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FOURNIÉ MICHEL*, NDIAYE MOCTAR*, AND RAYMOND JEAN-PIERRE*

Abstract. We study the numerical stabilization of a fluid-structure interaction system in a wind tunnel, around an unstable stationary solution. The goal is to find a feedback control, acting only in the structure equation, able to stabilize locally the full coupled nonlinear system. The Finite Element Approximation of this coupled system leads to a Differential Algebraic Equation (D.A.E.) for the fluid velocity, the structure displacement and its velocity, and a Lagrange multiplier taking into account together the incompressibility condition and the equality of the fluid velocity and the displacement velocity of the structure at the fluid-structure interface. We determine a feedback control law based on a spectral decomposition of the D.A.E. But the D.A.E. is not standard because the operator acting on the Lagrange multiplier in the differential equation is not the transpose of the operator involved in the algebraic constraints. We overcome these difficulties by proving new relationships between eigenvalue problems involving Lagrange multipliers and those without Lagrange multipliers. In numerical simulations, we show how the calculation of the degree of stabilizability of different invariant subspaces (generalized eigenspaces) may be helpful to determine an efficient control strategy. In particular, the determined feedback law is able to reject a perturbation (leading to a complete stabilization of the nonlinear fluid-structure system) whose magnitude is of order 15% of the inflow velocity.

Key words. Fluid-structure interaction, feedback control, stabilization, Navier-Stokes equations, beam equations, Finite element approximation.

AMS subject classifications. 93B52, 37M99, 37N10, 76D55, 74F10, 76D05

1. Introduction. We are interested in stabilizing a fluid-structure interaction model in a wind tunnel whose geometrical configuration is represented in Figure 1.1. The inflow velocity U_∞ generated by a powerful fan system is assumed to be constant and horizontal $U_\infty = (u_\infty, 0)^T$, with $u_\infty > 0$. A thick plate, with a straight base, is located in the center of the wind tunnel, and the fluid flows around this obstacle. The wind tunnel is formally divided into two subdomains. In the left hand side of the wind tunnel (LHS for short), the fluid flow is not precisely modeled. At the inflow boundary of the right hand side of the wind tunnel (RHS for short), the flow is assumed to be a known Blasius type profile g_s plus an unknown perturbation g_p. This unknown perturbation, relatively small with respect to g_s, takes into account the fact that the fluid flow in the LHS of the wind tunnel is not precisely known.

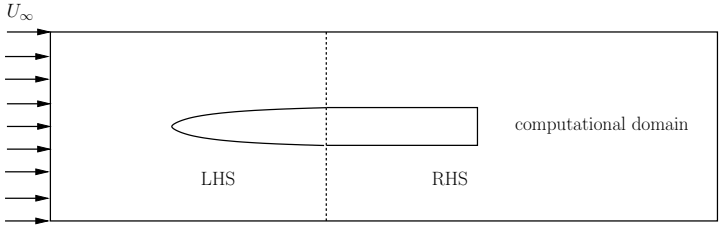

Fig. 1.1: Configuration of the wind tunnel.

In the absence of perturbation, that is when $g_p = 0$, assuming that the fluid flow is governed by the Navier-Stokes equations with some boundary conditions, the flow in the RHS is a stationary solution (u_s, p_s). For a Reynolds number $\mathrm{Re} = 200$, for which the numerical simulations are done, the stationary solution u_s is unstable. When $g_p \neq 0$, we observe a shedding of vortices behind the thick plate, see the vorticity profile in Figure 1.2. Our goal is to use a control device able to stabilize the fluid flow around the stationary solution u_s, in the presence of perturbations. This type of problem has been studied theoretically in [4, 21, 22] and numerically in [1, 2]. Here the novelty is that we would like to use the displacement of elastic structures, located at the upper and lower boundaries of the thick plate, to stabilize the coupled fluid-structure interaction system. Let us describe our model more precisely. The geometrical domain Ω, corresponding to the RHS of the wind tunnel, is defined by

$$\Omega = ([0, L] \times [-\ell, \ell]) \setminus ([0, \ell_s] \times [-e, e]),$$

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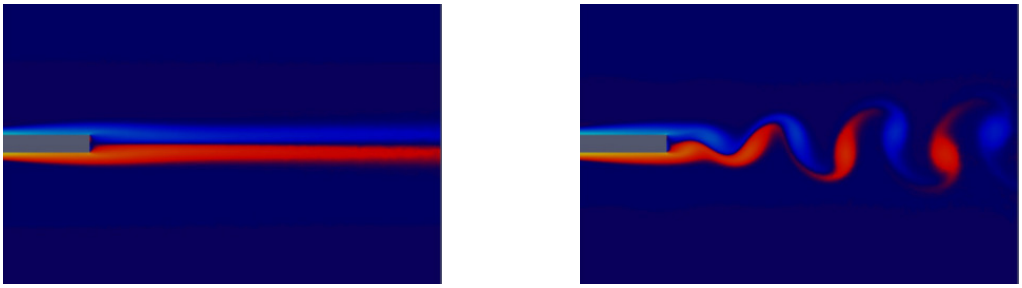


Fig. 1.2: Vorticities of the steady flow (left) and the perturbed flow (right).

where $L > 0$ is the length of the domain, 2ℓ is its height, ℓ_s is the length of the thick plate in the computational domain, and $2\,e$ is the thickness of the plate. The boundary of Ω, denoted by Γ, is split into different parts $\Gamma = \Gamma_s \cup \Gamma_i \cup \Gamma_e \cup \Gamma_n$ where

$$\Gamma_s = (0,\ell_s)\times\{-e\}\cup(0,\ell_s)\times\{e\},\quad \Gamma_e = (0,L)\times\{-\ell\}\cup(0,L)\times\{\ell\}\quad\text{and}\quad \Gamma_n = \{L\}\times(-\ell,\ell),$$
$$\Gamma_i = \Gamma_{i,1}\cup\Gamma_{i,2}\cup\Gamma_{i,3}\quad\text{with}\quad \Gamma_{i,1} = \{0\}\times(-\ell,-e),\quad \Gamma_{i,2} = \{0\}\times(e,\ell),\quad \Gamma_{i,3} = \{\ell_s\}\times(-e,e).$$

We also set $\Gamma_0 = \Gamma_s\cup\Gamma_i\cup\Gamma_e$ and $\Gamma_{i,e} = \Gamma_i\cup\Gamma_e$. The two elastic beams, used to stabilize the fluid flow, are located on Γ_s. The displacement η of the beams is assumed to be normal to Γ_s, and it satisfies an Euler-Bernoulli damped beam equation with clamped boundary conditions. Since the structure is deformed, the domain occupied by the fluid at time t depends on the displacement $\eta(t)$ of the structure, see Figure 1.3.

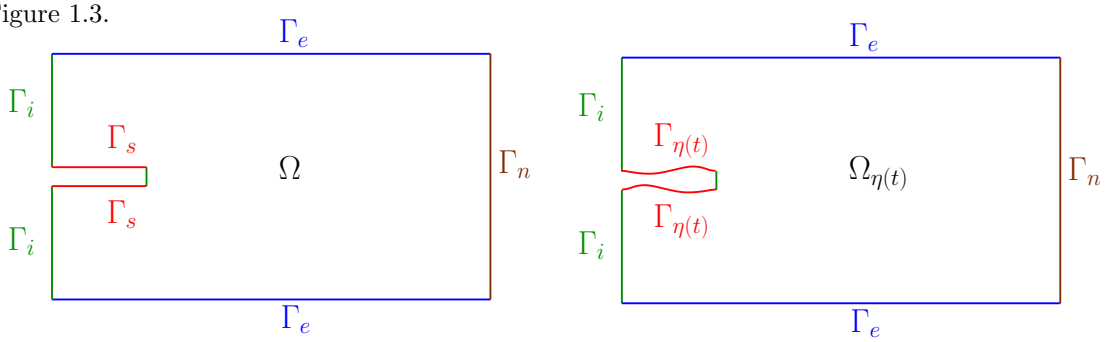


Fig. 1.3: Reference configuration (left) and deformed configuration (right).

The fluid domain at time t is denoted by $\Omega_{\eta(t)}$ and the fluid-structure interface by $\Gamma_{\eta(t)}$. We use the notations

$$Q_\eta^\infty = \bigcup_{t\in(0,\infty)}\big(\{t\}\times\Omega_{\eta(t)}\big),\quad \Sigma_\eta^\infty = \bigcup_{t\in(0,\infty)}\big(\{t\}\times\Gamma_{\eta(t)}\big),\quad e_1 = (1,0)\quad e_2 = (0,1),$$
$$Q^\infty = (0,\infty)\times\Omega,\quad \Sigma_s^\infty = (0,\infty)\times\Gamma_s, \Sigma_i^\infty = (0,\infty)\times\Gamma_i,\quad \Sigma_e^\infty = (0,\infty)\times\Gamma_e,\quad\text{and}\quad \Sigma_n^\infty = (0,\infty)\times\Gamma_n.$$

The Eulerian-Lagrangian system describing the evolution of the fluid-structure system [21] is

$$\begin{aligned}
&u_t - \operatorname{div}\sigma(u,p) + (u\cdot\nabla)u = 0 \text{ in } Q_\eta^\infty,\\
&\operatorname{div} u = 0 \text{ in } Q_\eta^\infty,\quad u = \eta_t n \text{ on } \Sigma_\eta^\infty,\quad u = g_s + g_p \text{ on } \Sigma_i^\infty,\\
&u\cdot n = 0 \text{ on } \Sigma_e^\infty,\quad \varepsilon(u)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty,\quad \sigma(u,p)n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{tt} - \beta\Delta_s\eta - \gamma\Delta_s\eta_t + \alpha\Delta_s^2\eta = -\sigma(u,p)|_{\Gamma_{\eta(t)}}\, n_{\eta(t)}\sqrt{1+\eta_x^2}\cdot n + f_s + f \text{ on } \Sigma_s^\infty,\\
&\eta = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\quad \eta_x = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&u(0) = u^0 \text{ on } \Omega,\quad \eta(0) = 0 \text{ on } \Gamma_s,\quad \eta_t(0) = \eta_2^0 \text{ on } \Gamma_s,
\end{aligned}\tag{1.1}$$

where u and p stand for the fluid velocity and pressure, $\sigma(u,p)$ is the Cauchy stress tensor

$$\sigma(u,p) = 2\,\nu\boldsymbol{\varepsilon}(u) - pI,\quad \boldsymbol{\varepsilon}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

ν is the fluid viscosity, $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ are parameters of the structure, $n_{\eta(t)}$ (resp. n) is the unit normal to $\Gamma_{\eta(t)}$ (resp. Γ_s) exterior to $\Omega_{\eta(t)}$ (resp. Ω), and u^0, η_2^0 are initial data. Here $\Delta_s = \partial_{xx}$ stands for the Laplace-Beltrami operator on Γ_s, the term $-\sigma(u,p)|_{\Gamma_{\eta(t)}} n_{\eta(t)} \sqrt{1+\eta_x^2} \cdot n$ represents the force exerted by the fluid on the structure, f_s is a stationary force defined below, g_s is a stationary boundary condition and f is the control function which will be used to stabilize the fluid-structure interaction system.

We notice that a Neumann boundary condition is prescribed on Σ_n^∞ while a Navier boundary condition is prescribed on Σ_e^∞. What is done in the present paper can be also adapted without any difficulty to the case where a Neumann boundary condition is imposed on Σ_e^∞. In that case, the transformation used to rewrite the system in the reference configuration must be defined differently. Other geometries could also be considered.

Let (u_s, p_s) be a stationary solution of the Navier-Stokes equations in the reference configuration Ω

$$\begin{aligned} &-\operatorname{div}\sigma(u_s,p_s) + (u_s\cdot\nabla)u_s = 0 \text{ in } \Omega,\\ &\operatorname{div} u_s = 0 \text{ in } \Omega, \quad u_s = 0 \text{ on } \Gamma_s, \quad u_s = g_s \text{ on } \Gamma_i,\\ &u_s\cdot n = 0 \text{ on } \Gamma_e, \quad \varepsilon(u_s)n\cdot\tau = 0 \text{ on } \Gamma_e, \quad \sigma(u_s,p_s)n = 0 \text{ on } \Gamma_n. \end{aligned} \tag{1.2}$$

We choose $f_s = -p_s|_{\Gamma_s}$ in (1.1). Thus, $(u,p,\eta) = (u_s, p_s, 0)$ is an unstable stationary solution of system (1.1), and it is the solution around which we want to stabilize (1.1). We choose the function f of the form

$$f(t,x,y) = \sum_{i=1}^{n_c} f_i(t) w_i(x,y), \tag{1.3}$$

where the functions w_i belongs to $L^2(\Gamma_s)$ and $\boldsymbol{f}(\cdot) = (f_1(\cdot), \cdots, f_{n_c}(\cdot))$ is the control variable. The functions w_i must be appropriately chosen so that the linearized system around $(u_s, p_s, 0)$ is stabilizable, see Section 7.3, where the stabilizability is studied. The goal of the paper is to find a control $\boldsymbol{f}$, in feedback form, able to stabilize system (1.1) around the stationary solution $(u_s, p_s, 0)$, with any prescribed exponential decay rate $-\omega < 0$, provided that g_p, $u^0 - u_s$ and η_2^0 are small enough in appropriate functional spaces.

The analysis of this stabilization problem has been done in [11]. Here we would like to develop a similar strategy for the semi-discrete system obtained by approximating by a finite element method the system (1.1) rewritten in the reference configuration.

To rewrite system $(1.1)_{1-4}$ in the reference configuration Q^∞, for all $t \geq 0$, we introduce the mapping

$$\mathcal{T}_{\eta(t)} : (x,y) \longmapsto (x,z) = \left(x, \frac{(\ell - e)y - \ell \mathcal{E}\eta(t,x,s(y)e)}{\ell - e - s(y)\mathcal{E}\eta(t,x,s(y)e)} \right),$$

where the sign function s is defined by $s(y) = -1$ if $y < 0$, $s(y) = 1$ if $y \geq 0$ and $\mathcal{E}$ is a continuous linear operator such that $\mathcal{E}\eta(t,x) = 0$ for $(t,x) \in (0,\infty) \times ((\ell_s + L)/2, L)$, see [11]. In the following, to simplify the notation, we write η in place of $\mathcal{E}\eta$.

For all $t \geq 0$, $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into Ω. We make the change of unknowns

$$\begin{aligned} &\tilde{u}(t,x,z) := u(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)), \quad \tilde{p}(t,x,z) := p(t, \mathcal{T}_{\eta(t)}^{-1}(x,z)),\\ &\eta_1(t,x) := \eta(t,x), \quad \eta_2(t,x) := \eta_t(t,x), \quad \text{and} \quad \tilde{u}^0 = u^0. \end{aligned} \tag{1.4}$$

The quadruplet $(\tilde{u}, \tilde{p}, \eta_1, \eta_2)$ satisfies the system

$$\begin{aligned} &\tilde{u}_t - \operatorname{div}\sigma(\tilde{u},\tilde{p}) + (\tilde{u}\cdot\nabla)\tilde{u} = F_f[\tilde{u},\tilde{p},\eta_1,\eta_2] \text{ in } Q^\infty,\\ &\operatorname{div}\tilde{u} = \operatorname{div} F_{\operatorname{div}}[\tilde{u},\eta_1] \text{ in } Q^\infty, \quad \tilde{u} = \eta_2 n \text{ on } \Sigma_s^\infty, \quad \tilde{u} = g_p + g_s \text{ on } \Sigma_i^\infty,\\ &\tilde{u}\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(\tilde{u})n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(\tilde{u},\tilde{p})n = 0 \text{ on } \Sigma_n^\infty,\\ &\eta_{1,t} - \eta_2 = 0 \text{ on } \Sigma_s^\infty,\\ &\eta_{2,t} - \beta\Delta_s\eta_1 - \gamma\Delta_s\eta_2 + \alpha\Delta_s^2\eta_1 = -\sigma(\tilde{u},\tilde{p})n\cdot n + F_s[\tilde{u},\eta_1] + f + f_s \text{ on } \Sigma_s^\infty,\\ &\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\ &\tilde{u}(0) = \tilde{u}^0 \text{ in } \Omega, \quad \eta_1(0) = 0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s, \end{aligned} \tag{1.5}$$

where the nonlinear terms F_f, $F_{\rm div}$ and F_s are given in Appendix A. The linearization of system (1.5) around the stationary solution $(u_s, p_s, 0, 0)$ is

$$\begin{aligned}
&v_t - \operatorname{div}\sigma(v,q) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0 \text{ in } Q^\infty,\\
&\operatorname{div} v = A_3\eta_1 \text{ in } Q^\infty, \quad v = \eta_2 n \text{ on } \Sigma_s^\infty, \quad v = g_p \text{ on } \Sigma_i^\infty,\\
&v\cdot n = 0 \text{ on } \Sigma_e^\infty, \quad \varepsilon(v)n\cdot\tau = 0 \text{ on } \Sigma_e^\infty, \quad \sigma(v,q)n = 0 \text{ on } \Sigma_n^\infty,\\
&\eta_{1,t} - \eta_2 = 0 \text{ on } \Sigma_s^\infty,\\
&\eta_{2,t} - \beta\Delta_s\eta_1 - \gamma\Delta_s\eta_2 + \alpha\Delta_s^2\eta_1 - A_4\eta_1 = -\sigma(v,q)n\cdot n + f \text{ on } \Sigma_s^\infty,\\
&\eta_1 = 0 \text{ on } (0,\infty)\times\partial\Gamma_s, \quad \eta_{1,x} = 0 \text{ on } (0,\infty)\times\partial\Gamma_s,\\
&v(0) = v^0 \text{ in } \Omega, \quad \eta_1(0) = 0 \text{ on } \Gamma_s, \quad \eta_2(0) = \eta_2^0 \text{ on } \Gamma_s.
\end{aligned} \tag{1.6}$$

The linear differential operators A_1, A_2, A_3 and A_4 are defined in Appendix A.

For the finite element approximation of systems (1.5) and (1.6), we shall take into account the Dirichlet boundary condition of the fluid flow on Γ_s, Γ_i and Γ_e by Lagrange multipliers. We introduce finite-dimensional subspaces $X_h \subset H^1(\Omega;\mathbb{R}^2)$ for the velocity, $P_h \subset L^2(\Omega)$ for the pressure, $S_h \subset H_0^2(\Gamma_s)$ for the displacement and its velocity, $D_h \subset \{g = (g_1, g_2) \in L^2(\Gamma_0;\mathbb{R}^2), g = 0 \text{ on } \Gamma_s\cup\Gamma_i, g_2 = 0 \text{ on } \Gamma_e\}$ for the multiplier associated to the Dirichlet boundary conditions on $\Gamma_0 = \Gamma_s\cup\Gamma_i\cup\Gamma_e$.

The finite dimensional approximation of the linearized system (1.6) is

$$\begin{aligned}
&\text{Find } v \in H^1_{\rm loc}((0,\infty);X_h),\ q \in L^2_{\rm loc}((0,\infty);P_h),\ \tau \in L^2_{\rm loc}((0,\infty);D_h),\\
&\eta_1 \in L^2_{\rm loc}((0,\infty);S_h), \eta_2 \in L^2_{\rm loc}((0,\infty);S_h), \quad \text{such that}\\
&\frac{d}{dt}\int_\Omega v(t)\phi\,dx = a_f(v(t),\eta_1(t),\eta_2(t),\phi) + b(\phi,q(t)) + \langle\tau(t),\phi\rangle_{\Gamma_0}, \quad \forall\phi\in X_h,\\
&b(v(t),\psi) = \int_\Omega A_3\eta_1(t)\,\psi\,dx, \quad \forall\psi\in P_h,\\
&\langle\mu, v(t)\rangle_{\Gamma_0} = \int_{\Gamma_i} g_p\cdot\,\mu\,dx + \int_{\Gamma_s}\eta_2(t)n\cdot\mu\,dx, \quad \forall\mu\in D_h,\\
&\frac{d}{dt}\int_{\Gamma_s}\eta_1(t)\zeta\,dx = \int_{\Gamma_s}\eta_2(t)\zeta\,dx, \quad \forall\zeta\in S_h,\\
&\frac{d}{dt}\int_{\Gamma_s}\eta_2(t)\zeta\,dx = a_s(\eta_1(t),\eta_2(t),\zeta) - \langle\tau(t),\zeta\,n\rangle_{\Gamma_s}, \quad \forall\zeta\in S_h,
\end{aligned} \tag{1.7}$$

where

$$a_f(v,\eta_1,\eta_2,\phi) = \int_\Omega\left(-2\nu\varepsilon(v):\varepsilon(\phi) - (u_s\cdot\nabla)\phi\cdot v - (\phi\cdot\nabla)u_s\cdot v + \int_\Omega A_1\eta_1\cdot\phi + \int_\Omega A_2\eta_1\cdot\phi\right)dx,$$

$$b(\phi,q) = \int_\Omega \operatorname{div}\phi\, q\,dx, \quad \langle\mu,\phi\rangle_{\Gamma_0} = \int_{\Gamma_s\cup\Gamma_i}\mu\cdot\phi\,dx + \int_{\Gamma_e}\mu_2\,\phi_2\,dx,$$

$$a_s(\eta_1,\eta_2,\zeta) = \int_{\Gamma_s}\left(-\beta\nabla\eta_1\cdot\nabla\zeta + \alpha\Delta\eta_1\cdot\Delta\zeta + A_4\eta_1\,\zeta - \gamma\nabla\eta_2\cdot\nabla\zeta\right)dx.$$

System (1.7) has to be completed by initial conditions. The Lagrange multiplier τ is introduced to take into account the boundary conditions $v = \eta_2 n$ on Γ_s, $v = g_p$ on Γ_i and $v\cdot n = 0$ on Γ_e.

