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Indifference Pricing in Reinsurance Using Coherent Monetary Criteria

Nabil KAZI-TANI*

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Abstract

This paper focuses on a non-proportional reinsurance pricing problem, for a layer contract with *reinstatements*. After defining the indifference price with respect to both a concave utility function and a convex risk measure, we prove that it is contained in some interval whose bounds are easily calculable. We provide numerical examples computed from real insurance data.

Key words: Insurance premium calculation, Coherent risk measures, Concave monetary utility functions, Reinsurance layers, Reinstatements.

AMS 2010 subject classifications: 91B30, 62P05

*Université de Lyon, Université Lyon 1, Institut de Science financière et d'Assurances, 50 avenue Tony Garnier, F-69007 Lyon, France. E-mail: nabil.kazi-tani@univ-lyon1.fr.

1 Introduction

Risk pricing is a major issue for insurance companies for at least two practical reasons: on the one hand, in the direct selling of insurance contracts to individual agents, companies have to compute and propose a premium to their clients. On the other hand, when insurance companies transfer a part of their subscribed risk through reinsurance, they also have to agree on a premium amount.

Premium calculation principles constitute a standard topic in actuarial science. It relies on objective (Von Neumann and Morgenstern [20]) or subjective (Savage [18]) expected utility theory. We refer to Borch [4], Gerber [10], [11], Goovaerts, De Vylder and Haezendonck [12] or Laeven and Goovaerts [9] and the references therein for details and surveys on the different premium calculation principles.

More precisely, for a given probability space, a premium principle is a mapping $\pi : \mathcal{X} \rightarrow \mathbb{R} \cup \{-\infty; +\infty\}$, assigning to each random variable in \mathcal{X} a price $\pi(X)$. For instance, among the many classic criteria, we have the variance principle: $\pi(X) := \mathbb{E}(X) + \beta \text{Var}(X)$ ($\beta > 0$), or the Esscher premium: $\pi(X) := \frac{\mathbb{E}[Xe^{\alpha X}]}{\mathbb{E}[e^{\alpha X}]}$, ($\alpha > 0$). A natural question consists in asking what are the properties (or axioms) that characterize a given pricing principle. The interest of the axiomatization is to give a small number of natural properties that a premium principle should satisfy. We refer again to [9] for a study of these properties.

More recently, a similar axiomatic study of risk measures has been undertaken, following the seminal paper of Artzner & al [2]. The idea here is to compute the minimal capital to join to a position modelled by a random variable X , in order to make it risk free. We summarize some of the axioms and important properties of risk measures in Section 2.

Let ρ be a convex monetary risk measure used by an insurance company to compute its regulatory capital $\rho(X)$. If we define a mapping U by $U(X) := -\rho(X)$, we obtain a monotone increasing and concave utility function. However if an economic agent uses a risk measure ρ to evaluate his risks, she usually do not use $U = -\rho$ to evaluate the utility associated to its profits.

In this paper, we work in the context of indifference pricing principles and, we focus on the case of reinsurance layers pricing. The indifference price of a reinsurance layer, relative to a utility function U , is the price at which the insurance company is indifferent between buying or not the given layer, with respect to the utility U . Nonetheless, buying the layer also impacts the minimal regulatory capital of the company. Let us assume that the firm supports a cost of capital \bar{c} , with $0 < \bar{c} < 1$. We denote R the net result of the insurance company without a given layer, and R^{XL} the net result with the layer. Then we define the indifference premium, relative to the pair (U, ρ) , as the solution of the equation

$$U [R - \bar{c}\rho(R)] = U [R^{XL} - \bar{c}\rho(R^{XL})].$$

This price takes into account both the change in utility because of the layer addition and the change in reserves because of the cost of capital.

When the reinsurance contract contains reinstatements (see Section 3.1 for a precise definition), the indifference price is in general not easy to compute explicitly. In Section 3, we give some bounds for the indifference price, that are easily computable using simulations of the underlying risks.

