Omega)$ such that

$$\int_{\Omega} \psi \zeta = 2\nu \langle \varepsilon(\phi), \nabla^2 \chi \rangle_{\mathbf{L}^2_{\delta_0}(\Omega), \mathbf{L}^2_{\delta_0}(\Omega)} - 2\nu \int_{\Gamma_0} \varepsilon(\phi) n \cdot \nabla \chi + \int_{\Omega} \left[-(u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi \right] \cdot \nabla \chi,$$

for all $\zeta \in L^2(\Omega)$, and where $\chi \in H_{\delta_0}^2(\Omega)$ is the solution of equation (4.8).

We also introduce the lifting operators $D \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s), \mathbf{H}_{\delta_0}^2(\Omega))$ and $D_p \in \mathcal{L}(H_0^{\frac{3}{2}}(\Gamma_s), H_{\delta_0}^1(\Omega))$ defined by

$$Dg = \phi \quad \text{and} \quad D_p g = \psi,$$

where (ϕ, ψ) is the solution of the system

$$\begin{aligned} \lambda_0 \phi - \operatorname{div} \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi &= 0, \quad \operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = gn \text{ on } \Gamma_s, \\ \phi &= 0 \text{ on } \Gamma_i, \quad \phi \cdot n = 0 \text{ and } \varepsilon(\phi) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi, \psi) n + u_s \cdot n \phi = 0 \text{ on } \Gamma_n. \end{aligned} \quad (6.1)$$

Finally, we introduce the unbounded operator $(\mathcal{A}^\#, D(\mathcal{A}^\#))$ in Z defined by

$$D(\mathcal{A}^\#) = \left\{ (P\phi, \xi_1, \xi_2) \in \mathbf{V}_{n, \Gamma_0}^{\frac{1}{2} + \varepsilon_0}(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \mid P(\phi - D\xi_2) \in D(A^*) \right\},$$

and

$$\mathcal{A}^\# = \begin{pmatrix} A^* & 0 & (\lambda_0 I - A^*)PD \\ (-A_{\alpha, \beta})^{-1} A_1^* & 0 & -I + (-A_{\alpha, \beta})^{-1} (A_4^* + 2\nu(\gamma_s A_3)^* + A_1^* \nabla N_s) \\ \gamma_s N_* + A_2^* & -A_{\alpha, \beta} & \gamma \Delta + \gamma_s N_* \nabla N_s + A_2^* \nabla N_s \end{pmatrix} + \mathcal{A}_p^\#.$$

The operator $\mathcal{A}_p^\#$ is defined in $D(\mathcal{A}^\#)$ by

$$\mathcal{A}_p^\# \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -(-A_{\alpha, \beta})^{-1} A_3^* \gamma_s N_* (P\phi + \nabla N_s \xi_1) \\ \gamma_s N_* (P\phi + \nabla N_s \xi_2) \end{pmatrix}, \quad \forall (P\phi, \xi_1, \xi_2) \in D(\mathcal{A}^\#).$$

LEMMA 6.1. *The adjoint of the operator M_s in Z and its inverse are defined by*

$$M_s^* = \begin{pmatrix} I & 0 & 0 \\ 0 & I & -(-A_{\alpha, \beta})^{-1} A_3^* N_s \\ 0 & 0 & I + \gamma_s N_s \end{pmatrix}, \quad (M_s^*)^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & (-A_{\alpha, \beta})^{-1} A_3^* N_s (I + \gamma_s N_s)^{-1} \\ 0 & 0 & (I + \gamma_s N_s)^{-1} \end{pmatrix}.$$

Proof. The proof follows from the fact that $(I + \gamma_s N_s) \in \operatorname{autom}(L^2(\Gamma_s))$. It is left to the reader. \square

PROPOSITION 6.2. *Let λ belong to \mathbb{C} and let (G_f, G_s^1, G_s^2) belong to H . A quadruplet $(\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}$ is solution of system*

$$\begin{aligned} \lambda \phi - \operatorname{div} \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi &= G_f, \quad \operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 n \text{ on } \Gamma_s, \\ \phi &= 0 \text{ on } \Gamma_i, \quad \phi \cdot n = 0 \text{ and } \varepsilon(\phi) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi, \psi) n + u_s \cdot n \phi = 0 \text{ on } \Gamma_n, \\ \lambda \xi_1 + \xi_2 - (-A_{\alpha, \beta})^{-1} (A_4^* + 2\nu(\gamma_s A_3)^*) \xi_2 - (-A_{\alpha, \beta})^{-1} A_1^* \phi + (-A_{\alpha, \beta})^{-1} A_3^* \psi &= G_s^1 \text{ on } \Gamma_s, \\ \lambda \xi_2 + \beta \Delta_s \xi_1 - \gamma \Delta_s \xi_2 - \alpha \Delta_s^2 \xi_1 - A_2^* \phi - \gamma_s \psi &= G_s^2 \text{ on } \Gamma_s, \\ \xi_1 &= 0 \text{ on } \partial \Gamma_s, \quad \xi_{1, x} = 0 \text{ on } \partial \Gamma_s, \end{aligned} \quad (6.2)$$

if and only if

$$\begin{aligned} \lambda M_s^* (P\phi, \xi_1, \xi_2)^T &= \mathcal{A}^\# (P\phi, \xi_1, \xi_2)^T + (PG_f, G_s^1, G_s^2)^T, \quad (I - P)\phi = \nabla N_s \xi_2, \\ \psi &= -\lambda N_s \xi_2 + N_* (\nabla N_s \xi_2 + P\phi) + N_p G_f. \end{aligned} \quad (6.3)$$

Proof. Let $(\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}$ be a solution of system (6.2). As in Theorem 4.12, we can prove that (ϕ, ψ) is a solution of system

$$\begin{aligned} (\lambda I - A^*)P\phi + (A^* - \lambda_0 I)PD\xi_2 &= PG_f, \quad (I - P)\phi = \nabla N_s \xi_2, \\ \psi &= -\lambda N_s \xi_2 + N_*(P\phi + \nabla N_s \xi_2) + N_p G_f. \end{aligned}$$

Replacing the pressure ψ by the above expression in the equations satisfied by ξ_1 and ξ_2 , we obtain

$$\begin{aligned} \lambda \xi_1 - \lambda(-A_{\alpha, \beta})^{-1} A_3^* N_s \xi_2 &= -\xi_2 + (-A_{\alpha, \beta})^{-1} (A_4^* + 2\nu(\gamma_s A_3)^* + A_1^* \nabla N_s) \xi_2 + (-A_{\alpha, \beta})^{-1} A_1^* P\phi \\ &\quad - (-A_{\alpha, \beta})^{-1} A_3^* \gamma_s N_*(P\phi + \nabla N_s \xi_1) + G_s^1, \end{aligned}$$

and

$$\lambda(I + \gamma_s N_s) \xi_2 = -A_{\alpha, \beta} \xi_1 + \gamma \Delta_s \xi_2 + \gamma_s N_* \nabla N_s \xi_2 + A_2^* P\phi + \gamma_s N_*(P\phi + \nabla N_s \xi_2) + G_s^2.$$

The proof is complete. \square

PROPOSITION 6.3. *The adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in Z is defined by*

$$D(\mathcal{A}^*) = \left\{ M_s^*(P\phi, \xi_1, \xi_2) \in Z \mid (P\phi, \xi_1, \xi_2) \in D(\mathcal{A}^\#) \right\} \quad \text{and} \quad \mathcal{A}^* = \mathcal{A}^\# M_s^{-*}.$$

Proof. We set $D(\widehat{\mathcal{A}}^*) = \left\{ M_s^*(P\phi, \xi_1, \xi_2) \in Z \mid (P\phi, \xi_1, \xi_2) \in D(\mathcal{A}^\#) \right\}$, and $\widehat{\mathcal{A}}^* = \mathcal{A}^\# M_s^{-*}$. With a Green's formula, we can prove that the solution (v, q, η_1, η_2) of system (5.1) and the solution $(\phi, \psi, \xi_1, \xi_2)$ of system (6.2) satisfy the identity

$$\left((F_f, F_s^1, F_s^2)^T, (\phi, \xi_1, \xi_2)^T \right)_H = \left((v, \eta_1, \eta_2)^T, (G_f, G_s^1, G_s^2)^T \right)_H. \quad (6.4)$$

This identity can be interpreted as follows. According to Proposition 5.6, (Pv, η_1, η_2) satisfies

$$(\lambda M_s - M_s \mathcal{A})(Pv, \eta_1, \eta_2)^T = (PF_f, F_s^1, F_s^2)^T.$$

Thanks to Proposition 6.2, $(P\phi, \xi_1, \xi_2)$ satisfies

$$(\lambda M_s^* - \mathcal{A}^\#)(P\phi, \xi_1, \xi_2)^T = (PG_f, G_s^1, G_s^2)^T.$$

Thus, identity (6.4) is equivalent to

$$\left((\lambda I - \mathcal{A})(Pv, \eta_1, \eta_2)^T, M_s^*(P\phi, \xi_1, \xi_2)^T \right)_H = \left((Pv, \eta_1, \eta_2)^T, (\lambda I - \widehat{\mathcal{A}}^*) M_s^*(P\phi, \xi_1, \xi_2)^T \right)_H.$$