In order to construct a linear feedback law, which is easy to compute, able to locally stabilize the nonlinear system (1.5), with any prescribed exponentially decay rate $-\omega < 0$, we are going to follow a strategy similar to that used in [11] for the continuous model. It can be summarized in several steps corresponding to each section.

– In section 2, we present the matrix formulation of the semi-discrete Finite Element approximations.

– In section 3, we reformulate the finite-dimensional linear system as a control system by eliminating the multiplier from the equations using a projector which plays a role similar to that of the Leray projector for the infinite-dimensional system.

– In section 4, we study the relationships between the eigenvalue problems involving Lagrange multipliers and those without Lagrange multipliers. Using these relationships, we are able to construct the feedback law without having to call the projector which is difficult to construct numerically.

– In section 5, we use the spectral decomposition to bring back the stabilization problem to the stabilization of a finite-dimensional linear system. Then, the feedback law is obtained by solving an Algebraic Riccati Equation of small dimension and then easy to solve.
– Finally, in section 6, thanks to numerical tests, we prove that the linear feedback law is able to stabilize the discrete nonlinear system. By choosing conveniently different parameters used to determine the feedback control law, we are able to stabilize perturbation amplitudes that are of order 15% of the stationary inflow boundary condition.

2. Semi-discrete approximations. In this section, we show that the matrix formulation of system (1.7) corresponds to system (2.3).

We denote by $(\phi_i)_{1\le i\le n_v}$ a basis of X_h, $(p_i)_{1\le i\le n_q}$ a basis of P_h, $(\zeta_i)_{1\le i\le n_s}$ a basis of S_h, $(\mu_i)_{1\le i\le n_\tau}$ a basis of D_h. We set

$$v=\sum_{i=1}^{n_v} v_i\phi_i,\quad v^0=\sum_{i=1}^{n_v} v_i^0\phi_i,\quad \eta_1=\sum_{i=1}^{n_s}\eta_1^i\zeta_i,\quad \eta_2=\sum_{i=1}^{n_s}\eta_2^i\zeta_i,\quad \eta_2^0=\sum_{i=1}^{n_s}\eta_{2,0}^i\zeta_i,$$
$$q=\sum_{i=1}^{n_q} q_ip_i,\quad g_p=\sum_{i=1}^{n_\tau} g_p^i\mu_i,\quad \tau=\sum_{i=1}^{n_\tau}\tau_i\mu_i,\quad w_j=\sum_{i=1}^{n_s} w_j^i\zeta_i.$$

If we denote with boldface letters the corresponding coordinate vectors, we have

$$\boldsymbol{v}=(v_1,\cdots,v_{n_v})^T,\quad \boldsymbol{v}^0=(v_1^0,\cdots,v_{n_v}^0)^T,\quad \boldsymbol{\eta}_1=(\eta_1^1,\cdots,\eta_1^{n_s})^T,\quad \boldsymbol{\eta}_2=(\eta_2^1,\cdots,\eta_2^{n_s})^T,$$
$$\boldsymbol{\eta}_2^0=(\eta_{2,0}^1,\cdots,\eta_{2,0}^{n_s})^T,\quad \boldsymbol{f}=(f_1,\cdots,f_{n_c})^T,\quad \boldsymbol{g}_p=(g_p^1,\cdots,g_p^{n^\tau})^T,\quad \boldsymbol{q}=(q_1,\cdots,q_{n_q})^T,$$
$$\boldsymbol{\tau}=(\tau_1,\cdots,\tau_{n_\tau})^T,\quad \boldsymbol{\theta}=(q_1,\cdots,q_{n_q},\tau_1,\cdots,\tau_{n_\tau})^T,\quad \boldsymbol{w}_j=(w_j^1,\cdots,w_j^{n_s})^T,\quad \mathbf{W}=[\boldsymbol{w}_1\ \cdots\ \boldsymbol{w}_{n_c}].$$

For the fluid, we introduce the stiffness matrices A_{vv}, A_{vq}, $A_{v\tau}$, $A_{v\theta}$, $A_{\theta g}$ and the mass matrix M_{vv} defined for all $1\le i\le n_v$, $1\le j\le n_v$, $1\le k\le n_q$, $1\le l\le n_\tau$, $1\le m\le n_\tau$ by

$$(A_{vv})_{ij}=-2\nu\int_\Omega \varepsilon(\phi_i):\varepsilon(\phi_j)-\int_\Omega(u_s\cdot\nabla)\phi_j\cdot\phi_i-\int_\Omega(\phi_j\cdot\nabla)u_s\cdot\phi_i,\qquad (M_{vv})_{ij}=\int_\Omega\phi_j\cdot\phi_i,$$
$$(A_{vq})_{ik}=\int_\Omega p_k\mathrm{div}\,\phi_i,\qquad (A_{v\tau})_{il}=\int_{\Gamma_s\cup\Gamma_i}\mu_l\cdot\phi_i+\int_{\Gamma_e}\mu_{l,2}\phi_{l,2},\qquad (M_{\tau\tau})_{ml}=\int_{\Gamma_m}\tau_l\cdot\tau_m,$$

$$A_{v\theta}=\begin{bmatrix}A_{vq} & A_{v\tau}\end{bmatrix},\qquad A_{\theta g}=\begin{bmatrix}0\\ M_{\tau\tau}\end{bmatrix}.$$

For the structure, we introduce the stiffness matrices $A_{\eta_1\eta_2}$, $A_{\eta_2\eta_1}$, $A_{\eta_2\eta_2}$ and the mass matrix $M_{\eta\eta}$ defined for all $1\le i\le n_s$ and $1\le j\le n_s$ by

$$(A_{\eta_1\eta_2})_{ij}=(M_{\eta\eta})_{ij}=\int_{\Gamma_s}\zeta_j\cdot\zeta_i,\qquad (A_{\eta_2\eta_1})_{ij}=-\int_{\Gamma_s}(\beta\nabla\zeta_j\cdot\nabla\zeta_i+\alpha\Delta\zeta_j\cdot\Delta\zeta_i)+\int_{\Gamma_s}A_4\zeta_j\cdot\zeta_i$$
$$(A_{\eta_2\eta_2})_{ij}=-\gamma\int_{\Gamma_s}\nabla\zeta_j\cdot\nabla\zeta_i,$$

We also introduce the coupling matrices $A_{v\eta_1}$, $A_{v\eta_2}$, $A_{\eta_1 q}$, $A_{\eta_2\tau}$, $A_{\eta_1\theta}$ and $A_{\eta_2\theta}$ defined for all $1\le i\le n_v$, $1\le j\le n_s$, $1\le k\le n_s$, $1\le l\le n_q$, $1\le m\le n_\tau$ by

$$(A_{v\eta_1})_{ij}=\int_\Omega A_1\zeta_j\cdot\phi_i,\ \ (A_{v\eta_2})_{ij}=\int_\Omega A_2\zeta_j\cdot\phi_i,\ \ (A_{\eta_1 q})_{kl}=-\int_\Omega A_3\zeta_k\cdot p_l,\ \ (A_{\eta_2\tau})_{km}=-\int_{\Gamma_s}\zeta_k n\cdot\mu_m,$$

$$A_{\eta_1\theta}=\begin{bmatrix}A_{\eta_1 q} & 0 & 0\end{bmatrix},\quad A_{\eta_2\theta}=\begin{bmatrix}0 & A_{\eta_2\tau} & 0\end{bmatrix}.$$

We set

$$n_z=n_v+2n_s,\quad n_\theta=n_q+n_\tau\quad\text{ and }\quad n=n_z+n_\theta.$$

We introduce the mass matrix M, the stiffness matrix A, and the control operator $B\in\mathcal{L}(\mathbb{R}^{n_c},\mathbb{R}^{n_z})$

$$M=\begin{bmatrix}M_{zz} & 0\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}A_{zz} & \widetilde{A}_{z\theta}\\ \widehat{A}_{z\theta}^T & 0\end{bmatrix}\quad\text{ and }\quad B=\begin{bmatrix}0\\ 0\\ M_{\eta\eta}\mathbf{W}\end{bmatrix},\tag{2.1}$$

where

$$M_{zz}=\begin{bmatrix}M_{vv} & 0 & 0\\ 0 & M_{\eta\eta} & 0\\ 0 & 0 & M_{\eta\eta}\end{bmatrix},\quad A_{zz}=\begin{bmatrix}A_{vv} & A_{v\eta_1} & A_{v\eta_2}\\ 0 & 0 & A_{\eta_1\eta_2}\\ 0 & A_{\eta_2\eta_1} & A_{\eta_2\eta_2}\end{bmatrix},\quad \widetilde{A}_{z\theta}=\begin{bmatrix}A_{v\theta}\\ 0\\ A_{\eta_2\theta}\end{bmatrix},\quad \widehat{A}_{z\theta}=\begin{bmatrix}A_{v\theta}\\ A_{\eta_1\theta}\\ A_{\eta_2\theta}\end{bmatrix}.\tag{2.2}$$

Thus, the matrix formulation of system (1.7) is

$$M\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\\ \boldsymbol{\theta}\end{bmatrix} = A\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\\ \boldsymbol{\theta}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ M_{\eta\eta}\mathbf{W}\boldsymbol{f}\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ -A_{\theta g}\boldsymbol{g}_p\end{bmatrix}, \qquad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix}, \tag{2.3}$$

or, equivalently

$$M_{zz}\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A_{zz}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \widetilde{A}_{z\theta}\boldsymbol{\theta} + B\boldsymbol{f}, \quad \widehat{A}_{z\theta}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A_{\theta g}\boldsymbol{g}_p, \quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix}, \tag{2.4}$$

where $\boldsymbol{v}$, $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{v}^0$, $\boldsymbol{\eta}_2^0$ and $\boldsymbol{g}_p$ are the coordinate vectors of v, η_1, η_2, v^0, η_2^0 and g_p respectively, $\boldsymbol{\theta} = (\boldsymbol{q}, \boldsymbol{\tau})^T$, $\boldsymbol{q}$ and $\boldsymbol{\tau}$ are the coordinate vectors of q and τ respectively. The initial conditions $\boldsymbol{v}^0$ and $\boldsymbol{\eta}_2^0$ are such that $(\boldsymbol{v}^0, 0, \boldsymbol{\eta}_2^0)$ belongs to $\mathrm{Ker}(\widehat{A}_{z\theta}^T)$.

3. Reformulation of the finite-dimensional linear system.

3.1. The stationary finite-dimensional linear system. The goal of this subsection is to rewrite the system

$$(\boldsymbol{\lambda}M - A)\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\\ \boldsymbol{\theta}\end{bmatrix} = \begin{bmatrix}F_f\\ F_{s,1}\\ F_{s,2}\\ 0\end{bmatrix}, \tag{3.1}$$

as a system satisfied by $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ in which the multiplier $\boldsymbol{\theta}$ is eliminated, and to characterize the multiplier $\boldsymbol{\theta}$ in terms of the solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and of the data $(F_f, F_{s,1}, F_{s,2})$. We are going to consider system (3.1) either when $\lambda \in \mathbb{C}$ and $(F_f, F_{s,1}, F_{s,2}) \in \mathbb{C}^{n_z}$, in which case the solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and the multiplier $\boldsymbol{\theta}$ respectively belong to $\mathbb{C}^{n_z}$ and $\mathbb{C}^{n_\theta}$, or when $\lambda \in \mathbb{R}$ and $(F_f, F_{s,1}, F_{s,2}) \in \mathbb{R}^{n_z}$, in which case the solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ and the multiplier $\boldsymbol{\theta}$ respectively belong to $\mathbb{R}^{n_z}$ and $\mathbb{R}^{n_\theta}$. In order to collect both the results stated for either real or complex solutions, below we state results valid for $\mathbb{K}^n$ where either $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Let us notice that the matrices involved in system (2.3) have real coefficients.

We first consider the following system involving only the velocity $\boldsymbol{v}$ and the multiplier $\boldsymbol{\theta}$

$$\boldsymbol{\lambda}M_{vv}\boldsymbol{v} = A_{vv}\boldsymbol{v} + A_{v\theta}\boldsymbol{\theta} + F_f, \quad A_{v\theta}^T\boldsymbol{v} = \boldsymbol{g}. \tag{3.2}$$

To eliminate the multiplier $\boldsymbol{\theta}$ from equation $(3.2)_1$, we are going to introduce the projection into $\mathrm{Ker}(A_{v\theta}^T)$ parallel to $\mathrm{Im}(M_{vv}^{-1}A_{v\theta})$.

PROPOSITION 3.1. *1. The projector $\mathbb{P}_v^T$ in $\mathbb{K}^{n_v}$ onto $\mathrm{Ker}(A_{v\theta}^T)$ parallel to $\mathrm{Im}(M_{vv}^{-1}A_{v\theta})$ is defined by*

$$\mathbb{P}_v^T = I - M_{vv}^{-1}A_{v\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\theta})^{-1}A_{v\theta}^T.$$

2. The projector $\mathbb{P}_v$ in $\mathbb{K}^{n_v}$ onto $\mathrm{Ker}(A_{v\theta}^T M_{vv}^{-1})$ parallel to $\mathrm{Im}(A_{v\theta})$ is

$$\mathbb{P}_v = I - A_{v\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\theta})^{-1}A_{v\theta}^T M_{vv}^{-1}.$$

Moreover, we have

$$\mathbb{P}_v A_{v\theta} = 0, \quad \mathbb{P}_v^2 = \mathbb{P}_v, \quad \mathbb{P}_v M_{vv} = M_{vv}\mathbb{P}_v^T, \quad \mathbb{P}_v M_{vv} = \mathbb{P}_v^2 M_{vv} = \mathbb{P}_v M_{vv}\mathbb{P}_v^T, \quad M_{vv}^{-1}\mathbb{P}_v = \mathbb{P}_v^T M_{vv}^{-1}.$$

Proof. It is similar to that of [1, Proposition 3.1]. ∎

REMARK 3.1. *From the Inf-Sup condition* (6.4)*, it follows that $A_{v\theta}$ is of rank n_θ. Thus, the matrix $A_{v\theta}^T A_{vv}^{-1} A_{v\theta}$ is invertible.* We set

$$\mathbb{A} = \mathbb{P}_v^T M_{vv}^{-1}A_{vv}, \quad A_{\theta\theta} = (A_{v\theta}^T A_{vv}^{-1}A_{v\theta})^{-1}, \quad M_{\theta\theta} = (A_{v\theta}^T M_{vv}^{-1}A_{v\theta})^{-1} \quad \text{and} \quad \mathbb{L} = A_{vv}^{-1}A_{v\theta}A_{\theta\theta}.$$

LEMMA 3.1. *We have* $(I - \mathbb{P}_v^T)\mathbb{L} = M_{vv}^{-1} A_{v\theta} M_{\theta\theta}$.

Proof. The proof follows from the fact that $I - \mathbb{P}_v^T = M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{v\theta}^T$. ∎

LEMMA 3.2. *Let* F_f *belong to* $\mathbb{K}^{n_v}$ *and* $\mathbf{g}$ *belong to* $\mathbb{K}^{n_\theta}$. *The couple* $(\boldsymbol{v}, \boldsymbol{\theta})$ *is a solution of system* (3.2) *if and only if*

$$\begin{aligned} &\boldsymbol{\lambda}\mathbb{P}_v^T \boldsymbol{v} = \mathbb{A}\mathbb{P}_v^T \boldsymbol{v} - \mathbb{A}\mathbb{P}_v^T \mathbb{L}\boldsymbol{g} + \mathbb{P}_v^T M_{vv}^{-1} F_f, \quad (I - \mathbb{P}_v^T)\boldsymbol{v} = M_{vv}^{-1} A_{v\theta} M_{\theta\theta} \boldsymbol{g}, \\ &\boldsymbol{\theta} = -M_{\theta\theta} \left(-\boldsymbol{\lambda}\boldsymbol{g} + N_{\theta\theta}\boldsymbol{g} + A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{v} + A_{v\theta}^T M_{vv}^{-1} F_f \right), \end{aligned} \tag{3.3}$$

where $N_{\theta\theta} = A_{v\theta}^T M_{vv}^{-1} A_{vv} M_{vv}^{-1} A_{v\theta} M_{\theta\theta}$.

Proof. The proof is similar to that of [1, Proposition 3.2]. ∎

In order to eliminate the multiplier $\boldsymbol{\theta}$ in system (3.1), we introduce the following operators

$$\mathbb{P} = \begin{bmatrix} \mathbb{P}_v & 0 & 0 \\ 0 & I_{\mathbb{K}^{n_s}} & 0 \\ 0 & 0 & I_{\mathbb{K}^{n_s}} \end{bmatrix}, \quad A_{z\theta} = \begin{bmatrix} A_{v\theta} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad M_s = \begin{bmatrix} M_{vv} & 0 & 0 \\ 0 & M_{\eta\eta} & 0 \\ 0 & N_{\eta_2\eta_1} & M_{\eta\eta} + N_{\eta_2\eta_2} \end{bmatrix}, \tag{3.4}$$

where

$$N_{\eta_2\eta_2} = A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T \quad \text{and} \quad N_{\eta_2\eta_1} = A_{\eta_2\theta} M_{\theta\theta} A_{\eta_1\theta}^T. \tag{3.5}$$

We have

$$M_s^{-1} = \begin{bmatrix} M_{vv}^{-1} & 0 & 0 \\ 0 & M_{\eta\eta}^{-1} & 0 \\ 0 & -(M_{\eta\eta} + N_{\eta_2\eta_2})^{-1} N_{\eta_2\eta_1} M_{\eta\eta}^{-1} & (M_{\eta\eta} + N_{\eta_2\eta_2})^{-1} \end{bmatrix}.$$

PROPOSITION 3.2. *The operator* $\mathbb{P}^T$ *is the projector in* $\mathbb{K}^{n_z}$ *onto* $\mathrm{Ker}(A_{z\theta}^T)$ *parallel to* $\mathrm{Im}(M_s^{-1} A_{z\theta})$ *and the operator* $\mathbb{P}$ *is the projector in* $\mathbb{K}^{n_z}$ *onto* $\mathrm{Ker}(A_{z\theta}^T M_s^{-1})$ *parallel to* $\mathrm{Im}(A_{z\theta})$. *Moreover, we have*

$$\mathbb{P}A_{z\theta} = 0, \quad \mathbb{P}^2 = \mathbb{P}, \quad \mathbb{P}M_s = M_s\mathbb{P}^T, \quad \mathbb{P}M_s = \mathbb{P}^2 M_s = \mathbb{P}M_s\mathbb{P}^T, \quad M_s^{-1}\mathbb{P} = \mathbb{P}^T M_s^{-1}.$$

Proof. It follows from Proposition 3.1. ∎

We introduce the matrix $\mathbf{A}$ in $\mathbb{R}^{n_z \times n_z}$ defined by

$$\mathbf{A} = \mathbb{P}^T M_s^{-1} \widetilde{A}_{zz}, \tag{3.6}$$

where

$$\widetilde{A}_{zz} = \begin{bmatrix} A_{vv} & A_{v\eta_1} + A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_1\theta}^T & A_{v\eta_2} + A_{vv}\mathbb{P}_v^T \mathbb{L} A_{\eta_2\theta}^T \\ 0 & 0 & A_{\eta_1\eta_2} \\ A_{\eta_2 v} & \widetilde{A}_{\eta_2\eta_1} & \widetilde{A}_{\eta_2\eta_2} \end{bmatrix}, \tag{3.7}$$

and

$$\begin{aligned} A_{\eta_2 v} &= -A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}, \\ \widetilde{A}_{\eta_2\eta_1} &= A_{\eta_1\eta_2} - A_{\eta_2\theta} M_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T), \\ \widetilde{A}_{\eta_2\eta_2} &= A_{\eta_2\eta_2} - A_{\eta_2\theta} M_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T). \end{aligned} \tag{3.8}$$

PROPOSITION 3.3. *Let* $(F_f, F_{s,1}, F_{s,2})$ *be in* $\mathbb{K}^{n_z}$. *A quadruplet* $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta}) \in \mathbb{K}^n$ *is solution of the equation*

$$(\boldsymbol{\lambda} M - A) \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} \\ 0 \end{bmatrix}, \tag{3.9}$$

if and only if

$$
\begin{aligned}
&(\boldsymbol{\lambda} M_s - M_s\mathbf{A})\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f \end{bmatrix}, \\
&(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2, \\
&\boldsymbol{\theta} = -M_{\theta\theta}(\boldsymbol{\lambda} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + \boldsymbol{\lambda} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 - N_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2)) \\
&\qquad -M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{v} - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1}(A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2 + F_f).
\end{aligned} \tag{3.10}
$$

If in addition,

$$
A_{v\theta}^T M_{vv}^{-1} F_f + A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2} = 0,
$$

then

$$
(\boldsymbol{\lambda} M_s - M_s\mathbf{A})\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = M_s \mathbb{P}^T M_{zz}^{-1} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} \end{bmatrix},
$$

and $\boldsymbol{\theta}$ *in* (3.10) *satisfies*

$$
\begin{aligned}
\boldsymbol{\theta} = &-M_{\theta\theta}(\boldsymbol{\lambda} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + \boldsymbol{\lambda} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 - N_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2)) \\
&-M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1}(A_{vv}\mathbb{P}_v^T \boldsymbol{v} + A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2) + M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2}.
\end{aligned}
$$

Proof. Step 1. Let $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ be a solution of (3.9). The couple $(\boldsymbol{v}, \boldsymbol{\theta})$ is solution of the system

$$
\boldsymbol{\lambda} M_{vv}\boldsymbol{v} = A_{vv}\boldsymbol{v} + A_{v\theta}\boldsymbol{\theta} + A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2 + F_f, \quad A_{v\theta}^T\boldsymbol{v} = -(A_{\eta_1\theta}^T\boldsymbol{\eta}_1 + A_{\eta_2\theta}^T\boldsymbol{\eta}_2).
$$