1.1 The collective risk theory

There are two different possible ways to apprehend the total loss of an insurance company in a given period of time (typically one year). First, one can consider the *individual* claim amounts S_i , that is to say the loss corresponding to the policy of the i -th insuree, and then, take the sum $I = \sum_{i=1}^n S_i$, where n is the number of policies. This kind of individual model of risk dates back to Cramér [5].

This individual model is numerically difficult to implement. Let us however mention the De Pril algorithm, which allows theoretically to obtain the law of I , but it is not widely used in practice (see [17], page 129).

A second possible way to proceed is to consider a random number of claims N in the given period and individual losses X_i , regardless to which policy it is attached. Then the sum $X = \sum_{i=1}^N X_i$ also denotes the total loss of the insurance company. It is the traditional *collective model* way to see the total loss (see [6] for a detailed account).

The collective risk model can be analysed dynamically in time, by writing

$$X_t = \sum_{i=1}^{N_t} X_i,$$

where N is a Poisson process counting the number of claims in the interval $[0, t]$. The variables (X_i) are traditionally assumed independent and identically distributed. In that case, X is a pure jump Levy process, for which we have powerful analytic tools.

The collective risk model is widely used among practitioners mainly because of its tractability. Panjer's recursion [16] is a popular technique to approximate the distribution function of the total loss in the collective model.

In Section 3, we will see how this risk modelization is affected by the presence of non-proportional reinsurance contracts.

1.2 The existing literature on reinstatements

Reinsurance is a mechanism allowing an insurer to transfer a part of its subscribed risk to a reinsurer.

There are several types of reinsurance contracts. The insurer can transfer its risk in a *proportional* or *linear* way, which means that he gives away a fixed percentage of its losses and of its premiums to the reinsurer: for a loss represented by a positive random variable X , he leaves $F = aX$ with $0 < a < 1$ to the reinsurer and supports $X - F = (1 - a)X$.

The insurer can also buy a *non proportional* layer contract, covering the losses that fall in the range $[l, l+m]$ ($l, m > 0$). In that case the reinsurer is exposed to $(X-l)^+ - (X-[l+m])^+$, which puts a ceiling on his losses, since the maximal reinsurer loss equals m (we used the standard notation $x^+ = \max(x, 0)$, for $x \in \mathbb{R}$).

Despite the huge literature on reinsurance problems, there are little references concerning the pricing of reinsurance contracts containing reinstatements. Sundt [19] obtained a formula for the pure premium of contracts with reinstatements ; this formula coincide with ours in this particular case. Sundt [19] and Wahlin [21] also give formulas corresponding to other premium principles, such that the standard deviation principle or the PH-transform. Mata [15] studies the properties of different pricing principles. Now that we better understand the properties and axioms of convex monetary risk measures, we can compare it

with the properties stated by Mata, who defines indifference prices with respect to various comonotonic risk measures, but using the approximation $\rho(-X) \approx -\rho(X)$. If ρ is convex and normalized, we can only say that $\rho(-X) \geq -\rho(X)$. That is why the values computed in [15] correspond, in the particular case where there is no cost of capital, to our upper bound (p_2 appearing in equation (3.6)). Hürlimann [13] provides a distribution free approximation to pure premiums of layers with reinstatements.

Let us also mention the paper of Albrecher and Haas [1], which gives a detailed analysis of ruin theory when there are possible reinstatements.

2 Preliminaries on Monetary Risk Measures and Utility Functions

2.1 Axioms

Motivated by some imperfections of traditional risk measures such as value-at-risk (which is recommended by the solvency II european regulatory requirement for insurance companies), Artzner & al. [2] and then Follmer and Schied [7] and Frittelli and Rosazza-Gianin [8] introduced the notions of coherent and convex risk measures. We will now recall their definitions, and some key properties. We state the given results for a generic space \mathcal{X} of random variables, typically we will take $\mathcal{X} = L^\infty(\mathbb{P})$ or $\mathcal{X} = L^p(\mathbb{P})$ with $p \geq 1$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space.