From that, we deduce that $D(\widehat{\mathcal{A}}^*) \subset D(\mathcal{A}^*)$. The reverse inclusion can be proved by standard arguments. Therefore, $(\widehat{\mathcal{A}}^*, D(\widehat{\mathcal{A}}^*))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in Z , and the proof is complete. \square

PROPOSITION 6.4. *Let λ belong to \mathbb{C} and let (G_f, G_s^1, G_s^2) belong to H . A quadruplet $(\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}$ is solution of system (6.2) if and only if*

$$\begin{aligned} \lambda M_s^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} &= \mathcal{A}^* M_s^* \begin{pmatrix} P\phi \\ \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} PG_f \\ G_s^1 - (-A_{\alpha, \beta})^{-1} A_3^* \gamma_s N_p G_f \\ G_s^2 + \gamma_s N_p G_f \end{pmatrix}, \quad (I - P)\phi = \nabla N_s \xi_2, \\ \psi &= -\lambda N_s \xi_2 + N_*(P\phi + \nabla N_s \xi_2) + N_p G_f. \end{aligned} \quad (6.5)$$

Proof. It is a direct consequence of Proposition 6.2 and of definition of $(\mathcal{A}^*, D(\mathcal{A}^*))$. \square

7. Eigenvalue problems. The goal of this section is to find relationships between the eigenvalue problem associated to \mathcal{A} and the eigenvalue problem associated to the linearized system associated to (3.3), i.e.

$$\lambda \in \mathbb{C}, \quad (Pv, \eta_1, \eta_2) \in D(\mathcal{A}), \quad \lambda(Pv, \eta_1, \eta_2)^T = \mathcal{A}(Pv, \eta_1, \eta_2)^T, \quad (7.1)$$

and

$$\begin{aligned}
& \lambda \in \mathbb{C}, \quad (v, q, \eta_1, \eta_2) \in H_{\delta_0}, \\
& \lambda v - \operatorname{div} \sigma(v, q) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1 \eta_1 - A_2 \eta_2 = 0, \quad \operatorname{div} v = A_3 \eta_1 \text{ in } \Omega, \\
& v = \eta_2 n \text{ on } \Gamma_s, \quad v = 0 \text{ on } \Gamma_i, \quad v \cdot n = 0 \text{ and } \varepsilon(v)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v, q)n = 0 \text{ on } \Gamma_n, \\
& \lambda \eta_1 - \eta_2 = 0 \text{ on } \Gamma_s, \\
& \lambda \eta_2 - \beta \Delta_s \eta_1 - \gamma \Delta_s \eta_2 + \alpha \Delta_s^2 \eta_1 - A_4 \eta_1 - \gamma_s q = 0 \text{ on } \Gamma_s, \\
& \eta_1 = 0 \text{ on } \partial \Gamma_s, \quad \eta_{1,x} = 0 \text{ on } \partial \Gamma_s.
\end{aligned} \tag{7.2}$$

We also consider both the adjoint eigenvalue problems

$$\lambda \in \mathbb{C}, \quad (P\phi, \xi_1, \xi_2) \in D(\mathcal{A}^*), \quad \lambda(P\phi, \xi_1, \xi_2)^T = \mathcal{A}^*(P\phi, \xi_1, \xi_2)^T, \tag{7.3}$$

and

$$\begin{aligned}
& \lambda \in \mathbb{C}, \quad (\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0}, \\
& \lambda \phi - \operatorname{div} \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi = 0, \quad \operatorname{div} \phi = 0 \text{ in } \Omega, \quad \phi = \xi_2 n \text{ on } \Gamma_s, \\
& \phi = 0 \text{ on } \Gamma_i, \quad \phi \cdot n = 0 \text{ and } \varepsilon(\phi)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\phi, \psi)n + u_s \cdot n \phi = 0 \text{ on } \Gamma_n, \\
& \lambda \xi_1 + \xi_2 - (-A_{\alpha, \beta})^{-1}(A_4^* + 2\nu(\gamma_s A_3)^*)\xi_2 - (-A_{\alpha, \beta})^{-1}A_1^* \phi + (-A_{\alpha, \beta})^{-1}A_3^* \psi = 0 \text{ on } \Gamma_s, \\
& \lambda \xi_2 + \beta \Delta_s \xi_1 - \gamma \Delta_s \xi_2 - \alpha \Delta_s^2 \xi_1 - A_2^* \phi - \gamma_s \psi = 0 \text{ on } \Gamma_s, \\
& \xi_1 = 0 \text{ on } \partial \Gamma_s, \quad \xi_{1,x} = 0 \text{ on } \partial \Gamma_s.
\end{aligned} \tag{7.4}$$

DEFINITION 7.1. A triplet $(Pv_k, \eta_{1,k}, \eta_{2,k}) \in D(\mathcal{A})$ is a generalized eigenfunction for problem (7.1) of order $k \geq 1$ associated to a solution $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ of (7.1) if $(Pv_k, \eta_{1,k}, \eta_{2,k})$ is obtained by solving the chain of equations

$$(\lambda I - \mathcal{A})(Pv_j, \eta_{1,j}, \eta_{2,j})^T = -(Pv_{j-1}, \eta_{1,j-1}, \eta_{2,j-1})^T, \quad \text{for } 1 \leq j \leq k.$$

A quadruplet $(v_k, q_k, \eta_{1,k}, \eta_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction for problem (7.2) of order $k \geq 1$ associated to a solution $(\lambda, (v_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (7.2) if $(v_k, q_k, \eta_{1,k}, \eta_{2,k})$ is obtained by solving, for $1 \leq j \leq k$, the chain of systems

$$\begin{aligned}
& \lambda v_j - \operatorname{div} \sigma(v_j, q_j) + (u_s \cdot \nabla)v_j + (v_j \cdot \nabla)u_s - A_1 \eta_{1,j} - A_2 \eta_{2,j} = -v_{j-1} \text{ in } \Omega, \\
& \operatorname{div} v_j = A_3 \eta_{1,j} \text{ in } \Omega, \quad v_j = \eta_{2,j} n \text{ on } \Gamma_s, \quad v_j = 0 \text{ on } \Gamma_i, \\
& v_j \cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(v_j)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(v_j, q_j)n = 0 \text{ on } \Gamma_n, \\
& \lambda \eta_{1,j} - \eta_{2,j} = -\eta_{1,j-1} \text{ on } \Gamma_s, \\
& \lambda \eta_{2,j} - \beta \Delta_s \eta_{1,j} - \gamma \Delta_s \eta_{2,j} + \alpha \Delta_s^2 \eta_{1,j} - A_4 \eta_{1,j} - \gamma_s q_j = -\eta_{2,j-1} \text{ on } \Gamma_s, \\
& \eta_{1,j} = 0 \text{ on } \partial \Gamma_s, \quad \eta_{1,j,x} = 0 \text{ on } \partial \Gamma_s.
\end{aligned}$$

We have similar statements for the adjoint eigenvalue problems (7.3) and (7.4).

7.1. Equivalence between direct eigenvalue problems.

THEOREM 7.2. A couple $(\lambda, (v, q, \eta_1, \eta_2)) \in \mathbb{C} \times H_{\delta_0}$ is a solution of eigenvalue problem (7.2) if and only if $(\lambda, (Pv, \eta_1, \eta_2)) \in \mathbb{C} \times D(\mathcal{A})$ is a solution of (7.1) and

$$\begin{aligned}
& (I - P)v = \nabla N_s \eta_2 + \nabla N_{\operatorname{div} A_3} \eta_1, \\
& q = -\lambda N_s \eta_2 - \lambda N_{\operatorname{div} A_3} \eta_1 + N_p A_1 \eta_1 + N_p A_2 \eta_2 + N_v (Pv + \nabla N_s \eta_2 + \nabla N_{\operatorname{div} A_3} \eta_1).
\end{aligned}$$

Proof. It follows from Proposition 5.6. □

THEOREM 7.3. A quadruplet $(v_k, q_k, \eta_{1,k}, \eta_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction associated with a solution $(\lambda, (v_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (7.2) if and only if $(Pv_k, \eta_{1,k}, \eta_{2,k}) \in D(\mathcal{A})$ is a generalized eigenfunction for (7.1) associated with a solution $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ and

$$\begin{aligned}
& (I - P)v_k = \nabla N_s \eta_{2,k} + \nabla N_{\operatorname{div} A_3} \eta_{1,k}, \\
& q_k = -\lambda N_s \eta_{2,k} - \lambda N_{\operatorname{div} A_3} \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v (Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\operatorname{div} A_3} \eta_{1,k}) \\
& \quad - N_s \eta_{2,k-1} - N_{\operatorname{div} A_3} \eta_{1,k-1}.
\end{aligned}$$

