



**HAL**  
open science

# A Stable Formulation of Resonant Maxwell's Equations in Cold Plasma

A Nicolopoulos, M Campos-Pinto, B. Després

► **To cite this version:**

A Nicolopoulos, M Campos-Pinto, B. Després. A Stable Formulation of Resonant Maxwell's Equations in Cold Plasma. 2018. hal-01743546v1

**HAL Id: hal-01743546**

**<https://hal.science/hal-01743546v1>**

Preprint submitted on 26 Mar 2018 (v1), last revised 7 Mar 2019 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Stable Formulation of Resonant Maxwell's Equations in Cold Plasma

A. Nicolopoulos\*, M. Campos-Pinto† and B. Després‡

March 26, 2018

## Abstract

We consider a boundary value problem (BVP) for a reduced system of time harmonic Maxwell equations in magnetized plasma. The dielectric tensor is strongly anisotropic and the system admits resonant solutions in the context of the limit absorption principle. In particular, in the vanishing viscosity limit the normal component of the electric field becomes infinite and non integrable at the resonant point, and the system becomes ill-posed. In this article we recast the problem in the framework of mixed variational problems and we propose a well-posed formulation that characterizes the singular limit solutions. A key tool is the method of manufactured solutions [7] to construct an integral variational characterization of the jump conditions at the resonance. The well posedness is demonstrated and basic numerical results illustrate the robustness of our approach.

**Keywords:** plasma heating, Maxwell's equations, hybrid resonance, mode coupling, singular solutions, manufactured solutions, mixed variational formulations, finite elements

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminary material</b>	<b>4</b>
2.1	A priori bounds . . . . .	5
2.2	Manufactured solutions . . . . .	7
2.3	An energy relation . . . . .	8
<b>3</b>	<b>A mixed variational formulation for the limit problem</b>	<b>10</b>
3.1	Euler-Lagrange equations and main Theorem . . . . .	11
3.2	Proof of the second part of the main Theorem . . . . .	12
3.3	Proof of the third part of the main Theorem . . . . .	14
<b>4</b>	<b>Numerical illustration</b>	<b>16</b>
4.1	The Whittaker test case: an analytical solution in normal incidence . . . . .	16
4.2	A more physical test-case . . . . .	18

## 1 Introduction

In this work, we propose and analyze an original mixed variational formulation for a singular second order system of equations of the form

$$-\frac{d^2}{dx^2}\mathbf{u}^\nu(x) + \frac{1}{\alpha(x) + i\nu}\mathbf{N}^\nu(x)\mathbf{u}^\nu(x) = 0 \quad \text{in } \Omega = (-1, 1), \quad \mathbf{u}^\nu(x) \in \mathbb{C}^2 \quad (1.1)$$

---

\*Sorbonne Université, Université Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris

†Sorbonne Université, Université Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris

‡Sorbonne Université, Université Paris-Diderot SPC, CNRS, Laboratoire Jacques-Louis Lions, LJLL, F-75005 Paris, Institut Universitaire de France.

with  $\alpha$  a smooth function vanishing at  $x = 0$ . Here the matrix  $\mathbf{N}^\nu(x) \in \mathcal{M}_2(\mathbb{C})$  is also a smooth function of  $x$ , and  $\nu \geq 0$  is a small regularizing parameter. At the limit  $\nu = 0$ , the matrix  $\mathbf{N} = \mathbf{N}^0$  satisfies an important condition

$$\text{rank}(\mathbf{N}(0)) = 1.$$

Equation (1.1) is completed with natural dissipative boundary conditions,

$$\frac{d}{dx} \mathbf{u}^\nu(\pm 1) \mp \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u}^\nu(\pm 1) = \mathbf{f}(\pm 1), \quad (1.2)$$

with  $\sigma > 0$  and  $\mathbf{f}$  a  $\mathbb{C}^2$ -valued field defined on  $\partial\Omega = \{-1, 1\}$ . The singularity arises for the limit problem corresponding to  $\nu = 0$ : as  $\alpha$  vanishes at  $x = 0$  and  $\mathbf{N}$  does not, the coefficients are unbounded and the limit boundary value problem (BVP) is ill-posed.

**Physical context.** Equation (1.1) can be obtained as a planar reformulation [16] of the linear cold plasma model which is routinely used to describe the propagation of radio-frequency electromagnetic (EM) waves in magnetized plasmas, with applications in the ionosphere and in tokamaks [5, 14, 15, 1]. In this model the time harmonic Maxwell's equations are coupled with a Newton law for the linearized response of the non-homogeneous electron plasma. After elementary manipulations the electron current density can be eliminated from the equations, and the time-harmonic EM field  $(\mathbf{E}^\nu, \mathbf{B}^\nu)$  satisfies a system of the form

$$\begin{cases} \mathbf{B}^\nu - \nabla \times \mathbf{E}^\nu = 0 \\ \nabla \times \mathbf{B}^\nu - \underline{\underline{\varepsilon}}^\nu \mathbf{E}^\nu = 0 \end{cases}. \quad (1.3)$$

Here  $\mathbf{E}^\nu$  is the unknown electric field. The physical magnetic field is actually  $\frac{1}{i\omega} \mathbf{B}^\nu$  with  $\omega$  the frequency of the wave sent into the plasma, but for convenience we call  $\mathbf{B}^\nu := \nabla \wedge \mathbf{E}^\nu$  the magnetic field. The presence of a bulk magnetic field results in a planar structure for the dielectric tensor. If the plasma density varies only in the  $x$  dimension and the bulk magnetic field is aligned with the  $z$  direction, we may consider a simplified tensor of the form

$$\underline{\underline{\varepsilon}}^\nu(\mathbf{x}) = \begin{pmatrix} \alpha(x) + i\nu & i\delta(x) & 0 \\ -i\delta(x) & \alpha(x) + i\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = (x, y, z). \quad (1.4)$$

Equation (1.1) then appears by considering fields of the form  $(\mathbf{E}^\nu, \mathbf{B}^\nu)(\mathbf{x}) = e^{ik_z z} (\hat{\mathbf{E}}^\nu, \hat{\mathbf{B}}^\nu)(x)$  which corresponds to waves being sent into the plasma with a wave vector  $\mathbf{k} = (k_x, 0, k_z)$ , see [16]. Writing  $\hat{\mathbf{E}}^\nu = (e_x^\nu, e_y^\nu, e_z^\nu)^t$  and  $\hat{\mathbf{B}}^\nu = (b_x^\nu, b_y^\nu, b_z^\nu)^t$ , we rewrite system (1.3) as

$$\begin{cases} b_x^\nu + ik_z e_y^\nu = 0 \\ b_y^\nu - ik_z e_x^\nu + e_z^{\nu'} = 0 \\ b_z^\nu - e_y^{\nu'} = 0 \end{cases}, \quad \begin{cases} -ik_z b_y^\nu - (\alpha + i\nu) e_x^\nu - i\delta e_y^\nu = 0 \\ ik_z b_x^\nu - b_z^{\nu'} + i\delta e_x^\nu - (\alpha + i\nu) e_y^\nu = 0 \\ b_y^{\nu'} - e_z^\nu = 0 \end{cases} \quad (1.5)$$

and we observe that all the components of the fields can be expressed in terms of  $e_y^\nu$  and  $b_y^\nu$ . The equations then yield (1.1) with  $\mathbf{u}^\nu = (e_y^\nu, b_y^\nu)^t$ , and a matrix

$$\mathbf{N}^\nu(x) = \begin{pmatrix} k_z^2(\alpha(x) + i\nu) + \delta(x)^2 - (\alpha(x) + i\nu)^2 & \delta(x)k_z \\ \delta(x)k_z & k_z^2 - (\alpha(x) + i\nu) \end{pmatrix} \quad \text{for } x \in \Omega. \quad (1.6)$$

Here the case  $k_z = 0$  corresponds to a wave being sent with a normal incidence with respect to the bulk magnetic field [16]. In this case the equations on  $e_y^\nu$  and  $b_y^\nu$  decouple: the latter wave is called an ordinary mode (O-mode) and in the limit  $\nu \rightarrow 0$  it satisfies a standard Helmholtz equation. The former one is called an extraordinary mode (X-mode). In the generic situation considered here of a function  $\alpha$  that vanishes locally, say at  $x = 0$ , and of a positive  $\delta(x) > 0$ , its limit equation is singular: the component  $e_x^\nu = -i\frac{\delta}{\alpha+i\nu} e_y^\nu$  may become non-integrable, and the problem needs to be addressed using a limit absorption principle with vanishing positive viscosity  $\nu \rightarrow 0^+$ . It has been mathematically analyzed in [9].

This singular behaviour corresponds to an interesting and fundamental resonance phenomenon that takes place at  $x = 0$ . In the context of controlled nuclear fusion this is one of the methods used to heat the plasma in a tokamak. The resonant heating is a manner to characterize the amplitude of the singularity. Letting  $\Pi^\nu = \text{Im}(\mathbf{E}^\nu \times \overline{\mathbf{B}^\nu})$  denote an ad-hoc Poynting vector, it can be written as

$$\lim_{\nu \rightarrow 0^+} \int_{\Omega} \nabla \cdot \Pi^\nu dx = \lim_{\nu \rightarrow 0^+} \nu \int_{\Omega} |\mathbf{E}^\nu(x)|^2 dx > 0 \quad (1.7)$$

where the positivity of the limit is already an indication of its singular nature. We refer to [8, 7] for additional mathematical results on the  $X$ -mode resonance, and to [6] for a numerical study.

In this article we develop a mathematical theory that covers the case  $k_z \neq 0$  corresponding to a wave being sent in the plasma with an *oblique incidence* with respect to the bulk magnetic field [16]. This case is notoriously more complicated to analyze since it brings what is called a mode coupling: as the extra-diagonal coefficients are non zero in the matrix (1.6), it is no longer possible to decouple the equations on  $e_y^\nu$  and  $b_y^\nu$ . Our goal is thus to characterize and analyze the reduced model (1.1) in the limit  $\nu \rightarrow 0^+$ . For that purpose we will use the Boffi-Brezzi-Fortin [2] framework for mixed variational formulations, and an important tool will be the method of manufactured solutions recently introduced in [7].

**Assumptions and notations.** Before stating the main results we need to particularize the class of matrices which is encompassed by our theory. As written above, for  $\nu \in \mathbb{R}$  the matrix-valued functions considered here are of the form (1.6).

**Assumption 1.** *We suppose that  $\alpha \in W^{3,\infty}(\Omega)$  is real-valued and such that 0 is its only root, with  $r = \alpha'(0) \neq 0$ . In addition we assume that  $\delta \in W^{3,\infty}(\Omega)$  is real and positive. Finally the Fourier variable is arbitrary,  $k_z \in \mathbb{R}$ , to handle the mode coupling phenomenon described in [16].*

**Notation 2.** *As  $N^\nu$  depends continuously on  $\nu$ , for  $\nu = 0$  we denote the limit matrix by  $N := N^0$ .*

The symmetry properties of (1.4) can be characterized as follows.

**Proposition 3.** *For  $\nu \in \mathbb{R}$ ,  $N^\nu$  is such that*

$$N^\nu = (N^\nu)^t, \quad \overline{N^\nu} = N^{-\nu}, \quad \text{and} \quad \ker N(0) = \text{Span}_{\mathbb{C}}\{(k_z, -\delta(0))^t\}.$$

*For  $\nu \geq 0$ , it also verifies the dissipation property*

$$\frac{1}{2i} \left( \frac{1}{\alpha + i\nu} N^\nu - \frac{1}{\alpha - i\nu} N^{-\nu} \right) = \text{Im} \left( \frac{1}{\alpha + i\nu} N^\nu \right) \leq 0, \quad (1.8)$$

*in the sense that it is negative semi-definite.*

**Proof.** Given (1.6), the three first properties are immediate as  $\alpha$ ,  $\delta$ ,  $k_z$  and  $\nu$  are real-valued and as  $\alpha(0) = 0$ . For the last one we use  $\text{Im} \left( \frac{1}{\alpha + i\nu} N^\nu \right) = \text{Im} \left( \frac{\alpha - i\nu}{\alpha^2 + \nu^2} N^\nu \right) = \frac{-\nu}{\alpha^2 + \nu^2} \begin{pmatrix} \delta^2 + \alpha^2 + \nu^2 & \delta k_z \\ \delta k_z & k_z^2 \end{pmatrix}$ . Because  $\delta^2 |c_1|^2 + 2\delta k_z c_1 \overline{c_2} + k_z^2 |c_2|^2 = |\delta c_1 + k_z c_2|^2$  for  $c_1, c_2 \in \mathbb{C}$ , it establishes  $\text{Im}(N^\nu/(\alpha + i\nu)) \leq 0$ . Finally as  $\overline{N^\nu}/(\alpha + i\nu) = N^{-\nu}/(\alpha - i\nu)$ , the announced property is verified.  $\square$

**Main results.** Our main results can now be formulated as follows.

**Theorem 4.** *Under the above assumptions, the unique solution  $\mathbf{u}^\nu$  of system (1.1)-(1.2) converges for  $\nu \rightarrow 0^+$  weakly in  $H^1(\Omega)^2$  towards a function  $\mathbf{u}^+$ . This function is a strong solution to the limit equation except at the resonance*

$$-\mathbf{u}''(x) + \frac{1}{\alpha(x)} \mathbf{N}(x) \mathbf{u}(x) = 0 \quad \text{in } \Omega - \{0\}, \quad (1.9)$$

*and satisfies strongly the boundary conditions (1.2).*

This first result establishes the existence of a limit for vanishing viscosity parameter  $\nu$ , but the limit equation (1.9) does not allow to completely characterize  $\mathbf{u}^+$  since the equation is not valid at  $x = 0$  due to the singularity. The next result establishes that  $\mathbf{u}^+$  is the solution of a well posed mixed variational formulation in the spaces

$$V = H^1(\Omega)^2 \times \mathbb{C} \quad \text{and} \quad Q = \{\mathbf{v} \in H^1(\Omega)^2, \mathbf{N}(0)\mathbf{v}(0) = 0\}.$$

**Theorem 5.** *The viscosity limit  $\mathbf{u}^+$  is the unique solution of a mixed variational formulation*

$$\begin{aligned} & \text{Find } ((\mathbf{u}, s), \boldsymbol{\lambda}) \in H \times Q \text{ such that} \\ & \begin{cases} a^+((\mathbf{u}, s), (\mathbf{v}, t)) - \overline{b((\mathbf{v}, t), \boldsymbol{\lambda})} = 0, & \forall (\mathbf{v}, t) \in H, \\ b((\mathbf{u}, s), \boldsymbol{\mu}) = \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q, \end{cases} \end{aligned} \quad (1.10)$$

*where the sesquilinear form  $b$  is defined in (2.3), the antilinear form  $\ell$  is defined in (2.5) and the sesquilinear form  $a^+$  is defined in (3.4).*

The proof heavily relies on the theory of mixed variational formulation [2] applied to a convenient characterization of the limit solution  $\mathbf{u}^+$ , where the sesquilinear form  $a^+$  appears to be exactly what is needed to complement the missing information stressed in Theorem 4. We will also propose in the core of this work another mixed variational formulation which is valid for  $\nu > 0$  and has the limit (1.10) for  $\nu = 0^+$ .