Definition 2.1. *An application $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called:*

- [MO] *Monotonic if $X \leq Y$ a.s. implies $\rho(X) \geq \rho(Y)$.*
- [CI] *Cash-invariant if $\forall c \in \mathbb{R}, \rho(X + c) = \rho(X) - c$.*
- [CO] *Convex if $\forall 0 \leq \lambda \leq 1, \rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.*
- [PH] *Positively homogeneous if $\forall \lambda \geq 0, \rho(\lambda X) = \lambda\rho(X)$.*

An application verifying the axioms [MO], [CI] is called a monetary risk measure. If it furthermore satisfies [CO] then it is called a convex risk measure, and if it also satisfies the axiom [PH] then we say it is a coherent risk measure.

The cash-invariance property, also called translation invariance, gives to the value $\rho(X)$ a monetary unit: $\rho(X)$ may be intuitively interpreted as the minimal capital amount to add to X , and to invest in a risk free manner, to make the position X acceptable or risk free. Indeed $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$.

The convexity axiom will describe the preference for diversity: it means that diversification should not increase risk.

Note that here positive random variables represent gains, then we consider *decreasing* risk measures, and cash-invariant with a minus sign.

2.2 Dual representation

An important kind of results are the Fenchel-Legendre type dual representation theorems for risk measures. It has been given in Artzner & al. [2] for coherent risk measures and Frittelli and Rosazza-Gianin [8] for convex risk measures defined on L^∞ , and after that by Biagini and Frittelli [3] in the case of convex risk measures defined on L^p .

We can now state the representation result:

Theorem 2.1 ([8]). *A convex risk measure ρ on L^∞ , which is continuous from below, admits the following dual representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{M}_1} \{E^{\mathbf{Q}}[-X] - \alpha(\mathbf{Q})\} \quad (2.1)$$

where \mathcal{M}_1 denotes the set of all probability measures on \mathcal{F} and $\alpha : \mathcal{M}_1 \rightarrow \mathbb{R} \cup \{+\infty\}$ is a penalty function given by:

$$\alpha(\mathbf{Q}) = \sup_{X \in L^\infty(\mathbb{P})} \{E^{\mathbf{Q}}[-X] - \rho(X)\}$$

If we interpret any probability measure \mathbf{Q} on (Ω, \mathcal{F}) as a possible model, or possible weighting of events, this representation tells us that every convex risk measure that is upper semi-continuous can be seen as a worst case average loss over a family of models minus a penalty, the penalty representing the likelihood of the given models.

2.3 Examples

The Value-at-Risk

The Value-at-Risk is the main risk measure used in practice and it is the measure recommended by the Solvency II european regulator for insurance companies. It consists in computing an α -quantile of the risk X :

$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} \text{ such that } \mathbb{P}(X + x \leq 0) \leq \alpha\},$$

where $\alpha \in [0, 1]$. α is considered as an "acceptable" bankruptcy probability, and then $VaR_\alpha(X)$ represents the losses that are attained only with that probability.

The measure $\rho = VaR_\alpha$ is not convex, and then do not satisfy a dual representation, but the following Average Value-at-Risk will.

The Average Value-at-Risk

To overcome the non-convexity of the monetary risk measure VaR_α , we consider the following Average Value-at-Risk at level $\alpha \in (0, 1]$ of a position $X \in \mathcal{X}$, given by:

$$AVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_u(X) du$$

The name Average Value-at-Risk is justified by the following equality, which holds if X has a continuous distribution, it says that the quantity $AVaR_\alpha(X)$ is an average of the losses greater or equal to $VaR_\alpha(X)$:

$$AVaR_\alpha(X) = \mathbb{E}[-X | -X \geq VaR_\alpha(X)].$$

$AVaR_\alpha$ is a coherent risk measure and its dual representation is given as follows:

$$AVaR_\alpha(X) = \sup_{Q \in \mathcal{M}_1} \{E^Q[-X] - \alpha(Q)\}, \quad X \in \mathcal{X}$$

where the penalty function α only takes the values $+\infty$ or 0 , it is given by the indicator function in the sense of convex analysis of the following set:

$$\mathcal{Q} := \{Q \in \mathcal{M}_1 \mid \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\alpha}\}$$