Proof. Thanks to Proposition 5.6, $(v_k, q_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (v_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (7.2) if and only if

$$\begin{aligned} \lambda(Pv_k, \eta_{1,k}, \eta_{2,k})^T &= \mathcal{A}(Pv_k, \eta_{1,k}, \eta_{2,k}) - M_s^{-1}(Pv_{k-1}, \eta_{1,k-1}, \eta_{2,k-1} + \gamma_s N_p v_{k-1})^T, \\ (I - P)v_k &= \nabla N_s \eta_{2,k} + \nabla N_{\text{div}} A_3 \eta_{1,k}, \\ q_k &= -\lambda N_s \eta_{2,k} - \lambda N_{\text{div}} A_3 \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v (Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\text{div}} A_3 \eta_{1,k}) \\ &\quad - N_p v_{k-1}, \end{aligned}$$

where $(v_{k-1}, q_{k-1}, \eta_{1,k-1}, \eta_{2,k-1})$ is a generalized eigenfunction of order $k - 1$. Since

$$N_p v_{k-1} = N_s \eta_{2,k-1} + N_{\text{div}} A_3 \eta_{1,k-1},$$

it follows that

$$\begin{aligned} \lambda(Pv_k, \eta_{1,k}, \eta_{2,k})^T &= \mathcal{A}(Pv_k, \eta_{1,k}, \eta_{2,k})^T - (Pv_{k-1}, \eta_{1,k-1}, \eta_{2,k-1})^T, \\ (I - P)v_k &= \nabla N_s \eta_{2,k} + \nabla N_{\text{div}} A_3 \eta_{1,k}, \\ q_k &= -\lambda N_s \eta_{2,k} - \lambda N_{\text{div}} A_3 \eta_{1,k} + N_p A_1 \eta_{1,k} + N_p A_2 \eta_{2,k} + N_v (Pv_k + \nabla N_s \eta_{2,k} + \nabla N_{\text{div}} A_3 \eta_{1,k}) \\ &\quad - N_s \eta_{2,k-1} - N_{\text{div}} A_3 \eta_{1,k-1}. \end{aligned}$$

We deduce, by induction, that $(v_k, q_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (v_0, q_0, \eta_{1,0}, \eta_{2,0}))$ of (7.2) if and only if $(Pv_k, \eta_{1,k}, \eta_{2,k})$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (Pv_0, \eta_{1,0}, \eta_{2,0}))$ of (7.1). \square

7.2. Equivalence between adjoint eigenvalue problems.

THEOREM 7.4. *A couple $(\lambda, (\phi, \psi, \xi_1, \xi_2)) \in \mathbb{C} \times H_{\delta_0}$ is a solution of eigenvalue problem (7.4) if and only if $(\lambda, M_s^*(P\phi, \xi_1, \xi_2)) \in \mathbb{C} \times D(\mathcal{A}^*)$ is a solution of (7.3) and*

$$(I - P)\phi = \nabla N_s \xi_2, \quad \psi = -\lambda N_s \xi_2 + N_*(P\phi + \nabla N_s \xi_2).$$

Proof. It follows from Proposition 6.4. \square

THEOREM 7.5. *A quadruplet $(\phi_k, \psi_k, \xi_{1,k}, \xi_{2,k}) \in H_{\delta_0}$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (\phi_0, \psi_0, \xi_{1,0}, \xi_{2,0}))$ of (7.4) if and only if $M_s^*(P\phi_k, \xi_{1,k}, \xi_{2,k}) \in D(\mathcal{A}^*)$ is a generalized eigenfunction of order $k \geq 1$ associated with a solution $(\lambda, (P\phi_0, \xi_{1,0}, \xi_{2,0}))$ of (7.3) and*

$$(I - P)\phi_k = \nabla N_s \xi_{2,k}, \quad \psi_k = -\lambda N_s \xi_{2,k} + N_*(P\phi_k + \nabla N_s \xi_{2,k}) + N_s \xi_{2,k-1}.$$

Proof. It is similar to that of Theorem 7.3 and relies on Proposition 6.4. \square

8. Stabilization of the linearized system.

8.1. Projected systems. Let $(\lambda_j)_{j \in \mathbb{N}^*}$ be the spectrum of \mathcal{A} . We denote by $G_{\mathbb{R}}(\lambda_j)$ the real generalized eigenspace of \mathcal{A} , that is the space generated by $\text{Re } G_{\mathbb{C}}(\lambda_j) \cup \text{Im } G_{\mathbb{C}}(\lambda_j)$, where $G_{\mathbb{C}}(\lambda_j)$ is the complex generalized eigenspace of \mathcal{A} , and $G_{\mathbb{R}}^*(\lambda_j)$ is the real generalized eigenspace of \mathcal{A}^* associated to the eigenvalue λ_j . Let $\omega > 0$ be such that $-\omega \notin \{\text{Re } \lambda_j \mid j \in \mathbb{N}^*\}$. We define the unstable subspaces

$$Z_u = \bigoplus_{j \in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{and} \quad Z_u^* = \bigoplus_{j \in J_u} G_{\mathbb{R}}^*(\lambda_j), \quad (8.1)$$

where J_u is a finite subset of \mathbb{N}^* such that $\text{Re } \lambda_j > -\omega$ for all $j \in J_u$, and $\text{Re } \lambda_j < -\omega$ for all $j \notin J_u$. There exist two subspaces Z_s and Z_s^* , invariant under $(e^{t\mathcal{A}})_{t \geq 0}$ and $(e^{t\mathcal{A}^*})_{t \geq 0}$, such that

$$Z = Z_u \oplus Z_s \quad \text{and} \quad Z = Z_u^* \oplus Z_s^*.$$

We have identified Z^* with Z . We denote by π_u the projection from Z onto Z_u along Z_s and by π_s the projection from Z onto Z_s along Z_u . We denote by d_u the dimension of the subspace Z_u . We characterize π_u in Proposition 8.2.

LEMMA 8.1. *Let (Pv, η_1, η_2) belong to Z , and $(P\phi, \xi_1, \xi_2)$ belong to Z^* . Let us set $\tilde{v} = Pv + \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1$ and $\tilde{\phi} = P\phi + \nabla N_s \xi_2$. Then $P\tilde{v} = Pv$, $(I - P)\tilde{v} = \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1$, $P\tilde{\phi} = P\phi$, $(I - P)\tilde{\phi} = \nabla N_s \xi_2$, and we have*

$$((\tilde{v}, \eta_1, \eta_2)^T, (\tilde{\phi}, \xi_1, \xi_2)^T)_H = ((Pv, \eta_1, \eta_2)^T, M_s^*(P\phi, \xi_1, \xi_2))_H. \quad (8.2)$$

Proof. It is clear that $P\tilde{v} = Pv$, $(I - P)\tilde{v} = \nabla N_s \eta_2 + \nabla N_{\text{div} A_3} \eta_1$, $P\tilde{\phi} = P\phi$, and $(I - P)\tilde{\phi} = \nabla N_s \xi_2$. Using the definitions of \tilde{v} and $\tilde{\phi}$, we have

$$(\tilde{v}, \tilde{\phi})_{\mathbf{L}^2(\Omega)} = (Pv, P\phi)_{\mathbf{V}_{n, \Gamma_0}^0(\Omega)} + (\eta_2, \gamma_s N_s \xi_2)_{L^2(\Gamma_s)} + (\eta_1, -(-A_{\alpha, \beta})^{-1} A_3^* N_s \xi_2)_{H_0^2(\Gamma_s)}.$$

We obtain (8.2) by adding $(\eta_1, \xi_1)_{H_0^2(\Gamma_s)} + (\eta_2, \xi_2)_{L^2(\Gamma_s)}$ in both sides of the above identity. \square

PROPOSITION 8.2. *There exist two families $(v_i, \eta_{1,i}, \eta_{2,i})_{1 \leq i \leq d_u}$ and $(\phi_i, \xi_{1,i}, \xi_{2,i})_{1 \leq i \leq d_u}$ of real valued functions satisfying the following statements:*

- (i) *The family $(Pv_i, \eta_{1,i}, \eta_{2,i})_{1 \leq i \leq d_u}$ is a basis of Z_u .*
- (ii) *The family $(M_s^*(P\phi_i, \xi_{1,i}, \xi_{2,i}))_{1 \leq i \leq d_u}$ is a basis of Z_u^* .*
- (iii) *We have the bi-orthogonality conditions*

$$((v_i, \eta_{1,i}, \eta_{2,i})^T, (\phi_j, \xi_{1,j}, \xi_{2,j})^T)_H = \delta_{i,j} \quad \text{and} \quad ((Pv_i, \eta_{1,i}, \eta_{2,i})^T, M_s^*(P\phi_j, \xi_{1,j}, \xi_{2,j})^T)_H = \delta_{i,j},$$

for all $1 \leq i, j \leq d_u$.