**Organization of this work.** Preliminary material, such as the sesquilinear form  $b$  and simple a priori bounds, are introduced in Section 2. The mixed variational formulation is constructed and studied in the following Section 3. Since the theory of mixed variational formulation is completed with a well established theory of numerical discretization, we take this opportunity to illustrate our main results with simple and reliable numerical results in Section 4. It also helps to better understand the physics which is behind the model problem (1.1). An application to an accurate calculation of the resonant heating is finally shown.

**Additional conventions.** Vectors will be written in bold lower-case letters as  $\mathbf{u}$  or  $\boldsymbol{\lambda}$ , matrices will be written in bold upper-case letters as  $\mathbf{N}$ . The dependency on  $\nu$  will be upper-indexed, as for  $\mathbf{N}^\nu$ . When the limit as  $\nu$  goes to 0 depends on the sign of  $\nu$ , it will be upper-indexed with a plus or a minus sign, as for  $\mathbf{u}^+$  or  $a^+$ . And when the limit does not depend on the way  $\nu$  goes to 0, it will not be indexed, as for  $\mathbf{N}$ , to simplify the notations. We will use the notations  $\{f\}_{-1}^1 = f(1) + f(-1)$  and  $[f]_{-1}^1 = f(1) - f(-1)$  for a scalar function  $f$  defined in  $-1$  and  $1$ . When the context makes it non ambiguous, the norm  $\|\cdot\|_{H^1(\Omega)^2}$  will often be denoted by  $\|\cdot\|_{H^1(\Omega)}$  or even simpler by  $\|\cdot\|_{H^1}$ . The dual spaces will be noted with a prime, for example  $Q'$  is the space of all continuous linear maps from  $Q$  to  $\mathbb{C}$ .

## 2 Preliminary material

Let us start the construction with two natural variational formulations associated to (1.1)-(1.2) and (1.2)-(1.9). For  $\nu > 0$ , the viscosity problem can be written in  $H^1(\Omega)^2$  as a first variational formulation:

$$\begin{aligned} & \text{Find } \mathbf{u} \in H^1(\Omega)^2 \text{ such that} \\ & b^\nu(\mathbf{u}, \mathbf{v}) = \ell^\nu(\mathbf{v}) \quad \text{for all test functions } \mathbf{v} \in H^1(\Omega)^2. \end{aligned} \quad (2.1)$$

Here the sesquilinear form is

$$b^\nu(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left( \mathbf{u}' \cdot \bar{\mathbf{v}}' + \mathbf{u} \cdot \frac{\mathbf{N}^\nu}{\alpha + i\nu} \bar{\mathbf{v}} \right) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \bar{\mathbf{v}} \right\}_{-1}^1, \quad \text{for } (\mathbf{u}, \mathbf{v}) \in H^1(\Omega)^2 \quad (2.2)$$

and the antilinear form is

$$\ell^\nu(\mathbf{v}) = [\mathbf{f} \cdot \bar{\mathbf{v}}]_{-1}^1, \quad \text{for } \mathbf{v} \in H^1(\Omega)^2.$$

In order to pass to the limit  $\nu = 0^+$ , in a way or another the kernel of  $\mathbf{N}(0)$  spanned by the vector  $(k_z, \delta(0))^t \neq 0$  has to be excluded. In this work, we decide to impose this constraint on the test functions.

**Definition 6.** For  $\nu = 0^+$ , we introduce a space of test functions  $Q = \{\mathbf{v} \in H^1(\Omega)^2, \mathbf{N}(0)\mathbf{v}(0) = 0\}$ .

The limit of (2.2) for  $\nu = 0^+$  is naturally written for  $\mathbf{u} \in H^1(\Omega)^2$  and  $\mathbf{v} \in Q \subset H^1(\Omega)^2$ :

$$\begin{aligned} & \text{Find } \mathbf{u} \in H^1(\Omega)^2 \text{ such that} \\ & b(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}) \quad \text{for all test functions } \mathbf{v} \in Q. \end{aligned} \quad (2.3)$$

where the sesquilinear form is

$$b(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left( \mathbf{u}' \cdot \bar{\mathbf{v}}' + \mathbf{u} \cdot \frac{\mathbf{N}}{\alpha} \bar{\mathbf{v}} \right) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \bar{\mathbf{v}} \right\}_{-1}^1, \quad \text{for } \mathbf{u} \in H^1(\Omega)^2, \mathbf{v} \in Q \quad (2.4)$$

and the antilinear form is the same

$$\ell = \ell^\nu \quad \text{on } Q. \quad (2.5)$$

**Remark 7.** The fact that (2.4) is well defined for  $\mathbf{u} \in H^1(\Omega)^2$  and  $\mathbf{v} \in Q$  is a consequence of Hardy's inequality [4]

$$\int_{\Omega} \frac{f^2(x)}{x^2} dx \leq 4 \int_{\Omega} f'^2(x) dx$$

for  $f$  a real-valued function of  $H^1(\Omega)$  vanishing at 0. It is sufficient to apply this inequality separately to the first and second components of the vector  $\mathbf{N}\mathbf{v}$  to show that  $\frac{\mathbf{N}}{\alpha}\mathbf{v} \in L^2(\Omega)^2$ .

It is easy to show that solutions  $\mathbf{u} = (e, b)^t \in H^1(\Omega)^2$  to (2.3) are in fact strong solutions except at  $x = 0$  of the following system of *four* second-order ODEs

$$\begin{cases} -e'' + \left(k_z^2 + \frac{\delta^2}{\alpha} - \alpha\right)e + \frac{\delta k_z}{\alpha}b = 0, & \text{in } (-1, 0) \text{ and } (0, 1), \\ -b'' + \frac{\delta k_z}{\alpha}e + \left(\frac{k_z^2}{\alpha} - 1\right)b = 0, & \text{in } (-1, 0) \text{ and } (0, 1), \end{cases} \quad (2.6)$$

for which we have two boundary conditions on the left at  $-1$  and two on the right at  $1$ . Now, because of the constraint imposed on the test function space  $Q$  it is only possible to show that *three* linear combination of the solutions, namely  $e$ ,  $b$  and  $k_z e' - \delta(0)b'$ , are in  $H^1(\Omega)$ , which yields three continuity relations at  $0$ . Thus we see that one constraint is missing to define uniquely a solution on  $\Omega$ . The problem here is similar to what is known for the  $X$ -mode equation corresponding to  $k_z = 0$ , as described in the introduction: for  $\nu = 0$  the system (2.6) completed with boundary conditions admits multiple solutions, see e.g. [9] or Proposition 31 below. In order to obtain the missing information (if there is one), we begin by gathering simple a priori bounds before passing to the limit.

## 2.1 A priori bounds

A priori bounds are derived in this section for the solution of problem (1.1)-(1.2) with positive viscosity  $\nu > 0$ . We remind that here  $\mathbf{u}^\nu = (e_y^\nu, b_y^\nu)^t$  consists of the second components of the electric and magnetic fields in the cold plasma model (1.5). As observed before the other components are easily recovered from these two ones, and we may point out that  $e_x^\nu = -i\frac{\delta}{\alpha+i\nu}e_y^\nu - i\frac{k_z}{\alpha+i\nu}b_y^\nu$ , so that a singularity of order  $1/\alpha$  is expected at the limit.

**Proposition 8.** *For  $\nu \in (0, 1]$ ,  $\sigma > 0$  and  $\mathbf{f}$  defined on  $\partial\Omega$  with values in  $\mathbb{C}^2$ , the weak formulation (2.1) of (1.1)-(1.2) has a unique solution in  $H^1(\Omega)^2$ . This solution is denoted  $\mathbf{u}^\nu$ .*

**Proof.** The right-hand side  $\ell^\nu$  is antilinear and continuous. Let us focus on the sesquilinear and continuous form  $b^\nu$ . From Proposition 3 we see that the real and imaginary parts of  $\frac{N^\nu}{\alpha+i\nu}$  are Hermitian. It follows that for  $\mathbf{v} = (v_1, v_2) \in H^1(\Omega)^2$ , the decomposition into real and imaginary parts of  $b^\nu(\mathbf{v}, \mathbf{v})$  writes

$$\begin{aligned} b^\nu(\mathbf{v}, \mathbf{v}) &= \int_{\Omega} (\mathbf{v}' \cdot \bar{\mathbf{v}}' + \mathbf{v} \cdot \frac{N^\nu \bar{\mathbf{v}}}{\alpha + i\nu}) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{v} \cdot \bar{\mathbf{v}} \right\}_{-1}^1 \\ &= \|\mathbf{v}'\|_{L^2}^2 + \int_{\Omega} \mathbf{v} \cdot \operatorname{Re} \left( \frac{N^\nu}{\alpha + i\nu} \right) \bar{\mathbf{v}} dx \\ &\quad + i \left( \int_{\Omega} \mathbf{v} \cdot \operatorname{Im} \left( \frac{N^\nu}{\alpha + i\nu} \right) \bar{\mathbf{v}} dx - \left\{ \sigma |v_1|^2 + \frac{|v_2|^2}{\sigma} \right\}_{-1}^1 \right). \end{aligned}$$

Since  $\alpha, \delta \in L^\infty(\Omega)$  and  $\nu > 0$ , it results that for a non-negative constant  $C_1 \geq 0$ ,

$$\begin{aligned} \operatorname{Re} b^\nu(\mathbf{v}, \mathbf{v}) &= \|\mathbf{v}'\|_{L^2}^2 + \int_{\Omega} \mathbf{v} \cdot \operatorname{Re} \left( \frac{N^\nu}{\alpha + i\nu} \right) \bar{\mathbf{v}} dx \\ &\geq \|\mathbf{v}'\|_{L^2}^2 - C_1 \|\mathbf{v}\|_{L^2}^2. \end{aligned} \quad (2.7)$$

One has

$$\begin{aligned} \operatorname{Im} b^\nu(\mathbf{v}, \mathbf{v}) &= \int_{\Omega} \mathbf{v} \cdot \operatorname{Im} \left( \frac{N^\nu}{\alpha + i\nu} \right) \bar{\mathbf{v}} dx - \left\{ \sigma |v_1|^2 + \frac{|v_2|^2}{\sigma} \right\}_{-1}^1 \\ &= \int_{\Omega} \frac{-\nu}{\alpha^2 + \nu^2} |(\delta, k_z)^t \cdot \mathbf{v}|^2 dx - \nu \|v_1\|_{L^2}^2 - \left\{ \sigma |v_1|^2 + \frac{|v_2|^2}{\sigma} \right\}_{-1}^1. \end{aligned} \quad (2.8)$$

Consider  $k_z \neq 0$ . In this case, one has the inequality

$$\operatorname{Im} b^\nu(\mathbf{v}, \mathbf{v}) \leq -C_2 \|\mathbf{v}\|_{L^2}^2$$

with a positive constant  $C_2 > 0$ . This way,

$$\begin{aligned} \operatorname{Re} \left( (C_2 + i(1 + C_1)) b^\nu(\mathbf{v}, \mathbf{v}) \right) &= C_2 \operatorname{Re} b^\nu(\mathbf{v}, \mathbf{v}) - (1 + C_1) \operatorname{Im} b^\nu(\mathbf{v}, \mathbf{v}) \\ &\geq C_2 (\|\mathbf{v}'\|_{L^2}^2 - C_1 \|\mathbf{v}\|_{L^2}^2) + (1 + C_1) C_2 \|\mathbf{v}\|_{L^2}^2 \\ &\geq C_2 \|\mathbf{v}\|_{H^1}^2. \end{aligned}$$

So the formulation (2.1) is coercive and the Lax-Milgram theorem [Corollary 5.8 in Brezis [4]] ensures that there exists a unique solution in  $H^1(\Omega)^2$ , denoted  $\mathbf{u}^\nu$ .

The second case is  $k_z = 0$ . The coercivity of  $\text{Im} b^\nu$  with respect to the second component  $u_2^\nu$  is lost. But it is not a problem because the system is decoupled in two scalar equations, as the matrix (1.6) is diagonal. The equation for  $u_1^\nu$  is still coercive, see computations above (2.7)-(2.8). The equation for  $u_2^\nu$  is the classical Helmholtz equation

$$-u_2^{\nu''} - u_2^\nu = 0 \quad \text{on } \Omega,$$

with dissipative boundary conditions, that admits a unique solution.  $\square$

**Lemma 9.** *There exists  $C > 0$  such that for all  $\nu \in (0, 1]$ ,  $\|\mathbf{u}^\nu\|_{H^1(\Omega)} \leq C$ .*

**Proof.** The proof is performed in two steps. Firstly we show that the boundary values  $\mathbf{u}^\nu(\pm 1)$  and  $\mathbf{u}^{\nu'}(\pm 1)$  are bounded uniformly with respect to  $\nu$ . This is the easy step. Secondly we show that these uniform bounds propagate inside  $\Omega$ .

**First step.** Taking the imaginary part of (2.1) with  $\mathbf{u} = \mathbf{v} = \mathbf{u}^\nu$ , it yields

$$\int_{\Omega} \mathbf{u}^\nu \cdot \text{Im} \left( \frac{1}{\alpha + i\nu} \mathbf{N}^\nu \right) \overline{\mathbf{u}^\nu} dx - \left\{ \sigma |u_1^\nu|^2 + \frac{1}{\sigma} |u_2^\nu|^2 \right\}_{-1}^1 = \text{Im} [\mathbf{f} \cdot \overline{\mathbf{u}^\nu}]_{-1}^1,$$

so that thanks to the dissipation property (1.8) and using  $\text{Im} [\mathbf{f} \cdot \overline{\mathbf{u}^\nu}]_{-1}^1 \geq -2[|\mathbf{f}| |\mathbf{u}^\nu|]_{-1}^1$ ,

$$\left\{ \sigma |u_1^\nu|^2 + \frac{1}{\sigma} |u_2^\nu|^2 \right\}_{-1}^1 - 2[|\mathbf{f}| |\mathbf{u}^\nu|]_{-1}^1 \leq 0.$$

This second-order polynomial on the four variables  $|u_1^\nu(-1)|$ ,  $|u_1^\nu(1)|$ ,  $|u_2^\nu(-1)|$  and  $|u_2^\nu(1)|$  has positive leading coefficients thus it can only be non-positive on a given compact set. And this compact set depends on the coefficients of the polynomial, which are  $\sigma$  and  $\mathbf{f}$ , but not on  $\nu$ .