Thanks to Lemma 3.2, we have

$$
\begin{aligned}
&\boldsymbol{\lambda}\mathbb{P}_v^T\boldsymbol{v} = \mathbb{A}\mathbb{P}_v^T\boldsymbol{v} + \mathbb{A}\mathbb{P}_v^T\mathbb{L}(A_{\eta_1\theta}^T\boldsymbol{\eta}_1 + A_{\eta_2\theta}^T\boldsymbol{\eta}_2) + \mathbb{P}_v^T M_{vv}^{-1}(A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2) + \mathbb{P}_v^T M_{vv}^{-1} F_f, \\
&(I - \mathbb{P}_v^T)\boldsymbol{v} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2, \\
&\boldsymbol{\theta} = -M_{\theta\theta}(\boldsymbol{\lambda} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + \boldsymbol{\lambda} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 - N_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2)) - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1}(A_{vv}\mathbb{P}_v^T\boldsymbol{v} + A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2 + F_f).
\end{aligned}
$$

The system on $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ is given by

$$
\boldsymbol{\lambda} M_{\eta\eta}\boldsymbol{\eta}_1 = A_{\eta_1\eta_2}\boldsymbol{\eta}_2 + F_{s,1} \quad \text{and} \quad \boldsymbol{\lambda} M_{\eta\eta}\boldsymbol{\eta}_2 = A_{\eta_2\theta}\boldsymbol{\theta} + A_{\eta_2\eta_1}\boldsymbol{\eta}_1 + A_{\eta_2\eta_2}\boldsymbol{\eta}_2 + F_{s,2}.
$$

We have

$$
\begin{aligned}
A_{\eta_2\theta}\boldsymbol{\theta} = &-A_{\eta_2\theta} M_{\theta\theta}(\boldsymbol{\lambda} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + \boldsymbol{\lambda} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 - N_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2)) \\
&-A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1}(A_{vv}\mathbb{P}_v^T\boldsymbol{v} + A_{v\eta_1}\boldsymbol{\eta}_1 + A_{v\eta_2}\boldsymbol{\eta}_2 + F_f).
\end{aligned}
$$

Or equivalently

$$
\begin{aligned}
A_{\eta_2\theta}\boldsymbol{\theta} = &-\boldsymbol{\lambda} A_{\eta_2\theta} M_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2) + (A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta} A_{\eta_1\theta}^T - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1})\boldsymbol{\eta}_1 \\
&+(A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta} A_{\eta_2\theta}^T - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2})\boldsymbol{\eta}_2 - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{v} \\
&-A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f.
\end{aligned}
$$

Replacing the above expression of $A_{\eta_2\theta}\boldsymbol{\theta}$ in the system satisfied by $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$, we obtain

$$
\begin{aligned}
&\boldsymbol{\lambda} M_{\eta\eta}\boldsymbol{\eta}_1 = A_{\eta_1\eta_2}\boldsymbol{\eta}_2 + F_{s,1}, \\
&\boldsymbol{\lambda}(M_{\eta\eta} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\eta}_2 + \boldsymbol{\lambda} A_{\eta_2\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 \\
&= (A_{\eta_2\eta_1} + A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta} A_{\eta_1\theta}^T - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1})\boldsymbol{\eta}_1 \\
&\quad +(A_{\eta_2\eta_2} + A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta} A_{\eta_2\theta}^T - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2})\boldsymbol{\eta}_2 \\
&\quad +F_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}\mathbb{P}_v^T\boldsymbol{v} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f.
\end{aligned}
$$

Thus we have

$$(\boldsymbol{\lambda} M_s - M_s \mathbb{P}^T M_s^{-1} \widetilde{A}_{zz}) \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f \end{bmatrix},$$

and (3.10) is proved. The converse can be proved by similar arguments.
Step 2. If in addition

$$A_{v\theta}^T M_{vv}^{-1} F_f + A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2} = 0,$$

then

$$\begin{aligned} &-M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f = M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2} \quad \text{and} \\ &F_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} F_f = F_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2}. \end{aligned}$$

Hence

$$(\boldsymbol{\lambda} M_s - M_s \mathbb{P}^T M_s^{-1} \widetilde{A}_{zz}) \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2} \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{\theta} = &-M_{\theta\theta}(\boldsymbol{\lambda} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + \boldsymbol{\lambda} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 - N_{\theta\theta}(A_{\eta_1\theta}^T \boldsymbol{\eta}_1 + A_{\eta_2\theta}^T \boldsymbol{\eta}_2)) \\ &-M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1}(A_{vv} \mathbb{P}_v^T \boldsymbol{v} + A_{v\eta_1} \boldsymbol{\eta}_1 + A_{v\eta_2} \boldsymbol{\eta}_2) + M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2}. \end{aligned}$$

We notice that

$$M_s^{-1} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_1\theta}^T M_{\eta\eta}^{-1} F_{s,1} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} F_{s,2} \end{bmatrix} = M_{zz}^{-1} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} \end{bmatrix}.$$

Thus we can conclude because

$$(\boldsymbol{\lambda} - \mathbb{P}^T M_s^{-1} \widetilde{A}_{zz}) \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P}^T M_{zz}^{-1} \begin{bmatrix} F_f \\ F_{s,1} \\ F_{s,2} \end{bmatrix}.$$

∎

3.2. Instationary systems.

PROPOSITION 3.4. *Let* $(\boldsymbol{v}^0, \boldsymbol{\eta}_2^0, \boldsymbol{g}_p) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_\tau}$ *such that* $(\boldsymbol{v}^0, 0, \boldsymbol{\eta}_2^0)$ *belongs to* $\mathrm{Ker}(\widehat{A}_{z\theta}^T)$ *and* $\boldsymbol{g}_p(0) = 0$. *System* (2.3) *or* (2.4) *is equivalent to the system*

$$\begin{aligned} &\frac{d}{dt} \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbf{A} \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} + \mathbf{B} \boldsymbol{f}, \quad \mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} (0) = \mathbb{P}^T \begin{bmatrix} \boldsymbol{v}^0 \\ 0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\ &(I - \mathbb{P}_v^T) \boldsymbol{v} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 + M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\theta g} \boldsymbol{g}_p, \\ &\boldsymbol{\theta} = -M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1' - M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2' - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T) \boldsymbol{\eta}_1 \\ &\quad -M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T) \boldsymbol{\eta}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{v} + M_{\theta\theta} A_{\theta g} \boldsymbol{g}_p' - M_{\theta\theta} N_{\theta\theta} A_{\theta g} \boldsymbol{g}_p, \end{aligned} \tag{3.11}$$

where

$$\mathbf{B} = M_s^{-1} B. \tag{3.12}$$

Proof. The proof follows from Proposition 3.3. ∎

3.3. Adjoint stationary systems. The goal of this section is to study the adjoint system

$$(\boldsymbol{\lambda} M - A^T) \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} G_f \\ G_{s,1} \\ G_{s,2} \\ 0 \end{bmatrix}, \tag{3.13}$$

and to establish results similar to those established for the direct system (3.9) in Proposition 3.3. For that we introduce the operator $\mathbf{A}^{\#}$ defined by

$$\mathbf{A}^{\#} = \mathbb{P}^T M_s^{-T} \widetilde{A}_{zz}^T.$$

PROPOSITION 3.5. *Let $(G_f, G_{s,1}, G_{s,2})$ be in $\mathbb{K}^{n_z}$. The quadruplet $(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}) \in \mathbb{K}^n$ is solution of equation* (3.13) *if and only if*

$$\begin{aligned}
&(\boldsymbol{\lambda} M_s^T - M_s^T \mathbf{A}^{\#}) \mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} G_f \\ G_{s,1} - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f \\ G_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f \end{bmatrix}, \\
&(I - \mathbb{P}_v^T) \boldsymbol{\psi} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2, \\
&\boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f,
\end{aligned} \tag{3.14}$$

where the operator $N_{\theta\theta}^{\#} \in \mathcal{L}(\mathbb{K}^{n_\theta}, \mathbb{K}^{n_v})$ is defined by

$$N_{\theta\theta}^{\#} = A_{v\theta}^T M_{vv}^{-1} A_{vv}^T M_{vv}^{-1} A_{v\theta} M_{\theta\theta}. \tag{3.15}$$

If in addition,

$$A_{v\theta}^T M_{vv}^{-1} G_f + A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} = 0,$$

then, we have

$$(\boldsymbol{\lambda} M_s^T - M_s^T \mathbf{A}^{\#}) \mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = M_s^T \mathbb{P}^T M_{zz}^{-1} \begin{bmatrix} G_f \\ G_{s,1} \\ G_{s,2} \end{bmatrix},$$

and

$$\boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} + M_{\theta\theta} A_{\eta_2\theta}^T G_{s,2}.$$

To prove the proposition, we will establish two preliminary results, Lemmas 3.3 and 3.4. We introduce the operator $\mathbb{L}^{\#} \in \mathcal{L}(\mathbb{K}^{n_\theta}, \mathbb{K}^{n_v})$ defined by

$$\mathbb{L}^{\#} \boldsymbol{h} = \boldsymbol{\psi}, \tag{3.16}$$

where $(\boldsymbol{\psi}, \boldsymbol{\rho})$ is the solution of the equation

$$A_{vv}^T \boldsymbol{\psi} + A_{v\theta} \boldsymbol{\rho} = 0, \quad A_{v\theta}^T \boldsymbol{\psi} = \boldsymbol{h}.$$

As in Lemma 3.2, we can prove that

$$\mathbb{L}^{\#} = A_{vv}^{-T} A_{v\theta} A_{\theta\theta}^T \quad \text{and} \quad (I - \mathbb{P}_v^T) \mathbb{L}^{\#} = M_{vv}^{-1} A_{v\theta} M_{\theta\theta}. \tag{3.17}$$

We introduce the operator $\mathbb{A}^{\#} \in \mathcal{L}(\mathbb{K}^{n_v})$ defined by

$$\mathbb{A}^{\#} = \mathbb{P}_v^T M_{vv}^{-1} A_{vv}^T. \tag{3.18}$$

LEMMA 3.3. *Let $G_f \in \mathbb{K}^{n_v}$ and $\boldsymbol{h} \in \mathbb{K}^{n_\theta}$. The couple $(\boldsymbol{\psi}, \boldsymbol{\rho})$ is a solution of system*

$$\boldsymbol{\lambda} M_{vv} \boldsymbol{\psi} = A_{vv}^T \boldsymbol{\psi} + A_{v\theta} \boldsymbol{\rho} + G_f, \quad A_{v\theta}^T \boldsymbol{\psi} = \boldsymbol{h}, \tag{3.19}$$

if and only if

$$\begin{aligned}
&\boldsymbol{\lambda}\mathbb{P}_v^T\boldsymbol{\psi} = \mathbb{A}^{\#}\mathbb{P}_v^T\boldsymbol{\psi} - \mathbb{A}^{\#}\mathbb{P}_v^T\mathbb{L}^{\#}\boldsymbol{h} + \mathbb{P}_v^T M_{vv}^{-1} G_f, \quad (I-\mathbb{P}_v^T)\boldsymbol{\psi} = M_{vv}^{-1}A_{v\theta}M_{\theta\theta}\boldsymbol{h},\\
&\boldsymbol{\rho} = -M_{\theta\theta}\left(-\boldsymbol{\lambda}\boldsymbol{h} + N_{\theta\theta}^{\#}\boldsymbol{h} + A_{v\theta}^T M_{vv}^{-1}A_{vv}^T\mathbb{P}_v^T\boldsymbol{\psi} + A_{v\theta}^T M_{vv}^{-1}G_f\right).
\end{aligned} \tag{3.20}$$

Proof. It is similar to that of Lemma 3.2. ∎

Now, we introduce the matrix

$$C := \begin{bmatrix} A_{vv}^T & 0 & A_{vv}^T\mathbb{P}_v^T\mathbb{L}^{\#}A_{\eta_2\theta}^T \\ A_{v\eta_1}^T + C_{\eta_1 v} & 0 & A_{\eta_2\eta_1}^T + C_{\eta_1\eta_2} \\ A_{v\eta_2}^T + C_{\eta_2 v} & A_{\eta_1\eta_2}^T & A_{\eta_2\eta_2}^T + C_{\eta_2\eta_2} \end{bmatrix},$$

where

$$\begin{aligned}
C_{\eta_1\eta_2} &:= A_{\eta_1\theta}M_{\theta\theta}N_{\theta\theta}^{\#}A_{\eta_2\theta}^T - A_{v\eta_1}^T M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T, \quad C_{\eta_1 v} := -A_{\eta_1\theta}M_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T,\\
C_{\eta_2\eta_2} &:= A_{\eta_2\theta}M_{\theta\theta}N_{\theta\theta}^{\#}A_{\eta_2\theta}^T - A_{v\eta_2}^T M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T, \quad C_{\eta_2 v} := -A_{\eta_2\theta}M_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}^T.
\end{aligned}$$

LEMMA 3.4. *We have* $\mathbb{P}^T M_s^{-T} C \mathbb{P}^T = \mathbf{A}^{\#}\mathbb{P}^T$.

Proof. We have

$$\widetilde{A}_{zz}^T = \begin{bmatrix} A_{vv}^T & 0 & A_{\eta_2 v}^T \\ A_{\eta_1 v}^T + (A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_1\theta}^T)^T & 0 & \widetilde{A}_{\eta_2\eta_1}^T \\ A_{v\eta_2}^T + (A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_2\theta}^T)^T & A_{\eta_1\eta_2}^T & \widetilde{A}_{\eta_2\eta_2}^T \end{bmatrix}.$$

Hence

$$(\widetilde{A}_{zz}^T - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & A_{\eta_2 v}^T - A_{vv}^T\mathbb{P}_v^T\mathbb{L}^{\#}A_{\eta_2\theta}^T \\ ((A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_1\theta}^T)^T - C_{\eta_1 v})\mathbb{P}_v^T & 0 & \widetilde{A}_{\eta_2\eta_1}^T - A_{\eta_2\eta_1}^T - C_{\eta_1\eta_2} \\ ((A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_2\theta}^T)^T - C_{\eta_2 v})\mathbb{P}_v^T & 0 & \widetilde{A}_{\eta_2\eta_2}^T - A_{\eta_2\eta_2}^T - C_{\eta_2\eta_2} \end{bmatrix}. \tag{3.21}$$

Thanks to equation (3.17), we obtain

$$\begin{aligned}
A_{vv}^T\mathbb{P}_v^T\mathbb{L}^{\#}A_{\eta_2\theta}^T &= A_{vv}^T\mathbb{L}^{\#}A_{\eta_2\theta}^T - A_{vv}^T(I-\mathbb{P}_v^T)\mathbb{L}^{\#}A_{\eta_2\theta}^T = A_{v\theta}A_{\theta\theta}^T A_{\eta_2\theta}^T - A_{vv}^T M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T\\
&= A_{v\theta}A_{\theta\theta}^T A_{\eta_2\theta}^T + A_{\eta_2 v}^T,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_1\theta}^T &= A_{vv}\mathbb{L}A_{\eta_1\theta}^T - A_{vv}(I-\mathbb{P}_v^T)\mathbb{L}A_{\eta_1\theta}^T = A_{v\theta}A_{\theta\theta}A_{\eta_1\theta}^T - A_{vv}M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_1\theta}^T\\
&= A_{v\theta}A_{\theta\theta}A_{\eta_1\theta}^T + C_{\eta_1 v}^T,
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_2\theta}^T &= A_{vv}\mathbb{L}A_{\eta_2\theta}^T - A_{vv}(I-\mathbb{P}_v^T)\mathbb{L}A_{\eta_2\theta}^T = A_{v\theta}A_{\theta\theta}A_{\eta_2\theta}^T - A_{vv}M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T\\
&= A_{v\theta}A_{\theta\theta}A_{\eta_2\theta}^T + C_{\eta_2 v}^T,
\end{aligned} \tag{3.24}$$

(the operator $A_{\eta_2\theta}$ is defined in Section 2). It follows that

$$\begin{aligned}
&A_{\eta_2 v}^T - A_{vv}^T\mathbb{P}_v^T\mathbb{L}^{\#}A_{\eta_2\theta}^T = -A_{v\theta}A_{\theta\theta}^T A_{\eta_2\theta}^T,\\
&((A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_1\theta}^T)^T - C_{\eta_1 v})\mathbb{P}_v^T = A_{\eta_1\theta}A_{\theta\theta}^T A_{v\theta}^T\mathbb{P}_v^T = 0,\\
&((A_{vv}\mathbb{P}_v^T\mathbb{L}A_{\eta_2\theta}^T)^T - C_{\eta_2 v})\mathbb{P}_v^T = A_{\eta_2\theta}A_{\theta\theta}^T A_{v\theta}^T\mathbb{P}_v^T = 0,
\end{aligned} \tag{3.25}$$

because $\mathbb{P}_v A_{v\theta} = 0$, see Proposition 3.1. We recall that

$$N_{\theta\theta} = A_{v\theta}^T M_{vv}^{-1} A_{vv} M_{vv}^{-1} A_{v\theta} M_{\theta\theta} \quad \text{and} \quad N_{\theta\theta}^{\#} = A_{v\theta}^T M_{vv}^{-1} A_{vv}^T M_{vv}^{-1} A_{v\theta} M_{\theta\theta},$$

then, we can easily prove that

$$A_{\eta_1\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T = A_{\eta_1\theta} N_{\theta\theta}^T M_{\theta\theta} A_{\eta_2\theta}^T \quad \text{and} \quad A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T = A_{\eta_2\theta} N_{\theta\theta}^T M_{\theta\theta} A_{\eta_2\theta}^T.$$

Hence

$$\begin{array}{l} \widetilde{A}_{\eta_2\eta_1}^T - A_{\eta_2\eta_1}^T - C_{\eta_1\eta_2} = A_{\eta_1\theta} N_{\theta\theta}^T M_{\theta\theta} A_{\eta_2\theta}^T - A_{\eta_1\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T = 0, \\ \widetilde{A}_{\eta_2\eta_2}^T - A_{\eta_2\eta_2}^T - C_{\eta_2\eta_2} = A_{\eta_2\theta} N_{\theta\theta}^T M_{\theta\theta} A_{\eta_2\theta}^T - A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T = 0. \end{array} \tag{3.26}$$

From equations (3.21), (3.25) and (3.26), we obtain

$$(\widetilde{A}_{zz}^T - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & -A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.27}$$

We deduce that

$$\mathbb{P}^T M_s^{-T} (\widetilde{A}_{zz}^T - C)\mathbb{P}^T = \begin{bmatrix} 0 & 0 & -\mathbb{P}_v^T M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0, \tag{3.28}$$

because $\mathbb{P}_v^T M_{vv}^{-1} A_{v\theta} = M_{vv} \mathbb{P}_v A_{v\theta} = 0$, see Proposition 3.1. Thus, we have $\mathbb{P}^T M_s^{-T} C \mathbb{P}^T = \mathbb{P}^T M_s^{-T} \widetilde{A}_{zz}^T \mathbb{P}^T$. □

Proof of Proposition 3.5.

Step 1. Let $(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}) \in \mathbb{C}^n$ be a solution of (3.13). Then, $(\boldsymbol{\psi}, \boldsymbol{\rho})$ is solution of the system

$$\boldsymbol{\lambda} M_{vv} \boldsymbol{\psi} = A_{vv}^T \boldsymbol{\psi} + A_{v\theta} \boldsymbol{\rho} + G_f, \quad A_{v\theta}^T \boldsymbol{\psi} = -A_{\eta_2\theta}^T \boldsymbol{\kappa}_2. \tag{3.29}$$

Thanks to Lemma 3.3, we have

$$\begin{array}{l} \boldsymbol{\lambda} \mathbb{P}_v^T \boldsymbol{\psi} = \mathbb{A}^{\#} \mathbb{P}_v^T \boldsymbol{\psi} + \mathbb{A}^{\#} \mathbb{P}_v^T \mathbb{L}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + \mathbb{P}_v^T M_{vv}^{-1} G_f, \quad (I - \mathbb{P}_v^T)\boldsymbol{\psi} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2, \\ \boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f. \end{array} \tag{3.30}$$

The couple $(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$ is a solution to the system

$$\begin{array}{l} \boldsymbol{\lambda} M_{\eta\eta} \boldsymbol{\kappa}_1 = A_{\eta_2\eta_1}^T \boldsymbol{\kappa}_2 + A_{v\eta_1}^T \boldsymbol{\psi} + A_{\eta_1\theta} \boldsymbol{\rho} + G_{s,1}, \\ \boldsymbol{\lambda} M_{\eta\eta} \boldsymbol{\kappa}_2 = A_{\eta_1\eta_2}^T \boldsymbol{\kappa}_1 + A_{\eta_2\eta_2}^T \boldsymbol{\kappa}_2 + A_{v\eta_2}^T \boldsymbol{\psi} + A_{\eta_2\theta} \boldsymbol{\rho} + G_{s,2}. \end{array}$$

We have

$$A_{\eta_2\theta} \boldsymbol{\rho} = -\boldsymbol{\lambda} A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f,$$

$$A_{\eta_1\theta} \boldsymbol{\rho} = -\boldsymbol{\lambda} A_{\eta_1\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + A_{\eta_1\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f.$$

Replacing these two expressions in the system satisfied by $(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)$, we obtain

$$\begin{array}{l} \boldsymbol{\lambda} M_{\eta\eta} \boldsymbol{\kappa}_1 + \boldsymbol{\lambda} A_{\eta_1\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 = (A_{\eta_2\eta_1}^T + A_{\eta_1\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T - A_{v\eta_1}^T M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T) \boldsymbol{\kappa}_2 \\ \qquad\qquad + (A_{v\eta_1}^T - A_{\eta_1\theta} A_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T) \mathbb{P}_v^T \boldsymbol{\psi} + G_{s,1} - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f, \\ \boldsymbol{\lambda} (M_{\eta\eta} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T) \boldsymbol{\kappa}_2 = A_{\eta_1\eta_2}^T \boldsymbol{\kappa}_1 + (A_{\eta_2\eta_2}^T + A_{\eta_2\theta} M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T - A_{v\eta_2}^T M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T) \boldsymbol{\kappa}_2 \\ \qquad\qquad + (A_{v\eta_2}^T - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T) \mathbb{P}_v^T \boldsymbol{\psi} + G_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f. \end{array}$$

Thus, thanks to Lemma 3.4, we have

$$
\begin{aligned}
&(\boldsymbol{\lambda} M_s^T - M_s^T \mathbf{A}^{\#})\mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} G_f \\ G_{s,1} - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f \\ G_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f \end{bmatrix}, \\
&(I - \mathbb{P}_v^T)\boldsymbol{\psi} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2, \\
&\boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f.
\end{aligned}
$$

The equivalence between (3.13) and (3.14) is proved.