- (iv) *The projector π_u is characterized by*

$$\pi_u(v, \eta_1, \eta_2)^T = \sum_{i=1}^{d_u} ((v, \eta_1, \eta_2)^T, M_s^*(P\phi_i, \xi_{1,i}, \xi_{2,i})^T)_H (Pv_i, \eta_{1,i}, \eta_{2,i})^T, \quad \forall (v, \eta_1, \eta_2)^T \in Z.$$

- (v) *The matrix Λ_u defined by*

$$\Lambda_u = [\Lambda_{i,j}]_{1 \leq i, j \leq d_u} \quad \text{with} \quad \Lambda_{i,j} = (\mathcal{A}(Pv_i, \eta_{1,i}, \eta_{2,i})^T, M_s^*(P\phi_j, \xi_{1,j}, \xi_{2,j})^T)_H,$$

is constituted of real Jordan blocks.

- (vi) *For all $1 \leq i \leq d_u$, there exists $q_i \in H_{\delta_0}^1(\Omega)$ such that $(v_i, q_i, \eta_{1,i}, \eta_{2,i}) \in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (7.2), and there exists $\psi_i \in H_{\delta_0}^1(\Omega)$ such that $(\phi_i, \psi_i, \xi_{1,i}, \xi_{2,i}) \in H_{\delta_0}$ is a real or imaginary part of a generalized eigenfunction of (7.4).*

Proof. Arguing as in [9, Lemma 6.2] or in [25], since \mathcal{A} is the infinitesimal generator of an analytic semigroup with compact resolvent, there exist a basis $(\tilde{v}_i, \eta_{1,i}, \eta_{2,i})_{1 \leq i \leq d_u}$ of Z_u constituted of real or imaginary parts of eigenfunctions and generalized eigenfunctions of \mathcal{A} and a basis $(M_s^*(\tilde{\phi}_i, \xi_{1,i}, \xi_{2,i}))_{1 \leq i \leq d_u}$ of Z_u^* constituted of the real or imaginary parts of eigenfunctions and generalized eigenfunctions of \mathcal{A}^* , such that

$$((\tilde{v}_i, \eta_{1,i}, \eta_{2,i})^T, M_s^*(\tilde{\phi}_j, \xi_{1,j}, \xi_{2,j})^T)_H = \delta_{i,j}, \quad \forall 1 \leq i, j \leq d_u,$$

$$\pi_u(v, \eta_1, \eta_2)^T = \sum_{i=1}^{d_u} ((v, \eta_1, \eta_2)^T, M_s^*(\tilde{\phi}_i, \xi_{1,i}, \xi_{2,i})^T)_H (\tilde{v}_i, \eta_{1,i}, \eta_{2,i})^T,$$

and the matrix Λ_u is constituted of real Jordan blocks.

We set $v_i = \tilde{v}_i + \nabla N_s \eta_{2,i} + \nabla N_{\text{div} A_3} \eta_{1,i}$ and $\phi_i = \tilde{\phi}_i + \nabla N_s \xi_{2,i}$. Thus, we have $Pv_i = \tilde{v}_i$, $P\phi_i = \tilde{\phi}_i$, and assertions (i), (ii) and (v) are proved. Assertion (iii) follows from Lemma 8.1.

Assertion (vi) comes from Theorems 7.3 and 7.5. \square

We introduce the matrix B_u defined by

$$B_u = [B_{i,j}]_{1 \leq i \leq d_u, 1 \leq j \leq n_c} \quad \text{with} \quad B_{i,j} = (w_i, \xi_{2,j})_{L^2(\Gamma_s)}. \quad (8.3)$$

We set $\mathcal{A}_{s,\omega} = \pi_s(\mathcal{A} + \omega I)$ and $\mathcal{B}_s = \pi_s \mathcal{B}$. We notice that

$$\|e^{t\mathcal{A}_{s,\omega}}\|_{\mathcal{L}(Z)} \leq Ce^{-\varepsilon_s t} \quad \forall t > 0, \quad \text{with } 0 < \varepsilon_s < \text{dist}(\text{Re } \sigma(\mathcal{A}_{s,\omega}), 0).$$

As in [21], we shall see that the stabilization of the linearized system is reduced to the stabilization of the pair $(\Lambda_u + \omega I, B_u)$.

8.2. Stabilizability of the pair $(\Lambda_u + \omega I, B_u)$. First, we introduce the unbounded operator $(A_s, D(A_s))$ in $H_0^2(\Gamma_s) \times L^2(\Gamma_s)$ defined by

$$D(A_s) = (H^4(\Gamma_s) \cap H_0^2(\Gamma_s)) \times H_0^2(\Gamma_s) \quad \text{and} \quad A_s = \begin{pmatrix} 0 & I \\ A_{\alpha,\beta} + A_4 + 2\nu\gamma_s A_3 & \gamma\Delta_s \end{pmatrix}.$$

Let λ be in \mathbb{C} . We introduce the unbounded operator $(L_{*,\lambda}, D(L_{*,\lambda}))$ defined on $L^2(\Gamma_s)$ by $D(L_{*,\lambda}) = H_0^2(\Gamma_s)$ and

$$L_{*,\lambda} = I + (-A_{\alpha,\beta})^{-1}(\lambda^2 - \gamma\lambda\Delta_s - A_4^* - 2\nu(\gamma_s A_3)^*).$$

LEMMA 8.3. *If λ does not belong to $\text{spect}(A_s)$, then $L_{*,\lambda}$ is an isomorphism from $H^4(\Gamma_s) \cap H_0^2(\Gamma_s)$ into $L^2(\Gamma_s)$ and we have*

$$(\lambda I - A_s^*)^{-1} = \begin{pmatrix} (-A_{\alpha,\beta})^{-1}(\lambda - \gamma\Delta)L_{*,\lambda}^{-1} & (-A_{\alpha,\beta})^{-1}(\lambda^2 - \gamma\lambda\Delta)L_{*,\lambda}^{-1}(-A_{\alpha,\beta})^{-1} - (-A_{\alpha,\beta})^{-1} \\ L_{*,\lambda}^{-1} & \lambda L_{*,\lambda}^{-1}(-A_{\alpha,\beta})^{-1} \end{pmatrix}.$$

For all $j \in J_u$ (J_u appears in (8.1)), we introduce the space $E(\lambda_j)$ defined by

$$E(\lambda_j) = \left\{ (\phi, \psi, \xi_1, \xi_2) \in H_{\delta_0} \mid (\lambda_j, (\phi, \psi, \xi_1, \xi_2)) \text{ is solution to the eigenvalue problem (7.4)} \right\}.$$

We choose the family $(w_i)_{1 \leq i \leq n_c}$ such that

$$\text{Vect}\{w_i \mid 1 \leq i \leq n_c\} = \text{Vect}\{\text{Re } \xi_{2,j}^k, \text{Im } \xi_{2,j}^k \mid j \in J_u, 1 \leq k \leq d_j\}, \quad (8.4)$$

where $(\phi_j^k, \psi_j^k, \xi_{1,j}^k, \xi_{2,j}^k)_{1 \leq i \leq d_j}$ is a basis of $E(\lambda_j)$ and "Vect" stands for the vector space spanned by the family within the brackets. The number d_j is the dimension of $E(\lambda_j)$. It also corresponds to the geometrical multiplicity of the eigenvalue λ_j of A .

ASSUMPTION 2. We assume that $-\omega \notin \text{spect}(A)$, $0 \notin \text{spect}(A)$ and

$$\{\lambda \in \text{spect}(A) \mid \text{Re } \lambda \geq -\omega\} \cap \{\lambda \in \text{spect}(A_s) \mid \text{Re } \lambda \geq -\omega\} = \emptyset. \quad (8.5)$$

We consider the following eigenvalue problem

$$\begin{aligned} \lambda \in \mathbb{C}^*, \quad \text{Re } \lambda \geq -\omega, \\ \lambda\phi - \text{div } \sigma(\phi, \psi) - (u_s \cdot \nabla)\phi + (\nabla u_s)^T \phi = 0 \text{ in } \Omega, \quad \text{div } \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \Gamma_s. \end{aligned} \quad (8.6)$$

ASSUMPTION 3. All solution (λ, ϕ, ψ) to (8.6) obeys the following unique continuation property:

$$\text{If } \lambda(A_2^* \phi + \gamma_s \psi) = A_3^* \psi - A_1^* \phi, \quad \text{then } (\phi, \psi) = (0, 0). \quad (8.7)$$

REMARK 8.4. *When $u_s = 0$ (i.e. in the case of the Stokes system), the unique continuation property stated in (8.7) is proved in [22] and [23]. Unfortunately, the proofs given in these two papers cannot be adapted to the case $u_s \neq 0$. Indeed, a necessary and may be not sufficient condition to adapt those proofs should be to assume that u_s is analytic. However, this property can be verified numerically, see [8].*

THEOREM 8.5. *We assume that Assumptions 1, 2, and 3 are satisfied, and that $(w_i)_{1 \leq i \leq n_c}$ is given by (8.4). Then, the pair $(\Lambda_u + \omega I_{\mathbb{R}^{n_c}}, B_u)$ is stabilizable.*

Proof. Thanks to [4, Part III, Chapter 1, Proposition 3.3], the pair $(\Lambda_u + \omega I_{\mathbb{R}^{n_c}}, B_u)$ is stabilizable if and only if

$$\text{Ker}(\lambda I - \mathcal{A}^*) \cap \text{Ker}(\mathcal{B}^*) = \{0\} \text{ for all } \lambda \in \mathbb{C} \text{ such that } \text{Re } \lambda \geq -\omega.$$

Let $M_s^*(P\phi, \xi_1, \xi_2)$ belong to $\text{Ker}(\lambda I - \mathcal{A}^*) \cap \text{Ker}(\mathcal{B}^*)$. We set $(I - P)\phi = \nabla N_s \xi_2$. From Proposition 6.4, it follows that there exists $\psi \in H_{\delta_0}^1(\Omega)$ such that $(\lambda, (\phi, \psi, \xi_1, \xi_2)) \in \mathbb{C} \times H_{\delta_0}$ is solution to the eigenvalue problem (7.4). Since $\mathcal{B}^* M_s^*(P\phi, \xi_1, \xi_2)^T = 0$, we have

$$\left(\int_{\Gamma_s} w_i \cdot \xi_2 \right)_{1 \leq i \leq n_c} = 0. \quad (8.8)$$

We are going to prove that $(\phi, \psi, \xi_1, \xi_2) = (0, 0, 0, 0)$. From equation (8.8) and due to the construction of $(w_i)_{1 \leq i \leq n_c}$, it follows that $\xi_2 = 0$. Next, we can distinguish two cases.