**Second step.** One has

$$|\mathbf{u}^{\nu''}(x)| = \left| \frac{\mathbf{N}^\nu(x)}{\alpha(x) + i\nu} \mathbf{u}^\nu(x) \right| \leq \frac{C}{|x|} |\mathbf{u}^\nu(x)|, \quad x \neq 0, \quad (2.9)$$

with  $C > 0$  a positive constant depending only on  $\alpha$ ,  $\delta$  and  $k_z$ , so independent of  $\nu$  and  $x$ . Introduce the auxiliary function  $g$

$$\begin{cases} g''(x) &= \frac{C}{|x|} |\mathbf{u}^\nu(x)| \geq |\mathbf{u}^{\nu''}(x)| \quad \text{in } (-1, 0), \\ g'(-1) &= |\mathbf{u}^{\nu'}(-1)|, \\ g(-1) &= |\mathbf{u}^\nu(-1)|. \end{cases} \quad (2.10)$$

We notice that  $g'(-1)$  and  $g(-1)$  are bounded uniformly with respect to  $\nu$ . The functions  $g$ ,  $g'$  and  $g''$  are non-negative for  $-1 \leq x < 0$ . One has  $g'(x) = g'(-1) + \int_{-1}^x g''(y) dy$  so relation (2.9) ensures

$$g'(x) \geq |\mathbf{u}^{\nu'}(-1)| + \int_{-1}^x |\mathbf{u}^{\nu''}(y)| dy \geq |\mathbf{u}^{\nu'}(x)| \quad \text{in } (-1, 0). \quad (2.11)$$

Integrating a second times yields

$$g(x) = g(-1) + \int_{-1}^x g'(y) dy = |\mathbf{u}^\nu(-1)| + \int_{-1}^x g'(y) dy \geq |\mathbf{u}^\nu(-1)| + \int_{-1}^x |\mathbf{u}^{\nu'}(y)| dy \geq |\mathbf{u}^\nu(x)|, \quad (2.12)$$

for  $x$  in  $(-1, 0)$ . Use again (2.10) to get

$$g(x) \geq \frac{|x|}{C} g''(x) \quad \text{in } (-1, 0).$$

This last inequality is used to obtain a bound on  $g$ . Indeed  $(\ln g)'' = g''/g - (g'/g)^2 \leq g''/g \leq C/|x|$  for  $x$  in  $(-1, 0)$ . Since the primitive of  $1/x$  is the logarithm, which is an integrable function, a double integration on  $(-1, x)$  gives a  $L^\infty(-1, 0)$  bound on  $g$ . Therefore relation (2.12) guarantees

$$|\mathbf{u}^\nu(x)| \leq C \quad \text{in } (-1, 0), \quad \text{for } 0 < \nu \leq 1,$$

for a positive constant  $C > 0$  independent of  $\nu$ . And it follows

$$|\mathbf{u}^\nu(x)| \leq \tilde{C}(1 + |\ln|x||) \quad \text{in } (-1, 0), \quad \text{for } 0 < \nu \leq 1,$$

for another positive constant  $\tilde{C} > 0$  independent of  $\nu$ . Therefore  $\mathbf{u}^\nu$  is in fact bounded in  $H^1(-1, 0)^2$  independently of  $\nu$ . A similar bound uniform with respect to  $\nu$  holds in  $H^1(0, 1)^2$ . Finally since  $\mathbf{u}^\nu \in H^1(\Omega)$  is continuous at  $x = 0$ , it establishes

$$\|\mathbf{u}^\nu\|_{H^1(\Omega)} = \|\mathbf{u}^\nu\|_{H^1(-1,0)} + \|\mathbf{u}^\nu\|_{H^1(0,1)} \leq C$$

for a positive constant independent of  $\nu$ . The proof is ended.  $\square$

**Corollary 10.** *As  $\nu \rightarrow 0^+$  and up to a subsequence,  $\mathbf{u}^\nu$  admits a weak limit in  $H^1(\Omega)^2$ , that we denote  $\mathbf{u}^+$ . This limit is a solution of (2.3) for all  $\mathbf{v} \in Q$ .*

The goal is now to derive a variational formulation satisfied by  $\mathbf{u}^+$ . We will see that  $\mathbf{u}^+$  is actually the solution of a well-posed formulation, in the sense that it has a unique solution. Therefore  $\mathbf{u}^+$  will be the weak limit in  $H^1(\Omega)^2$  of the whole sequence  $\mathbf{u}^\nu$  as  $\nu \rightarrow 0^+$ , and not only of a subsequence.

**Remark 11.** *Choosing  $\nu < 0$  and  $\nu \rightarrow 0^-$  leads to another limit denoted  $\mathbf{u}^-$ . A priori  $\mathbf{u}^+ \neq \mathbf{u}^-$ . The analytical solution (4.8) at the end of this paper is an example where indeed  $\mathbf{u}^+ \neq \mathbf{u}^-$ .*

## 2.2 Manufactured solutions

In this Section we consider the diagonal coefficient of the dielectric tensor has a vanishing second order derivative at the resonance, that is  $\alpha''(0) = 0$  and  $\alpha(x) = rx + O(x^3)$ . This is only for the simplicity of notations, the general situation  $\alpha''(0) \neq 0$  is treated in Remark 14.

For  $\nu > 0$  define

$$\begin{aligned} \mathbf{w}_1^\nu &= \begin{pmatrix} \frac{i}{\delta} \left( 1 - \frac{k_z^2}{\alpha + i\nu} + \frac{k_z^2}{rx + i\nu} \right) \\ \frac{ik_z}{\alpha + i\nu} - \frac{ik_z}{rx + i\nu} \end{pmatrix}, \\ \mathbf{w}_2^\nu &= \begin{pmatrix} \delta(0) \\ k_z \end{pmatrix} \frac{i}{r} \left( \frac{\log(r^2 x^2 + \nu^2)}{2} - i \operatorname{atan}\left(\frac{rx}{\nu}\right) \right). \end{aligned} \quad (2.13)$$

These functions intend to approximate the electromagnetic field  $\mathbf{w}_1^\nu \approx (e_y^\nu, b_y^\nu)^t$  and  $\mathbf{w}_2^\nu \approx (b_z^\nu, e_z^\nu)^t = (e_y^{\nu'}, b_y^{\nu'})^t$  at the singularity. They are solutions of the non-homogeneous system

$$\begin{cases} -\mathbf{w}_2^{\nu'} + \frac{1}{\alpha + i\nu} \mathbf{N}^\nu \mathbf{w}_1^\nu &= \mathbf{z}_1^\nu, \\ \mathbf{w}_2^\nu - \mathbf{w}_1^{\nu'} &= \mathbf{z}_2^\nu, \end{cases} \quad \text{in } \Omega, \quad (2.14)$$

with right hand sides

$$\begin{aligned} \mathbf{z}_1^\nu &= \begin{pmatrix} \frac{i\delta}{\alpha + i\nu} - \frac{i\delta(0)}{rx + i\nu} + i \frac{k_z^2 - (\alpha + i\nu)}{\delta} \left( 1 - \frac{k_z^2}{\alpha + i\nu} + \frac{k_z^2}{rx + i\nu} \right) \\ 0 \end{pmatrix}, \\ \mathbf{z}_2^\nu &= \begin{pmatrix} \frac{i\delta(0)}{r} \left( \frac{\log(r^2 x^2 + \nu^2)}{2} - i \operatorname{atan}\left(\frac{rx}{\nu}\right) \right) + \frac{i\delta'}{\delta^2} \left( 1 - \frac{k_z^2}{\alpha + i\nu} + \frac{k_z^2}{rx + i\nu} \right) - \frac{ik_z^2}{\delta} \left( \frac{\alpha'}{(\alpha + i\nu)^2} - \frac{r}{(rx + i\nu)^2} \right) \\ \frac{ik_z \alpha'}{(\alpha + i\nu)^2} - \frac{ik_z r}{(rx + i\nu)^2} + \frac{ik_z}{r} \left( \frac{\log(r^2 x^2 + \nu^2)}{2} - i \operatorname{atan}\left(\frac{rx}{\nu}\right) \right) \end{pmatrix}. \end{aligned} \quad (2.15)$$

**Proposition 12.** *For  $\nu \in (0, 1]$ , the manufactured solution  $(\mathbf{w}_1^\nu, \mathbf{w}_2^\nu)$  and the right hand side  $(\mathbf{z}_1^\nu, \mathbf{z}_2^\nu)$  are bounded in  $L^2(\Omega)^2$  uniformly with respect to  $\nu$ .*

**Proof.** The non trivial part concerns  $1/(\alpha + i\nu) - 1/(rx + i\nu)$  and  $\alpha'/(\alpha + i\nu)^2 - r/(rx + i\nu)^2$ . First,

$$\left| \frac{1}{\alpha + i\nu} - \frac{1}{rx + i\nu} \right| \leq \left| \frac{\alpha - rx}{\alpha rx} \right| = O(1) \quad \text{for small } x, \quad (2.16)$$

because  $\alpha - rx = O(x^2)$  thanks to Assumption 1. So  $\frac{1}{\alpha + i\nu} - \frac{1}{rx + i\nu} \in L^\infty(\Omega)$  with a bound uniform with respect to  $\nu$ . For the second estimation,

$$\frac{\alpha'}{(\alpha + i\nu)^2} - \frac{r}{(rx + i\nu)^2} = \frac{\alpha' r^2 x^2 - \alpha^2 r + 2i\nu(\alpha' rx - \alpha r) - \nu^2(\alpha' - r)}{(\alpha + i\nu)^2 (rx + i\nu)^2}.$$



Assuming  $\alpha - rx = O(x^3)$  and keeping track of  $\nu$ ,

$$\left| \frac{\alpha' r^2 x^2 - \alpha^2 r}{(\alpha + i\nu)^2 (rx + i\nu)^2} \right| \leq \frac{|\alpha' r^2 x^2 - \alpha^2 r|}{(\alpha r x)^2} + \frac{O(\nu x^3)}{\nu \alpha (rx)^2} + \frac{O(\nu^2 x^2)}{\nu^2 (rx)^2} = O(1) \text{ for small } x \text{ and } \nu.$$

Each term is again in  $L^\infty(\Omega)$ , and the dependency on  $\nu$  is cancelled as it is of the same order at the numerator and denominator of each fraction.  $\square$

**Remark 13.** According to (2.14), we have the sharper bound  $\|\mathbf{w}_1^\nu\|_{H^1} \leq C$  for a positive constant  $C > 0$  independent of  $\nu$ .

**Remark 14.** General coefficients are  $\alpha = rx + px^2 + O(x^3)$  with  $p$  non necessarily zero. A solution is to replace in (2.15) occurrences of  $r/(rx + i\nu)^2$  by  $(r + 2px)/(rx + px^2 + \frac{p^2}{r}x^3 + i\nu)^2$ . Indeed one can check that

$$\left| \frac{\alpha'}{(\alpha + i\nu)^2} - \frac{r + 2px}{(rx + px^2 + \frac{p^2}{r}x^3 + i\nu)^2} \right| = O(1) \text{ for small } x \text{ and } \nu.$$

The  $p^2 x^3/r$  term is here to filter out the non zero root of  $rx + px^2$ .

**Lemma 15.** As  $\nu \rightarrow 0^+$ , the manufactured functions defined above admit the following limits in  $L^2(\Omega)^2$

$$\begin{aligned} \mathbf{w}_1^+ &= \begin{pmatrix} \frac{i}{\delta} \left(1 - \frac{k_z^2}{\alpha} + \frac{k_z^2}{rx}\right) \\ \frac{ik_z}{\alpha} - \frac{ik_z}{rx} \end{pmatrix}, \\ \mathbf{w}_2^+ &= \begin{pmatrix} \delta(0) \\ k_z \end{pmatrix} \frac{i}{r} (\log |rx| - i \operatorname{sign}(rx) \frac{\pi}{2}), \\ \mathbf{z}_1^+ &= \begin{pmatrix} \frac{i\delta}{\alpha} - \frac{i\delta(0)}{rx} + i \frac{k_z^2 - \alpha}{\delta} \left(1 - \frac{k_z^2}{\alpha} + \frac{k_z^2}{rx}\right) \\ 0 \end{pmatrix}, \\ \mathbf{z}_2^+ &= \begin{pmatrix} \frac{i\delta(0)}{r} (\log |rx| - i \operatorname{sign}(rx) \frac{\pi}{2}) + \frac{i\delta'}{\delta^2} \left(1 - \frac{k_z^2}{\alpha} + \frac{k_z^2}{rx}\right) - \frac{ik_z^2}{\delta} \left(\frac{\alpha'}{\alpha^2} - \frac{r}{(rx)^2}\right) \\ \frac{ik_z \alpha'}{\alpha^2} - \frac{ik_z r}{(rx)^2} + \frac{ik_z}{r} (\log |rx| - i \operatorname{sign}(rx) \frac{\pi}{2}) \end{pmatrix}. \end{aligned} \quad (2.17)$$

**Proof.** This is immediate using Proposition 12.  $\square$

**Remark 16.** To characterize  $\mathbf{u}^-$ , we could define a family of manufactured functions for a negative viscosity by the same formula and have similar  $\nu$ -independent bounds. The limits as  $\nu \rightarrow 0^-$  of  $\mathbf{w}_2^\nu$  and  $\mathbf{z}_2^\nu$  would then be different as  $\operatorname{atan}(rx/\nu) \xrightarrow{\nu \rightarrow 0^-} \operatorname{sign}(-rx) \frac{\pi}{2}$ .

### 2.3 An energy relation

As introduced at the beginning of the paper, a key observation is the energy identity (1.7). We make this remark instrumental in our context by considering specific quadratic forms associated to the Poynting vector of the scaled difference between the electromagnetic field and the corresponding manufactured solutions. This is performed introducing the space of non negative test-functions that do not vanish at the singularity location

$$\mathcal{C}_{0,+}^1(\Omega) = \{\psi \in \mathcal{C}_0^1(\Omega), \psi \geq 0, \psi(0) > 0\}. \quad (2.18)$$

This technical tool is essential in our method.

**Definition 17.** For  $\nu > 0$ ,  $\varphi \in \mathcal{C}_{0,+}^1(\Omega)$  and  $(\mathbf{u}, s) \in H^1(\Omega)^2 \times \mathbb{C}$ , set the quadratic form  $\mathcal{J}^\nu$

$$\mathcal{J}^\nu(\mathbf{u}, s) = -\operatorname{Im} \int_{\Omega} (\mathbf{u} - s\mathbf{w}_1^\nu) \cdot \overline{(\mathbf{u}' - s\mathbf{w}_2^\nu)} \varphi' dx + \operatorname{Im} \int_{\Omega} (s\mathbf{z}_1^\nu \cdot \overline{(\mathbf{u} - s\mathbf{w}_1^\nu)} - s\mathbf{z}_2^\nu \cdot \overline{(\mathbf{u}' - s\mathbf{w}_2^\nu)}) \varphi dx. \quad (2.19)$$

Denote the limit quadratic form  $\mathcal{J}^+ = \lim_{\nu \rightarrow 0^+} \mathcal{J}^\nu$  defined on the same space  $H^1(\Omega)^2 \times \mathbb{C}$ .

We will pass to the limit in  $\mathcal{J}^\nu(\mathbf{u}^\nu, s)$  as  $\nu \rightarrow 0^+$ . In this direction it will be imperative in our analysis to establish that the quantities arising from (2.19)

$$\mathbf{u}^\nu \cdot \overline{\mathbf{u}^{\nu'}} \varphi', \quad \mathbf{w}_1^\nu \cdot \overline{\mathbf{u}^{\nu'}} \varphi', \quad \mathbf{u}^\nu \cdot \overline{\mathbf{w}_2^\nu} \varphi', \quad \mathbf{u}^\nu \cdot \overline{\mathbf{z}_1^\nu} \varphi, \quad \mathbf{w}_1^\nu \cdot \overline{\mathbf{z}_1^\nu} \varphi, \quad \mathbf{u}^{\nu'} \cdot \overline{\mathbf{z}_2^\nu} \varphi, \quad \mathbf{w}_2^\nu \cdot \overline{\mathbf{z}_2^\nu} \varphi,$$

are bounded in  $L^1(\Omega)$  independently of  $\nu$ .