The Entropic risk measure

As mentioned before, the penalty function α appearing in the representation (2.1) account for the likelihood of a given model \mathbf{Q} . Suppose that we measure the likelihood of \mathbf{Q} using the information theoretic entropic function $H(\mathbf{Q}|\mathbb{P})$ defined by

$$H(\mathbf{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbf{Q}}{d\mathbb{P}} \log \left(\frac{d\mathbf{Q}}{d\mathbb{P}} \right) \right].$$

The corresponding risk measure

$$\rho_e(X) = \sup_{\mathbf{Q} \in \mathcal{M}_1(\mathbb{P})} \{E^{\mathbf{Q}}[-X] - H(\mathbf{Q}|\mathbb{P})\}$$

is called the entropic risk measure.

The variational principle for the entropic function H gives

$$\rho_e(X) = \log \mathbb{E}_{\mathbb{P}}[e^{-X}].$$

The Semi-deviation risk measure

For $0 \leq \delta \leq 1$ and $1 \leq p \leq \infty$, we define the semi-deviation risk measure as follows :

$$\rho(X) := \mathbb{E}[-X] + \delta \mathbb{E}[(X - \mathbb{E}(X))^p]^{1/p}.$$

ρ is a positively homogeneous law-invariant monetary risk measure. If we define $U(X) := -\rho(X)$, we obtain the semi-deviation utility function, whose properties are studied in [14] for example.

3 Pricing of Layers with Reinstatements

The indifference price p of a reinsurance layer is solution of an equation indicating that the utility of an insurance company, when it buys the contract and pays the price p , is equal to its utility when it does not enter the transaction. When the contract contains reinstatements, the total premium paid is random (see equation (3.3)). This can complicate the calculation of the indifference price. In the next sections we will bound this price by two easily computable values.

3.1 The contract payoff

Let u_0 be the initial surplus of an insurance company, and let β be the gross premium income per time unit, so that the surplus process R without reinsurance is given at time t by

$$R(t) = u_0 + \beta t - \sum_{i=1}^{N_t} X_i$$

where $\{N_t, t \geq 0\}$ is a non decreasing integer valued pure jump process modelling the number of claims up to time t , and $\{X_i, i \in \mathbb{N}^*\}$ is a sequence of independent and identically distributed (i.i.d.) random variables modelling the individual claim sizes of the insurance portfolio.

In an XL reinsurance contract with retention l and limit m , the reinsurer covers the part of each claim that exceeds l , up to the upper bound $l + m$. The reinsurer's part Z_i and the insurer's part C_i related to a claim X_i are then given by

$$Z_i = (X_i - l)^+ - (X_i - l - m)^+ = \begin{cases} 0 & \text{if } X_i \leq l \\ X_i - l, & \text{if } l \leq X_i \leq l + m \\ m & \text{if } X_i \geq l + m \end{cases} \quad (3.1)$$

and

$$C_i = (X_i \wedge l) + (X_i - l - m)^+ = \begin{cases} X_i & \text{if } X_i \leq l \\ l, & \text{if } l \leq X_i \leq l + m \\ X_i - m & \text{if } X_i \geq l + m \end{cases} \quad (3.2)$$

Let

$$Z(t) := \sum_{i=1}^{N_t} Z_i$$

be the aggregate liability of the reinsurer. It is usually limited to a maximal amount M .

When M is a multiple of the individual maximum cover m , i.e when $M = (k + 1)m$, we say that the reinsurance contract contains k reinstatements. In that case the aggregate liability of the reinsurer is given by $R_k(t) := Z(t) \wedge ((k + 1)m)$.