Step 2. If in addition $A_{v\theta}^T M_{vv}^{-1} G_f + A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} = 0$, we have $M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f = -M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2}$. Hence

$$
\begin{aligned}
&G_{s,1} - A_{\eta_1\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f = G_{s,1} + A_{\eta_1\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} \\
&\text{and} \quad G_{s,2} - A_{\eta_2\theta} M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} G_f = G_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2}.
\end{aligned}
$$

Moreover $\boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv}^T \mathbb{P}_v^T \boldsymbol{\psi} + M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2}$, and

$$
(\boldsymbol{\lambda} M_s^T - M_s^T \mathbf{A}^{\#})\mathbb{P}^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \mathbb{P} \begin{bmatrix} G_f \\ G_{s,1} + A_{\eta_1\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} \\ G_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} \end{bmatrix}.
$$

Notice that

$$
M_s^{-T} \begin{bmatrix} G_f \\ G_{s,1} + A_{\eta_1\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} \\ G_{s,2} + A_{\eta_2\theta} M_{\theta\theta} A_{\eta_2\theta}^T M_{\eta\eta}^{-1} G_{s,2} \end{bmatrix} = M_{zz}^{-1} \begin{bmatrix} G_f \\ G_{s,1} \\ G_{s,2} \end{bmatrix}.
$$

The proof is complete. □

4. Equivalence between eigenvalue problems. We are going to study the links between the eigenvalue problems associated to the operator $\mathbf{A}$ and the eigenvalue problem associated to the pair (A, M), i.e.

$$
\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \in \mathrm{Ker}(A_{z\theta}^T), \quad \mathbf{A} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{bmatrix} = \boldsymbol{\lambda} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{bmatrix}, \tag{4.1}
$$

and

$$
\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\theta}) \in \mathbb{C}^n, \quad A \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \\ \boldsymbol{\theta} \end{bmatrix} = \boldsymbol{\lambda} M \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \\ \boldsymbol{\theta} \end{bmatrix}. \tag{4.2}
$$

We will also study the links between the adjoint eigenvalue problems

$$
\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathrm{Ker}(A_{z\theta}^T), \quad \mathbf{A}^{\#} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix} = \boldsymbol{\lambda} \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \end{bmatrix}, \tag{4.3}
$$

and

$$
\boldsymbol{\lambda} \in \mathbb{C}, \quad (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}) \in \mathbb{C}^n, \quad A^T \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix} = \boldsymbol{\lambda} M \begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa}_1 \\ \boldsymbol{\kappa}_2 \\ \boldsymbol{\rho} \end{bmatrix}. \tag{4.4}
$$

We recall that $A_{z\theta}$ is defined in (3.4) and that $\mathrm{Ker}(A_{z\theta}^T) = \mathrm{Im}(\mathbb{P}^T)$.

DEFINITION 4.1. *A vector $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i) \in \mathrm{Ker}(A_{z\theta}^T)$ is a generalized eigenvector of order i for problem (4.1) associated with a solution $(\boldsymbol{\lambda}, (\boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0))$ of (4.1) if $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)$ is obtained by solving the chain of equations*

$$
(\boldsymbol{\lambda} I - \mathbf{A}) \begin{bmatrix} \boldsymbol{u}^j \\ \boldsymbol{\delta}_1^j \\ \boldsymbol{\delta}_2^j \end{bmatrix} = - \begin{bmatrix} \boldsymbol{u}^{j-1} \\ \boldsymbol{\delta}_1^{j-1} \\ \boldsymbol{\delta}_2^{j-1} \end{bmatrix}, \quad \textit{for } 1 \le j \le i.
$$

A vector $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}^i) \in \mathbb{C}^n$ *is a generalized eigenvector of order* i *for problem (4.2) associated with a solution* $(\boldsymbol{\lambda}, (\boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0, \boldsymbol{\theta}^0))$ *of (4.2) if it is obtained by solving the chain of equations*

$$(\boldsymbol{\lambda} M - A) \begin{bmatrix} \boldsymbol{u}^j \\ \boldsymbol{\delta}_1^j \\ \boldsymbol{\delta}_2^j \\ \boldsymbol{\theta}^j \end{bmatrix} = -M \begin{bmatrix} \boldsymbol{u}^{j-1} \\ \boldsymbol{\delta}_1^{j-1} \\ \boldsymbol{\delta}_2^{j-1} \\ \boldsymbol{\theta}^{j-1} \end{bmatrix}, \quad \text{for } 1 \leq j \leq i.$$

We have similar definitions for the problems (4.3) and (4.4).

4.1. Equivalence between the direct eigenvalue problems.

THEOREM 4.1. *A pair* $(\boldsymbol{\lambda}, (\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\theta}))$ *is a solution of the eigenvalue problem (4.2) if and only if* $(\boldsymbol{\lambda}, (\mathbb{P}_v^T \boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{z\theta}^T)$ *is a solution of the eigenvalue problem (4.1) and*

$$\begin{aligned} &(I - \mathbb{P}_v^T)\boldsymbol{u} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1 - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2, \\ &\boldsymbol{\theta} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1 - \boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2 - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T)\boldsymbol{\delta}_1 \\ &\quad - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\delta}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{u}. \end{aligned} \tag{4.5}$$

Proof. It follows from Proposition 3.3. ∎

THEOREM 4.2. *A quadruplet* $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}^i)$ *is a generalized eigenvector of order* i *associated with the solution* $(\boldsymbol{\lambda}, (\boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0, \boldsymbol{\theta}^0))$ *of problem (4.2) if and only if* $(\mathbb{P}_v^T \boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i) \in \mathrm{Ker}(A_{z\theta}^T)$ *is a generalized eigenvector of order* i *associated with the solution* $(\boldsymbol{\lambda}, (\mathbb{P}_v^T \boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{z\theta}^T)$ *of (4.1) and*

$$\begin{aligned} &(I - \mathbb{P}_v^T)\boldsymbol{u}^i = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^i - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^i, \\ &\boldsymbol{\theta}^i = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^i - \boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^i - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T)\boldsymbol{\delta}_1^i \\ &\quad - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\delta}_2^i - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{u}^i - M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^{i-1} - M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^{i-1}, \end{aligned}$$

where the vector $(\mathbb{P}_v^T \boldsymbol{u}^{i-1}, \boldsymbol{\delta}_1^{i-1}, \boldsymbol{\delta}_2^{i-1}) \in \mathrm{Ker}(A_{z\theta}^T)$ *is a generalized eigenvector of order* $i-1$ *associated with* $(\boldsymbol{\lambda}, (\mathbb{P}_v^T \boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0))$.

Proof. Thanks to Proposition 3.3, it follows that $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}^i)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{u}^0, \boldsymbol{\delta}_1^0, \boldsymbol{\delta}_2^0, \boldsymbol{\theta}^0))$ of problem (4.2) if and only if

$$(\boldsymbol{\lambda} I - \mathbf{A})\mathbb{P}^T \begin{bmatrix} \boldsymbol{u}^i \\ \boldsymbol{\delta}_1^i \\ \boldsymbol{\delta}_2^i \end{bmatrix} = -\mathbb{P}^T \begin{bmatrix} \boldsymbol{u}^{i-1} \\ \boldsymbol{\delta}_1^{i-1} \\ \boldsymbol{\delta}_2^{i-1} \end{bmatrix},$$

$$\begin{aligned} &(I - \mathbb{P}_v^T)\boldsymbol{u}^i = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^i - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^i, \\ &\boldsymbol{\theta}^i = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^i - \boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^i - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T)\boldsymbol{\delta}_1^i \\ &\quad - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T)\boldsymbol{\delta}_2^i - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{u}^i - M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\delta}_1^{i-1} - M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\delta}_2^{i-1}. \end{aligned}$$

We deduce the theorem by induction. ∎

4.2. Equivalence between the adjoint eigenvalue problems. Now, we are going to prove the equivalence between the adjoint eigenvalue problems.

THEOREM 4.3. *A pair* $(\boldsymbol{\lambda}, (\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\rho}))$ *is a solution of the eigenvalue problem (4.4) if and only if* $(\boldsymbol{\lambda}, (\mathbb{P}_v^T \boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{z\theta}^T)$ *is a solution of the eigenvalue problem (4.3) and*

$$\begin{aligned} &(I - \mathbb{P}_v^T)\boldsymbol{\psi} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2, \\ &\boldsymbol{\rho} = -\boldsymbol{\lambda} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 + M_{\theta\theta} N_{\theta\theta}^{\#} A_{\eta_2\theta}^T \boldsymbol{\kappa}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{\psi}. \end{aligned}$$

Proof. It follows from Proposition 3.5 ∎

THEOREM 4.4. *A quadruplet $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\boldsymbol{\psi}^0, \boldsymbol{\kappa}_1^0, \boldsymbol{\kappa}_2^0, \boldsymbol{\rho}^0))$ of problem (4.4) if and only if $(\mathbb{P}_v^T\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i) \in \mathrm{Ker}(A_{z\theta}^T)$ is a generalized eigenvector of order i associated with the solution $(\boldsymbol{\lambda}, (\mathbb{P}_v^T\boldsymbol{\psi}^0, \boldsymbol{\kappa}_1^0, \boldsymbol{\kappa}_2^0)) \in \mathbb{C}^* \times \mathrm{Ker}(A_{z\theta}^T)$ of (4.3) and*

$$(I - \mathbb{P}_v^T)\boldsymbol{\psi}^i = -M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i,$$
$$\boldsymbol{\rho}^i = -\boldsymbol{\lambda}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i + M_{\theta\theta}N_{\theta\theta}^{\#}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i - M_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}A_{vv}\mathbb{P}_v^T\boldsymbol{\psi}^i + M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^{i-1},$$

where the vector $(\mathbb{P}_v^T\boldsymbol{\psi}^{i-1}, \boldsymbol{\kappa}_1^{i-1}, \boldsymbol{\kappa}_2^{i-1}) \in \mathrm{Ker}(A_{z\theta}^T)$ is a generalized eigenvector of order $i-1$.

Proof. It relies on Proposition 3.5 and is similar to the proof of Theorem 4.2. ∎

5. Stabilization of the finite-dimensional linear system.

5.1. Spectral decomposition of the operator $\mathbf{A}\mathbb{P}^T$. We are looking for a decomposition of $\mathbb{R}^{n_z}$ into a sum of the generalized eigenspaces of the operator $\mathbf{A}\mathbb{P}^T$. It is convenient to assume that $0 \notin \mathrm{spect}(\mathbf{A})$. If it is not the case, we can replace $\mathbf{A}$ by $\mathbf{A} - \lambda_0\mathbf{I}$ with $\lambda_0 \notin \mathrm{spect}(\mathbf{A})$ before the analysis. We can decompose $\mathbb{R}^{n_z}$ as follows

$$\mathbb{R}^{n_z} = \mathrm{Ker}(A_{z\theta}^T) \oplus \mathrm{Im}(M_s^{-1}A_{z\theta}),$$
$$\text{with} \quad \mathrm{Ker}(A_{z\theta}^T) = \mathrm{Im}(\mathbb{P}^T) \text{ and } \mathrm{Im}(M_s^{-1}A_{z\theta}) = \mathrm{Ker}(\mathbb{P}^T).$$

Since $0 \notin \mathrm{spect}(\mathbf{A})$, we have $\mathrm{Ker}(\mathbf{A}\mathbb{P}^T) = \mathrm{Ker}(\mathbb{P}^T) = \mathrm{Im}(M_s^{-1}A_{z\theta})$. Therefore, 0 is an eigenvalue of the operator $\mathbf{A}\mathbb{P}^T$ and $\mathrm{Im}(M_s^{-1}A_{z\theta})$ is the corresponding eigenspace. In order to decompose $\mathrm{Ker}(A_{z\theta}^T)$ into the other eigenspaces of the operator $\mathbf{A}\mathbb{P}^T$, we consider the eigenvalue problem (4.1) and the adjoint problem (4.3). We are going to prove that the spectral decomposition of the operators $\mathbf{A}\mathbb{P}^T$ and $\mathbf{A}^{\#}\mathbb{P}^T$ can be deduced from the study of the eigenvalue problems (4.2) and (4.4). First, we show the following result.

LEMMA 5.1. *Let $(\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2) \in \mathbb{C}^{n_z}$ and $(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) \in \mathbb{C}^{n_z}$ be such that*

$$A_{v\theta}^T\boldsymbol{u} + A_{\eta_1\theta}^T\boldsymbol{\delta}_1 + A_{\eta_2\theta}^T\boldsymbol{\delta}_2 = 0 \quad \text{and} \quad A_{v\theta}^T\boldsymbol{\psi} + A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 = 0.$$

Then, we have

$$(\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2)^T(\mathbb{P}M_s^T - M_{zz})(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) = 0.$$

Proof. We have

$$\mathbb{P}M_s^T - M_{zz} = \begin{bmatrix} (\mathbb{P}_v - I)M_{vv} & 0 & 0 \\ 0 & 0 & N_{\eta_2\eta_1}^T \\ 0 & 0 & N_{\eta_2\eta_2} \end{bmatrix} = \begin{bmatrix} -A_{v\theta}M_{\theta\theta}A_{v\theta}^T & 0 & 0 \\ 0 & 0 & N_{\eta_2\eta_1}^T \\ 0 & 0 & N_{\eta_2\eta_2} \end{bmatrix},$$

because $\mathbb{P}_v - I = -A_{v\theta}(A_{v\theta}^TM_{vv}^{-1}A_{v\theta})^{-1}A_{v\theta}^TM_{vv}^{-1} = -A_{v\theta}M_{\theta\theta}A_{v\theta}^TM_{vv}^{-1}$, see Proposition 3.1. It follows that

$$\begin{aligned}(\boldsymbol{u}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2)^T(\mathbb{P}M_s^T - M_{zz})(\boldsymbol{\psi}, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) &= -\boldsymbol{u}^TA_{v\theta}M_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} + \boldsymbol{\delta}_1^TA_{\eta_1\theta}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 + \boldsymbol{\delta}_2^TA_{\eta_2\theta}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2 \\ &= -\boldsymbol{u}^TA_{v\theta}M_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} - \boldsymbol{\delta}_1^TA_{\eta_1\theta}M_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} - \boldsymbol{\delta}_2^TA_{\eta_2\theta}M_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} \\ &= -(\boldsymbol{u}^TA_{v\theta} + \boldsymbol{\delta}_1^TA_{\eta_1\theta} + \boldsymbol{\delta}_2^TA_{\eta_2\theta})M_{\theta\theta}A_{v\theta}^T\boldsymbol{\psi} = 0\end{aligned}$$

∎

We set

$$n_0 = \dim(\mathrm{Ker}(A_{z\theta}^T)). \tag{5.1}$$

THEOREM 5.1. *There exist two families $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i)_{1\le i\le n_z}$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i)_{1\le i\le n_z}$ of $\mathbb{C}^{n_z}$ constituted of eigenvectors and generalized eigenvectors of (4.2) and (4.4) respectively, such that*

- *The union of the families $(\mathbb{P}_v^T\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{1\le i\le n_0}$ and $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{n_0+1\le i\le n_z}$ is a basis of $\mathbb{C}^{n_z}$, $(\mathbb{P}_v^T\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{1\le i\le n_0}$ is constituted of eigenvectors and generalized eigenvectors of problem (4.1) and $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{n_0+1\le i\le n_z}$ is a family of vectors belonging to $\mathrm{Ker}(\mathbb{P}^T)$.*

- *The union of the families* $(\mathbb{P}_v^T\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ *and* $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_z}$ *is a basis of* $\mathbb{C}^{n_z}$, $(\mathbb{P}_v^T\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ *is constituted of eigenvectors and generalized eigenvectors of problem (4.3) and* $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_z}$ *is a family of vectors belonging to* $\mathrm{Ker}(\mathbb{P}^T)$.
- *We have the decompositions*

$$\Lambda_{\mathbb{C}} = \boldsymbol{F}^{-1}\mathbf{A}\mathbb{P}^T\boldsymbol{F} \quad \text{and} \quad \Lambda_{\mathbb{C}}^T = \boldsymbol{\Phi}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\boldsymbol{\Phi},$$

where $\Lambda_{\mathbb{C}}$ *is a decomposition of* $\mathbf{A}\mathbb{P}^T$ *into complex Jordan blocks. The matrices* $\boldsymbol{F}$ *and* $\boldsymbol{\Phi}$ *are in the form* $\boldsymbol{F} = [\mathbb{P}^T\boldsymbol{F}_1|\boldsymbol{F}_0]$ *and* $\boldsymbol{\Phi} = [\mathbb{P}^T\boldsymbol{\Phi}_1|\boldsymbol{\Phi}_0]$ *where* $\boldsymbol{F}_1$, $\boldsymbol{F}_0$, $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_0$ *are the matrices whose columns are* $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{1\le i\le n_0}$, $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{n_0+1\le i\le n_z}$, $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$, $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_z}$ *respectively. Moreover, we have the following bi-orthogonality conditions*

$$\boldsymbol{F}^* M_s^T\boldsymbol{\Phi} = I_{\mathbb{C}^{n_z}} \quad \text{and} \quad \boldsymbol{F}_1^* M_{zz}\boldsymbol{\Phi}_1 = I_{\mathbb{R}^{n_z}},$$

where $\boldsymbol{F}^*$ *is the complex transpose conjugate of* $\boldsymbol{F}$.