**Proposition 18.** For  $\nu > 0$ ,  $s \in \mathbb{C}$  and  $\mathbf{u}^\nu$  the solution considered in Proposition 8, the following identity is verified

$$\mathcal{J}^\nu(\mathbf{u}^\nu, s) = \nu \int_{\Omega} \left( \frac{1}{\alpha^2 + \nu^2} |(\delta, k_z)^t \cdot (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)|^2 + |(1, 0)^t \cdot (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)|^2 \right) \varphi dx \geq 0. \quad (2.20)$$

Moreover, up to a subsequence,

$$\begin{aligned} \lim_{\nu \rightarrow 0^+} \mathcal{J}^\nu(\mathbf{u}^\nu, s) &= \mathcal{J}^+(\mathbf{u}^+, s) \\ &= \frac{\pi}{|r|} |(\delta(0), k_z)^t \cdot (\mathbf{u}^+(0) - s\mathbf{w}_1^+(0))|^2 \varphi(0). \end{aligned} \quad (2.21)$$

**Proof.** As  $\mathbf{u}^\nu$  verifies (1.1),  $(\mathbf{w}_1^\nu, \mathbf{w}_2^\nu)$  verifies (2.14), it follows

$$\begin{cases} -(\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'})' + \frac{\mathbf{N}^\nu}{\alpha + i\nu} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) &= -s\mathbf{z}_1^\nu, \\ (\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'}) - (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)' &= -s\mathbf{z}_2^\nu. \end{cases}$$

Since  $\varphi$  is compactly supported in  $\Omega$ , integrating by parts and elementary manipulations give that

$$\begin{aligned} -\operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \cdot \overline{(\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'})} \varphi' dx &= \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \overline{(\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'})'} dx \\ &\quad + \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)' \varphi \cdot \overline{(\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'})} dx \\ &= \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \overline{\frac{\mathbf{N}^\nu}{\alpha + i\nu} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)} dx \\ &\quad + \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \overline{s\mathbf{z}_1^\nu} dx \\ &\quad - \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)' \varphi \cdot \overline{s\mathbf{z}_2^\nu} dx \\ &= \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \overline{\frac{\mathbf{N}^\nu}{\alpha + i\nu} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)} dx \\ &\quad - \operatorname{Im} \int_{\Omega} \left( \overline{(\mathbf{u}^\nu - s\mathbf{w}_1^\nu)} \cdot s\mathbf{z}_1^\nu - \overline{(\mathbf{u}^{\nu'} - s\mathbf{w}_2^{\nu'})} \cdot s\mathbf{z}_2^\nu \right) \varphi dx. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{J}^\nu(\mathbf{u}^\nu, s) &= \operatorname{Im} \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \cdot \overline{\frac{\mathbf{N}^\nu}{\alpha + i\nu} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)} \varphi dx \\ &= - \int_{\Omega} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \cdot \operatorname{Im} \left( \frac{\mathbf{N}^\nu}{\alpha + i\nu} \right) \overline{(\mathbf{u}^\nu - s\mathbf{w}_1^\nu)} \varphi dx \\ &= \nu \int_{\Omega} \left( \frac{1}{\alpha^2 + \nu^2} |(\delta, k_z)^t \cdot (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)|^2 + |(1, 0)^t \cdot (\mathbf{u}^\nu - s\mathbf{w}_1^\nu)|^2 \right) \varphi dx. \end{aligned} \quad (2.22)$$

The second part of the integral is bounded by  $\nu$  and converges towards 0. To tackle the first part with coefficient  $\nu/(\alpha^2 + \nu^2)$ , observe that there exists a continuous function  $\epsilon$  defined on  $\mathbb{R}_+$  and vanishing at 0, such that

$$\nu \int_{\Omega} \frac{dx}{\alpha^2 + \nu^2} = \frac{1}{|r|} \int_{-|r|/\nu}^{|r|/\nu} \frac{dx}{x^2 + 1} + \epsilon(\nu) \xrightarrow{\nu \rightarrow 0^+} \frac{\pi}{|r|}.$$

Since  $\mathbf{u}^\nu$  is bounded in  $H^1(\Omega)^2$  uniformly with respect to  $\nu$ , there exists  $C > 0$  independent of  $\nu$  such that

$$|\mathbf{u}^\nu(0) - \mathbf{u}^+(0)|^2 = \int_0^1 ((1-x)|\mathbf{u}^\nu - \mathbf{u}^+|^2)' dx \leq 3\|\mathbf{u}^\nu - \mathbf{u}^+\|_{L^2} \|\mathbf{u}^\nu - \mathbf{u}^+\|_{H^1} \leq C\|\mathbf{u}^\nu - \mathbf{u}^+\|_{L^2}.$$

Up to a subsequence the right hand side tends to 0, therefore  $\mathbf{u}^\nu(0) \rightarrow \mathbf{u}^+(0)$  as  $\nu \rightarrow 0^+$ . The same reasons imply  $\mathbf{w}_1^\nu(0) \rightarrow \mathbf{w}_1^+(0)$  as  $\nu \rightarrow 0^+$ .

Consequently  $\mathcal{J}^\nu(\mathbf{u}^\nu, s)$  converges towards  $\frac{\pi}{|r|} |(\delta(0), k_z)^t \cdot (\mathbf{u}^+(0) - s\mathbf{w}_1^+(0))|^2 \varphi(0)$ , and the result is proven.  $\square$

**Lemma 19.** For  $\mathbf{u} \in H^1(\Omega)^2$ , the quadratic forms can be expanded as second order polynomials with respect to  $s \in \mathbb{C}$ . One has

$$\begin{aligned} \mathcal{J}^\nu(\mathbf{u}, s) &= -\operatorname{Im} \int_{\Omega} \mathbf{u} \cdot \overline{\mathbf{u}'} \varphi \, dx + \operatorname{Im} \left( s \int_{\Omega} ((\overline{\mathbf{u}'} \cdot \mathbf{w}_1^\nu - \overline{\mathbf{u}} \cdot \mathbf{w}_2^\nu) \varphi' + (\overline{\mathbf{u}} \cdot \mathbf{z}_1^\nu - \overline{\mathbf{u}'} \cdot \mathbf{z}_2^\nu) \varphi) \, dx \right) \\ &\quad - |s|^2 \int_{\Omega} \mathbf{w}_1^\nu \cdot \operatorname{Im} \left( \frac{1}{\alpha + i\nu} \mathbf{N}^\nu \right) \overline{\mathbf{w}_1^\nu} \varphi \, dx, \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} \mathcal{J}^+(\mathbf{u}, s) &= -\operatorname{Im} \int_{\Omega} \mathbf{u} \cdot \overline{\mathbf{u}'} \varphi \, dx + \operatorname{Im} \left( s \int_{\Omega} ((\overline{\mathbf{u}'} \cdot \mathbf{w}_1^+ - \overline{\mathbf{u}} \cdot \mathbf{w}_2^+) \varphi' + (\overline{\mathbf{u}} \cdot \mathbf{z}_1^+ - \overline{\mathbf{u}'} \cdot \mathbf{z}_2^+) \varphi) \, dx \right) \\ &\quad + |s|^2 \frac{\pi \varphi(0)}{|r|}. \end{aligned} \quad (2.24)$$

**Proof.** Develop the formula (2.19) with respect to  $s$

$$\begin{aligned} \mathcal{J}^\nu(\mathbf{u}, s) &= -\operatorname{Im} \int_{\Omega} \mathbf{u} \cdot \overline{\mathbf{u}'} \varphi \, dx + \operatorname{Im} s \int_{\Omega} ((\overline{\mathbf{u}'} \cdot \mathbf{w}_1^\nu - \overline{\mathbf{u}} \cdot \mathbf{w}_2^\nu) \varphi' + (\overline{\mathbf{u}} \cdot \mathbf{z}_1^\nu - \overline{\mathbf{u}'} \cdot \mathbf{z}_2^\nu) \varphi) \, dx \\ &\quad - |s|^2 \operatorname{Im} \int_{\Omega} (\mathbf{w}_1^\nu \cdot \overline{\mathbf{w}_2^\nu} \varphi' + (\mathbf{z}_1^\nu \cdot \overline{\mathbf{w}_1^\nu} - \mathbf{z}_2^\nu \cdot \overline{\mathbf{w}_2^\nu}) \varphi) \, dx. \end{aligned}$$

As in the proof of Proposition 18, it follows by an integration by parts that

$$\operatorname{Im} \int_{\Omega} (\mathbf{w}_1^\nu \cdot \overline{\mathbf{w}_2^\nu} \varphi' + (\mathbf{z}_1^\nu \cdot \overline{\mathbf{w}_1^\nu} - \mathbf{z}_2^\nu \cdot \overline{\mathbf{w}_2^\nu}) \varphi) \, dx = \int_{\Omega} \mathbf{w}_1^\nu \cdot \operatorname{Im} \left( \frac{\mathbf{N}^\nu}{\alpha + i\nu} \right) \overline{\mathbf{w}_1^\nu} \varphi \, dx, \quad (2.25)$$

which yields relation (2.23). Again, according to the definition of  $\mathbf{w}_1^+$  and  $\mathbf{w}_2^+$ , when  $\nu$  goes to  $0^+$

$$\begin{aligned} \int_{\Omega} \mathbf{w}_1^\nu \cdot \operatorname{Im} \left( \frac{\mathbf{N}^\nu}{\alpha + i\nu} \right) \overline{\mathbf{w}_1^\nu} \varphi \, dx &= -\nu \int_{\Omega} \left( \frac{1}{\alpha^2 + \nu^2} |(\delta, k_z)^t \cdot \mathbf{w}_1^\nu|^2 + |(1, 0)^t \cdot \mathbf{w}_1^\nu|^2 \right) \varphi \, dx \\ &\rightarrow \frac{\pi}{|r|} |(\delta(0), k_z)^t \cdot \mathbf{w}_1^+(0)|^2 \varphi(0) \end{aligned} \quad (2.26)$$

as detailed at the end of the proof of Proposition 18. Finally, see (2.17),  $\mathbf{w}_1^+(0) = (\frac{i}{\delta(0)}, 0)^t$  so (2.24) is established.  $\square$

Another integral relation, which will be used in the numerical section, is the following.

**Proposition 20.** The limit solution satisfies a second integral relation: for any  $\varphi \in C_0^1(\Omega)$ , it holds

$$\int_{\Omega} (\mathbf{u}^+ \cdot \mathbf{w}_2^+ - \mathbf{u}^{+'} \cdot \mathbf{w}_1^+) \varphi' \, dx = \int_{\Omega} (\mathbf{u}^+ \cdot \mathbf{z}_1^+ - \mathbf{u}^{+'} \cdot \mathbf{z}_2^+) \varphi \, dx. \quad (2.27)$$

*Proof.* For  $\nu > 0$ , using again (2.1) and (2.14)

$$\begin{aligned} \int_{\Omega} (\mathbf{u}^\nu \cdot \mathbf{w}_2^\nu - \mathbf{u}^{\nu'} \cdot \mathbf{w}_1^\nu) \varphi' \, dx &= \int_{\Omega} ((\mathbf{u}^\nu \varphi)' \cdot \mathbf{w}_2^\nu - \mathbf{u}^{\nu'} \varphi \cdot \mathbf{w}_2^\nu) \, dx \\ &\quad - \int_{\Omega} (\mathbf{u}^{\nu'} \cdot (\mathbf{w}_1^\nu \varphi)' - \mathbf{u}^{\nu'} \cdot \mathbf{w}_1^{\nu'} \varphi) \, dx \\ &= \int_{\Omega} (\mathbf{u}^\nu \varphi \cdot \left( -\frac{\mathbf{N}^\nu}{\alpha + i\nu} \mathbf{w}_1^\nu + \mathbf{z}_1^\nu \right) - \mathbf{u}^{\nu'} \varphi \cdot \mathbf{w}_2^\nu) \, dx \\ &\quad - \int_{\Omega} \left( -\frac{\mathbf{N}^\nu}{\alpha + i\nu} \mathbf{u}^\nu \cdot (\mathbf{w}_1^\nu \varphi) - \mathbf{u}^{\nu'} \cdot \mathbf{w}_1^{\nu'} \varphi \right) \, dx \\ &= \int_{\Omega} (\mathbf{u}^\nu \varphi \cdot \mathbf{z}_1^\nu - \mathbf{u}^{\nu'} \varphi \cdot \mathbf{z}_2^\nu) \, dx. \end{aligned}$$

Now, up to a subsequence, as  $\mathbf{u}^\nu$  converges towards  $\mathbf{u}^+$  in  $H^1(\Omega)^2$  and the manufactured functions converge in  $L^2(\Omega)^2$ , (2.27) is obtained passing to the limit  $\nu \rightarrow 0^+$ .  $\square$

### 3 A mixed variational formulation for the limit problem

The minimization of the quantity  $\mathcal{J}^+$  on the product space of weak solutions of (1.2)-(1.9) and of complex scalars yields a mixed variational formulation in the Hilbert spaces equipped with natural norm

$$\begin{aligned} V &= H^1(\Omega)^2 \times \mathbb{C}, & Q &= \{ \mathbf{v} \in H^1(\Omega)^2, \mathbf{N}(0)\mathbf{v}(0) = 0 \}, \\ \|(\mathbf{u}, s)\|_V &= \|\mathbf{u}\|_{H^1(\Omega)^2} + |s|, & \|\mathbf{v}\|_Q &= \|\mathbf{v}\|_{H^1(\Omega)^2}. \end{aligned}$$

We extend the form  $b$  defined in (2.4) from  $H^1(\Omega)^2 \times Q$  to  $V \times Q$  by

$$b((\mathbf{u}, s), \boldsymbol{\lambda}) = \int_{\Omega} (\mathbf{u}' \cdot \bar{\boldsymbol{\lambda}}' + \mathbf{u} \cdot \frac{\mathbf{N}\bar{\boldsymbol{\lambda}}}{\alpha}) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \bar{\boldsymbol{\lambda}} \right\}_{-1}^1, \text{ for } (\mathbf{u}, s) \in V \text{ and } \boldsymbol{\lambda} \in Q, \quad (3.1)$$

and recall that  $\ell$  is the antilinear form such that for all  $\boldsymbol{\lambda} \in Q$ ,  $\ell(\boldsymbol{\lambda}) = [\mathbf{f} \cdot \bar{\boldsymbol{\lambda}}]_{-1}^1$ .

We are able to write the Lagrangian associated to the minimization of  $\mathcal{J}^+$  on the space of weak solutions of (1.2)-(1.9).

**Definition 21.** For  $(\mathbf{u}, s) \in V$  and  $\boldsymbol{\lambda} \in Q$ , let  $\mathcal{L}^+$  be defined as

$$\mathcal{L}^+((\mathbf{u}, s), \boldsymbol{\lambda}) = \mathcal{J}^+(\mathbf{u}, s) + \text{Im} (b((\mathbf{u}, s), \boldsymbol{\lambda}) - \ell(\boldsymbol{\lambda})). \quad (3.2)$$

### 3.1 Euler-Lagrange equations and main Theorem

The Euler-Lagrange equations associated to the extremalization of  $\mathcal{L}^+$  are

$$\begin{cases} d\mathcal{J}_{(\mathbf{u}, s)}^+(\mathbf{v}, t) + \text{Im} b((\mathbf{v}, t), \boldsymbol{\lambda}) = 0, & \forall (\mathbf{v}, t) \in V, \\ \text{Im} b((\mathbf{u}, s), \boldsymbol{\mu}) = \text{Im} \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q. \end{cases} \quad (3.3)$$

Let  $a^+$  be the sesquilinear form defined by  $\text{Im} a^+((\mathbf{u}, s), (\mathbf{v}, t)) = d\mathcal{J}_{(\mathbf{u}, s)}^+(\mathbf{v}, t)$  for all  $(\mathbf{u}, s), (\mathbf{v}, t) \in H$ .