The cover of the j -th reinstatement is given by

$$r_j := \min [\max(Z(t) - jm, 0), m]$$

For the j -th reinstatement, the cedent pays a premium P_j given by

$$P_j := c_j p_0 \frac{1}{m} \min [\max(Z(t) - (j - 1)m, 0), m]$$

Let us consider a concrete example, allowing to better understand these contracts payoffs. Assume that the insurance company enters an XL contract with one year maturity, including two reinstatements and covering the losses between 100 and 200 ($l = 100$ and $m = 100$). The initial premium required by the reinsurer is given by $p_0 = 10$. The costs of the first and second reinstatements correspond respectively to 80% and 50% of the initial premium ($c_1 = 0.8$ and $c_2 = 0.5$).

Assume the first four losses correspond to the following amounts:

1. $X_1 = 150$, we have then $C_1 = 100$ and $Z_1 = 50$. The insurance company has to reinstate half of the layer, to do so it pays $50 * \frac{1}{100} * 80\% * 10 = 4$.
2. $X_2 = 190$, we have then $C_1 = 100$ and $Z_1 = 90$. The ceding company has to reinstate 90% of the layer. For this, it uses what is left of the first reinstatement (50) and a part of the second one (40). So it pays :
 $50 * \frac{1}{100} * 80\% * 10 + 40 * \frac{1}{100} * 50\% * 10 = 6$

3. $X_3 = 200$, in that case all the layer has to be reinstated, but only 60% of the second reinstatement is available, the layer is then reinstated only up to this limit and the company pays: $60 * \frac{1}{100} * 50\% * 10 = 3$.
4. $X_4 = 200$, the insurance company was only covered for the losses between 100 and 160, we have then here $Z_4 = 60$ and $C_4 = 140$. All the reinstatements have been used, the contract stops and the ceding company is not reinsured for the rest of the year.

The total premium paid in this example amounts to $\bar{P} = 10 + 4 + 6 + 3 = 23$. This value could have been lower if all the reinstatements had not been used before the end of the contract.

We can write the random total premium paid by the cedent during the considered period in the following way:

$$\bar{P}(t) := p_0 \left(1 + \frac{1}{m} \sum_{j=1}^k c_j \min [\max(Z(t) - (j-1)m, 0), m] \right) \quad (3.3)$$

and the wealth process of the cedent is given by

$$\begin{aligned} R(t) &= u_0 + \beta t - \sum_{i=1}^{N_t} C_i - \bar{P}(t) \\ &= u_0 + \beta t - \sum_{i=1}^{N_t} X_i + \sum_{i=1}^{N_t} Z_i - \bar{P}(t). \end{aligned}$$

In what follows, we will denote for simplicity

$$\tilde{N}(t) := \frac{1}{m} \sum_{j=1}^k c_j \min [\max(Z(t) - (j-1)m, 0), m]$$

Finally, let us now consider the more realistic situation where the cedent has already bought reinsurance contracts which are still running, and is considering to add a given layer $[l, l+m]$ with k reinstatements to its reinsurance program. Then we will denote by $R(t)$ the reference net result of the cedent (without the additional layer) and we denote F^{XL} the different quantities concerning the additional XL layer.

We have for instance

$$\begin{aligned} R^{XL}(t) &= u_0 + \beta t - \sum_{i=1}^{N_t} X_i + \sum_{i=1}^{N_t} Z_i - \bar{P} + \sum_{i=1}^{N_t} Z_i^{XL} - \bar{P}^{XL} \\ &= R(t) + \sum_{i=1}^{N_t} Z_i^{XL} - \bar{P}^{XL}, \end{aligned} \quad (3.4)$$

where \bar{P}^{XL} is the total premium paid by the cedent's company for the additional XL layer.

3.2 The pricing bounds

We assume that the cedent uses a concave monetary utility function U , and a convex monetary risk measure ρ to compute its needed regulatory capital.