Proof. Step 1. Decomposition of $\mathbf{A}\mathbb{P}^T$ *and* $\mathbf{A}^{\#}\mathbb{P}^T$. Using the complex Jordan decomposition of real matrices, we know that there exists a matrix $\widehat{\boldsymbol{F}} \in \mathbb{C}^{n_z\times n_z}$ constituted of eigenvectors and generalized eigenvectors of Problem (4.1) such that $\Lambda_{\mathbb{C}} = \widehat{\boldsymbol{F}}^{-1}\mathbf{A}\mathbb{P}^T\widehat{\boldsymbol{F}}$. We set $\widehat{\boldsymbol{\Phi}} = M_s^{-T}\widehat{\boldsymbol{F}}^{-*}$, where $M_s^{-T} = (M_s^T)^{-1}$, $\widehat{\boldsymbol{F}}^{-*} = (\widehat{\boldsymbol{F}}^*)^{-1}$ and $\widehat{\boldsymbol{F}}^*$ is the conjugate transpose of $\widehat{\boldsymbol{F}}$. Thus, the matrices $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{\Phi}}$ satisfy the bi-orthogonality condition $\widehat{\boldsymbol{F}}^* M_s^T\widehat{\boldsymbol{\Phi}} = I_{\mathbb{C}^{n_z}}$.
Now, we are going to prove that $\Lambda_{\mathbb{C}}^* = \widehat{\boldsymbol{\Phi}}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\widehat{\boldsymbol{\Phi}}$.
From the identities $M_s^{-1}\widehat{\boldsymbol{\Phi}}^{-*} = \widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{F}}\Lambda_{\mathbb{C}} = \mathbf{A}\mathbb{P}^T\widehat{\boldsymbol{F}} = \mathbb{P}^T M_s^{-1}\widetilde{A}_{zz}\mathbb{P}^T\widehat{\boldsymbol{F}}$, we deduce that $M_s^{-1}\widehat{\boldsymbol{\Phi}}^{-*}\Lambda_{\mathbb{C}} = \mathbb{P}^T M_s^{-1}\widetilde{A}_{zz}\mathbb{P}^T M_s^{-1}\widehat{\boldsymbol{\Phi}}^{-*}$ and $\Lambda_{\mathbb{C}}\widehat{\boldsymbol{\Phi}}^* = \widehat{\boldsymbol{\Phi}}^* M_s\mathbb{P}^T M_s^{-1}\widetilde{A}_{zz}\mathbb{P}^T M_s^{-1}$. Since $M_s\mathbb{P}^T = \mathbb{P}M_s$ and $\mathbb{P}^T M_s^{-1} = M_s^{-1}\mathbb{P}$, we have $\Lambda_{\mathbb{C}}\widehat{\boldsymbol{\Phi}}^* = \widehat{\boldsymbol{\Phi}}^*\mathbb{P}\widetilde{A}_{zz}M_s^{-1}\mathbb{P}$. Taking the complex conjugate transpose,
we obtain $\widehat{\boldsymbol{\Phi}}\Lambda_{\mathbb{C}}^* = \mathbb{P}^T M_s^{-T}\widetilde{A}_{zz}^T\mathbb{P}^T\widehat{\boldsymbol{\Phi}} = \mathbf{A}^{\#}\mathbb{P}^T\widehat{\boldsymbol{\Phi}}$. Hence $\Lambda_{\mathbb{C}}^* = \widehat{\boldsymbol{\Phi}}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\widehat{\boldsymbol{\Phi}}$.
Step 2. Construction of $\boldsymbol{F}$ *and* $\boldsymbol{\Phi}$. We denote by $(\hat{\boldsymbol{u}}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_i$ (resp. $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_i$) the columns of the matrix $\widehat{\boldsymbol{F}}$ (resp. $\widehat{\boldsymbol{\Phi}}$). Without loss of generality, we assume that $(\hat{\boldsymbol{u}}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{1\le i\le n_0}$ and $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$ belong to $\mathrm{Im}(\mathbb{P}^T)$. We denote by $\boldsymbol{F}_0$ (resp. $\boldsymbol{\Phi}_0$) the matrix whose columns are $(\hat{\boldsymbol{u}}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{n_0+1\le i\le n_z}$ (resp. $(\hat{\boldsymbol{\psi}}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{n_0+1\le i\le n_z}$). Finally, we denote by $\boldsymbol{\lambda}_i$ the eigenvalue of $\mathbf{A}\mathbb{P}^T$ associated to $(\hat{\boldsymbol{u}}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)$. Thanks to Theorem 4.2, $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i)$ with

$$\begin{aligned}
\boldsymbol{u}^i &= \hat{\boldsymbol{u}}^i - M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\delta}_1^i - M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\delta}_2^i,\\
\boldsymbol{\theta}_{\mathbf{c}}^i &= -\boldsymbol{\lambda}_i M_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\delta}_1^i - \boldsymbol{\lambda}_i M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\delta}_2^i - M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_1} - N_{\theta\theta}A_{\eta_1\theta}^T)\boldsymbol{\delta}_1^i\\
&\quad -M_{\theta\theta}(A_{v\theta}^T M_{vv}^{-1}A_{v\eta_2} - N_{\theta\theta}A_{\eta_2\theta}^T)\boldsymbol{\delta}_2^i - M_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{u}}^i - M_{\theta\theta}A_{\eta_1\theta}^T\boldsymbol{\delta}_1^{i-1} - M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\delta}_2^{i-1},
\end{aligned}$$

is a generalized eigenvector of problem (4.2). From Theorem 4.4, $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}^i)$ with

$$\begin{aligned}
\boldsymbol{\psi}^i &= \hat{\boldsymbol{\psi}}^i - M_{vv}^{-1}A_{v\theta}M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i,\\
\boldsymbol{\rho}^i &= -\boldsymbol{\lambda}_i M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i + M_{\theta\theta}N_{\theta\theta}^{\#}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^i - M_{\theta\theta}A_{v\theta}^T M_{vv}^{-1}A_{vv}\mathbb{P}_v^T\hat{\boldsymbol{\psi}}^i + M_{\theta\theta}A_{\eta_2\theta}^T\boldsymbol{\kappa}_2^{i-1},
\end{aligned}$$

is a generalized eigenvector of problem (4.4). We denote by $\boldsymbol{F}_1$ (resp. $\boldsymbol{\Phi}_1$) the matrix whose columns are $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i)_{1\le i\le n_0}$ (resp. $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i)_{1\le i\le n_0}$). By construction, we have

$$\boldsymbol{F} := [\mathbb{P}^T\boldsymbol{F}_1 \,|\, \boldsymbol{F}_0] = \widehat{\boldsymbol{F}} \quad \text{and} \quad \boldsymbol{\Phi} := [\mathbb{P}^T\boldsymbol{\Phi}_1 \,|\, \boldsymbol{\Phi}_0] = \widehat{\boldsymbol{\Phi}}.$$

Step 3. Bi-orthogonality conditions. We have to show that $\boldsymbol{F}^* M_s^T\boldsymbol{\Phi} = I_{\mathbb{C}^{n_z}}$ and $\boldsymbol{F}_1^* M_{zz}\boldsymbol{\Phi}_1 = I_{\mathbb{C}^{n_z}}$.
Due to the bi-orthogonality condition satisfied by $\widehat{\boldsymbol{F}}$ and $\widehat{\boldsymbol{\Phi}}$, we have

$$\boldsymbol{F}^* M_s^T\boldsymbol{\Phi} = I_{\mathbb{C}^{n_z}} \quad \text{and} \quad (\mathbb{P}^T\boldsymbol{F}_1)^* M_s^T\mathbb{P}^T\boldsymbol{\Phi}_1 = \boldsymbol{F}_1^*\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Phi}_1 = I_{\mathbb{C}^{n_z}}. \tag{5.2}$$

Since $\mathbb{P}M_s^T\mathbb{P}^T = \mathbb{P}M_s^T$, we have $\boldsymbol{F}_1^* M_{zz}\boldsymbol{\Phi}_1 = \boldsymbol{F}_1^*\mathbb{P}M_s^T\boldsymbol{\Phi}_1 + \boldsymbol{F}_1^*(M_{zz} - \mathbb{P}M_s^T)\boldsymbol{\Phi}_1 = \boldsymbol{F}_1^*\mathbb{P}M_s^T\mathbb{P}^T\boldsymbol{\Phi}_1 + \boldsymbol{F}_1^*(M_{zz} - \mathbb{P}M_s^T)\boldsymbol{\Phi}_1$. Therefore, thanks to Lemma 5.1 and equation (5.2), we get $\boldsymbol{F}_1^* M_{zz}\boldsymbol{\Phi}_1 = I_{\mathbb{C}^{n_z}}$.
□

THEOREM 5.2. *There exist two families* $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}^i)_{1\le i\le n_z}$ *and* $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}^i)_{1\le i\le n_z}$ *of* $\mathbb{R}^n$ *such that*
- *The union of the families* $(\mathbb{P}_v^T\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le n_0}$ *and* $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1\le i\le n_z}$ *is a basis of* $\mathbb{R}^{n_z}$.
- *The union of the families* $(\mathbb{P}_v^T\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le n_0}$ *and* $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{n_0+1\le i\le n_z}$ *is a basis of* $\mathbb{R}^{n_z}$.
- *We have the decompositions*

$$\Lambda = \boldsymbol{E}^{-1}\mathbf{A}\mathbb{P}^T\boldsymbol{E} \quad \textit{and} \quad \Lambda^T = \boldsymbol{\Xi}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\boldsymbol{\Xi},$$

where Λ *is a decomposition of* $\mathbf{A}\mathbb{P}^T$ *into real Jordan blocks. The matrices* $\boldsymbol{E}$ *and* $\boldsymbol{\Xi}$ *are in the form* $\boldsymbol{E} = [\mathbb{P}^T\boldsymbol{E}_1 \,|\, \boldsymbol{E}_0]$ *and* $\boldsymbol{\Xi} = [\mathbb{P}^T\boldsymbol{\Xi}_1 \,|\, \boldsymbol{\Xi}_0]$ *where* $\boldsymbol{E}_1$, $\boldsymbol{E}_0$, $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_0$ *are the matrices whose columns are* $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1\le i\le n_0}$, $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1\le i\le n_z}$, $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1\le i\le n_0}$, $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{n_0+1\le i\le n_z}$ *respectively. Moreover, we have the following bi-orthogonality conditions*

$$\boldsymbol{E}^T M_s^T\boldsymbol{\Xi} = I_{\mathbb{R}^{n_z}} \quad \textit{and} \quad \boldsymbol{E}_1^T M_{zz}\boldsymbol{\Xi}_1 = I_{\mathbb{R}^{n_z}}.$$

Proof. We denote by $(\boldsymbol{\lambda}_j)_j$ the eigenvalues of the pair (M, A). To construct $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}^i)_{1\le i\le n_z}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}^i)_{1\le i\le n_z}$, we proceed in the following way.
Case 1. $\boldsymbol{\lambda}_j$ *is real.* If $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_j$, we can assume that they are real vectors. Thus, we set

$$(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}^i) = (\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}^i) = (\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i).$$

Case 2. $\boldsymbol{\lambda}_j$ *is complex.* There exists k such that $\boldsymbol{\lambda}_k = \bar{\boldsymbol{\lambda}}_i$ is also an eigenvalue of (M, A). If $(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i)$ and $(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i)$ are generalized eigenvectors associated to $\boldsymbol{\lambda}_i$, $(\boldsymbol{u}^m, \boldsymbol{\delta}_1^m, \boldsymbol{\delta}_2^m, \boldsymbol{\theta}_{\mathbf{c}}^m)$ and $(\boldsymbol{\psi}^m, \boldsymbol{\kappa}_1^m, \boldsymbol{\kappa}_2^m, \boldsymbol{\rho}_{\mathbf{c}}^m)$ are eigenvectors or generalized eigenvectors associated to $\boldsymbol{\lambda}_k$, then we may assume that $(\bar{\boldsymbol{u}}^i, \bar{\boldsymbol{\delta}}_1^i, \bar{\boldsymbol{\delta}}_2^i, \bar{\boldsymbol{\theta}}_{\mathbf{c}}^i) = (\boldsymbol{u}^m, \boldsymbol{\delta}_1^m, \boldsymbol{\delta}_2^m, \boldsymbol{\theta}_{\mathbf{c}}^m)$ and $(\bar{\boldsymbol{\psi}}^i, \bar{\boldsymbol{\kappa}}_1^i, \bar{\boldsymbol{\kappa}}_2^i, \bar{\boldsymbol{\rho}}_{\mathbf{c}}^i) = (\boldsymbol{\psi}^m, \boldsymbol{\kappa}_1^m, \boldsymbol{\kappa}_2^m, \boldsymbol{\rho}_{\mathbf{c}}^m)$. We set

$$\begin{aligned}
(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}^i) &= \sqrt{2}\mathrm{Re}(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i) \quad \text{and} \quad (\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}^i) = \sqrt{2}\mathrm{Re}(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i),\\
(\boldsymbol{v}^m, \boldsymbol{\eta}_1^m, \boldsymbol{\eta}_2^m, \boldsymbol{\theta}^m) &= \sqrt{2}\mathrm{Im}(\boldsymbol{u}^i, \boldsymbol{\delta}_1^i, \boldsymbol{\delta}_2^i, \boldsymbol{\theta}_{\mathbf{c}}^i) \quad \text{and} \quad (\boldsymbol{\phi}^m, \boldsymbol{\zeta}_1^m, \boldsymbol{\zeta}_2^m, \boldsymbol{\rho}^m) = \sqrt{2}\mathrm{Im}(\boldsymbol{\psi}^i, \boldsymbol{\kappa}_1^i, \boldsymbol{\kappa}_2^i, \boldsymbol{\rho}_{\mathbf{c}}^i).
\end{aligned}$$

The two families $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i, \boldsymbol{\theta}^i)_{1\le i\le n_z}$ and $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i, \boldsymbol{\rho}^i)_{1\le i\le n_z}$ of $\mathbb{R}^n$ satisfying the bi-orthogonality conditions $\boldsymbol{E}^T M_s^T\boldsymbol{\Xi} = I_{\mathbb{R}^{n_z}}$ and $\boldsymbol{E}_1^T M_{zz}\boldsymbol{\Xi}_1 = I_{\mathbb{R}^{n_z}}$, and such that $\Lambda = \boldsymbol{E}^{-1}\mathbf{A}\mathbb{P}^T\boldsymbol{E}$ and $\Lambda^T = \boldsymbol{\Xi}^{-1}\mathbf{A}^{\#}\mathbb{P}^T\boldsymbol{\Xi}$. ∎

COROLLARY 5.3. *We have the following decompositions of* $\mathbf{A}\mathbb{P}^T$ *and* $\mathbf{A}^{\#}\mathbb{P}^T$

$$\Lambda = \boldsymbol{\Xi}^T M_s\mathbf{A}\mathbb{P}^T\boldsymbol{E} \quad \textit{and} \quad \Lambda^T = \boldsymbol{E}^T M_s^T\mathbf{A}^{\#}\mathbb{P}^T\boldsymbol{\Xi}.$$

Proof. It follows from the bi-orthogonality of $\boldsymbol{E}$ and $\boldsymbol{\Xi}$. ∎

5.2. Projected systems. Let $(\boldsymbol{\lambda}_j)_{j\in\mathbb{N}}$ be the eigenvalues of the operator $\mathbf{A}\mathbb{P}^T$. We denote by $\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}\mathbb{P}^T$ and $\mathbf{G}^*_{\mathbb{R}}(\boldsymbol{\lambda}_j)$ the real generalized eigenspace of $\mathbf{A}^{\#}\mathbb{P}^T$ associated to the eigenvalue $\boldsymbol{\lambda}_j$. We choose $\alpha_s > 0$ and $\mathbf{J}_u$ a finite subset of $\mathbb{N}$ such that $\{j \in \mathbb{N} \mid \mathrm{Re}(\boldsymbol{\lambda}_j) \ge -\alpha_s\} \subset \mathbf{J}_u$. We define the unstable subspaces

$$\mathbb{Z}_u = \oplus_{j\in\mathbf{J}_u}\mathbf{G}_{\mathbb{R}}(\boldsymbol{\lambda}_j) \quad \text{and} \quad \mathbb{Z}_u^* = \oplus_{j\in\mathbf{J}_u}\mathbf{G}^*_{\mathbb{R}}(\boldsymbol{\lambda}_j).$$

Without loss of generality, we assume that

$$\mathbb{Z}_u = \mathrm{Vect}\left\{(\mathbb{P}_v^T\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) \mid 1 \le i \le d_u\right\} \text{ and } \mathbb{Z}_s = \mathrm{Vect}\left\{(\mathbb{P}_v^T\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) \mid d_u + 1 \le i \le n_0\right\},$$

where d_u is the dimension of $\mathbb{Z}_u$. Introducing

$$\mathbb{Z}_0 = \mathrm{Ker}(\mathbf{A}\mathbb{P}^T) = \mathrm{Vect}\left\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) \mid n_0 + 1 \le i \le n_z\right\},$$

we have

$$\mathbb{R}^{n_z} = \mathbb{Z}_u \oplus \mathbb{Z}_s \oplus \mathbb{Z}_0.$$

We introduce the subspaces

$$\widetilde{\mathbb{Z}}_u = \mathrm{Vect}\left\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) \mid 1 \le i \le d_u\right\} \quad \text{and} \quad \widetilde{\mathbb{Z}}_s = \mathrm{Vect}\left\{(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i) \mid d_u + 1 \le i \le n_0\right\}.$$

PROPOSITION 5.4. *We have the decomposition* $\mathbb{R}^{n_z} = \widetilde{\mathbb{Z}}_u \oplus \widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0$.

Proof. It follows from the fact that

$$\mathbb{Z}_u \oplus \mathbb{Z}_s \oplus \mathbb{Z}_0 = \mathbb{R}^{n_z}, \quad \dim(\widetilde{\mathbb{Z}}_u) = \dim(\mathbb{Z}_u) \quad \text{and} \quad \dim(\widetilde{\mathbb{Z}}_s) = \dim(\mathbb{Z}_s).$$

∎

We set $d_s = n_0 - d_u$ and $d_0 = n_z - n_0$. We denote by

- $\boldsymbol{E}_u \in \mathbb{R}^{n_z \times d_u}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{1 \le i \le d_u}$,
- $\boldsymbol{E}_s \in \mathbb{R}^{n_z \times d_s}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{d_u+1 \le i \le n_0}$,
- $\boldsymbol{E}_0 \in \mathbb{R}^{n_z \times d_0}$ the matrix whose columns are $(\boldsymbol{v}^i, \boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i)_{n_0+1 \le i \le n_z}$,
- $\boldsymbol{\Xi}_u \in \mathbb{R}^{n_z \times d_u}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{1 \le i \le d_u}$,
- $\boldsymbol{\Xi}_s \in \mathbb{R}^{n_z \times d_s}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{d_u+1 \le i \le n_0}$,
- $\boldsymbol{\Xi}_0 \in \mathbb{R}^{n_z \times d_0}$ the matrix whose columns are $(\boldsymbol{\phi}^i, \boldsymbol{\zeta}_1^i, \boldsymbol{\zeta}_2^i)_{n_0+1 \le i \le n_z}$,
- $\boldsymbol{\Theta}_u \in \mathbb{R}^{n_\theta \times d_u}$ the matrix whose columns are $(\boldsymbol{\theta}^i)_{1 \le i \le d_u}$,
- $\boldsymbol{\Theta}_s \in \mathbb{R}^{n_\theta \times d_s}$ the matrix whose columns are $(\boldsymbol{\theta}^i)_{d_u+1 \le i \le n_0}$,
- $\boldsymbol{\chi}_u \in \mathbb{R}^{n_\theta \times d_u}$ the matrix whose columns are $(\boldsymbol{\rho}^i)_{1 \le i \le d_u}$,
- $\boldsymbol{\chi}_s \in \mathbb{R}^{n_\theta \times d_s}$ the matrix whose columns are $(\boldsymbol{\rho}^i)_{d_u+1 \le i \le n_0}$.

LEMMA 5.2. *We have the relationships*

$$\begin{aligned}
&\boldsymbol{E}_u^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_u = I_{\mathbb{R}^{d_u}}, && \boldsymbol{E}_s^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_u = 0_{\mathbb{R}^{d_u \times d_s}}, && \boldsymbol{E}_0^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_u = 0_{\mathbb{R}^{d_u \times d_0}},\\
&\boldsymbol{E}_u^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_s = 0_{\mathbb{R}^{d_s \times d_u}}, && \boldsymbol{E}_s^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_s = I_{\mathbb{R}^{d_s}}, && \boldsymbol{E}_0^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_s = 0_{\mathbb{R}^{d_s \times d_0}},\\
&\boldsymbol{E}_u^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_0 = 0_{\mathbb{R}^{d_0 \times d_u}}, && \boldsymbol{E}_s^T \mathbb{P} M_s^T \mathbb{P}^T \boldsymbol{\Xi}_0 = 0_{\mathbb{R}^{d_0 \times d_s}}, && \boldsymbol{E}_0^T M_s^T \boldsymbol{\Xi}_0 = I_{\mathbb{R}^{d_0}}.
\end{aligned}$$

Proof. It follows from the fact the matrices $\boldsymbol{E} = [\mathbb{P}^T \boldsymbol{E}_u \,|\, \mathbb{P}^T \boldsymbol{E}_s \,|\, \boldsymbol{E}_0]$ and $\boldsymbol{\Xi} = [\mathbb{P}^T \boldsymbol{\Xi}_u \,|\, \mathbb{P}^T \boldsymbol{\Xi}_s \,|\, \boldsymbol{\Xi}_0]$ satisfy the bi-orthogonality condition $\boldsymbol{E}^T M_s^T \boldsymbol{\Xi} = I_{\mathbb{R}^{n_z}}$. ∎

LEMMA 5.3. *We have*

$$\Lambda = \begin{bmatrix} \Lambda_u & 0_{\mathbb{R}^{d_u \times d_s}} & 0_{\mathbb{R}^{d_u \times d_0}} \\ 0_{\mathbb{R}^{d_s \times d_u}} & \Lambda_s & 0_{\mathbb{R}^{d_s \times d_0}} \\ 0_{\mathbb{R}^{d_0 \times d_u}} & 0_{\mathbb{R}^{d_0 \times d_s}} & 0_{\mathbb{R}^{d_0}} \end{bmatrix} \quad \textit{and} \quad \Lambda^T = \begin{bmatrix} \Lambda_u^T & 0_{\mathbb{R}^{d_u \times d_s}} & 0_{\mathbb{R}^{d_u \times d_0}} \\ 0_{\mathbb{R}^{d_s \times d_u}} & \Lambda_s^T & 0_{\mathbb{R}^{d_s \times d_0}} \\ 0_{\mathbb{R}^{d_0 \times d_u}} & 0_{\mathbb{R}^{d_0 \times d_s}} & 0_{\mathbb{R}^{d_0}} \end{bmatrix},$$

where

$$\begin{aligned}
&\Lambda_u = \boldsymbol{\Xi}_u^T \mathbb{P} M_s \mathbf{A} \mathbb{P}^T \boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T \mathbb{P} M_s^T \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi}_u,\\
&\Lambda_s = \boldsymbol{\Xi}_s^T \mathbb{P} M_s \mathbf{A} \mathbb{P}^T \boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T \mathbb{P} M_s^T \mathbf{A}^\# \mathbb{P}^T \boldsymbol{\Xi}_s.
\end{aligned} \tag{5.3}$$

Proof. It follows from Corollary 5.3. ∎

LEMMA 5.4. *We have the relationships*

$$\boldsymbol{E}_u^T M_{zz} \boldsymbol{\Xi}_u = I_{\mathbb{R}^{d_u}}, \quad \boldsymbol{E}_s^T M_{zz} \boldsymbol{\Xi}_s = I_{\mathbb{R}^{d_s}}, \quad \boldsymbol{E}_u^T M_{zz} \boldsymbol{\Xi}_s = 0_{\mathbb{R}^{d_u \times d_s}}, \quad \boldsymbol{E}_s^T M_{zz} \boldsymbol{\Xi}_u = 0_{\mathbb{R}^{d_s \times d_u}}.$$

Proof. It follows from the fact the matrices $\boldsymbol{E}_1 = [\boldsymbol{E}_u \,|\, \boldsymbol{E}_s]$ and $\boldsymbol{\Xi}_1 = [\boldsymbol{\Xi}_u \,|\, \boldsymbol{\Xi}_s]$ satisfy the bi-orthogonality condition $\boldsymbol{E}_1^T M_{zz} \boldsymbol{\Xi}_1 = I_{\mathbb{R}^{n_z}}$. ∎