**Lemma 22.** One has for  $(\mathbf{u}, s), (\mathbf{v}, t) \in H$

$$\begin{aligned} a^+((\mathbf{u}, s), (\mathbf{v}, t)) &= \int_{\Omega} (\bar{\mathbf{v}} \cdot \mathbf{u}' - \mathbf{u} \cdot \bar{\mathbf{v}}') \varphi' dx - s \int_{\Omega} ((\mathbf{w}_2^+ \cdot \bar{\mathbf{v}} - \mathbf{w}_1^+ \cdot \bar{\mathbf{v}}') \varphi' + (\mathbf{z}_2^+ \cdot \bar{\mathbf{v}}' - \mathbf{z}_1^+ \cdot \bar{\mathbf{v}}) \varphi) dx \\ &\quad + \bar{t} \int_{\Omega} ((\bar{\mathbf{w}}_2^+ \cdot \mathbf{u} - \bar{\mathbf{w}}_1^+ \cdot \mathbf{u}') \varphi' + (\bar{\mathbf{z}}_2^+ \cdot \mathbf{u}' - \bar{\mathbf{z}}_1^+ \cdot \mathbf{u}) \varphi) dx + 2 \frac{\pi \varphi(0)}{|r|} i s \bar{t}. \end{aligned} \quad (3.4)$$

*Proof.* Differentiate (2.24) to get

$$\begin{aligned} d\mathcal{J}_{(\mathbf{u}, s)}^+(\mathbf{v}, t) &= -\text{Im} \int_{\Omega} (\mathbf{u} \cdot \bar{\mathbf{v}}' + \mathbf{v} \cdot \bar{\mathbf{u}}') \varphi dx + \text{Im} \left( s \int_{\Omega} ((\bar{\mathbf{v}}' \cdot \mathbf{w}_1^+ - \bar{\mathbf{v}} \cdot \mathbf{w}_2^+) \varphi' + (\bar{\mathbf{v}} \cdot \mathbf{z}_1^+ - \bar{\mathbf{v}}' \cdot \mathbf{z}_2^+) \varphi) dx \right) \\ &\quad + \text{Im} \left( t \int_{\Omega} ((\bar{\mathbf{u}}' \cdot \mathbf{w}_1^+ - \bar{\mathbf{u}} \cdot \mathbf{w}_2^+) \varphi' + (\bar{\mathbf{u}} \cdot \mathbf{z}_1^+ - \bar{\mathbf{u}}' \cdot \mathbf{z}_2^+) \varphi) dx \right) + 2 \text{Re}(s \bar{t}) \frac{\pi \varphi(0)}{|r|}. \end{aligned}$$

Defining  $a^+$  the sesquilinear form such that  $\text{Im} a^+((\mathbf{u}, s), (\mathbf{v}, t)) = d\mathcal{J}_{(\mathbf{u}, s)}^+(\mathbf{v}, t)$ , it yields (3.4).  $\square$

Our problem (3.3) can be recast as

$$\begin{aligned} &\text{Find } ((\mathbf{u}, s), \boldsymbol{\lambda}) \in V \times Q \text{ such that} \\ &\begin{cases} \text{Im} a^+((\mathbf{u}, s), (\mathbf{v}, t)) + \text{Im} b((\mathbf{v}, t), \boldsymbol{\lambda}) = 0, & \forall (\mathbf{v}, t) \in V, \\ \text{Im} b((\mathbf{u}, s), \boldsymbol{\mu}) = \text{Im} \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q. \end{cases} \end{aligned} \quad (3.5)$$

The interest of this formulation is that it fits into the frame of classical mixed variational formulations, see [2]. We are now able to state the main result of this paper.

**Theorem 23.** The variational formulation (3.5) is equivalent to

$$\begin{aligned} &\text{Find } ((\mathbf{u}, s), \boldsymbol{\lambda}) \in V \times Q \text{ such that} \\ &\begin{cases} a^+((\mathbf{u}, s), (\mathbf{v}, t)) - \overline{b((\mathbf{v}, t), \boldsymbol{\lambda})} = 0, & \forall (\mathbf{v}, t) \in V, \\ b((\mathbf{u}, s), \boldsymbol{\mu}) = \ell(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q. \end{cases} \end{aligned} \quad (3.6)$$

There exists a unique solution in the space  $V \times Q$ . In addition, the first component of the solution is the limit  $\mathbf{u}^+$  defined in Corollary 10.

*Proof.* The equivalence between (3.6) and (3.5) is immediate since  $\text{Im} b = -\text{Im} \bar{b}$  and that  $V$  and  $Q$  are both complex-valued. The second part, namely the well-posedness, is proven in Section 3.2. The last part is proven in Section 3.3, where the expressions of  $s^+$  and  $\boldsymbol{\lambda}^+$  are also precised.  $\square$

**Corollary 24.** The whole sequence  $\mathbf{u}^\nu$  converges weakly in  $H^1(\Omega)^2$  towards  $\mathbf{u}^+$  as  $\nu \rightarrow 0^+$ .

### 3.2 Proof of the second part of the main Theorem

First of all, note that problem (3.6) can be reformulated using the operators  $A^+ : V \rightarrow V'$ ,  $B : V \rightarrow Q'$  and  $h \in Q'$  such that for all  $(\mathbf{u}, s), (\mathbf{v}, t) \in V$  and  $\boldsymbol{\lambda} \in Q$ ,

$$(A^+(\mathbf{u}, s), (\mathbf{v}, t))_{V', V} = a^+((\mathbf{u}, s), (\mathbf{v}, t)), \quad (B(\mathbf{u}, s), \boldsymbol{\lambda})_{Q', Q} = b((\mathbf{u}, s), \boldsymbol{\lambda}), \quad (h, \boldsymbol{\lambda})_{Q', Q} = \ell(\boldsymbol{\lambda}). \quad (3.7)$$

The variational problem (3.6) then writes

$$\begin{aligned} & \text{Find } ((\mathbf{u}, s), \boldsymbol{\lambda}) \in V \times Q \text{ such that} \\ & \begin{cases} A^+(\mathbf{u}, s) - \overline{B^t \boldsymbol{\lambda}} = 0, & \text{in } V', \\ B(\mathbf{u}, s) = h, & \text{in } Q'. \end{cases} \end{aligned} \quad (3.8)$$

Set  $K = \ker B$ . Define the *restriction* of  $A^+$  on  $K \subset V$  with values in  $K'$  which contains  $V'$ . This operator  $A_{KK'}^+ : K \rightarrow K'$  is such that for all  $(\mathbf{u}, s), (\mathbf{v}, t) \in K$ ,  $(A_{KK'}^+(\mathbf{u}, s), (\mathbf{v}, t))_{K', K} = (A^+(\mathbf{u}, s), (\mathbf{v}, t))_{V', V}$ . An important well-posedness result for mixed systems we will use is the following.

**Theorem 25** (Theorem 4.2.2 in Boffi-Brezzi-Fortin [2]). *The mixed system*

$$\begin{aligned} & \text{Find } ((\mathbf{u}, s), \boldsymbol{\lambda}) \in V \times Q \text{ such that} \\ & \begin{cases} A^+(\mathbf{u}, s) - \overline{B^t \boldsymbol{\lambda}} = \kappa, & \text{in } V', \\ B(\mathbf{u}, s) = h, & \text{in } Q'. \end{cases} \end{aligned} \quad (3.9)$$

admits a unique solution for any  $\kappa \in V'$  and  $h \in Q'$  if and only if  $A_{KK'}^+$  is bijective and  $B$  is onto.

The verification of these conditions for our problem is given in three steps. Firstly we characterize  $K$ , then we prove that  $A_{KK'}^+$  is bijective and finally we prove that  $B$  is onto. Therefore it proves the first part of Theorem 23.

**Proposition 26.** *There exists  $\mathbf{v}, \mathbf{w} \in H^1(\Omega)^2$  such that a basis of  $K$  is  $\mathcal{B}_K = \{(\mathbf{v}, 0), (\mathbf{w}, 0), (0, 1)\}$  with  $\mathbf{v}(\pm 1) \neq 0$  and  $\mathbf{w}(\pm 1) \neq 0$ . In particular  $\dim(K) = 3$ .*

**Proof.** Since the bilinear form  $b$  defined in (3.1) has no dependence with respect to the scalar  $s$ , the space spanned by  $(0, 1)$  is in  $K$ . Let us now consider  $(\mathbf{u}, 0)$  belonging to  $K \subset H^1(\Omega)^2 \times \mathbb{C}$ . The function  $\mathbf{u}$  is a continuous function, and the continuity in 0 will have its importance in the sequel.

The claim will be proved if and only if we can show that such functions span a vectorial space of dimension 2. This part of the proof is as follows. For any  $\varepsilon \in (0, 1)$ , one has that  $\mathbf{u} \in H^2(-1, -\varepsilon)^2 \cup H^2(\varepsilon, 1)^2$  and that outside of 0,  $-\mathbf{u}'' + \frac{\mathbf{N}}{\alpha} \mathbf{u} = 0$ . Also, it verifies the boundary conditions

$$\mathbf{u}'(\pm 1) \mp \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u}(\pm 1) = 0.$$

Define  $\mathbf{v}_L$  and  $\mathbf{w}_L$  the solutions to the Cauchy problems associated to that ODE on  $(-1, 0)$  for the boundary conditions

$$\begin{cases} \mathbf{v}_L(-1) = (1, 0)^t, \\ \mathbf{v}'_L(-1) = (-i\sigma, 0)^t, \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{w}_L(-1) = (0, 1)^t, \\ \mathbf{w}'_L(-1) = (0, -i/\sigma)^t. \end{cases}$$

Similarly, define  $\mathbf{v}_R$  and  $\mathbf{w}_R$  the solutions to the Cauchy problems associated to that ODE on  $(0, 1)$  for the boundary conditions

$$\begin{cases} \mathbf{v}_R(1) = (1, 0)^t, \\ \mathbf{v}'_R(1) = (i\sigma, 0)^t, \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{w}_R(1) = (0, 1)^t, \\ \mathbf{w}'_R(1) = (0, i/\sigma)^t. \end{cases}$$

Then  $\mathbf{u}$  is a complex linear combination of  $\mathbf{v}_L$  and  $\mathbf{w}_L$  on  $(-1, 0)$ , and of  $\mathbf{v}_R$  and  $\mathbf{w}_R$  on  $(0, 1)$ .

These solutions to Cauchy problems on the left and right hand sides can all be extended continuously in 0, reasoning as in the proof of Proposition 8 for the  $H^1(-1, 0)^2$  and  $H^1(0, 1)^2$  bounds for  $\mathbf{u}^\nu$ , using the ODE on  $(-1, 0)$  and on  $(0, 1)$  respectively.

On the left, it defines an operator  $\phi_L : \mathbf{c} \in \mathbb{C}^2 \mapsto \mathbf{u}(0^-) \in \mathbb{C}^2$  by solving the Cauchy problem

$$\begin{cases} -\mathbf{u}'' + \frac{\mathbf{N}}{\alpha} \mathbf{u} = 0, & \text{in } (-1, 0), \\ \mathbf{u}(-1) = \mathbf{c}, \\ \mathbf{u}'(-1) = - \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{c}. \end{cases}$$

On the right, it defines an operator  $\phi_R : \mathbf{d} \in \mathbb{C}^2 \mapsto \mathbf{u}(0^+) \in \mathbb{C}^2$  by solving the Cauchy problem

$$\begin{cases} -\mathbf{u}'' + \frac{\mathbf{N}}{\alpha} \mathbf{u} = 0, & \text{in } (0, 1), \\ \mathbf{u}(1) = \mathbf{d}, \\ \mathbf{u}'(1) = \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{d}. \end{cases} \quad (3.10)$$

The condition that  $\mathbf{u} \in H^1(\Omega)^2$  is now equivalent to the continuity condition  $\mathbf{u}(0^-) = \mathbf{u}(0^+)$ , that is  $T(\mathbf{u}(-1), \mathbf{u}(1)) = 0$  where the linear mapping  $T : \mathbb{C}^4 \rightarrow \mathbb{C}^2$  is defined by  $T(\mathbf{c}, \mathbf{d}) = \phi_L(\mathbf{c}) - \phi_R(\mathbf{d})$ .

It is in fact easy to show that the dimension of the range of  $T$  is equal to 2: a sufficient and simpler condition is to show that the dimension of the range of  $\phi_R$  is also equal to 2, and this is equivalent to saying that  $\phi_R$  is one-to-one. The condition  $\phi_R(\mathbf{d}) = 0$  is equivalent to say that  $\mathbf{u}$  in (3.10) is such that  $\mathbf{u}(0^+) = 0$ . For  $\varepsilon \in (0, 1)$ , integrating the equation against  $\bar{\mathbf{u}}$  on  $(\varepsilon, 1)$  yields

$$\int_{\varepsilon}^1 (|\mathbf{u}'|^2 + \mathbf{u} \cdot \frac{\mathbf{N}}{\alpha} \bar{\mathbf{u}}) dx - \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{d} \cdot \bar{\mathbf{d}} + \mathbf{u}'(\varepsilon) \cdot \overline{\mathbf{u}(\varepsilon)} = 0.$$

Refer again to the proof of Proposition 8 to get  $|\mathbf{u}'(\varepsilon)| \leq C(1 + |\ln \varepsilon|) = C(1 - \ln \varepsilon)$ . One has

$$|\mathbf{u}(\varepsilon)| \leq \int_0^{\varepsilon} |\mathbf{u}'(y)| dy \leq C \int_0^{\varepsilon} (1 - \ln y) dy = C\varepsilon(2 - \ln \varepsilon).$$

So one can pass to the limit  $\mathbf{u}'(\varepsilon) \cdot \overline{\mathbf{u}(\varepsilon)} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . It yields

$$\int_0^1 (|\mathbf{u}'|^2 + \mathbf{u} \cdot \frac{\mathbf{N}}{\alpha} \bar{\mathbf{u}}) dx - \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{d} \cdot \bar{\mathbf{d}} = 0,$$

which is well-defined thanks to Hardy's inequality. Taking the imaginary part, it yields  $\mathbf{d} = 0$ . So  $\phi_R$  is one-to-one, and bijective. A similar computation establishes  $\phi_L$  is bijective. So the dimension of the range of  $\phi_R$  is equal to 2, and the dimensions of the range and kernel of  $T$  are also equal to 2.