Case 1. $\lambda \notin \text{spect}(A)$. Then, we have $(\phi, \psi) = (0, 0)$ and, by using equation (7.4)₄, we also have $\xi_1 = 0$.

Case 2. $\lambda \in \text{spect}(A)$. In that case, λ does not belong to $\text{spect}(A_s)$. Therefore, we have

$$(\xi_1, \xi_2)^T = (\lambda I - A_s^*)^{-1} (-A_{\alpha, \beta})^{-1} (A_1^* \phi - A_3^* \psi, A_2^* \phi + \gamma_s \psi)^T,$$

which implies that $\xi_2 = L_{*, \lambda}^{-1} (-A_{\alpha, \beta})^{-1} (A_1^* \phi - A_3^* \psi + \lambda A_2^* \phi + \lambda \gamma_s \psi)$. Since $\xi_2 = 0$, we have $\lambda (A_2^* \phi + \gamma_s \psi) = A_3^* \psi - A_1^* \phi$. Thus, thanks to Assumption 3, we have $(\phi, \psi) = (0, 0)$. Equation (7.4)₄ gives $\xi_1 = 0$. \square

8.3. The feedback law. We are going to determine a feedback law of finite dimension able to stabilize the linearized system associated to (3.3). For that, it is sufficient to find a feedback law stabilizing the pair $(\Lambda_u + \omega I, B_u)$. Since $(\Lambda_u + \omega I, B_u)$ is stabilizable and $-\Lambda_u - \omega I$ is stable, the following Algebraic Riccati Equation

$$P_u \in \mathbb{R}^{d_u \times d_u}, \quad P_u = P_u^T > 0, \quad P_u (\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}}) P_u - P_u B_u B_u^T P_u = 0,$$

admits a unique solution P_u . Moreover, the operator $K_u = [K_u^{i,j}]_{1 \leq i \leq n_c, 1 \leq j \leq d_u}$ defined by $K_u = -B_u^T P_u$ provides a stabilizing feedback law for $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, B_u)$. We introduce the operator $\mathcal{K}_0 \in \mathcal{L}(Z, \mathbb{R}^{n_c})$ defined by

$$\mathcal{K}_0(Pv, \eta_1, \eta_2)^T = \left(\sum_{j=1}^{d_u} K_u^{i,j} ((Pv, \eta_1, \eta_2)^T, (P\phi_j, \xi_{1,j}, \xi_{2,j})^T)_H \right)_{1 \leq i \leq n_c}.$$

THEOREM 8.6. *We assume that Assumptions 1, 2, and 3 are satisfied. Then, the operator \mathcal{K}_0 provides a stabilizing feedback for $(\mathcal{A} + \omega I, \mathcal{B})$. Moreover, the operator $\mathcal{A} + \omega I + \mathcal{B}\mathcal{K}_0$, with domain $D(\mathcal{A} + \omega I + \mathcal{B}\mathcal{K}_0) = D(\mathcal{A})$, is the infinitesimal generator of an exponentially stable analytic semigroup on Z .*

Proof. The proof is similar to that of [14, Theorem 3]. \square

PROPOSITION 8.7. *Let (Pv, η_1, η_2) belong to Z . If we choose $(I - P)v = \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1 \in \mathbf{L}^2(\Omega)$, then we have*

$$\mathcal{K}_0(Pv, \eta_1, \eta_2)^T = \mathcal{K}(Pv + (I - P)v, \eta_1, \eta_2)^T,$$

where the operator $\mathcal{K} \in \mathcal{L}(H, \mathbb{R}^{n_c})$ is defined by

$$\mathcal{K}(v, \eta_1, \eta_2)^T = \left(\sum_{j=1}^{d_u} K_u^{i,j} ((v, \eta_1, \eta_2)^T, (\phi_j, \xi_{1,j}, \xi_{2,j})^T)_H \right)_{1 \leq i \leq n_c}, \quad \forall (v, \eta_1, \eta_2) \in H. \quad (8.9)$$

Proof. It is a consequence of Lemma 8.1. \square

9. Stabilization of the non-homogeneous linearized system. We are going to prove that \mathcal{K} , defined in (8.7), stabilizes the following non-homogeneous closed-loop system

$$\begin{aligned} v_t - \text{div } \sigma(v, q) + (u_s \cdot \nabla)v + (v \cdot \nabla)u_s - A_1 \eta_1 - A_2 \eta_2 - \omega v &= F_f \text{ in } Q^\infty, \\ \text{div } v &= A_3 \eta_1 + \text{div } F_{\text{div}} \text{ in } Q^\infty, \quad v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad v = g_p \text{ on } \Sigma_i^\infty, \\ v \cdot n &= 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v, q)n = 0 \text{ on } \Sigma_n^\infty, \\ \eta_{1,t} - \eta_2 - \omega \eta_1 &= 0 \text{ on } \Sigma_s^\infty, \\ \eta_{2,t} - \beta \Delta_s \eta_1 - \gamma \Delta_s \eta_2 + \alpha \Delta_s^2 \eta_1 - \omega \eta_2 &= \gamma_s q + F_s + \sum_{i=1}^{n_c} [\mathcal{K}(v, \eta_1, \eta_2)^T]_i w_i \text{ on } \Sigma_s^\infty, \\ \eta_1 &= 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \\ v(0) &= v^0 \text{ in } \Omega, \quad \eta_1(0) = 0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \quad (9.1)$$

where $[\mathcal{K}(v, \eta_1, \eta_2)^T]_i$ is the i -th component of the vector $\mathcal{K}(v, \eta_1, \eta_2)^T \in \mathbb{R}^{n_c}$.

REMARK 9.1. *The feedback law \mathcal{K} does not depend on the Leray projector P .*

THEOREM 9.2. *We assume that Assumptions 1, 2, and 3 are satisfied. Let $(F_f, F_{\text{div}}, F_s)$ belong to Y , g_p belong to $H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$ and (v^0, η_2^0) belong to $\mathbf{V}_{\Gamma_i, e}^1(\Omega) \times H_0^1(\Gamma_s)$ such that $v^0|_{\Gamma_s} = \eta_2^0 n$. We assume that F_{div} satisfies the following boundary conditions*

$$\begin{aligned} F_{\text{div}} &= 0 \text{ on } \Sigma_s^\infty, \quad F_{\text{div}} = 0 \text{ on } \Sigma_i^\infty, \quad F_{\text{div}} \cdot n = 0 \text{ on } \Sigma_e^\infty, \\ \varepsilon(F_{\text{div}})n \cdot \tau &= 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(F_{\text{div}})n = 0 \text{ on } \Sigma_n^\infty. \end{aligned} \quad (9.2)$$

The solution (v, q, η_1, η_2) of system (9.1) satisfies the estimate

$$\|(v, q, \eta_1, \eta_2)\|_{X_{\delta_0}} \leq C \|(v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s)\|,$$

where $\|(v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s)\| := \|v^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(\Gamma_s)} + \|g_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))} + \|(F_f, F_{\text{div}}, F_s)\|_Y$.