PROPOSITION 5.5. *We have*

$$\Lambda_u = \boldsymbol{\Xi}_u^T A_{zz} \boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T A_{zz}^T \boldsymbol{\Xi}_u, \quad \Lambda_s = \boldsymbol{\Xi}_s^T A_{zz} \boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T A_{zz}^T \boldsymbol{\Xi}_s.$$

Proof. By construction of the matrices $\boldsymbol{E}_u$, $\boldsymbol{\Xi}_u$, $\boldsymbol{\Theta}_u$ and $\boldsymbol{\chi}_u$, we have

$$\begin{bmatrix} A_{zz} & \widetilde{A}_{z\theta} \\ \widehat{A}_{z\theta}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_u \\ \boldsymbol{\Theta}_u \end{bmatrix} = \begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_u \\ \boldsymbol{\Theta}_u \end{bmatrix} \Lambda_u \quad \text{i.e.} \quad A_{zz} \boldsymbol{E}_u + \widetilde{A}_{z\theta} \boldsymbol{\Theta}_u = M_{zz} \boldsymbol{E}_u \Lambda_u, \quad \widehat{A}_{z\theta}^T \boldsymbol{E}_u = 0,$$

$$\begin{bmatrix} A_{zz}^T & \widehat{A}_{z\theta} \\ \widetilde{A}_{z\theta}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi}_u \\ \boldsymbol{\chi}_u \end{bmatrix} = \begin{bmatrix} M_{zz} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Xi}_u \\ \boldsymbol{\chi}_u \end{bmatrix} \Lambda_u^T \quad \text{i.e.} \quad A_{zz}^T \boldsymbol{\Xi}_u + \widehat{A}_{z\theta} \boldsymbol{\chi}_u = M_{zz} \boldsymbol{\Xi}_u \Lambda_u^T, \quad \widetilde{A}_{z\theta}^T \boldsymbol{\Xi}_u = 0.$$

We deduce that

$$\boldsymbol{\Xi}_u^T M_{zz} \boldsymbol{E}_u \Lambda_u = \boldsymbol{\Xi}_u^T A_{zz} \boldsymbol{E}_u + \boldsymbol{\Xi}_u^T \widetilde{A}_{z\theta} \boldsymbol{\Theta}_u = \boldsymbol{\Xi}_u^T A_{zz} \boldsymbol{E}_u + (\widetilde{A}_{z\theta}^T \boldsymbol{\Xi}_u)^T \boldsymbol{\Theta}_u = \boldsymbol{\Xi}_u^T A_{zz} \boldsymbol{E}_u,$$
$$\boldsymbol{E}_u^T M_{zz} \boldsymbol{\Xi}_u \Lambda_u^T = \boldsymbol{E}_u^T A_{zz}^T \boldsymbol{\Xi}_u + \boldsymbol{E}_u^T \widehat{A}_{z\theta} \boldsymbol{\chi}_u = \boldsymbol{E}_u^T A_{zz}^T \boldsymbol{\Xi}_u + (\widehat{A}_{z\theta}^T \boldsymbol{E}_u)^T \boldsymbol{\chi}_u = \boldsymbol{E}_u^T A_{zz}^T \boldsymbol{\Xi}_u.$$

From Lemma 5.4, it follows the relations for (Λ_u, Λ_u^T) and similarly for (Λ_s, Λ_s^T)

$$\Lambda_u = \boldsymbol{\Xi}_u^T A_{zz} \boldsymbol{E}_u, \quad \Lambda_u^T = \boldsymbol{E}_u^T A_{zz}^T \boldsymbol{\Xi}_u, \quad \text{and} \quad \Lambda_s = \boldsymbol{\Xi}_s^T A_{zz} \boldsymbol{E}_s, \quad \Lambda_s^T = \boldsymbol{E}_s^T A_{zz}^T \boldsymbol{\Xi}_s.$$

∎

We introduce the operators $\Pi_u \in \mathcal{L}(\mathbb{R}^{n_z}, \widetilde{\mathbb{Z}}_u)$ and $\Pi_s \in \mathcal{L}(\mathbb{R}^{n_z}, \widetilde{\mathbb{Z}}_s)$ defined by

$$\Pi_u = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T M_{zz} \quad \text{and} \quad \Pi_s = \boldsymbol{E}_s \boldsymbol{\Xi}_s^T M_{zz}.$$

PROPOSITION 5.6. *The operator Π_u is the projection of $\mathbb{R}^{n_z}$ onto $\widetilde{\mathbb{Z}}_u$ parallel to $\widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0$ and the operator Π_s is the projection of $\mathbb{R}^{n_z}$ onto $\widetilde{\mathbb{Z}}_s$ parallel to $\widetilde{\mathbb{Z}}_u \oplus \mathbb{Z}_0$.*

Proof. We have

$$\Pi_u^2 = \boldsymbol{E}_u (\boldsymbol{\Xi}_u^T M_{zz} \boldsymbol{E}_u) \boldsymbol{\Xi}_u^T M_{zz} = \boldsymbol{E}_u \boldsymbol{\Xi}_u^T M_{zz} = \Pi_u,$$

because $\boldsymbol{\Xi}_u^T M_{zz} \boldsymbol{E}_u = I_{\mathbb{R}^{d_u}}$, see Lemma 5.4. Therefore, Π_u is the projection onto $\mathrm{Im}(\Pi_u)$ parallel to $\mathrm{Ker}(\Pi_u)$. Thus, to prove that Π_u is the projection onto $\widetilde{\mathbb{Z}}_u$ parallel to $\widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0$, we have to show that $\mathrm{Im}(\Pi_u) = \widetilde{\mathbb{Z}}_u$ and $\mathrm{Ker}(\Pi_u) = \widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0$. It is immediate that $\mathrm{Im}(\Pi_u) = \widetilde{\mathbb{Z}}_u$ and $\widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0 \subset \mathrm{Ker}(\Pi_u)$, and since $\dim(\widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0) = \dim(\mathrm{Ker}(\Pi_u))$, we have $\mathrm{Ker}(\Pi_u) = \widetilde{\mathbb{Z}}_s \oplus \mathbb{Z}_0$. We prove similarly that Π_s is the projection of $\mathbb{R}^{n_z}$ onto $\widetilde{\mathbb{Z}}_s$ parallel to $\widetilde{\mathbb{Z}}_u \oplus \mathbb{Z}_0$. ∎

PROPOSITION 5.7. *The operator $\mathbb{P}^T \boldsymbol{E}_u \boldsymbol{\Xi}_u^T \mathbb{P} M_s$ is the projection of $\mathbb{R}^{n_z}$ onto $\mathbb{Z}_u$ parallel to $\mathbb{Z}_s \oplus \mathbb{Z}_0$ and the operator $\mathbb{P}^T \boldsymbol{E}_s \boldsymbol{\Xi}_s^T \mathbb{P} M_s$ is the projection of $\mathbb{R}^{n_z}$ onto $\mathbb{Z}_s$ parallel to $\mathbb{Z}_u \oplus \mathbb{Z}_0$. Moreover, we have the relationships*

$$\mathbb{P}^T \boldsymbol{E}_u \boldsymbol{\Xi}_u^T \mathbb{P} M_s = \mathbb{P}^T \Pi_u M_{zz}^{-1} M_s \mathbb{P}^T \quad \text{and} \quad \mathbb{P}^T \boldsymbol{E}_s \boldsymbol{\Xi}_s^T \mathbb{P} M_s = \mathbb{P}^T \Pi_s M_{zz}^{-1} M_s \mathbb{P}^T.$$

Proof. It is similar to that of Proposition 5.6. ∎

PROPOSITION 5.8. *If $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ is solution of system (2.4), then the triplets $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$ and $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})$ defined by*

$$(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = \boldsymbol{\Xi}_u^T M_{zz} (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T \quad \text{and} \quad (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M_{zz} (\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T,$$

obey the system

$$\begin{aligned}
\frac{d}{dt} \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} &= \Lambda_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} - \boldsymbol{\chi}_u^T \widehat{A}_{z\theta}^T A_{\theta g} \boldsymbol{g}_p + \mathbb{B}_u \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} (0) = \boldsymbol{\Xi}_u^T M_{zz} \begin{bmatrix} \boldsymbol{v}^0 \\ 0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix}, \\
\frac{d}{dt} \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} &= \Lambda_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} - \boldsymbol{\chi}_s^T \widehat{A}_{z\theta}^T A_{\theta g} \boldsymbol{g}_p + \mathbb{B}_s \boldsymbol{f}, \quad \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix} (0) = \boldsymbol{\Xi}_s^T M_{zz} \begin{bmatrix} \boldsymbol{v}^0 \\ 0 \\ \boldsymbol{\eta}_2^0 \end{bmatrix},
\end{aligned} \tag{5.4}$$

where

$$\mathbb{B}_u = \boldsymbol{\Xi}_u^T B \quad \text{and} \quad \mathbb{B}_s = \boldsymbol{\Xi}_s^T B. \tag{5.5}$$

Conversely, if the pair $((\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u}), (\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s}))$ is solution to system (5.4), then $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$, with

$$\begin{aligned}
&\mathbb{P}^T \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} = \mathbb{P}^T \boldsymbol{E}_u \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{\eta}_{1,u} \\ \boldsymbol{\eta}_{2,u} \end{bmatrix} + \mathbb{P}^T \boldsymbol{E}_s \begin{bmatrix} \boldsymbol{v}_s \\ \boldsymbol{\eta}_{1,s} \\ \boldsymbol{\eta}_{2,s} \end{bmatrix}, \\
&(I - \mathbb{P}_v^T) \boldsymbol{v} = -M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_1 - M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_2 + M_{vv}^{-1} A_{v\theta} M_{\theta\theta} A_{\theta g} \boldsymbol{g}_p, \\
&\boldsymbol{\theta} = -M_{\theta\theta} A_{\eta_1\theta}^T \boldsymbol{\eta}_{1,t} - M_{\theta\theta} A_{\eta_2\theta}^T \boldsymbol{\eta}_{2,t} - M_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_1} - N_{\theta\theta} A_{\eta_1\theta}^T) \boldsymbol{\eta}_1 \\
&\qquad - M_{\theta\theta} (A_{v\theta}^T M_{vv}^{-1} A_{v\eta_2} - N_{\theta\theta} A_{\eta_2\theta}^T) \boldsymbol{\eta}_2 - M_{\theta\theta} A_{v\theta}^T M_{vv}^{-1} A_{vv} \mathbb{P}_v^T \boldsymbol{v} + M_{\theta\theta} A_{\theta g} \boldsymbol{g}_p' - M_{\theta\theta} N_{\theta\theta} A_{\theta g} \boldsymbol{g}_p,
\end{aligned}$$

is solution of system (2.4).

Proof. The proof can be done as in [1, Proof of Proposition 3.8]. ∎

5.3. Linear feedback law and degree of stabilizability. The goal is to find a feedback law able to stabilize system (2.3) or equivalently (2.4) with a prescribed exponential decay rate $-\omega$. Then after fixing $\mathbb{Z}_u$ and due to Proposition 5.8, it is necessary and sufficient to find a feedback law able to stabilize the equation satisfied by $(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})$. That is why we make the following assumption.

$$\text{For all } \omega > -\text{Re}\,\text{spect}(\Lambda_u) \text{ the pair } (\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u) \text{ is stabilizable.} \tag{5.6}$$

Let us recall that $\mathbb{B}_u$ defined in (5.5) depends to the choice of the family $(\boldsymbol{w}_i)_{1\le i\le n_c}$. Therefore, the choice of this family is crucial to check the condition (5.6) which is satisfied if and only if the stabilizability Gramian of the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u)$ is positive definite. In that case, the stabilizability Gramian is nothing but $\mathbb{P}_{u,\omega}^{-1}$, where $\mathbb{P}_{u,\omega}$ is the solution of the Algebraic Riccati Equation

$$\begin{aligned} &\mathbb{P}_{u,\omega} \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \mathbb{P}_{u,\omega} = \mathbb{P}_{u,\omega}^T > 0, \\ &\mathbb{P}_{u,\omega}(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega I_{\mathbb{R}^{d_u}})\mathbb{P}_{u,\omega} - \mathbb{P}_{u,\omega}\mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega} = 0. \end{aligned} \tag{5.7}$$

We could define the degree of stabilizability as the lowest eigenvalue of $\mathbb{P}_{u,\omega}^{-1}$. Moreover, we recall that $\mathbb{B}_u$ depends on the basis chosen for $\mathbb{Z}_u^*$. Thus $\mathbb{P}_{u,\omega}$ and $\mathbb{P}_{u,\omega}^{-1}$ also depend on this basis. In order to obtain a degree of stabilizability $d_{u,\omega}$ independent of the basis chosen for $\mathbb{Z}_u^*$, as in [1], we define it by

$$d_{u,\omega} = \max(\text{spect}(\mathbb{M}_u\mathbb{P}_{u,\omega}\mathbb{M}_u))^{-1} \quad \text{with} \quad \mathbb{M}_u = (\boldsymbol{\Xi}_u^T M_{zz}\boldsymbol{\Xi}_u)^{\frac{1}{2}}.$$

Different choices of $\mathbb{Z}_u$ may give different degrees of stabilizability. Among different possible choices, we choose the subspace $\mathbb{Z}_u$ with the greatest degree of stabilizability. Next, we choose the feedback law able to stabilize the pair $(\Lambda_u + \omega I_{\mathbb{R}^{d_u}}, \mathbb{B}_u)$ equal to

$$\mathbb{K}_{u,\omega} = -\mathbb{B}_u^T\mathbb{P}_{u,\omega}. \tag{5.8}$$

5.4. Stability of the closed-loop linear system. Let us set $K_\omega = -B^T\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz}$ where $\mathbb{P}_{u,\omega}$ is the solution of (5.7). We are going to prove that the linear closed-loop system

$$M_{zz}\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = (A_{zz} + BK_\omega)\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \widetilde{A}_{z\theta}\boldsymbol{\theta}, \quad \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{v}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix}, \quad \widehat{A}_{z\theta}^T\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = A_{\theta g}\boldsymbol{g}_p, \tag{5.9}$$

is exponentially stable.

PROPOSITION 5.9. *If $\mathbb{P}_{u,\omega}$ is solution of (5.7) then $\mathbf{P}_{u,\omega} = M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz}$ is the solution of*

$$\begin{aligned} &\mathbf{P}_{u,\omega} \in \mathcal{L}(\widetilde{\mathbb{Z}}_u), \quad \mathbf{P}_{u,\omega} = \mathbf{P}_{u,\omega}^T > 0, \\ &\mathbf{P}_{u,\omega}(\mathbf{A}_u + \omega\Pi_u) + (\mathbf{A}_u^T + \omega\Pi_u^T)\mathbf{P}_{u,\omega} - \mathbf{P}_{u,\omega}\mathbf{B}_u\mathbf{B}_u^T\mathbf{P}_{u,\omega} = 0, \end{aligned} \tag{5.10}$$

with $\mathbf{A}_u = \Pi_u M_{zz}^{-1}A_{zz}\Pi_u$ and $\mathbf{B}_u = \Pi_u M_{zz}^{-1}B$.

Proof. Let $\mathbb{P}_{u,\omega}$ be a solution of (5.7). Using the identities

$$\boldsymbol{\Xi}_u^T A_{zz}\boldsymbol{E}_u = \Lambda_u, \quad \boldsymbol{E}_u^T A_{zz}^T\boldsymbol{\Xi}_u = \Lambda_u^T, \quad \boldsymbol{\Xi}_u^T M_{zz} = \boldsymbol{\Xi}_u^T M_{zz}\Pi_u, \quad M_{zz}\boldsymbol{\Xi}_u = \Pi_u^T M_{zz}\boldsymbol{\Xi}_u,$$

we get

$$\begin{aligned} &M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\Lambda_u\boldsymbol{\Xi}_u^T M_{zz} = M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T A_{zz}\boldsymbol{E}_u\boldsymbol{\Xi}_u^T M_{zz} = \mathbf{P}_{u,\omega}\mathbf{A}_u, \\ &\text{and} \quad M_{zz}\boldsymbol{\Xi}_u\Lambda_u^T\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz} = M_{zz}\boldsymbol{\Xi}_u\boldsymbol{E}_u^T A_{zz}^T\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz} = \mathbf{A}_u^T\mathbf{P}_{u,\omega}. \end{aligned} \tag{5.11}$$

We also have

$$M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}I_{d_u}\boldsymbol{\Xi}_u^T M_{zz} = \mathbf{P}_{u,\omega}\Pi_u \quad \text{and} \quad M_{zz}\boldsymbol{\Xi}_u I_{d_u}\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz} = \Pi_u^T\mathbf{P}_{u,\omega}. \tag{5.12}$$

Finally, we have $M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\mathbb{B}_u\mathbb{B}_u^T\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz} = M_{zz}\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T BB^T\boldsymbol{\Xi}_u\mathbb{P}_{u,\omega}\boldsymbol{\Xi}_u^T M_{zz} = \mathbf{P}_{u,\omega}\mathbf{B}_u\mathbf{B}_u^T\mathbf{P}_{u,\omega}$, combined with the equations (5.11), (5.12), we prove that $\mathbf{P}_{u,\omega}$ is solution of (5.10). ∎

REMARK 5.1. *Since $\mathbf{P}_{u,\omega}$ does not depend on the choice of the basis of $\mathbb{Z}_u^*$, the feedback law K_ω does not depend on the basis of $\mathbb{Z}_u^*$. Indeed, we have*

$$K_\omega = -B^T M_{zz}^{-1} M_{zz} \boldsymbol{\Xi}_u \mathbb{P}_{u,\omega} \boldsymbol{\Xi}_u^T M_{zz} = -B^T M_{zz}^{-1} \mathbf{P}_{u,\omega}.$$

THEOREM 5.10. *Let $\omega > -\mathrm{Re}\,\mathrm{spect}(\Lambda_u)$, $(\boldsymbol{v}^0, \boldsymbol{\eta}_2^0, \boldsymbol{g}_p) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_\tau}$ be such that $(\boldsymbol{v}^0, 0, \boldsymbol{\eta}_2^0)$ belongs to $\mathrm{Ker}(\widehat{A}_{z\theta}^T)$ and $e^{\omega t}\boldsymbol{g}_p \in H_0^1(0,\infty;\mathbb{R}^{n_\tau})$. For all $t > 0$, the solution $(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\theta})$ of system (5.9) satisfies the following estimates for some $C > 0$*

$$\begin{aligned}
&\|\Pi_u(\boldsymbol{v}(t), \boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t))\|_{\mathbb{R}^{n_z}} \le Ce^{-\omega t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{L^2(0,\infty;\mathbb{R}^{n_\tau})}),\\
&\|\Pi_s(\boldsymbol{v}(t), \boldsymbol{\eta}_1(t), \boldsymbol{\eta}_2(t))\|_{\mathbb{R}^{n_z}} \le Ce^{-\min(\alpha_s,\omega)t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{L^2(0,\infty;\mathbb{R}^{n_\tau})}),\\
&\|(I - \mathbb{P}_v^T)\boldsymbol{v}(t)\|_{\mathbb{R}^{n_v}} \le Ce^{-\min(\alpha_s,\omega)t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{H^1(0,\infty;\mathbb{R}^{n_\tau})}).
\end{aligned}$$

Proof. Setp 1. From Proposition 5.8, $e^{\omega t}(\boldsymbol{v}_u, \boldsymbol{\eta}_{1,u}, \boldsymbol{\eta}_{2,u})^T = e^{\omega t}\boldsymbol{\Xi}_u^T M_{zz}(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ is the solution of

$$\frac{d}{dt}e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} = (\Lambda_u + \omega I_{\mathbb{R}^{d_u}} + \mathbb{B}_u\mathbb{K}_{u,\omega})e^{\omega t}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} - e^{\omega t}\boldsymbol{\chi}_u^T\widehat{A}_{z\theta}^T A_{\theta g}\boldsymbol{g}_p,\quad \begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix}(0) = \boldsymbol{\Xi}_u^T M\begin{bmatrix}\boldsymbol{v}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.$$

Since $\Lambda_u + \omega I_{\mathbb{R}^{d_u}} + \mathbb{B}_u\mathbb{K}_{u,\omega}$ is exponentially stable, there exists $C > 0$ such that

$$\|(\boldsymbol{v}_u(t), \boldsymbol{\eta}_{1,u}(t), \boldsymbol{\eta}_{2,u}(t))\|_{\mathbb{R}^{d_u}} \le Ce^{-\omega t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{L^2(0,\infty;\mathbb{R}^{n_\tau})}).$$

Setp 2. The triplet $(\boldsymbol{v}_s, \boldsymbol{\eta}_{1,s}, \boldsymbol{\eta}_{2,s})^T = \boldsymbol{\Xi}_s^T M_{zz}(\boldsymbol{v}, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2)^T$ obeys the system

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} = \Lambda_s\begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix} + \mathbb{B}_s\mathbb{K}_{u,\omega}\begin{bmatrix}\boldsymbol{v}_u\\ \boldsymbol{\eta}_{1,u}\\ \boldsymbol{\eta}_{2,u}\end{bmatrix} - \boldsymbol{\chi}_s^T\widehat{A}_{z\theta}^T A_{\theta g}\boldsymbol{g}_p,\quad \begin{bmatrix}\boldsymbol{v}_s\\ \boldsymbol{\eta}_{1,s}\\ \boldsymbol{\eta}_{2,s}\end{bmatrix}(0) = \boldsymbol{\Xi}_s^T M_{zz}\begin{bmatrix}\boldsymbol{v}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix}.$$

Since $\mathrm{Re}(\mathrm{spect}(\Lambda_s)) < -\alpha_s$, there exists $C > 0$ such that

$$\|(\boldsymbol{v}_s(t), \boldsymbol{\eta}_{1,s}(t), \boldsymbol{\eta}_{2,s}(t))\|_{\mathbb{R}^{d_s}} \le Ce^{-\min(\alpha_s,\omega)t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{L^2(0,\infty;\mathbb{R}^{n_\tau})}).$$

Setp 3. Finally, we have

$$\|(I - \mathbb{P}_v^T)\boldsymbol{v}(t)\|_{\mathbb{R}^{n_v}} \le Ce^{-\min(\alpha_s,\omega)t}(\|\boldsymbol{v}^0\|_{\mathbb{R}^{n_v}} + \|\boldsymbol{\eta}_2^0\|_{\mathbb{R}^{n_s}} + \|e^{\omega t}\boldsymbol{g}_p\|_{H^1(0,\infty;\mathbb{R}^{n_\tau})}).$$

∎

6. Numerical simulations. Numerical simulations for fluid-structure interaction problems remain a challenge, see the review [14, 6]. A broad spectrum of methods exists and can be adapted to the present work. There are mainly two approaches to numerically solve fluid-structure interaction systems. The first one, called the *monolithic* approach, consists of solving the coupled system as a unique problem, see [8, 24, 13]. The second one, called the *partitioned* approach, uses two different codes to solve separately the equations of the fluid and of the structure, see [9, 10]. The *monolithic* approach is generally stable contrary to the *partitioned* approach.