Take  $(\mathbf{v}, 0)$  and  $(\mathbf{w}, 0) \in K$  defined such that  $\mathbf{v}(-1) = (1, 0)^t$  and  $\mathbf{w}(-1) = (0, 1)^t$ . These functions coincide with  $\mathbf{v}_L$  and  $\mathbf{w}_L$  respectively on the interval  $(-1, 0)$ , and they span a 2 dimensional subspace of  $K$ . Necessarily  $\phi_R(\mathbf{v}(1)) = \phi_L(\mathbf{v}(-1)) \neq 0$  so  $\mathbf{v}(1) \neq 0$ . For the same reason  $\mathbf{w}(1) \neq 0$ . The claim is proved.  $\square$

**Proposition 27.**  $A_{KK'}^+$  is a bijection between  $K$  and  $K'$ .

**Proof.** The space  $K$  being of finite dimension, it is sufficient to prove  $A_{KK'}^+$  is one-to-one. Consider the basis  $\mathcal{B}_K$  of  $K$  defined in Proposition 26. The operator  $A_{KK'}^+$  is associated to a matrix  $M$

$$M = \begin{pmatrix} \int (\mathbf{v}' \cdot \bar{\mathbf{v}} - \mathbf{v} \cdot \bar{\mathbf{v}}') \varphi' & \int (\mathbf{w}' \cdot \bar{\mathbf{v}} - \mathbf{w} \cdot \bar{\mathbf{v}}') \varphi' & a_1 \\ \int (\mathbf{v}' \cdot \bar{\mathbf{w}} - \mathbf{v} \cdot \bar{\mathbf{w}}') \varphi' & \int (\mathbf{w}' \cdot \bar{\mathbf{w}} - \mathbf{w} \cdot \bar{\mathbf{w}}') \varphi' & a_2 \\ -\bar{a}_1 & -\bar{a}_2 & 2i\pi \frac{\varphi(0)}{|r|} \end{pmatrix}$$

for  $a_1, a_2 \in \mathbb{C}$  two given scalars, see (3.4). Let  $(c_1, c_2, c_3) \in \mathbb{C}^3$  be in the kernel of this matrix. In particular

$$\overline{(c_1, c_2, -c_3)} M (c_1, c_2, c_3)^t = 0.$$

That is

$$|c_1|^2 \operatorname{Im} \int_{\Omega} \mathbf{v}' \cdot \bar{\mathbf{v}} \varphi' dx + |c_2|^2 \operatorname{Im} \int_{\Omega} \mathbf{w}' \cdot \bar{\mathbf{w}} \varphi' dx - |c_3|^2 \frac{\pi \varphi(0)}{|r|} = 0.$$

Remark that for  $\mathbf{u} \in K$ ,

$$\begin{aligned} \operatorname{Im} \int_{\Omega} \mathbf{u}' \cdot \bar{\mathbf{u}} \varphi' dx &= -\operatorname{Im} \int_{\Omega} \mathbf{u} \cdot \frac{\mathbf{N}}{\alpha} \bar{\mathbf{u}} (\varphi - \varphi(0)) dx + \operatorname{Im} \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \bar{\mathbf{u}} (\varphi - \varphi(0)) \right\}_{-1}^1 \\ &= -(\sigma \{|u_1|^2\}_{-1}^1 + \frac{1}{\sigma} \{|u_2|^2\}_{-1}^1) \varphi(0) \leq 0. \end{aligned}$$

For  $\mathbf{u} = \mathbf{v}$  or  $\mathbf{u} = \mathbf{w}$ , this quantity does not vanish as  $\mathbf{v}(\pm 1) \neq 0$  and  $\mathbf{w}(\pm 1) \neq 0$ . Moreover  $\varphi(0) > 0$ . Necessarily,  $(c_1, c_2, c_3) = 0$ , and  $A_{KK'}^+$  is one-to-one.  $\square$

**Proposition 28.**  $B$  is onto from  $V$  to  $Q'$ .

**Proof.** Proving  $B$  is onto from  $Q \times \{0\} \subset V$  to  $Q'$  is sufficient. In the sequel of the proof there will be the abuse of notation that  $B$  is defined from  $Q$  to  $Q'$ .

For all  $\mathbf{u}, \boldsymbol{\lambda} \in Q$ , decompose  $b((\mathbf{u}, 0), \boldsymbol{\lambda})$  as the sum of two sesquilinear forms  $b_0(\mathbf{u}, \boldsymbol{\lambda}) + b_1(\mathbf{u}, \boldsymbol{\lambda})$  with

$$b_0(\mathbf{u}, \boldsymbol{\lambda}) = \int_{\Omega} (\mathbf{u}' \cdot \overline{\boldsymbol{\lambda}'} + \mathbf{u} \cdot \overline{\boldsymbol{\lambda}}) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \overline{\boldsymbol{\lambda}} \right\}_{-1}^1, \quad b_1(\mathbf{u}, \boldsymbol{\lambda}) = \int_{\Omega} \mathbf{u} \cdot \left( \frac{N}{\alpha} - 1 \right) \overline{\boldsymbol{\lambda}} dx.$$

The sesquilinear form  $b_0$  is coercive, as for all  $\mathbf{u} \in Q$ ,  $\operatorname{Re} b_0(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_{H^1}^2$ . So  $B_0 : \mathbf{u} \in Q \mapsto (\boldsymbol{\lambda} \mapsto b_0(\mathbf{u}, \boldsymbol{\lambda})) \in Q'$  is positive and bounded below in the sense of [11]. Denote by  $B_1$  the operator  $\mathbf{u} \in Q \mapsto (\boldsymbol{\lambda} \mapsto b_1(\mathbf{u}, \boldsymbol{\lambda})) \in Q'$ . All bounded sequences  $(\mathbf{u}_n)_{n \in \mathbb{N}} \subset Q$  admit a subsequence strongly converging in  $L^2(\Omega)^2$  towards a limit  $\mathbf{u} \in Q$ . Besides, Cauchy-Schwarz and Hardy's inequalities imply that

$$|(B_1 \mathbf{u}_n, \boldsymbol{\lambda})_{Q', Q} - (B_1 \mathbf{u}, \boldsymbol{\lambda})_{Q', Q}| = \left| \int_{\Omega} (\mathbf{u}_n - \mathbf{u}) \cdot \left( \frac{N}{\alpha} - 1 \right) \overline{\boldsymbol{\lambda}} dx \right| \leq C \|\mathbf{u}_n - \mathbf{u}\|_{L^2} \|\boldsymbol{\lambda}\|_{H^1}.$$

Hence  $B_1$  is compact. Therefore  $B = B_0 + B_1$  is a Fredholm operator of order 0 since it is a compact perturbation of a positive and bounded below operator  $B_0$ , see Theorem 2.33 in [11]. The Fredholm's alternative establishes  $B$  is onto provided it is injective. This part is verified as follows. Take  $\mathbf{u} \in \ker B$ . Then

$$(B\mathbf{u}, \mathbf{u})_{Q', Q} = \int_{\Omega} (|\mathbf{u}'|^2 + \mathbf{u} \cdot \frac{N}{\alpha} \overline{\mathbf{u}}) dx - \{i\sigma|\mathbf{u}_1|^2 + \frac{i}{\sigma}|\mathbf{u}_2|^2\}_{-1}^1 = 0.$$

Once again, taking the imaginary part yields  $\mathbf{u}(\pm 1) = 0$  on the boundary of the domain. The boundary condition  $\mathbf{u}'(\pm 1) = \pm \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u}(\pm 1)$  yields  $\mathbf{u}'(\pm 1) = 0$ . This is propagated by the equation on  $\Omega$ , see Proposition 26, so  $\mathbf{u} = 0$ . The injectivity of  $B$  on  $Q \times \{0\}$  is proven and the proof is ended.  $\square$

### 3.3 Proof of the third part of the main Theorem

The third part of Theorem 23 states that  $\mathbf{u}^+$ , the weak  $H^1$  limit of  $\mathbf{u}^\nu$  defined in Corollary 10, is the solution to the variational formulation (3.6). In order to establish this result we derive a new variational formulation for  $\nu > 0$  which tends to the limit problem as  $\nu \rightarrow 0^+$ . It will also yield additional information about the Lagrange multipliers  $(s^+, \boldsymbol{\lambda}^+)$ .

For  $\nu \in \mathbb{R}$ ,  $(\mathbf{u}, s), (\mathbf{v}, t) \in V$  and  $\boldsymbol{\lambda} \in H^1(\Omega)^2$ , define the sesquilinear form  $a^\nu$  such that

$$\operatorname{Im} a^\nu((\mathbf{u}, s), (\mathbf{v}, t)) = d\mathcal{J}_{(\mathbf{u}, s)}^\nu(\mathbf{v}, t) \quad (3.11)$$

by

$$\begin{aligned} a^\nu((\mathbf{u}, s), (\mathbf{v}, t)) &= \int_{\Omega} (\overline{\mathbf{v}} \cdot \mathbf{u}' - \mathbf{u} \cdot \overline{\mathbf{v}'}) \varphi' dx - s \int_{\Omega} ((\mathbf{w}_2^\nu \cdot \overline{\mathbf{v}} - \mathbf{w}_1^\nu \cdot \overline{\mathbf{v}'}) \varphi' + (\mathbf{z}_2^\nu \cdot \overline{\mathbf{v}'} - \mathbf{z}_1^\nu \cdot \overline{\mathbf{v}}) \varphi) dx \\ &\quad + \overline{t} \int_{\Omega} ((\overline{\mathbf{w}}_2^\nu \cdot \mathbf{u} - \overline{\mathbf{w}}_1^\nu \cdot \mathbf{u}') \varphi' + (\overline{\mathbf{z}}_2^\nu \cdot \mathbf{u}' - \overline{\mathbf{z}}_1^\nu \cdot \mathbf{u}) \varphi) dx \\ &\quad - 2i s \overline{t} \int_{\Omega} \mathbf{w}_1^\nu \cdot \operatorname{Im} \left( \frac{1}{\alpha + i\nu} N^\nu \right) \overline{\mathbf{w}}_1^\nu \varphi dx. \end{aligned}$$

We extend trivially the form  $b^\nu$  defined in (2.2) from  $H^1(\Omega)^2 \times H^1(\Omega)^2$  to  $V \times H^1(\Omega)^2$  by

$$b^\nu((\mathbf{u}, s), \boldsymbol{\lambda}) = \int_{\Omega} (\mathbf{u} \cdot \overline{\boldsymbol{\lambda}} + \mathbf{u} \cdot \frac{N^\nu \overline{\boldsymbol{\lambda}}}{\alpha + i\nu}) dx - \left\{ \begin{pmatrix} i\sigma & 0 \\ 0 & i/\sigma \end{pmatrix} \mathbf{u} \cdot \overline{\boldsymbol{\lambda}} \right\}_{-1}^1, \quad \text{for } (\mathbf{u}, s) \in V, \boldsymbol{\lambda} \in H^1(\Omega)^2.$$

Now that  $\nu$  regularizes the equations, the form  $b^\nu$  is defined on all  $H^1(\Omega)^2$  without any difficulty.

**Proposition 29.** *For any  $\nu > 0$ ,  $s \in \mathbb{C}$ ,  $\mathbf{v} \in H^1(\Omega)^2$  and  $\boldsymbol{\mu} \in H^1(\Omega)^2$ , the solution  $\mathbf{u}^\nu$  of (2.1) satisfies*

$$b^\nu((\mathbf{u}^\nu, s), \boldsymbol{\mu}) = \ell^\nu(\boldsymbol{\mu}) \quad \text{and} \quad a^\nu((\mathbf{u}^\nu, s), (\mathbf{v}, 0)) = \overline{b^{-\nu}((\mathbf{v}, 0), -(\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi)}. \quad (3.12)$$

Moreover  $a^\nu((\mathbf{u}^\nu, s), (0, t)) = 0$  for all  $t \in \mathbb{C}$  is equivalent to

$$\nu \int_{\Omega} \left( \frac{1}{\alpha^2 + \nu^2} (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \begin{pmatrix} \delta^2 & \delta k_z \\ \delta k_z & k_z^2 \end{pmatrix} \overline{\mathbf{w}}_1^\nu + (\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \overline{\mathbf{w}}_1^\nu \right) dx = 0. \quad (3.13)$$

**Proof.** The first relation of (3.12) is just a reformulation of (2.1), with the extension of  $b^\nu$  on the whole space  $V$ . Now for  $s \in \mathbb{C}$  and  $\mathbf{v} \in H^1(\Omega)^2$ ,

$$\begin{aligned} a^\nu((\mathbf{u}^\nu, s), (\mathbf{v}, 0)) &= \int_{\Omega} (\bar{\mathbf{v}} \cdot \mathbf{u}' - \mathbf{u} \cdot \bar{\mathbf{v}}') \varphi' dx - s \int_{\Omega} ((\mathbf{w}_2^\nu \cdot \bar{\mathbf{v}} - \mathbf{w}_1^\nu \cdot \bar{\mathbf{v}}') \varphi' + (\mathbf{z}_2^\nu \cdot \bar{\mathbf{v}}' - \mathbf{z}_1^\nu \cdot \bar{\mathbf{v}}) \varphi) dx \\ &= - \int_{\Omega} ((\mathbf{u} - s\mathbf{w}_1^\nu) \varphi)' \cdot \bar{\mathbf{v}}' dx + \int_{\Omega} (\mathbf{u} - s\mathbf{w}_1^\nu)' \cdot (\bar{\mathbf{v}} \varphi)' dx - s \int_{\Omega} (\mathbf{z}_2^\nu \cdot (\bar{\mathbf{v}} \varphi)' - \mathbf{z}_1^\nu \cdot \bar{\mathbf{v}} \varphi) dx \\ &= - \int_{\Omega} ((\mathbf{u} - s\mathbf{w}_1^\nu) \varphi)' \cdot \bar{\mathbf{v}}' dx - \int_{\Omega} (\mathbf{u} - s\mathbf{w}_1^\nu) \varphi \cdot \frac{N^\nu}{\alpha + i\nu} \bar{\mathbf{v}} dx, \end{aligned} \quad (3.14)$$

which is exactly  $\overline{b^{-\nu}((\mathbf{v}, 0), -(\mathbf{u}^\nu - s\mathbf{w}_1^\nu) \varphi)}$ : it yields the second relation of (3.12).