Definition 3.1. We say that \tilde{P}_0 is the indifference price of a given XL layer relatively to the pair (U, ρ) , if \tilde{P}_0 solves the equation

$$U(R(t) - \bar{c}\rho(R(t))) = U(R^{XL}(t) - \bar{c}\rho(R^{XL}(t))) \quad (3.5)$$

where \bar{c} is a given cost of capital.

When the insurance company does not buy the XL layer, its net result is given by $R(t)$, and its regulatory needed capital is expressed by $\rho(R(t))$. If the XL layer is added to the current reinsurance program, then the net result becomes $R^{XL}(t)$ and there is possibly a regulatory capital saving thanks to the layer that we can write as $\rho(R(t)) - \rho(R^{XL}(t))$.

Remark 3.1. If we define $\bar{U}(R) := U(R - \bar{c}\rho(R))$, then we have

$$\bar{U}(R + m) = \bar{U}(R) + (1 + \bar{c})m \geq \bar{U}(X) + m,$$

and \bar{U} is a cash super-additive utility function. This is analogous to the presence of a capitalization factor, which is here given by \bar{c} .

We have the following estimate for the indifference price of an XL layer with reinstatements

Proposition 3.1. If \tilde{P}_0 is the indifference price of a given XL layer relatively to the pair (U, ρ) , then

$$p_1 \leq \tilde{P}_0 \leq p_2,$$

where

$$p_1 := \frac{A}{-\bar{U}(-1 - \tilde{N})}, \quad p_2 := \frac{A}{\bar{U}(1 + \tilde{N})}, \quad (3.6)$$

and

$$A := \bar{U}\left(R(t) + \sum_{i=1}^{N_t} Z_i\right) - \bar{U}(R(t)).$$

Proof.

We use the decomposition (3.4) of R^{XL} and the cash invariance property to deduce that equation (3.5) is equivalent to

$$\begin{aligned} & U\left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t))\right) - \bar{c}\rho\left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t))\right) \\ &= U(R(t)) - \bar{c}\rho(R(t)). \end{aligned} \quad (3.7)$$

We can use the fact that U is super-additive whereas ρ is subadditive, and that both are positively homogeneous to write

$$\begin{aligned} U\left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t))\right) &\geq U\left(R(t) + \sum_{i=1}^{N_t} Z_i\right) + \tilde{P}_0 U\left(-1 + \tilde{N}(t)\right), \\ \rho\left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t))\right) &\leq \rho\left(R(t) + \sum_{i=1}^{N_t} Z_i\right) + \tilde{P}_0 \rho\left(-1 + \tilde{N}(t)\right). \end{aligned}$$

By plugging this into (3.7) we obtain

$$\begin{aligned} & \tilde{P}_0 \left[U \left(-(1 + \tilde{N}(t)) \right) - \bar{c} \rho \left(-(1 + \tilde{N}(t)) \right) \right] \\ & \leq U(R(t)) - U \left(R(t) + \sum_{i=1}^{N_t} Z_i \right) - \bar{c} \left[\rho(R(t)) - \rho \left(R(t) + \sum_{i=1}^{N_t} Z_i \right) \right] \end{aligned} \quad (3.8)$$

We notice that $-(1 + \tilde{N}(t)) < 0$ a.s. and since U is monotone increasing and ρ is monotone decreasing, we have

$$U \left(-(1 + \tilde{N}(t)) \right) - \bar{c} \rho \left(-(1 + \tilde{N}(t)) \right) < 0.$$

Then we deduce from inequality (3.8) that $p_1 \leq \tilde{P}_0$.

To prove the other inequality, we start again from equation (3.7), and notice this time that since U is super-additive whereas ρ is subadditive, we have $U(X - Y) \leq U(X) - U(Y)$ and $\rho(X - Y) \geq \rho(X) - \rho(Y)$, for any bounded random variables X and Y . This gives

$$\begin{aligned} U \left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t)) \right) & \leq U \left(R(t) + \sum_{i=1}^{N_t} Z_i \right) - \tilde{P}_0 U \left(1 + \tilde{N}(t) \right), \\ \rho \left(R(t) + \sum_{i=1}^{N_t} Z_i - \tilde{P}_0(1 + \tilde{N}(t)) \right) & \geq \rho \left(R(t) + \sum_{i=1}^{N_t} Z_i \right) - \tilde{P}_0 \rho \left(1 + \tilde{N}(t) \right). \end{aligned}$$

By plugging this again into (3.7), and noticing that

$$U \left(1 + \tilde{N}(t) \right) - \bar{c} \rho \left(1 + \tilde{N}(t) \right) > 0,$$

we obtain $\tilde{P}_0 \leq p_2$.