Proof. Step 1. Reformulation of the system. Due to Remark 4.2, the solution (w, π) of the system

$$\begin{aligned} \lambda_0 w - \text{div } \sigma(w, \pi) + (u_s \cdot \nabla)w + (w \cdot \nabla)u_s &= 0, \quad \text{div } w = 0 \text{ in } \Omega, \\ w &= 0 \text{ on } \Gamma_s, \quad w = g_p \text{ on } \Gamma_i, \quad w \cdot n = 0 \text{ and } \varepsilon(w)n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(w, \pi)n = 0 \text{ on } \Gamma_n. \end{aligned} \quad (9.3)$$

satisfies

$$\|w\|_{H_0^1(0, \infty; H_{\delta_0}^2(\Omega))} + \|\pi\|_{L^2(0, \infty; H_{\delta_0}^1(\Omega))} \leq C \|g_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))}. \quad (9.4)$$

We set $\hat{v} = v - F_{\text{div}} - w$ and $\hat{q} = q - \pi$. The quadruplet $(\hat{v}, \hat{q}, \eta_1, \eta_2)$ is solution to the system

$$\begin{aligned} \hat{v}_t - \text{div } \sigma(\hat{v}, \hat{q}) + (u_s \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)u_s - A_1 \eta_1 - A_2 \eta_2 - \omega \hat{v} &= \tilde{F}_f \text{ in } Q^\infty, \\ \text{div } \hat{v} &= A_3 \eta_1 \text{ in } Q^\infty, \quad \hat{v} = \eta_2 n \text{ on } \Sigma_s^\infty, \quad \hat{v} = 0 \text{ on } \Sigma_i^\infty, \\ \hat{v} \cdot n &= 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\hat{v})n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\hat{v}, \hat{q})n = 0 \text{ on } \Sigma_n^\infty, \\ \eta_{1,t} - \eta_2 - \omega \eta_1 &= 0 \text{ on } \Sigma_s^\infty, \\ \eta_{2,t} - \beta \Delta_s \eta_1 - \gamma \Delta_s \eta_2 + \alpha \Delta_s^2 \eta_1 - \omega \eta_2 &= \gamma_s \hat{q} + \tilde{F}_s + \sum_{i=1}^{n_c} [\mathcal{K}(\hat{v}, \eta_1, \eta_2)^T]_i w_i \text{ on } \Sigma_s^\infty, \\ \eta_1 &= 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \\ \hat{v}(0) &= v^0 \text{ in } \Omega, \quad \eta_1(0) = 0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \quad (9.5)$$

where

$$\begin{aligned} \tilde{F}_f &= F_f - F_{\text{div},t} + 2\nu \text{div } \varepsilon(F_{\text{div}}) - (u_s \cdot \nabla)F_{\text{div}} - (F_{\text{div}} \cdot \nabla)u_s + \omega F_{\text{div}} + \lambda_0 w + w_t, \\ \text{and } \tilde{F}_s &= F_s + \sum_{i=1}^{n_c} [\mathcal{K}(F_{\text{div}} + w, 0, 0)^T]_i w_i + \gamma_s \pi. \end{aligned}$$

We can verify that $(\hat{v}, \eta_1, \eta_2) = (P\hat{v}, \eta_1, \eta_2) + ((I - P)\hat{v}, 0, 0)$ is solution to the system

$$\begin{aligned} \frac{d}{dt}(P\hat{v}, \eta_1, \eta_2)^T &= (\mathcal{A} + \omega I + \mathcal{BK}_0)(P\hat{v}, \eta_1, \eta_2)^T + (P\tilde{F}_f, 0, \tilde{F}_s + \gamma_s N\tilde{F}_f), \\ (P\hat{v}, \eta_1, \eta_2)^T(0) &= (Pv^0, 0, \eta_2^0)^T, \quad (I - P)\hat{v} = \nabla N_s \eta_2 + \nabla N_{\text{div}} A_3 \eta_1. \end{aligned} \quad (9.6)$$

This is the analogue of Proposition 5.6 for time dependent problems.

Step 2. Regularity of solutions to system (9.5). Thanks to Propositions 5.16 and 5.1, we prove that $(Pv^0, 0, \eta_2^0)$ belongs to $[D(\mathcal{A}), Z]_{\frac{1}{2}}$, $(P\tilde{F}_f, 0, F_s + \gamma_s N_p \tilde{F}_f)$ belongs to $L^2(0, \infty; Z)$ and we have the estimate

$$\|(Pv^0, 0, \eta_2^0)\|_{[D(\mathcal{A}), Z]_{\frac{1}{2}}} + \|(P\tilde{F}_f, 0, F_s + \gamma_s N_p \tilde{F}_f)\|_{L^2(0, \infty; Z)} \leq C \|(v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s)\|. \quad (9.7)$$

Since $(\mathcal{A} + \omega I + \mathcal{BK}_0, D(\mathcal{A} + \omega I + \mathcal{BK}_0))$ is the infinitesimal generator of an exponentially stable analytic semigroup on Z , it follows that

$$\|(P\hat{v}, \eta_1, \eta_2)\|_{L^2(0, \infty; D(\mathcal{A}) \cap H^1(0, \infty; Z))} \leq C \|(v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s)\|,$$

(see [3, Chapter 1, Theorem 3.1]). In particular, we have

$$\|P\hat{v}\|_{H^1(0, \infty; L^2(\Omega))} + \|\eta_1\|_{H^{4,2}(\Sigma_s^\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma_s^\infty)} \leq C \|(v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s)\|. \quad (9.8)$$

From (9.6) and (9.8), it follows that $(I - P)\widehat{v}$ belongs to $H^1(0, \infty; \mathbf{L}^2(\Omega))$. It implies that $\widehat{v} = P\widehat{v} + (I - P)\widehat{v} \in H^1(0, \infty; \mathbf{L}^2(\Omega))$ and

$$\|\widehat{v}\|_{H^1(0, \infty; \mathbf{L}^2(\Omega))} \leq C \left\| \left\| (v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s) \right\| \right\|. \quad (9.9)$$

For all $t > 0$, $(\widehat{v}(t), \widehat{q}(t))$ is solution to the system

$$\begin{aligned} \lambda_0 \widehat{v}(t) - \text{div} \sigma(\widehat{v}(t), \widehat{q}(t)) + (u_s \cdot \nabla) \widehat{v}(t) + (\widehat{v}(t) \cdot \nabla) u_s &= G(t) \text{ in } \Omega, \\ \text{div} \widehat{v}(t) &= A_3 \eta_1(t) \text{ in } \Omega, \quad \widehat{v}(t) = \eta_2(t) n \text{ on } \Gamma_s, \quad \widehat{v}(t) = 0 \text{ on } \Gamma_i, \\ \widehat{v}(t) \cdot n &= 0 \text{ on } \Gamma_e, \quad \varepsilon(\widehat{v}(t)) n \cdot \tau = 0 \text{ on } \Gamma_e, \quad \sigma(\widehat{v}(t), \widehat{q}(t)) n = 0 \text{ on } \Gamma_n, \end{aligned} \quad (9.10)$$

where

$$G = -\widehat{v}_t + (\lambda_0 + \omega) \widehat{v} - (u_s \cdot \nabla) \widehat{v} - (\widehat{v} \cdot \nabla) u_s + A_1 \eta_1 + A_2 \eta_2 + \widetilde{F}_f.$$

From (9.7), (9.8) and (9.9), it follows that G belongs to $L^2(0, \infty; \mathbf{L}^2(\Omega))$ and

$$\|G\|_{L^2(0, \infty; \mathbf{L}^2(\Omega))} \leq C \left\| \left\| (v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s) \right\| \right\|.$$

Thus, thanks to Theorem 4.1, $(\widehat{v}(t), \widehat{q}(t))$ belongs to $\mathbf{H}_{\delta_0}^2(\Omega) \times H_{\delta_0}^1(\Omega)$ and

$$\|\widehat{v}(t)\|_{\mathbf{H}_{\delta_0}^2(\Omega)} + \|\widehat{q}(t)\|_{H_{\delta_0}^1(\Omega)} \leq C (\|G(t)\|_{\mathbf{L}^2(\Omega)} + \|\eta_2(t)\|_{H_0^{\frac{3}{2}}(\Gamma_s)} + \|A_3 \eta_1(t)\|_{H^1(\Omega)}). \quad (9.11)$$

We deduce that $(\widehat{v}, \widehat{p})$ belongs to $L^2(0, \infty; \mathbf{H}_{\delta_0}^2(\Omega)) \times L^2(0, \infty; H_{\delta_0}^1(\Omega))$ and

$$\|\widehat{v}\|_{L^2(0, \infty; \mathbf{H}_{\delta_0}^2(\Omega))} + \|\widehat{p}\|_{L^2(0, \infty; H_{\delta_0}^1(\Omega))} \leq C \left\| \left\| (v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s) \right\| \right\|. \quad (9.12)$$

By combining inequalities (9.4), (9.8) and (9.12), we finally prove that

$$\|(v, q, \eta_1, \eta_2)\|_{X_{\delta_0}} \leq C \left\| \left\| (v^0, \eta_2^0, g_p, F_f, F_{\text{div}}, F_s) \right\| \right\|.$$