In most of numerical codes, the equations of the fluid are rewritten in an Arbitrary Lagrangian Eulerian (ALE) formulation. However, our analysis and the definition of the feedback law expressed in the reference configuration suggest that the natural framework for numerical simulation corresponds to the Lagrangian formulation for the fluid. This is why we are going to use the discrete approximation of system (1.5) in the numerical simulations.

The goal of this section is to explain how to solve numerically the full nonlinear system (1.5) with and without control.

6.1. Full discretization of the nonlinear system. The spatial semi-discretization of the nonlinear system (1.5) is similar to that of the linearized system (1.6), see Section 2. The time discretization is treated by a classical Backward Difference Formula of order 2 (BDF2) using three time levels (t^{n+2}, t^{n+1}, t^n) with a time step Δt. In the nonlinear system, two types of nonlinearities appear. The

first ones, corresponding to geometrical nonlinearities, are treated explicitly contrary to the convective term in the fluid part which is treated implicitly with a Newton method. This approach significantly reduces the computational complexity, and it is usually used without compromising stability and accuracy [15]. For time advancing scheme, we retain the monolithic approach. The contribution of the feedback control is treated explicitly (this choice is more consistent with feedback strategy in particular when an estimator is introduced).

With this approach, the full discretization of the nonlinear system (1.5) is given by

$$\begin{aligned}
&\frac{3}{2\Delta t}M_{zz}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}^{n+2} - \frac{2}{\Delta t}M_{zz}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}^{n+1} + \frac{1}{\Delta t}M_{zz}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}^{n} = \widehat{A}_{zz}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}^{n+2} + \widetilde{A}_{z\theta}\boldsymbol{\theta}^{n+2}\\
&\qquad + \begin{bmatrix}N_f(\tilde{\boldsymbol{u}})\\ 0\\ 0\end{bmatrix}^{n+2} + \begin{bmatrix}N_v(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\theta})\\ 0\\ N_{\eta_2}(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1)\end{bmatrix}^{n+1} + \begin{bmatrix}0\\ 0\\ M_{\eta\eta}\mathbf{W}\boldsymbol{f}\end{bmatrix}^{n+1} + \begin{bmatrix}0\\ 0\\ M_{\eta\eta}\boldsymbol{f}_s\end{bmatrix}, \qquad (6.1)\\
&\widetilde{A}_{z\theta}^T\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}^{n+2} = N_\theta(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1)^{n+1} + A_{\theta g}\boldsymbol{g}_p^{n+1} + A_{\theta g}\boldsymbol{g}_s, \quad \begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}(0) = \begin{bmatrix}\boldsymbol{u}^0\\ 0\\ \boldsymbol{\eta}_2^0\end{bmatrix},
\end{aligned}$$

where $$\widehat{A}_{zz} = \begin{bmatrix}\widetilde{A}_{vv} & 0 & 0\\ 0 & 0 & A_{\eta_1\eta_2}\\ 0 & \widetilde{A}_{\eta_2\eta_1} & A_{\eta_2\eta_2}\end{bmatrix},$$

with $\widetilde{A}_{vv} = \left(-2\nu\int_\Omega \varepsilon(\phi_i):\varepsilon(\phi_j)\right)_{1\le i,j\le n_v}$, $\widetilde{A}_{\eta_2\eta_1} = \left(-\int_{\Gamma_s}(\beta\nabla\zeta_j\cdot\nabla\zeta_i + \alpha\Delta\zeta_j\cdot\Delta\zeta_i)\right)_{1\le i,j\le n_s}$,

and $\tilde{\boldsymbol{u}}$, $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$, $\boldsymbol{f}_s$, $\boldsymbol{g}_s$, $\boldsymbol{g}_p$ and $\boldsymbol{\theta}$ are the coordinate vectors of the approximations of $\tilde{u}$, η_1, η_2, f_s, g_s, g_p and of the multipliers. We refer to Section 2 for details on the other elementary matrices to assemble. The nonlinear terms N_f, N_v, N_{η_2} and N_θ (depending on F_f, F_s, F_{div} given in appendix) are defined by

$$\begin{aligned}
N_f(\tilde{\boldsymbol{u}}) &= -\left(\int_\Omega \sum_{i=1}^{n_v}\sum_{j=1}^{n_v}\tilde{u}_i\tilde{u}_j(\phi_i\cdot\nabla)\phi_j\cdot\phi_k\right)_{1\le k\le n_v},\\
N_v(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\theta}) &= \left(\int_\Omega F_f\left[\sum_{i=1}^{n_v}\tilde{u}_i\phi_i, \sum_{j=1}^{n_p}q_jp_j, \sum_{l=1}^{n_s}\eta_{1,l}\zeta_l, \sum_{m=1}^{n_s}\eta_{2,m}\zeta_m\right]\cdot\phi_k\right)_{1\le k\le n_v},\\
N_{\eta_2}(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1) &= \left(\int_{\Gamma_s} F_s\left[\sum_{i=1}^{n_v}\tilde{u}_i\phi_i, \sum_{j=1}^{n_s}\eta_{1,j}\zeta_j\right]\cdot\zeta_k\right)_{1\le k\le n_s},\\
N_\theta(\tilde{\boldsymbol{u}},\boldsymbol{\eta}_1) &= \left[\left(\int_\Omega \operatorname{div} F_{\mathrm{div}}\left[\sum_{i=1}^{n_v}\tilde{u}_i\phi_i, \sum_{j=1}^{n_s}\eta_{1,j}\zeta_j\right]\cdot q_k\right)_{1\le k\le n_p}, 0, \cdots, 0\right].
\end{aligned}$$

Now, let us verify that system (6.1) is well-posed. For that, we have to prove that the system

$$\frac{3}{2\Delta t}M_{zz}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = \widehat{A}_{zz}\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} + \begin{bmatrix}L_{vv}\tilde{\boldsymbol{u}}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}F_f\\ F_s^1\\ F_s^2\end{bmatrix} + \begin{bmatrix}A_{v\theta}\boldsymbol{\theta}\\ 0\\ A_{\eta_2\theta}\boldsymbol{\theta}\end{bmatrix}, \quad \widetilde{A}_{z\theta}^T\begin{bmatrix}\tilde{\boldsymbol{u}}\\ \boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix} = F_\theta, \qquad (6.2)$$

admits a unique solution for all $(F_f, F_s^1, F_s^2, F_\theta)\in\mathbb{R}^n$ and for all matrix $L_{vv}\in\mathbb{R}^{n_v\times n_v}$ corresponding to a linearization of the nonlinear term N_f. In order to study the above system, we decompose the multiplier associated to the Dirichlet boundary conditions $\boldsymbol{\tau}$ into the form $\boldsymbol{\tau} = (\boldsymbol{\tau}_{tang}, \boldsymbol{\tau}_n)$ where $\boldsymbol{\tau}_{tang}$ and $\boldsymbol{\tau}_n$ are the tangential part and the normal part of $\boldsymbol{\tau}$ respectively. Thus, the matrix $A_{\eta_2\tau}$ can be rewritten into the form, see Section 2

$$A_{\eta_2\tau} = [0 \quad A_{\eta_2\tau_n}].$$

THEOREM 6.1. *We assume that $A_{\eta_2\tau_n}$ is of maximal rank. Then, for all $(F_f, F_s^1, F_s^2, F_\theta) \in \mathbb{R}^n$ and for all $L_{vv} \in \mathbb{R}^{n_v \times n_v}$, there exists Δt small enough such that the system (6.2) admits a unique solution.*

Proof. From the equations of the structure, it is possible to express $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in terms of the multiplier $\boldsymbol{\theta}$ if Δt is small enough. Finally, it is possible to rewrite the system on $(\tilde{\boldsymbol{u}}, \boldsymbol{\theta})$ in such a way that the system admits a unique solution under the Inf-Sup condition (6.4). □

REMARK 6.1. *The rank condition on $A_{\eta_2\tau_n}$ assumed in Theorem 6.1 is satisfied when we approximate, with the same grid, the structure displacements by Hermite finite elements and the multiplier associated to the Dirichlet boundary conditions by $\mathbb{P}_1$ finite elements.*

6.2. Numerical tests for the nonlinear system without control for $Re = 200$.

6.2.1. Data of the numerical experiments. We recall, see Section 1, that the reference configuration of the fluid domain is

$$\Omega = ([0, L] \times [-\ell, \ell]) \setminus ([0, \ell_s] \times [-e, e]),$$

We choose $e = 0.05$, $L = 50e$, $\ell = 21e$ and $\ell_s = 10e$.

The stationary inflow condition $g_s = (g_{s,1}, g_{s,2})$ imposed on the boundary $\Gamma_i = \{0\} \times [-21e, -e] \cup \{10e\} \times [-e, e] \cup \{0\} \times [e, 21e]$ is an approximation of the Blasius boundary layer profile defined by $g_{s,2} = 0$ and

$$g_{s,1}(z) = \begin{cases} U_m & \forall\, e + b \le z \le 21e, \\ U_m \left(2\dfrac{z-e}{b} - 2\left(\dfrac{z-e}{b}\right)^3 + \left(\dfrac{z-e}{b}\right)^4 \right) & \forall\, e \le z \le e + b, \\ 0 & \forall\, -e \le z \le e, \\ U_m \left(2\dfrac{z+e}{b} - 2\left(\dfrac{z+e}{b}\right)^3 + \left(\dfrac{z+e}{b}\right)^4 \right) & \forall\, -e - b \le z \le -e, \\ U_m & \forall\, -21e \le z \le -e - b, \end{cases}$$

where U_m is the maximum velocity at the inflow, $\mathrm{Re} = \frac{2eU_m}{\nu}$ is the Reynolds number of the fluid and $b = \frac{14e}{\sqrt{\mathrm{Re}}}$ is the thickness of the boundary layer. For the numerical tests realized with the free GetFEM++ library [23], we choose the following parameters $\mathrm{Re} = 200, U_m = 1.0$ for the fluid and for the structure

$$\alpha = 10^{-3}, \quad \beta = 0, \quad \gamma = 10^{-1}.$$

Mesh and finite element approximation. We use a triangular mesh of 89418 cells, symmetric with respect to the horizontal axis $z = 0$, locally refined near the inflow boundary and near the structure. The mesh on Γ_s used for S_h and D_h is that induced by the mesh on $\overline{\Omega}$. For the space discretization X_h, P_h and D_h, we use generalized Taylor-Hood finite elements $\mathbb{P}_2$-$\mathbb{P}_1$-$\mathbb{P}_1$ for the velocity, the pressure and the Lagrange multipliers satisfying the Inf-Sup condition

$$\inf_{p \in P_h, p \neq 0} \sup_{v \in X_h, v \neq 0} \frac{b(v, p)}{\|v\|_{X_h} \|p\|_{P_h}} \ge a_0, \tag{6.3}$$

for some $a_0 > 0$ independent of h, see [5]. Thus, according to [12], the Inf-Sup condition

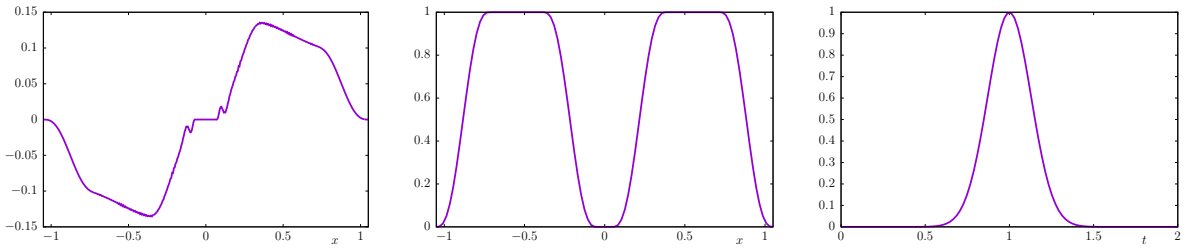
$$\inf_{(p,\tau) \in P_h \times D_h, (p,\tau) \neq 0} \sup_{v \in X_h, v \neq 0} \frac{b(v, p) - \langle v, \tau \rangle_{\Gamma_0}}{\|v\|_{X_h} \|(p, \tau)\|_{P_h \times D_h}} \ge a_1 > 0, \tag{6.4}$$

is satisfied under some conditions on the mesh size used for the boundary. For the displacement of the structure, we use Hermite finite elements. Then the resulting total degrees of freedom is equal to 406339.

6.2.2. The boundary perturbation. We test the efficiency of the feedback laws for inflow boundary perturbations on Γ_i of the form

$$g_p(t, z) = \beta_p \mu(t) g(z) \quad \text{with} \quad \mu(t) = e^{-30(t-1)^2} \quad \text{and} \quad g(z) = \left(m\sigma(\boldsymbol{\psi}^1, \boldsymbol{p}^1) n \cdot n, 0 \right)^T. \tag{6.5}$$

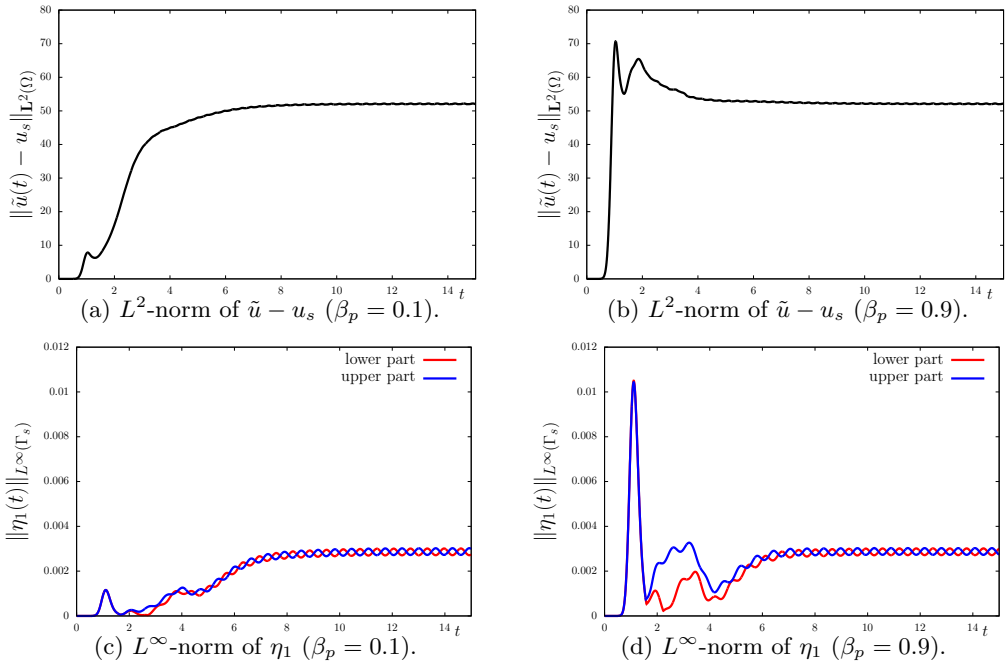
The quadruplet $(\boldsymbol{\psi}^1, \boldsymbol{\kappa}^1, \boldsymbol{\kappa}^2, \boldsymbol{\rho}^1)$ is the real part of an eigenvector associated to the most unstable eigenvalue of the adjoint problem (4.4) and $\boldsymbol{p}^1$ is the contribution of the pressure in the multiplier $\boldsymbol{\rho}^1$, see Theorem 5.1. Thus, the perturbation g_p is one of the most destabilizing normal boundary perturbations for the fluid. The truncation function m is used to impose the compatibility conditions at the junction between Γ_i and Γ_s and is defined by $m(z) = g(\frac{3(z-e)}{l-e})$ for $z \ge 0$, $m(z) = g(\frac{-3(z-e)}{l-e})$ for $z < 0$ where $g(x) = G(z)$ for $z \le \frac{3}{2}$, $G(3-z)$ for $z > \frac{3}{2}$ with $G(z) = 0$ for $z < 0, G(z) = z^3(6z^2 - 15z + 10)$ for $0 \le z \le 1$ and $G(z) = 1$ for $z > 1$. The function μ is used to localize the perturbation in time. See Figure 6.1 for illustrations. Finally, $\beta_p > 0$ is a parameter used to vary the amplitude of the perturbation.



(a) Graph of $m\sigma(\boldsymbol{\phi}^1, \boldsymbol{p}^1)n \cdot n$. (b) Graph of m. (c) Graph of μ.

Fig. 6.1: Graphs of the functions $m\sigma(\boldsymbol{\phi}^1, \boldsymbol{p}^1)n \cdot n$, m and μ.

6.2.3. Evolution of the perturbed nonlinear system without control. We choose two values of the amplitude $\beta_p = 0.1$ and $\beta_p = 0.9$. For $\beta_p = 0.1$, the maximal value of the perturbation g_p corresponds to 1.35% of that of g_s. In Figure 6.2, we plot the evolution of the L^2-norm of the difference between the velocity $\tilde{u}$ of the perturbed nonlinear system (1.1) and the velocity u_s of the steady flow, the evolution of the L^∞-norm of the displacements of the lower part and upper part of the structure. The results confirm that the stationary solution is unstable. Moreover, we remark that, for $t \geq 6$, the norms of the solutions for $\beta_p = 0.1$ and $\beta_p = 0.9$ asymptotically tend to the same value.



(a) L^2-norm of $\tilde{u} - u_s$ ($\beta_p = 0.1$). (b) L^2-norm of $\tilde{u} - u_s$ ($\beta_p = 0.9$).

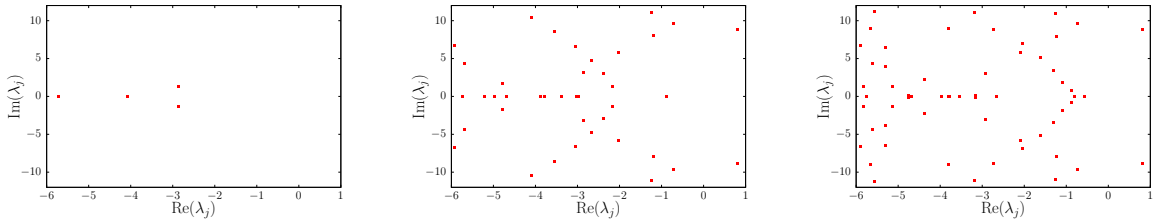
(c) L^∞-norm of η_1 ($\beta_p = 0.1$). (d) L^∞-norm of η_1 ($\beta_p = 0.9$).

Fig. 6.2: Perturbed nonlinear systems without control for $\beta_p = 0.1$ and $\beta_p = 0.9$.

6.3. Numerical stabilization of the nonlinear system for $Re = 200$. We are going to show, by numerical experiments, that we are able to locally stabilize the nonlinear system (1.1) for inflow boundary perturbations. We recall that the fluid equations are rewritten in the reference configuration via a change of variables. Coming back to the moving domain is guaranteed only if the displacements of the structure are less than 0.05, see [11, Theorem 3.1]. Thus, we have to construct feedback laws generating admissible displacements of the structure. To satisfy that constraint, we can vary the shift parameter ω, the family of control functions $(\boldsymbol{w}_i)_{1 \leq i \leq n_c}$ and the choice of the unstable subspace $\mathbb{Z}_u$. In order to find a feedback law efficient in presence of inflow boundary perturbations, we use the following strategy. First, we compute the eigenvalues of the linearized system. Then, we fix a value for ω and we compute the degrees of stabilizability of the linearized system for different choices of $(\boldsymbol{w}_i)_i$ and of $\mathbb{Z}_u$. Finally, using the degrees of stabilizability and other numerical tests described below, we fix the family $(\boldsymbol{w}_i)_{1 \leq i \leq n_c}$, and next the unstable subspace $\mathbb{Z}_u$.

6.3.1. Computation of the spectra. We compute the spectra of the structure, the fluid and the fluid-structure systems, close to the imaginary axis, see Figure 6.3. For that, we use an Arnoldi method combined with a *shift and inverse* transformation implemented in the Arpack Library. The results are reported in Table 6.1. We have solved the eigenvalue problems for a mesh of 89418 cells, for a coarser mesh of 75846 cells and for a very fine mesh of 283956 cells. The values are reported in Table 6.2. We observe that, in comparison with the results obtained using the very fine mesh, the solutions of the eigenvalue problems for the mesh of 89418 cells are accurate up to 10^{-2}.

6.3.2. The choice of ω. The parameter ω permits to improve the decay rate of the controlled solutions towards the stationary solution. However, the L^∞-norm of the control and then the L^∞-norm of the displacements of the structure increase with ω. Thus, we have to choose ω in order to realize a compromise between the efficiency of the control and small enough displacements of the structure. We fix $\omega = 2$. In Section 6.3.5, we will see how to modify ω in order to reduce the displacements of the structure.