□

Remark 3.2. *The monotonicity properties of U and ρ implies that $0 \leq p_1$. Furthermore, the concavity and convexity properties of U and ρ implies that we have indeed $p_1 \leq p_2$.*

In figure 1 below, we represented the upper and lower bounds p_1 and p_2 obtained for the layers with retention equal to $10^8(n + 1)$ euros and limit equal to 10^8 euros, for n varying from 0 to 32. In other words, we computed the bound prices for 33 layer contracts, with the same limit value, for a retention ranging from one hundred million euros to 3.3 billion euros. All the contracts contain $k = 4$ possible reconstitutions, and each reconstitution when it happens, is paid at $c = 100\%$ of the initial price. The prices are expressed as a percentage of $m = 10^8$, which correspond to rates on line in the reinsurance terminology.

These reinsurance contracts cover an insurance portfolio of risk of natural catastrophes type. The calculations have been made using 25000 years of simulations of losses from a real life insurance portfolio.

We took as a matter of example a utility function given by the semi-deviation utility with parameter $\delta = \frac{1}{2}$ and as risk measures, the $AVaR_\alpha$ with $\alpha = 1/200$.

We can see that the gap between p_1 and p_2 is important for the first contracts. This is due to the fact that the first contracts correspond to low retentions, for which all the

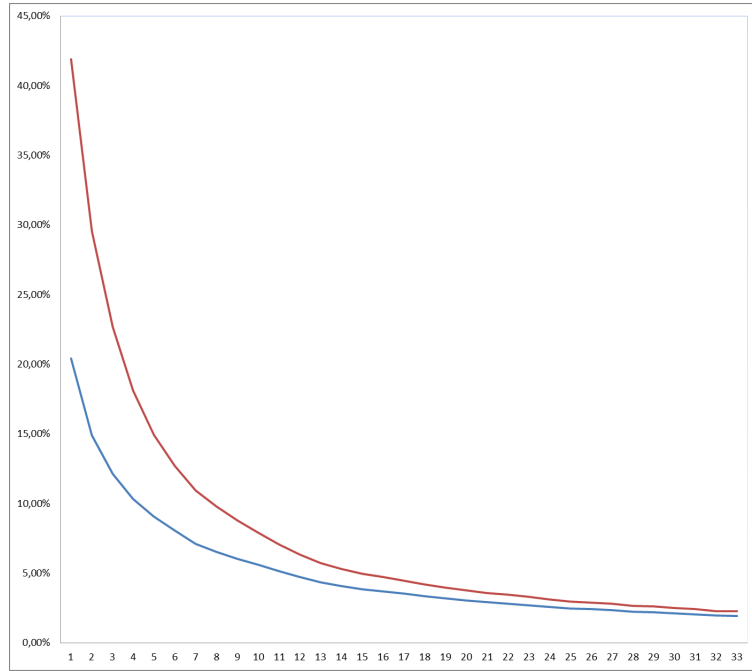


Figure 1: Semi-deviation utility function and $AVaR_\alpha$ risk measure

reconstitutions are systematically used. More precisely, if $k = 4$, the value \tilde{N} appearing in the definitions of p_1 and p_2 will be very often equal to 4 for the contracts with a low retention, and the different quantities $-\bar{U}(-1-4)$ and $U(1+4)$ will contribute considerably to the gap between p_1 and p_2 .

However for the contracts with a high retention, beginning from 1.2 billion, the bounds are rather satisfying.

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