□

10. Stabilization of the nonlinear system. The goal of this section is to prove Theorems 3.3 and 3.2. For that, we are going to show that the feedback law \mathcal{K} , stabilizing the linearized system associated to (3.3), is also able to locally stabilize the nonlinear system (3.3). We consider the nonlinear closed-loop system

$$\begin{aligned} \widehat{u}_t - \text{div} \sigma(\widehat{u}, \widehat{p}) + (u_s \cdot \nabla) \widehat{u} + (\widehat{u} \cdot \nabla) u_s - A_1 \widehat{\eta}_1 - A_2 \widehat{\eta}_2 - \omega \widehat{u} &= \mathcal{F}_f[\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2] \text{ in } Q^\infty, \\ \text{div} \widehat{u} &= A_3 \widehat{\eta}_1 + \text{div} \mathcal{F}_{\text{div}}[\widehat{u}, \widehat{\eta}_1] \text{ in } Q^\infty, \quad \widehat{u} = \widehat{\eta}_2 n \text{ on } \Sigma_s^\infty, \quad \widehat{u} = \widehat{g}_p \text{ on } \Sigma_i^\infty, \\ \widehat{u} \cdot n &= 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\widehat{u}) n \cdot \tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\widehat{u}, \widehat{p}) n = 0 \text{ on } \Sigma_n^\infty, \\ \widehat{\eta}_{1,t} - \widehat{\eta}_2 - \omega \widehat{\eta}_1 &= 0 \text{ on } \Sigma_s^\infty, \\ \widehat{\eta}_{2,t} - \beta \Delta_s \widehat{\eta}_1 - \gamma \Delta_s \widehat{\eta}_2 + \alpha \Delta_s^2 \widehat{\eta}_1 - A_4 \widehat{\eta}_1 - \omega \widehat{\eta}_2 &= \gamma_s \widehat{p} + \mathcal{F}_s[\widehat{u}, \widehat{\eta}_1] + \sum_{i=1}^{n_c} [\mathcal{K}(\widehat{u}, \widehat{\eta}_1, \widehat{\eta}_2)^T]_i w_i \text{ on } \Sigma_s^\infty, \\ \widehat{\eta}_1 &= 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \quad \widehat{\eta}_{1,x} = 0 \text{ on } (0, \infty) \times \partial \Gamma_s, \\ \widehat{u}(0) &= \widehat{u}^0 \text{ in } \Omega, \quad \widehat{\eta}_1(0) = 0 \text{ on } \Gamma_s, \quad \widehat{\eta}_2(0) = \eta_2^0 \text{ on } \Gamma_s. \end{aligned} \quad (10.1)$$

Due to the compatibility condition $\widehat{u}^0|_{\Gamma_s} = u^0 - u_s|_{\Gamma_s} = \eta_2^0 n$, and since $\widehat{\eta}_1(0) = 0$, the initial condition $\widehat{u}(0) = \widehat{u}^0$ is equivalent to $P\widehat{u}(0) = P\widehat{u}^0$.

We are going to use the results on the non-homogeneous linear system (9.1) and a fixed point argument to prove that system (10.1) is locally exponentially stable. For that, we introduce the mapping \mathcal{G} from X_{δ_0} into itself, defined by

$$\mathcal{G}(\widetilde{v}, \widetilde{q}, \widetilde{\eta}_1, \widetilde{\eta}_2) = (v, q, \eta_1, \eta_2),$$

where (v, q, η_1, η_2) is the solution of system (9.1) with right hand side

$$(F_f, F_{\text{div}}, F_s) = e^{-\omega t} (\mathcal{F}_f[\widetilde{v}, \widetilde{q}, \widetilde{\eta}_1, \widetilde{\eta}_2], \mathcal{F}_{\text{div}}[\widetilde{v}, \widetilde{\eta}_1], \mathcal{F}_s[\widetilde{v}, \widetilde{\eta}_1]),$$

initial data (\hat{u}^0, η_2^0) in $\mathbf{V}_{\Gamma_i, e}^1(\Omega) \times H_0^1(\Gamma_s)$ such that $\hat{u}^0 = \eta_2^0 n$ on Γ_s and boundary perturbation \hat{g}_p in $H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$. We are going to prove that there exists $R > 0$ such that \mathcal{G} is a contraction in

$$B_{\delta_0, R} = \{(v, q, \eta_1, \eta_2) \in X_{\delta_0} \mid \|(v, q, \eta_1, \eta_2)\|_{X_{\delta_0}} \leq R \quad \text{and} \quad \|e^{-\omega \cdot} \eta_1\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max}\},$$

where $0 < \eta_{\max} < e$.

LEMMA 10.1. *There exists $C > 0$ such that, for all η be $H^{4,2}(\Sigma_s^\infty)$, we have*

$$\begin{aligned} \|\eta\|_{C_b([0, \infty); C^1(\bar{\Gamma}_s))} + \|\eta_{xx}\|_{L^\infty(\Sigma_s^\infty)} &\leq C\|\eta\|_{H^{4,2}(\Sigma_s^\infty)}, \\ \|\eta_t\|_{L^2(0, \infty; L^\infty(\Gamma_s))} + \|\eta_{tx}\|_{L^2(0, \infty; L^\infty(\Gamma_s))} + \|\eta_{xxx}\|_{L^2(0, \infty; L^\infty(\Gamma_s))} &\leq C\|\eta\|_{H^{4,2}(\Sigma_s^\infty)}. \end{aligned}$$

Proof. The results are standard, and their proofs are left to the reader. \square

PROPOSITION 10.2. *There exists $C > 0$ such that, for all $(v, q, \eta_1, \eta_2) \in B_{\delta_0, R}$, we have*

$$\|(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\text{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1])\|_Y \leq C(1+R)R^2. \quad (10.2)$$

If (v, q, η_1, η_2) and $(\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)$ belong to $B_{\delta_0, R}$, then we have

$$\begin{aligned} &\|(\mathcal{F}_f[v, p, \eta_1, \eta_2], \mathcal{F}_{\text{div}}[v, \eta_1], \mathcal{F}_s[v, \eta_1]) - (\mathcal{F}_f[\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2], \mathcal{F}_{\text{div}}[\tilde{v}, \tilde{\eta}_1], \mathcal{F}_s[\tilde{v}, \tilde{\eta}_1])\|_Y \\ &\leq CR(1+R)\|(v, q, \eta_1, \eta_2) - (\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{X_{\delta_0}}. \end{aligned} \quad (10.3)$$

The mapping \mathcal{G} is well-defined. For all (v, q, η_1, η_2) and $(\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)$ in $B_{\delta_0, R}$, we have

$$\begin{aligned} \|\mathcal{G}(v, q, \eta_1, \eta_2)\|_{X_{\delta_0}} &\leq C\left(\|\hat{u}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(\Gamma_s)} + \|\hat{g}_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))} + (1+R)R^2\right), \\ \|\mathcal{G}(v, q, \eta_1, \eta_2) - \mathcal{G}(\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{X_{\delta_0}} &\leq CR(1+R)\|(v, q, \eta_1, \eta_2) - (\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{X_{\delta_0}}. \end{aligned} \quad (10.4)$$

Proof. Estimate of $\mathcal{F}_{\text{div}}[v, \eta_1]$. The estimates of $\mathcal{F}_{\text{div}}[v, \eta_1] = -\eta_1 v_1 e_1 + z \eta_{1,x} v_1 e_2$ may be obtained by computing $\mathcal{F}_{\text{div}, x}$, $\mathcal{F}_{\text{div}, z}$, $\mathcal{F}_{\text{div}, xx}$, $\mathcal{F}_{\text{div}, zz}$, and $\mathcal{F}_{\text{div}, t}$, and by using Lemma 10.1.

Estimate of $\mathcal{F}_f[v, p, \eta_1, \eta_2]$. To prove that $\mathcal{F}_f[v, p, \eta_1, \eta_2]$ belongs to $L^2(Q^\infty)$, we are going to show that all its terms belong to $L^2(Q^\infty)$. By using Lemma 10.1, we can prove that all the nonlinear terms involving either η_1 or η_2 and v belong to $L^2(Q^\infty)$. For example, we have

$$\|\eta_{1,xx} v_z\|_{\mathbf{L}^2(Q^\infty)} \leq \|\eta_{1,xx}\|_{L^\infty(\Sigma_s^\infty)} \|v_z\|_{\mathbf{L}^2(Q^\infty)} \leq C(1+R)R^2.$$

Let us prove that $\eta_{1,x} v_{xz}$ belongs to $L^2(Q^\infty)$. We have $\eta_{1,x} \in H^1(0, \infty; H_0^1(\Gamma_s)) \hookrightarrow L^\infty(0, \infty; H_0^1(\Gamma_s))$ and $v_{xz} \in L^2(0, \infty; \mathbf{L}_{\delta_0}^2(\Omega))$. Therefore, thanks to Lemma 5.3, we have

$$\|\eta_{1,x} v_{xz}\|_{L^2(Q^\infty)} \leq C\|\eta_{1,x}\|_{L^\infty(0, \infty; H_0^1(\Gamma_s))} \|v_{xz}\|_{L^2(0, \infty; \mathbf{L}_{\delta_0}^2(\Omega))} \leq C(1+R)R^2.$$

Similarly, we can prove that the involving η_1 and second order derivatives of v belong to $L^2(Q^\infty)$.