(a) Structure. (b) Fluid. (c) Fluid-structure.

Fig. 6.3: Spectra.

$\boldsymbol{\lambda}_{1,2}$	$\boldsymbol{\lambda}_3$	$\boldsymbol{\lambda}_4$	$\boldsymbol{\lambda}_{5,6}$	$\boldsymbol{\lambda}_7$	$\boldsymbol{\lambda}_8$	$\boldsymbol{\lambda}_{9,10}$
$-2.87 \pm 1.27i$	-4.08	-5.71	$-12.93 \pm 3.26i$	-16.32	-19.96	$-30.32 \pm 5.49i$

(a) First eigenvalues of the structure system.

$\boldsymbol{\lambda}_{1,2}$	$\boldsymbol{\lambda}_{3,4}$	$\boldsymbol{\lambda}_5$	$\boldsymbol{\lambda}_{6,7}$	$\boldsymbol{\lambda}_{8,9}$	$\boldsymbol{\lambda}_{10,11}$
$0.81 \pm 8.80i$	$-0.70 \pm 9.64i$	-0.88	$-1.18 \pm 7.99i$	$-1.23 \pm 11.05i$	$-2.03 \pm 5.83i$

(b) First eigenvalues of the fluid system.

$\boldsymbol{\lambda}_{1,2}$	$\boldsymbol{\lambda}_3$	$\boldsymbol{\lambda}_{4,5}$	$\boldsymbol{\lambda}_6$	$\boldsymbol{\lambda}_{7,8}$	$\boldsymbol{\lambda}_{9,10}$
$0.82 \pm 8.79i$	-0.57	$-0.72 \pm 9.60i$	-0.80	$-0.86 \pm 0.80i$	$-1.08 \pm 1.88i$

(c) First eigenvalues of the fluid-structure system.

Table 6.1: First eigenvalues of the structure, the fluid and the coupled systems.

cells	$\boldsymbol{\lambda}_{1,2}$	$\boldsymbol{\lambda}_3$	$\boldsymbol{\lambda}_{4,5}$	$\boldsymbol{\lambda}_6$	$\boldsymbol{\lambda}_{7,8}$	$\boldsymbol{\lambda}_{9,10}$
75846	$0.820 \pm 8.789i$	-0.574	$-0.722 \pm 9.602i$	-0.807	$-0.864 \pm 0.809i$	$-1.088 \pm 1.883i$
89418	$0.822 \pm 8.792i$	-0.574	$-0.722 \pm 9.604i$	-0.806	$-0.864 \pm 0.809i$	$-1.089 \pm 1.883i$
283956	$0.824 \pm 8.794i$	-0.574	$-0.722 \pm 9.604i$	-0.806	$-0.864 \pm 0.809i$	$-1.089 \pm 1.883i$

Table 6.2: First eigenvalues of the fluid-structure system for different meshes.

6.3.3. The choice of the controls functions, the degrees of stabilizability and the unstable subspace $\mathbb{Z}_u$. We choose the control functions $(\boldsymbol{w}_i)$ in the family $(\boldsymbol{\zeta}_2^i)_i$ defined in Theorem 5.2 (the functions $\boldsymbol{\zeta}_2^i$ are the approximations of the control functions used to stabilize the infinite dimensional system, see [11]). More precisely, we consider control families of the form $(\boldsymbol{w}_i)_{1 \le i \le n_c} = (\boldsymbol{\zeta}_2^i)_{1 \le i \le n_c}$ with $n_c \ge 2$. We have computed the degrees of stabilizability of the linearized system for different values of n_c and for different choices of $\mathbb{Z}_u$. We considered $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2})$, $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_3)$, $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_{4,5})$ and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$. The results are reported in Table 6.3. We observe that the degrees of stabilizability vary slightly with n_c. That is why, in the sequel, we choose $n_c = 2$. The unstable subspaces $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2})$ and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_3)$ are the most stabilizable ones and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_{4,5})$ is the worst one. However, due to the nonlinearities, it is not sure that these conclusions remain valid for the nonlinear system. That is why we compute the controlled nonlinear system for the different unstable subspaces and for a small perturbation amplitude ($\beta_p = 0.1$). The L^2-norm of the controls are reported in Table 6.4 and the difference between the controlled solutions and

the stationary solution are plotted in Figure 6.4. The results confirm that the worst choice of $\mathbb{Z}_u$ is $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_{4,5})$ even if it corresponds to the best decay rate. We notice that $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$ corresponds to the best degree of stabilizability. Moreover, it corresponds to the smallest displacements of the structure. Thus, we choose $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$ in the following tests.

n_c	2	3	5	10
$10^5 d$	9.59	9.59	9.59	9.59

(a) $\mathbb{Z}_u = G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2})$.

n_c	2	3	5	10
$10^5 d$	9.59	9.59	9.59	9.59

(b) $\mathbb{Z}_u = G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_3)$.

n_c	2	3	5	10
$10^5 d$	4.29	4.30	4.30	4.30

(c) $\mathbb{Z}_u = G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_{4,5})$.

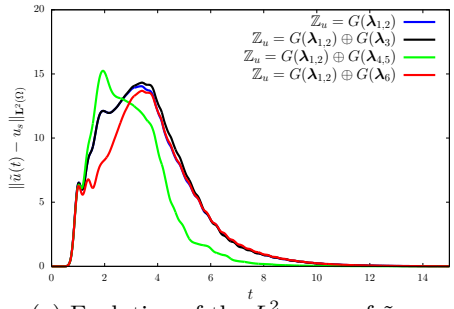
n_c	2	3	5	10
$10^5 d$	8.44	8.45	8.45	8.45

(d) $\mathbb{Z}_u = G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_6)$.

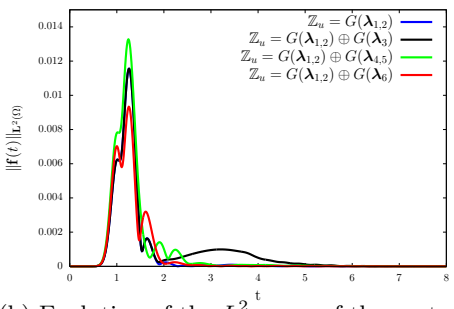
Table 6.3: Influence of $(\boldsymbol{w}_i)_{1 \le i \le n_c}$ and of $\mathbb{Z}_u$ on the degrees of stabilizability.

$\mathbb{Z}_u$	$G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2})$	$G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_3)$	$G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_{4,5})$	$G_{\mathbb{R}}(\boldsymbol{\lambda}_{1,2}) \oplus G_{\mathbb{R}}(\boldsymbol{\lambda}_6)$
$\|\boldsymbol{f}\|$	0.087	0.088	0.104	0.076

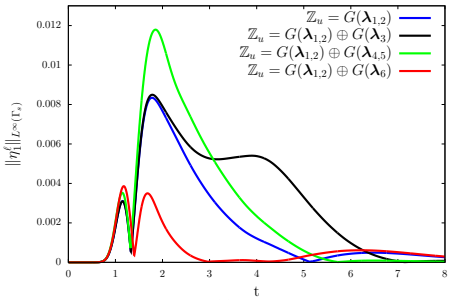
Table 6.4: L^2-Norm of the controls for different choices of $\mathbb{Z}_u$ ($\beta_p = 0.1$).



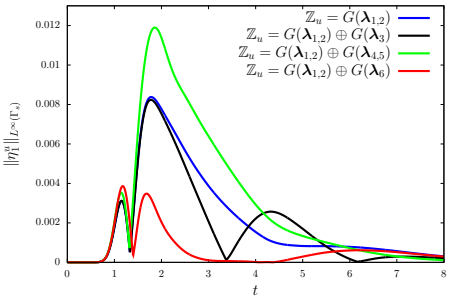
(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolution of the L^2-norm of the controls.



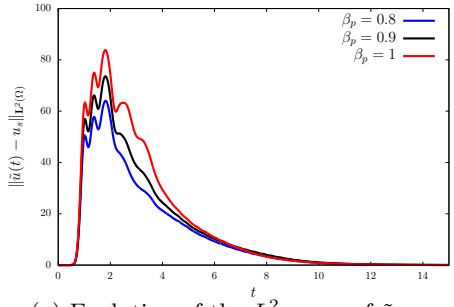
(c) Evolution of the L^∞-norm of η_1 (lower part).



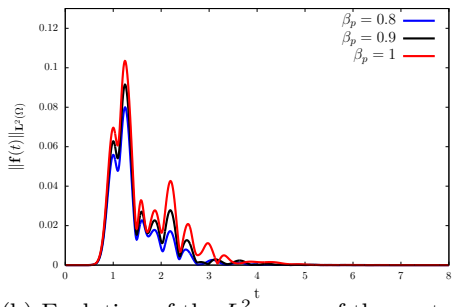
(d) Evolution of the L^∞-norm of η_1 (upper part).

Fig. 6.4: Influence of $\mathbb{Z}_u$ on the stabilization of the perturbed nonlinear system for $\beta_p = 0.1$.

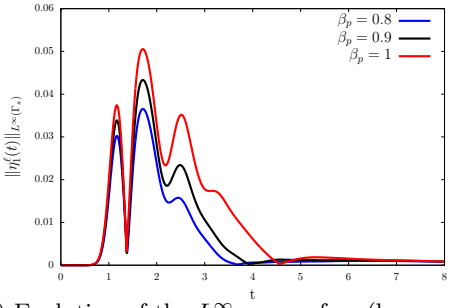
6.3.4. Efficiency of the feedback law for $\omega = 2$, $n_c = 2$ and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$. We test the efficiency of the feedback law for increasing perturbation amplitudes. The results are plotted in Figure 6.5. We see that we are able to stabilize the nonlinear system for perturbation amplitudes smaller than $\beta_p = 1$. Moreover, as expected, the control and the displacements of the structure increase with the amplitude of the perturbation. With this strategy, we are not able to stabilize the nonlinear system for $\beta_p = 1$ because the displacements of the structure are slightly larger than the limit displacement equal to 0.05. However, we notice two control phases. The first one, from $t = 0$ to $t = 1.5$, corresponds to the time interval where the perturbation passes through the fluid domain, and a second one for $t > 1.5$. We know that the more we act in the first phase (by increasing ω) the less we shall need to act thereafter. That implies that the displacements of the structure increase in the first phase and decrease in the second one. Thus, since the displacements are smaller than 0.05 in the first phase, we can increase the parameter ω. In the next section, we will see that we are able to stabilize the nonlinear system for perturbation amplitudes greater than $\beta_p = 1$ with $\omega = 2.5$.



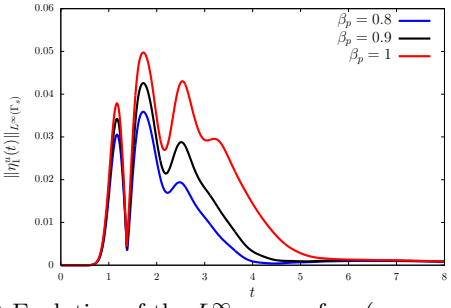
(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolution of the L^2-norm of the controls.



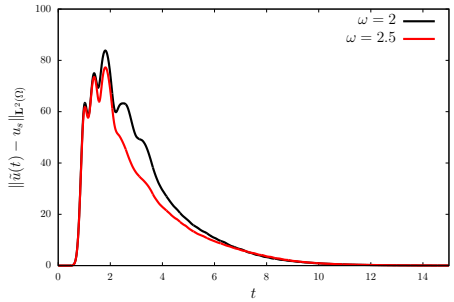
(c) Evolution of the L^∞-norm of η_1 (lower part).



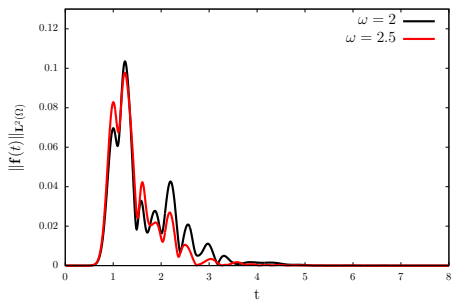
(d) Evolution of the L^∞-norm of η_1 (upper part).

Fig. 6.5: Controlled nonlinear system for $\omega = 2$, $n_c = 2$, $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$ and for different β_p.

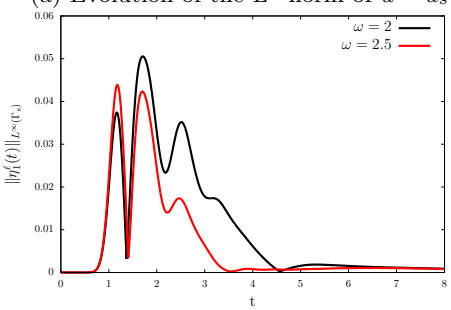
6.3.5. Efficiency of the feedback law for $\omega = 2.5$, $n_c = 2$ and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$. The goal is to stabilize the nonlinear system for perturbation amplitudes greater than $\beta_p = 1$. We recall that, for $\omega = 2$, $n_c = 2$ and $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$, we are not able to stabilize the original nonlinear system for $\beta_p = 1$ because the displacements of the structure are greater than 0.05. That is why we increase ω by choosing $\omega = 2.5$. We compute the solutions of the corresponding controlled system. The results are reported in Figure 6.6. The feedback law is able to stabilize the nonlinear system for $\beta_p = 1$ and the displacements of the structure are smaller that 0.05. Moreover, as expected, we observe that by increasing ω, the control and the displacements of the structure increase in the first period and decrease in the second one.



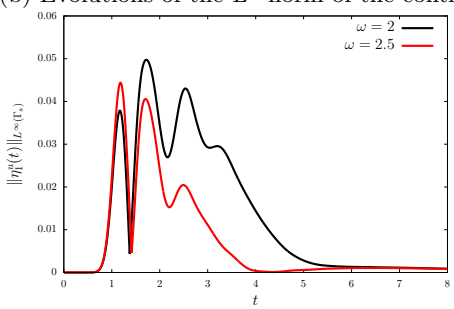
(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolutions of the L^2-norm of the controls.



(c) Evolution of the L^∞-norm of η_1 (lower part).



(d) Evolution of the L^∞-norm of η_1 (upper part).

Fig. 6.6: Influence of ω on the stabilization of the perturbed nonlinear system for $\beta_p = 1$.

The feedback law obtained by choosing $\omega = 2.5$ is also able to stabilize the nonlinear system for $\beta_p = 1.1$ corresponding to an amplitude perturbation of order 15% of g_s, see Figure 7.1. The displacements of the structure are close to the limit 0.05 in the two phases. Thus, we cannot stabilize the nonlinear system for larger perturbation amplitudes and we cannot hope to stabilize it by increasing the value of ω.

7. Conclusion. For a Reynolds number $Re = 200$ and for a boundary perturbation, localized in time around $t = 1$, and of amplitude of order 15% of the stationary inflow boundary condition, we propose a strategy for choosing a feedback control able to stabilize the fluid-structure system coupling the Navier–Stokes equations with an Euler-Bernoulli damped beam equation.

The control is obtained by stabilizing the projected linearized system onto an invariant subspace containing the unstable subspace of the linearized dynamical system. The feedback law is determined by solving a Riccati equation of small dimension. This feedback law, which is constructed on the linearized model, is applied to the nonlinear fluid–structure model. In order to improve the performance of the control law, we choose the invariant subspace based on the analysis of the degrees of stabilizability. Next we vary the exponential decay rate involved in the Riccati equation in order that the structure displacements of the nonlinear closed-loop system remain sufficiently small. This is important for practical applications.

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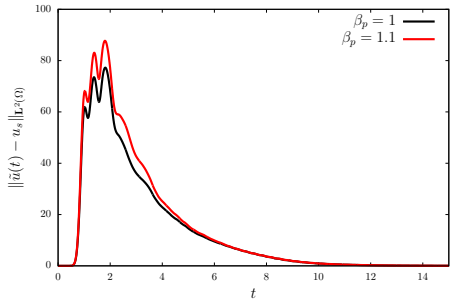
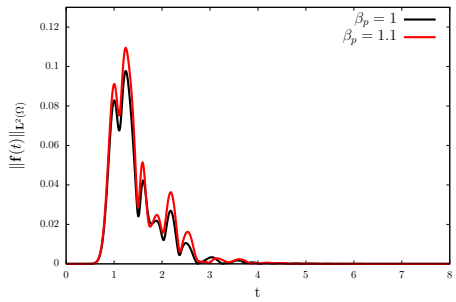
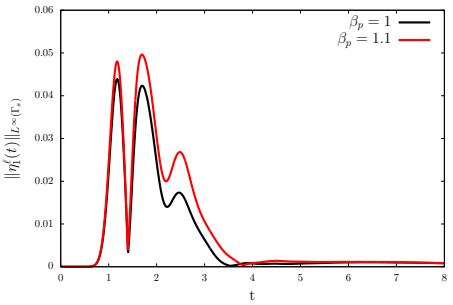


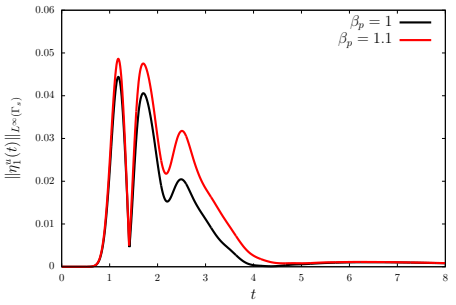
(a) Evolution of the L^2-norm of $\tilde{u} - u_s$.



(b) Evolution of the L^2-norm of the controls.



(c) Evolution of the L^∞-norm of η_1 (lower part).



(d) Evolution of the L^∞-norm of η_1 (upper part).

Fig. 7.1: Perturbed nonlinear system for $\omega = 2.5$, $n_c = 2$, $\mathbb{Z}_u = G_{\mathbb{R}}(\lambda_{1,2}) \oplus G_{\mathbb{R}}(\lambda_6)$, $\beta_p = 1$ and $\beta_p = 1.1$.

Appendix A. The geometrical terms.

We recall that the sign function s is defined by $s(y) = -1$ if $y < 0$ and $s(y) = 1$ if $y \geq 0$. The nonlinear terms F_f, F_{div} and F_s in system (1.5) are defined by

$$
\begin{aligned}
F_f[\tilde{u}, \tilde{p}, \eta_1, \eta_2] = & \frac{(\ell - s(z)z)\eta_2}{\ell - e - s(z)\eta_1}\tilde{u}_z - \frac{2\nu(\ell - s(z)z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_{xz} + \frac{\nu((z-\ell)^2\eta_{1,x}^2 - \eta_1^2)}{(\ell - e + s(z)\eta_1)^2}\tilde{u}_{zz} \\
& + \frac{2\nu(\ell - s(z)z)\eta_{1,x}^2}{(\ell - e - s(z)\eta_1)^2}\tilde{u}_z - \frac{\nu(\ell - s(z)z)\eta_{1,xx}}{\ell - e - s(z)\eta_1}\tilde{u}_z - \frac{\nu(\ell - e)\eta_1}{(\ell - e - s(z)\eta_1)^2}\tilde{u}_{zz} + \frac{(\ell - s(z)z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{p}_z e_1 \\
& - \frac{s(z)\eta_1}{\ell - e - s(z)\eta_1}\tilde{p}_z e_2 + \frac{(\ell + z)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_1\tilde{u}_z - \frac{s(z)\eta_1}{\ell - e - s(z)\eta_1}\tilde{u}_2\tilde{u}_z - \nu\nabla\text{div}\, F_{\text{div}}[\tilde{u}, \eta_1],
\end{aligned}
$$

$$F_{\rm div}[\tilde{u},\eta_1] = \frac{1}{\ell - e}\left(s(z)\eta_1\tilde{u}_1 e_1 + (\ell - s(z)z)\eta_{1,x}\tilde{u}_1 e_2\right),$$
$$F_s[\tilde{u},\eta_1] = \nu\left(\frac{(\ell-e)\eta_{1,x}}{\ell - e - s(z)\eta_1}\tilde{u}_{1,z} + \eta_{1,x}\tilde{u}_{2,x} - s(z)\frac{2\eta_1 - (\ell - s(z)z)\eta_{1,x}^2}{\ell - e - s(z)\eta_1}\tilde{u}_{2,z}\right).$$

The linear differential operators A_1, A_2, A_3 and A_4 in (1.6) are defined by

$$A_1\eta_1 = \frac{s(z)\eta_1}{\ell - e}\left(p_{s,z}e_2 + u_{s,2}u_{s,z} - 2\nu u_{s,zz} + \nu u_{s,1,xx}e_1 + \nu u_{s,1,xz}e_2\right) + \frac{\nu\eta_{1,x}}{\ell - e}\left(u_{s,1,z}e_2 - u_{s,1,x}e_1\right)$$
$$+\frac{(\ell - s(z)z)}{\ell - e}\left(\left(p_{s,z}e_1 - 2\nu u_{s,xz} + u_{s,1}u_{s,z} - \nu u_{s,1,zz}e_2 - \nu u_{s,1,xz}e_1\right)\eta_{1,x} - \nu\left(u_{s,z} + u_{s,1,z}e_1\right)\eta_{1,xx}\right)$$
$$A_2\eta_2 = \frac{(\ell - s(z)z)\eta_2}{\ell - e}u_{s,z}, \quad A_3\eta_1 = \frac{1}{\ell - e}(s(z)\eta_1 u_{s,1,x} + (\ell - s(z)z)\eta_{1,x}u_{s,1,z}),$$
$$A_4\eta_1 = \nu\left(\frac{s(z)\eta_1}{\ell - e}u_{s,2,z} - \eta_{1,x}u_{s,1,z}\right).$$

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