Finally if for all  $t \in \mathbb{C}$

$$a^\nu((\mathbf{u}^\nu, s), (0, t)) = \overline{-a^\nu((0, t), (\mathbf{u}^\nu, s))} = 0,$$

because of (3.11)  $\partial_s \mathcal{J}^\nu(\mathbf{u}^\nu, s) = 0$ . Then (3.13) follows from Proposition 18 and (2.20).  $\square$

For  $\nu > 0$ , define for all  $\mathbf{v} \in H^1(\Omega)^2$

$$\Gamma^\nu(\mathbf{v}) = \nu \int_{\Omega} \left( \frac{1}{\alpha^2 + \nu^2} \mathbf{v} \cdot \begin{pmatrix} \delta^2 & \delta k_z \\ \delta k_z & k_z^2 \end{pmatrix} \overline{\mathbf{w}_1^\nu} + \mathbf{v} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \overline{\mathbf{w}_1^\nu} \right) dx.$$

As it appears in (3.12)-(3.13), the candidates for the Lagrange multipliers are

$$s^\nu = \Gamma^\nu(\mathbf{u}^\nu \varphi) / \Gamma^\nu(\mathbf{w}_1^\nu \varphi) \quad \text{and} \quad \boldsymbol{\lambda}^\nu = -(\mathbf{u}^\nu - s^\nu \mathbf{w}_1^\nu) \varphi. \quad (3.15)$$

Indeed, by construction, for  $Q^\nu = \{\mathbf{v} \in H^1(\Omega)^2, \Gamma^\nu(\mathbf{v}) = 0\}$ ,  $(\mathbf{u}^\nu, s^\nu) \in V$  and  $\boldsymbol{\lambda}^\nu \in Q^\nu$  are a solution to the problem

$$\begin{aligned} &\text{Find } (\mathbf{u}, s) \in V \text{ and } \boldsymbol{\lambda} \in Q^\nu \text{ such that} \\ &\begin{cases} a^\nu((\mathbf{u}, s), (\mathbf{v}, t)) - \overline{b^{-\nu}((\mathbf{v}, t), \boldsymbol{\lambda})} = 0, & \forall (\mathbf{v}, t) \in V, \\ b^\nu((\mathbf{u}, s), \boldsymbol{\mu}) = \ell^\nu(\boldsymbol{\mu}), & \forall \boldsymbol{\mu} \in Q^\nu. \end{cases} \end{aligned} \quad (3.16)$$

And (3.16) continuously matches (3.6) as  $\nu \rightarrow 0^+$  in the sense that

$$\mathbf{v} \in Q^\nu \xrightarrow{\nu \rightarrow 0^+} \mathbf{v} \in Q, \quad (3.17)$$

that for all  $(\mathbf{u}, s), (\mathbf{v}, t) \in V$ ,

$$a^\nu((\mathbf{u}, s), (\mathbf{v}, t)) \xrightarrow{\nu \rightarrow 0^+} a((\mathbf{u}, s), (\mathbf{v}, t)), \quad (3.18)$$

and that for all  $(\mathbf{u}, s) \in V$ , if  $\boldsymbol{\mu}_\nu \in Q^\nu$  is such that  $\boldsymbol{\mu}_\nu \xrightarrow{\nu \rightarrow 0^+} \boldsymbol{\mu} \in Q$ , then

$$b^{\pm\nu}((\mathbf{u}, s), \boldsymbol{\mu}_\nu) \xrightarrow{\nu \rightarrow 0^+} b((\mathbf{u}, s), \boldsymbol{\mu}). \quad (3.19)$$

**Proposition 30.** *The solution to problem (3.6) is  $(\mathbf{u}^+, s^+) \in V$  and  $\boldsymbol{\lambda}^+ \in Q$  for  $s^+ = -i(\delta(0), k_z)^t \cdot \mathbf{u}^+(0)$  and  $\boldsymbol{\lambda}^+ = -(\mathbf{u}^+ - s^+ \mathbf{w}_1^+) \varphi$ .*

**Proof.** It has been established in Corollary 24 that  $\mathbf{u}^\nu \xrightarrow{H^1} \mathbf{u}^+$  as  $\nu \rightarrow 0^+$ . Secondly,  $s^\nu = \Gamma^\nu(\mathbf{u}^\nu \varphi) / \Gamma^\nu(\mathbf{w}_1^\nu \varphi)$  and as in (2.21),

$$\begin{cases} \Gamma^\nu(\mathbf{u}^\nu \varphi) \rightarrow \frac{\pi}{|r|} \mathbf{u}^+(0) \cdot \begin{pmatrix} \delta^2(0) & \delta(0)k_z \\ \delta(0)k_z & k_z^2 \end{pmatrix} \overline{\mathbf{w}_1^+(0)} \varphi(0) = -i \frac{\pi}{|r|} (\delta(0), k_z)^t \cdot \mathbf{u}^+(0) \varphi(0), \\ \Gamma^\nu(\mathbf{w}_1^\nu \varphi) \rightarrow \frac{\pi}{|r|} \mathbf{w}_1^+(0) \cdot \begin{pmatrix} \delta^2(0) & \delta(0)k_z \\ \delta(0)k_z & k_z^2 \end{pmatrix} \overline{\mathbf{w}_1^+(0)} \varphi(0) = \frac{\pi}{|r|} \varphi(0), \end{cases}$$

and  $s^\nu \rightarrow -i(\delta(0), k_z)^t \cdot \mathbf{u}^+(0)$  in  $\mathbb{C}$ . Finally,  $\boldsymbol{\lambda}^\nu \xrightarrow{H^1} \boldsymbol{\lambda}^+$  as  $\nu \rightarrow 0^+$ . Now as (3.16) is verified by all  $(\mathbf{u}^\nu, s^\nu)$  and  $\boldsymbol{\lambda}^\nu$  and because of (3.17)-(3.18)-(3.19),  $(\mathbf{u}^+, s^+)$  and  $\boldsymbol{\lambda}^+$  verify the variational formulation (3.6).  $\square$



## 4 Numerical illustration

In order to illustrate the qualitative behavior of the solutions and their dependence with respect to  $\nu$ , we present some numerical results obtained for these variational formulations using Lagrange finite elements of order 1. We refer to [2] for a description of standard discretization methods for such mixed variational problems.

Our numerical solutions are obtained through convenient approximations of (3.6) and (3.16), which are the new formulations using the manufactured solutions. For the purpose of comparison, we also present the approximation of the more classical formulation of the initial problem (2.1). It will show the gain of accuracy of our method in the regime of small  $\nu$ .

The particular case  $k_z = 0$  is the normal incidence and the general case  $k_z \neq 0$  is the oblique incidence. In normal incidence, the system of equations is decoupled. Denote  $\mathbf{u}^\nu = (e^\nu, b^\nu)$ . For  $b^\nu$ , it is a Helmholtz equation. For  $e^\nu$ , it writes

$$-e^{\nu''}(x) + \left( \frac{\delta^2(x)}{\alpha(x) + i\nu} - (\alpha(x) + i\nu) \right) e^\nu(x) = 0 \quad \text{in } \Omega, \quad (4.1)$$

with boundary conditions

$$e^{\nu'}(\pm 1) \mp i\sigma e^\nu(\pm 1) = f(\pm 1). \quad (4.2)$$

In the case where Maxwell's equations (1.5) are decoupled, equation (4.1) concerns  $e_y^\nu = e^\nu$ ,  $e_x^\nu = -i\frac{\delta}{\alpha+i\nu}e_y^\nu$  and  $b_z^\nu = e_y^{\nu'}$ . The equations on  $e_x^\nu$ ,  $e_y^\nu$  and  $b_z^\nu$  are called the X-mode equations, for extraordinary mode. The equations that concern  $e_z^\nu$ ,  $b_x^\nu$  and  $b_y^\nu$  are called the O-mode equations, for ordinary mode. We are also able to compute a numerical value of the resonant heating. This quantity is based on the divergence of the Poynting vector  $\Pi^\nu = \text{Im}(\mathbf{E}^\nu \times \mathbf{B}^\nu)$ . As computed in [7],  $\nabla \cdot \Pi^\nu = \nu \|\mathbf{E}^\nu\|_2^2$ , so that

$$\int_\Omega \nabla \cdot \Pi^\nu \varphi \, dx = \nu \int_\Omega \left( \frac{1}{\alpha^2 + \nu^2} |(\delta, k_z)^t \cdot \mathbf{u}^\nu|^2 + |e^\nu|^2 + |b^{\nu'}|^2 \right) \varphi \, dx \xrightarrow{\nu \rightarrow 0^+} \frac{\pi}{|r|} \varphi(0) |(\delta(0), k_z) \cdot \mathbf{u}^+(0)|^2. \quad (4.3)$$

We will present in Fig. 2 a comparison of the values of the resonant heating for three different approximation methods .

### 4.1 The Whittaker test case: an analytical solution in normal incidence

In normal incidence, we construct an analytical reference solution. Take the coefficients

$$\alpha = -x \quad \text{and} \quad \delta = \sqrt{1 - x/4 + x^2}. \quad (4.4)$$

With these coefficients, the limit of equation (4.1) as  $\nu \rightarrow 0^+$  is the Whittaker equation with unknown  $e^+$

$$-e^{+''}(x) + \left( \frac{1}{4} - \frac{1}{x} \right) e^+(x) = 0 \quad \text{in } (-1, 0) \text{ and } (0, 1). \quad (4.5)$$

General solutions of the Whittaker equation outside  $x = 0$  are linear combinations of the elementary solutions

$$u : x \mapsto x e^{-x/2}, \quad v : x \mapsto -e^{x/2} + \left( \ln|x| + \int_1^x \frac{e^y - 1}{y} dy \right) x e^{-x/2}. \quad (4.6)$$

To get a unique solution two additional constraints are missing. This information can be recovered using the fact the solution we are interested in is the  $H^1$  weak limit of (4.1) as  $\nu \rightarrow 0^+$ . First, we have the continuity of  $e^+$  in 0. Second, we have the integral relation (2.27). As for  $k_z = 0$ , second components of  $\mathbf{w}_1^+$ ,  $\mathbf{w}_2^+$ ,  $\mathbf{z}_1^+$  and  $\mathbf{z}_2^+$  are zero, see (2.13)-(2.15), denote the first components  $w_1^+$ ,  $w_2^+$ ,  $z_1^+$  and  $z_2^+$ . Relation (2.27) then rewrites

$$\int_\Omega (e^+ w_2^+ - e^{+'} w_1^+) \varphi' \, dx = \int_\Omega (e^+ z_1^+ - e^{+'} z_2^+) \varphi \, dx. \quad (4.7)$$

**Proposition 31.** *The limit solution  $e^+$  of (4.1) as  $\nu \rightarrow 0^+$  is such that*

$$e^+ = \begin{cases} a_L u + cv, & -1 \leq x \leq 0 \\ a_R u + cv, & 1 \leq x \leq 0 \end{cases}. \quad (4.8)$$

for  $a_L, a_R$  and  $c \in \mathbb{C}$ , with the jump condition

$$a_R - a_L = -\frac{i\pi\delta(0)^2}{|r|} \frac{v(0)}{u'(0)} c. \quad (4.9)$$

One can check that the limit solution  $e^-$  is such that

$$a_{\bar{R}} - a_{\bar{L}} = -(a_R - a_L). \quad (4.10)$$

**Proof.** For some  $a_L, a_R, c_L$  and  $c_R \in \mathbb{C}$ ,  $e^+ = a_L u + c_L v$  in  $(-1, 0)$  and  $e^+ = a_R u + c_R v$  in  $(0, 1)$ . The continuity of  $e^+$  in 0 yields  $c_L = c_R$ . Denote  $c$  that coefficient. As for any  $1 > \varepsilon > 0$ , (4.5) and (2.14) yield

$$\begin{aligned} \int_{\varepsilon}^1 (e^+ w_2^+ - e^{+'} w_1^+) \varphi' dx &= \int_{\varepsilon}^1 (e^+ z_1^+ - e^{+'} z_2^+) \varphi dx + (e^{+'}(\varepsilon) w_1^+(\varepsilon) - e^+(\varepsilon) w_2^+(\varepsilon)) \varphi(\varepsilon), \\ \int_{-1}^{-\varepsilon} (e^+ w_2^+ - e^{+'} w_1^+) \varphi' dx &= \int_{-1}^{-\varepsilon} (e^+ z_1^+ - e^{+'} z_2^+) \varphi dx - (e^{+'}(-\varepsilon) w_1^+(-\varepsilon) - e^+(-\varepsilon) w_2^+(-\varepsilon)) \varphi(-\varepsilon), \end{aligned}$$

relation (4.7) is equivalent to

$$\int_{-\varepsilon}^{\varepsilon} (e^+ w_2^+ - e^{+'} w_1^+) \varphi' dx = \int_{-\varepsilon}^{\varepsilon} (e^+ z_1^+ - e^{+'} z_2^+) \varphi dx - [(e^{+'} w_1^+ - e^+ w_2^+) \varphi]_{-\varepsilon}^{\varepsilon}. \quad (4.11)$$

Both integrals on  $(-\varepsilon, \varepsilon)$  vanish as  $\varepsilon \rightarrow 0$ , since  $e^+, w_1^+, z_2^+ \in H^1(\Omega)$ ,  $w_2^+, z_1^+ \in L^2(\Omega)$  and  $\varphi \in C^1(\Omega)$ :

$$\begin{aligned} \left| \int_{\varepsilon}^{\varepsilon} (e^+ w_2^+ - e^{+'} w_1^+) \varphi' dx \right| &\leq \left( \|e^+\|_{L^\infty} \|w_2^+\|_{L^2} \|\varphi'\|_{L^\infty} + \|e^{+'}\|_{L^2} \|w_1^+\|_{L^\infty} \|\varphi'\|_{L^\infty} \right) \sqrt{2\varepsilon}, \\ \left| \int_{-\varepsilon}^{-\varepsilon} (e^+ z_1^+ - e^{+'} z_2^+) \varphi dx \right| &\leq \left( \|e^+\|_{L^\infty} \|z_1^+\|_{L^2} \|\varphi\|_{L^\infty} + \|e^{+'}\|_{L^2} \|z_2^+\|_{L^\infty} \|\varphi\|_{L^\infty} \right) \sqrt{2\varepsilon}. \end{aligned} \quad (4.12)$$

The scalar difference converges towards 0 as  $\varepsilon \rightarrow 0$  because of (4.11) and (4.12). It also rewrites

$$\begin{aligned} [(e^{+'} w_1^+ - e^+ w_2^+) \varphi]_{-\varepsilon}^{\varepsilon} &= w_1^+(\varepsilon) \varphi(\varepsilon) (e^{+'}(\varepsilon) - e^{+'}(-\varepsilon)) + e^{+'}(-\varepsilon) \int_{-\varepsilon}^{\varepsilon} (w_1^+ \varphi)' dx \\ &\quad - e^+(\varepsilon) \varphi(\varepsilon) (w_2^+(\varepsilon) - w_2^+(-\varepsilon)) - w_2^+(-\varepsilon) \int_{-\varepsilon}^{\varepsilon} (e^+ \varphi)' dx. \end{aligned}$$

Both integrals can be bounded again by  $\sqrt{\varepsilon}$  up to a multiplicative constant. Since  $\sqrt{\varepsilon} \ln(\varepsilon) \rightarrow 0$  with  $\varepsilon$ ,  $\sqrt{\varepsilon} e^{+'}(-\varepsilon)$ ,  $\sqrt{\varepsilon} w_2^+(-\varepsilon)$  and  $v'(\varepsilon) - v'(-\varepsilon)$  also vanish in 0. So

$$[(e^{+'} w_1^+ - e^+ w_2^+) \varphi]_{-\varepsilon}^{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \frac{i\varphi(0)}{\delta(0)} (a_R - a_L) u'(0) - cv(0) \varphi(0) \frac{\delta(0)\pi}{|r|} = 0$$

and the jump condition is obtained.  $\square$

Now that we have (4.8)-(4.9), the two boundary conditions are sufficient to determine these three coefficients. The numerical results presented here have been obtained for the parameters

$$\sigma = 1, f(-1) = 1, f(1) = 2, \text{ and } \varphi(x) = e^{\frac{1}{2x-1} - \frac{1}{2x+1}} \mathbf{1}_{(-\frac{1}{2}, \frac{1}{2})}. \quad (4.13)$$

Note that  $\varphi \in \mathcal{C}_{0,+}^1(\Omega)$ .

We observe in Fig. 1 that for a coarse grid, the discretization of the limit problem is accurate, and that for a small  $\nu$ , the discretization of our new formulation of (1.1)-(1.2) using manufactured solutions is more satisfying than the one of the classical formulation (2.1).

In Fig. 2, we also consider the oblique case  $k_z = 4$ , with boundary conditions

$$\mathbf{f}(-1) = (1, 3)^t, \mathbf{f}(1) = (2, 5i)^t.$$

The resonant heating (4.3) from the approximate solutions can be computed and it is presented in the right part of Fig. 2 for the three different methods in function of the number of cells. The result is very typical of convergence tables with respect to two small parameters which are  $\nu$  and  $h = \frac{1}{N_{\text{cell}}}$  in our case. The classical finite element method is very sensitive to small  $\nu$  since the exact solution and the limit problem become singular or ill-posed at the limit. It explains why an important number of cells is necessary to compute the resonant heating. On the contrary the discretization of the limit problem (3.6) captures the correct resonant heating for a very small number of cells. The intermediate formulation (3.16) displays an intermediate behavior, with respect to resonant heating. These results are a direct consequence that a correct numerical value of the resonant heating is, through formula (4.3), function of getting a correct numerical value of the solution at the resonance point  $x = 0$ . It is already visible in Fig. 1 that the new formulation (3.6) is much better for the computation of the solution at the resonance point  $x = 0$ . This result has its own physical interest in the context of fusion plasmas, but it also illustrates the mathematical interest of having a correct formulation of the limit problem.