Now, we estimate $(v \cdot \nabla)v$. Since $v \in L^2(0, \infty; \mathbf{H}^{\frac{3}{2} + \varepsilon_0}(\Omega)) \cap H^1(0, \infty; \mathbf{L}^2(\Omega))$, v belongs to $L^\infty(0, \infty; \mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega))$ and ∇v belongs to $L^2(0, \infty; \mathbf{H}^{\frac{1}{2} + \varepsilon_0}(\Omega))$. Thanks to [13, Proposition B.1], we have

$$\|(v \cdot \nabla)v\|_{\mathbf{L}^2(Q^\infty)} \leq C\|v\|_{L^\infty(0, \infty; \mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega))} \|\nabla v\|_{L^2(0, \infty; \mathbf{H}^{\frac{1}{2} + \varepsilon_0}(\Omega))} \leq C(1+R)R^2.$$

Thus, $\mathcal{F}_f[v, p, \eta_1, \eta_2]$ belongs to $\mathbf{L}^2(Q^\infty)$ and $\|\mathcal{F}_f[v, p, \eta_1, \eta_2]\|_{\mathbf{L}^2(Q^\infty)} \leq C(1+R)R^2$.

Estimate of $\mathcal{F}_s[v, \eta_1]$. We can easily prove that $\mathcal{F}_s[v, \eta_1]$ belongs to $L^2(\Sigma_s^\infty)$ when v belongs to $H_\delta^{2,1}(Q_\infty)$ and η_1 belongs to $H^{4,2}(\Sigma_s^\infty)$.

The Lipschitz estimates in (10.3) can be obtained with the same arguments as above. They follow from the fact that all the terms involved in the calculations are at least quadratic with respect to (v, q, η_1, η_2) and $(\tilde{v}, \tilde{q}, \tilde{\eta}_1, \tilde{\eta}_2)$.

The estimates in (10.4) follows from estimates (10.2) and (10.3), and from Theorem 9.2. \square

THEOREM 10.3. *Let η_{\max} be in $(0, e)$ and ω be positive. There exists $r > 0$ such that, for all $(\hat{u}^0, \eta_2^0) \in \mathbf{V}_{\Gamma_i, e}^1(\Omega) \times H_0^1(\Gamma_s)$ and $\hat{g}_p \in H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$ satisfying $\hat{u}^0 = \eta_2^0 n$ on Γ_s , $\hat{g}_p \in H_0^1(0, \infty; \mathbf{H}(\Gamma_i))$ and*

$$\|\hat{u}^0\|_{\mathbf{H}^1(\Omega)} + \|\eta_2^0\|_{H^1(\Gamma_s)} + \|\hat{g}_p\|_{H_0^1(0, \infty; \mathbf{H}(\Gamma_i))} \leq r,$$

system (10.1) admits a solution $(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2) \in X_{\delta_0}$ satisfying the estimates

$$\|e^{-\omega t} \hat{\eta}_1(t)\|_{L^\infty(\Gamma_s)} \leq \eta_{\max} \quad \text{and} \quad \|(\hat{u}(t), \hat{\eta}_1(t), \hat{\eta}_2(t))\|_{\mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega) \times H^3(\Gamma_s) \times H^1(\Gamma_s)} \leq C, \quad \forall t > 0.$$

Proof. We choose $R_0 > 0$ small enough such that

$$\frac{R_0}{2} + C(1 + R_0)R_0^2 < R_0, \quad CR_0(1 + R_0) < 1, \quad \text{and} \quad CR_0 < \eta_{\max},$$

and we set $r = 2CR_0$, where C is the constant appearing in (10.4). Due to (10.4), with such a choice, \mathcal{G} is a contraction in B_{δ_0, R_0} , and system (10.1) admits a unique solution $(\hat{u}, \hat{q}, \hat{\eta}_1, \hat{\eta}_2) \in B_{\delta_0, R_0}$ satisfying the estimates

$$\|e^{-\omega \cdot} \hat{\eta}_1\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max} \quad \text{and} \quad \|(\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2)\|_{X_{\delta_0}} \leq R_0.$$

From [18, Theorem 2.1] and Lemma 4.3, it follows that

$$\|\hat{u}(t)\|_{\mathbf{H}^{\frac{3}{4} + \frac{\varepsilon_0}{2}}(\Omega)} + \|\hat{\eta}_1(t)\|_{H^3(\Gamma_s)} + \|\hat{\eta}_2(t)\|_{H^1(\Gamma_s)} \leq CR_0.$$

□

Proof of Theorem 3.2. Thanks to Theorem 10.3, we have to check that, for all $\eta \in H^{4,2}(\Sigma_s^\infty)$ such that $\|\eta\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max}$, and all $t \in (0, \infty)$, $\mathcal{T}_{\eta(t)}$ is a \mathcal{C}^1 -diffeomorphism from $\Omega_{\eta(t)}$ into Ω . For that, it is enough to use the continuous embedding $H^{4,2}(\Sigma_s^\infty) \hookrightarrow \mathcal{C}_b([0, \infty); \mathcal{C}^1(\Gamma_s))$ (see Lemma 10.1) and the estimate $\|\eta\|_{L^\infty(\Sigma_s^\infty)} \leq \eta_{\max}$.

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Appendix A. Nonlinear terms. The nonlinear terms \mathcal{F}_f and \mathcal{F}_s in system (3.3) are defined by

$$\begin{aligned} \mathcal{F}_f[\hat{u}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2] = & \frac{(\ell - s(z)z)\hat{\eta}_2\hat{u}_z}{\ell - e} - \frac{\nu(\ell - s(z)z)\hat{\eta}_{1,xx}\hat{u}_z}{\ell - e} - \frac{2\nu(\ell - s(z)z)\hat{\eta}_{1,x}\hat{u}_{xz}}{\ell - e} \\ & + \frac{2\nu s(z)\hat{\eta}_1\hat{u}_{zz}}{\ell - e} - \nu \nabla \operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1] + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}\hat{p}_z e_1}{\ell - e} - \frac{s(z)\hat{\eta}_1\hat{p}_z e_2}{\ell - e} \\ & + \frac{(\ell - s(z)z)e^{-\omega t}\hat{\eta}_{1,x}\hat{u}_1\hat{u}_z}{\ell - e} + \frac{(\ell - s(z)z)\hat{\eta}_{1,x}\hat{u}_1 u_{s,z}}{\ell - e} + \frac{(\ell - s(z)z)\hat{\eta}_{1,x} u_{s,1}\hat{u}_z}{\ell - e} \\ & - \frac{e^{-\omega t} s(z)\hat{\eta}_1\hat{u}_2\hat{u}_z}{\ell - e} - \frac{s(z)\hat{\eta}_1\hat{u}_2 u_{s,z}}{\ell - e} - \frac{s(z)\hat{\eta}_1 u_{s,2}\hat{u}_z}{\ell - e} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_2(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\ & + \frac{\nu(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,xx}(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{2\nu(\ell - s(z)z)\hat{\eta}_{1,x}^2(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} \\ & - \frac{2\nu(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_{xz} + u_{s,xz})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{\nu((\ell - s(z)z)^2\hat{\eta}_{1,x}^2 - \hat{\eta}_1^2)(e^{-\omega t}\hat{u}_{zz} + u_{s,zz})}{(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} \\ & + \frac{2\nu(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)\hat{\eta}_1^2(e^{-\omega t}\hat{u}_{zz} + u_{s,zz})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)^2} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{p}_z + p_{s,z})e_1}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\ & - \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{p}_z + p_{s,z})e_2}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} + \frac{(\ell - s(z)z)s(z)\hat{\eta}_1\hat{\eta}_{1,x}(e^{-\omega t}\hat{u}_1 + u_{s,1})(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} \\ & - \frac{\hat{\eta}_1^2(e^{-\omega t}\hat{u}_2 + u_{s,2})(e^{-\omega t}\hat{u}_z + u_{s,z})}{(\ell - e)(\ell - e - s(z)e^{-\omega t}\hat{\eta}_1)} - e^{-\omega t}(\hat{u} \cdot \nabla)\hat{u}, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_s[\hat{u}, \hat{\eta}_1] = & \nu \left(\hat{\eta}_{1,x} \hat{u}_{1,z} + \hat{\eta}_{1,x} \hat{u}_{2,x} + \frac{2s(z) \hat{\eta}_1 \hat{u}_{2,z}}{\ell - e} \right) - \frac{\nu \hat{\eta}_1 \hat{\eta}_{1,x} (e^{-\omega t} \hat{u}_{1,z} + u_{s,1,z})}{\ell - e - s(z) e^{-\omega t} \hat{\eta}_1} \\ & - \frac{2\nu \hat{\eta}_1^2 (e^{-\omega t} \hat{u}_{2,z} + u_{s,2,z})}{(\ell - e)(\ell - e - s(z) e^{-\omega t} \hat{\eta}_1)} - \frac{\nu(\ell - e) \eta_{1,x}^2 (e^{-\omega t} \hat{u}_{2,z} + u_{s,2,z})}{\ell - e - s(z) e^{-\omega t} \hat{\eta}_1} - 2\nu \gamma_s (\operatorname{div} \mathcal{F}_{\operatorname{div}}[\hat{u}, \hat{\eta}_1]). \end{aligned}$$

In the definition of the nonlinear terms, as indicated in Remark 3.1, the extension operator \mathcal{E} is implicitly present in front of all the terms involving $\hat{\eta}_1$ or $\hat{\eta}_2$.

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