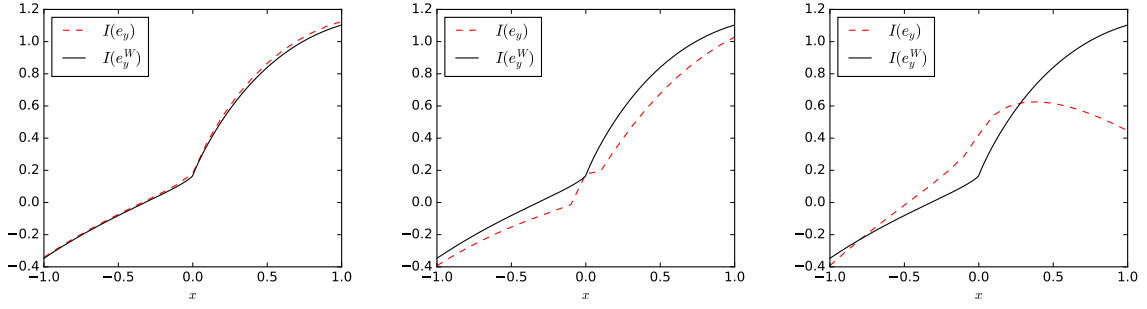


Figure 1: From left to right, imaginary parts of the first component of the numerical solutions on a mesh of 40 cells for the limit problem with manufactured solutions, for the problem with small  $\nu = 10^{-7}$  and manufactured solutions, and for the classical problem (Finite Element method) with small  $\nu = 10^{-7}$ . Analytical solution in solid line, approximations in dashed lines.

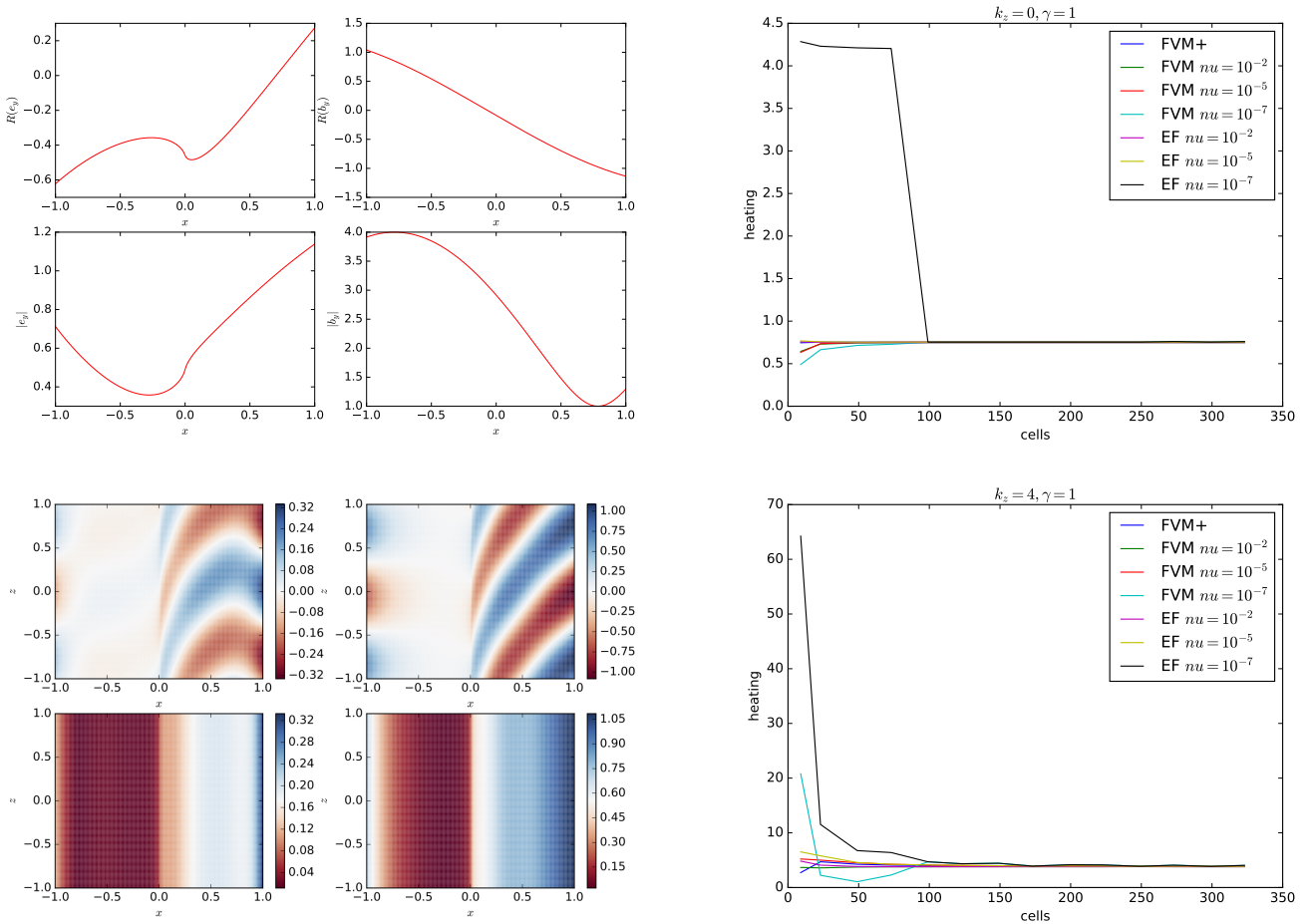


Figure 2: Mesh with 200 cells. Above: solution for  $k_z = 0$ . Below: solution for  $k_z = 4$ . On the left, figure above: from left to right, real parts of the approximation of fields  $e$  and  $b$  by the limit problem above, modulus of the corresponding fields below. On the left, figure below: same but on the 2D reconstruction of the solution  $\mathbf{u}^{2D} = \mathbf{u}e^{ik_z z}$ . On the right: discrete heating for the three different methods in function of the number of cells  $N_{\text{cell}}$ .

## 4.2 A more physical test-case

Let us finally consider a configuration for which a very simplified antenna sends a time-harmonic plane wave into the plasma at  $x = -1$ . We restore the physical dimension of all coefficients of the tensor (1.4)

by considering

$$\|e\|^\nu = \begin{pmatrix} \alpha + i\nu & i\delta & 0 \\ -i\delta & \alpha + i\nu & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

We take parameters such that the plasma is propagative on the first sixth part from the left for half a period, and resonant at  $x = 0$ . The parameters are

$$\alpha(x) = \begin{cases} (6\pi)^2, & x < -\frac{2}{3} \\ -\frac{3x}{2}(6\pi)^2, & -\frac{2}{3} \leq x < \frac{1}{3} \\ -\frac{(6\pi)^2}{2}, & x \geq \frac{1}{3} \end{cases}, \delta(x) = \begin{cases} 0, & x < -\frac{2}{3} \\ 120x + \frac{240}{3}, & -\frac{2}{3} \leq x < \frac{1}{3} \\ 120, & x \geq \frac{1}{3} \end{cases}, \gamma = (6\pi)^2. \quad (4.14)$$

The boundary conditions are now

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\gamma \end{pmatrix} \mathbf{u}'(\pm 1) \mp \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i/\sigma_2 \end{pmatrix} \mathbf{u}(\pm 1) = \mathbf{f}(\pm 1).$$

The dispersion relation  $k_x^2 + k_z^2 = \alpha^2(-1)$  characterizes a plane wave propagating on  $(-1, -2/3)$ , where the coefficients of the equation are constant. We take  $k_x = k_z = \frac{\alpha(-1)}{\sqrt{2}}$ . In the tests

$$\mathbf{f}(-1) = e^{-ik_z}(2ik_z, 20)^t, \mathbf{f}(1) = (0, 0)^t,$$

and

$$\sigma_1 = k_z, \sigma_2 = \alpha(-1)/k_z.$$

The cut-off function  $\varphi$  is the same as in (4.13).

In Fig. 3, for normal incidence, approximation of the three fields concerned by the X-mode solution to the limit problem are plotted, as for  $\mathbf{u} = (e, b)^t$ ,

$$e_y^+ = e, e_x^+ = \frac{-i\delta}{\alpha}e \text{ and } b_z^+ = e'.$$

The discontinuity of  $e_x$  in  $1/x$  appears clearly.

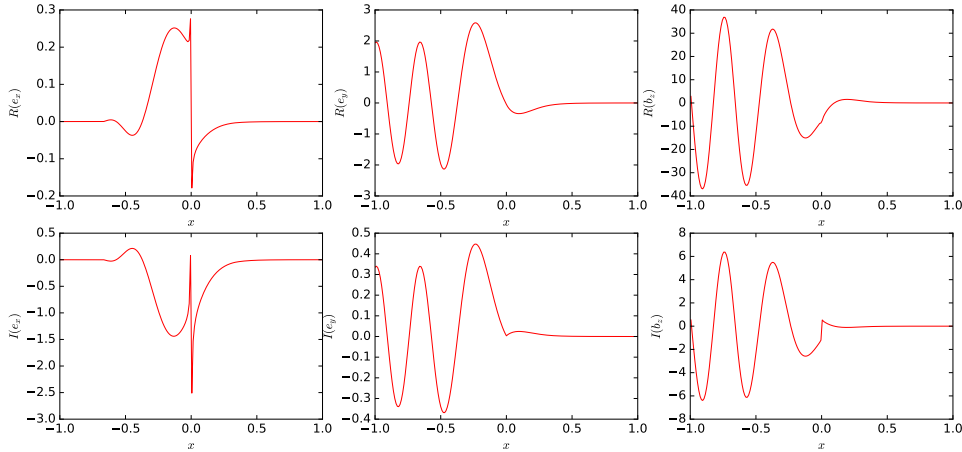


Figure 3: From left to right, approximation of the X-mode fields  $e_x^+$ ,  $e_y^+$  and  $b_z^+$  using the discretization of the limit formulation. Real parts above, imaginary parts below, for a mesh of 200 cells.

In Fig. 4, for oblique incidence taking  $k_z = \frac{6\pi}{\sqrt{2}}$ , the singular field  $e_x^+$ , and the two regular fields  $e_y^+$  and  $b_y^+$  are plotted, as for  $\mathbf{u} = (e, b)^t$ ,

$$e_y^+ = e, b_y^+ = b \text{ and } e_x^+ = -\frac{i\delta}{\alpha}e - \frac{ik_z}{\alpha}b.$$

We observe on figure 4 that the second component of the wave is propagated almost until  $x = 0$ , that its first component behaves as if it was influenced by the singularity in such a way it corresponds to a reflection of the incident plane wave, that the field  $e_x^+$  does present a singularity at  $x = 0$ , and that all fields are absorbed on the right side of the singularity.

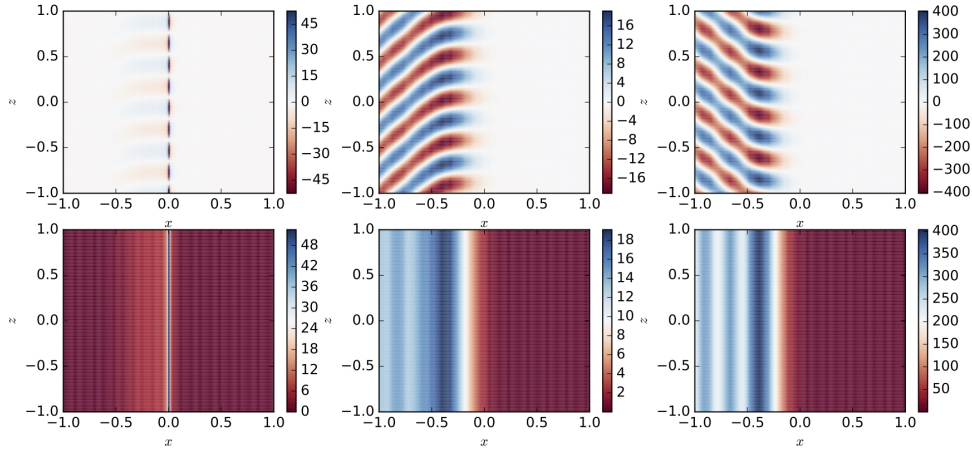


Figure 4: From left to right, 2D reconstruction of the approximation of the fields  $e_x^+$ ,  $e_y^+$  and  $b_y^+$  using the discretization of the limit formulation, and using again  $\mathbf{u}^{2D} = \mathbf{u}e^{ik_z z}$ . Real parts above, modulus below, for a mesh of 200 cells.

## References

- [1] A. Back, T. Hattori, S. Labrunie, J.-R. Roche and P. Bertrand, Electromagnetic wave propagation and absorption in magnetized plasmas: variational formulations and domain decomposition. *ESAIM: Mathematical Modeling and Numerical Analysis*, EDP Sciences, 49 (2015), no. 5, 1239-1260.
- [2] D. Boffi, F. Brezzi and M. Fortin *Mixed finite element methods and applications*. Springer-Verlag, Berlin, 2013.
- [3] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr, T-coercivity for scalar interface problems between dielectrics and metamaterials. *ESAIM: Mathematical Modeling and Numerical Analysis*, EDP Sciences, 46 (2012), no. 6, 1363-1387.
- [4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, 2011.
- [5] K.G. Budden, *Radio Waves in the Ionosphere*. Cambridge University Press, 1961.
- [6] C. Caldini-Queiros, B. Després, L.-M. Imbert-Gérard, M. Kachanovska, A numerical study of the solution of X-mode equations around the hybrid resonance. *ESAIM: Proceedings and Surveys*, EDP Sciences, 53 (2016), 1-21.
- [7] M. Campos-Pinto and B. Després, Constructive formulations of resonant Maxwell's equations. *SIAM J. Math. Anal.* 49 (2017), no. 5, 3637-3670.
- [8] B. Després, L.M. Imbert-Gérard and O. Lafitte, Singular solutions for the plasma at the resonance. *Journal de l'École polytechnique - Mathématiques*, 4 (2017), 177-222.
- [9] B. Després, L.M. Imbert-Gérard and R. Weder, Hybrid resonance of Maxwell's equations in slab geometry. *J. Math. Pures Appl.*, Elsevier, 101 (2014), no. 5, 623-659.
- [10] L. Lu, K. Crombé, D. Van Eester, L. Colas, J. Jacquot and S. Heuraux, Ion cyclotron wave coupling in the magnetized plasma edge of tokamaks: impact of a finite, inhomogeneous density inside the antenna box. *Plasma Physics and Controlled Fusion*, 58 (2016), no. 5.
- [11] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, New York, 2000.
- [12] P. Monk, *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

- [13] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients. *J. Math. Pures Appl.*, Elsevier, 106 (2016), no. 2, 342-374.
- [14] T.H. Stix, *The Theory of Plasma Waves*. McGraw-Hill, 1962.
- [15] D.G. Swanson, *Plasma Waves*. Institute of Physics Pub., Bristol and Philadelphia, 2003.
- [16] R.B. White and F.F. Chen, Amplification and absorption of electromagnetic waves in overdense plasmas. *Plasma Physics*, 16 (1974), pp. 565-587.