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# Inf-Convolution of Choquet Integrals and Applications in Optimal Risk Transfer

Nabil KAZI-TANI\*

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## Abstract

Motivated by reinsurance optimization, we study in this paper some particular optimal risk transfer problems, between two economic agents who do not share the same risk vision and anticipation. More precisely, we conduct an analysis of Choquet integrals, as non necessarily law invariant monetary risk measures. We first establish a new representation result of convex comonotone risk measures, then we give a representation result of Choquet integrals by introducing the notion of *local distortion*. This allows us to compute in an explicit manner the inf-convolution of two Choquet integrals, with examples illustrating the impact of the absence of the law invariance property.

**Key words:** Capacity, Choquet Integrals, Risk Measures, Inf-convolution, Risk transfer.

**AMS 2010 subject classifications:** 91B30, 91B69

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## 1 Introduction

### 1.1 Choquet integrals

One possible generalization of the Lebesgue integral with respect to a (probability) measure is the Choquet integral with respect to a capacity (definition 2.5). Thus, it provides in particular a generalization of the mathematical expectation. It was first introduced by Choquet in 1953 ([12]).

Choquet asked the following natural question: what are the properties of a measure  $\mu$  that preserve the interior regular approximation property: for each borelian  $B$ ,

$$\mu(B) = \sup\{\mu(K), K \text{ compact}, K \subset B\}.$$

He showed in [12] that additivity properties do not play any role, and that by contrast monotonicity and continuity properties are crucial. That is how the additivity property  $\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B)$  is dropped and replaced by property 3 in definition 2.4. Then Choquet introduced a notion of integral with respect to set functions not having additivity properties (see [15] for a monography).

Capacities and Choquet integrals were first developed and used in the potential theory. Then Choquet integrals have been applied intensively as criteria in economic decisions under uncertainty after the demonstrations by Greco [21] and Schmedler [30] of Choquet integral representations of increasing and comonotonic functionals, under additional continuity assumptions. Since then, an important literature has developed on the study of

economic beliefs and preference relations ([20]), their representations using non-additive capacities ([16], [17], [10]) and the equilibria that it generate ([9], [11], [7]).

We adopt in this paper the point of view of convex risk measures and focus on risk measures that we can represent as Choquet integrals, which are not necessarily law invariant. In [25], Jouini & al. showed that the optimal risk sharing between two agents having law invariant and comonotonic monetary risk measures is given by a sum of stop loss contracts. We extend their result in several directions, since we consider law dependent, and cash sub-additive risk measures, and show that the bounds of the stop loss contracts correspond to quantile values of the total risk. Moreover, we treat the case of  $L^p$  random variables, which is more suitable to real cases and theoretical examples than bounded random variables.

## 1.2 Reinsurance and risk transfer

Reinsurance is a mechanism allowing an insurer to transfer a part of its subscribed risk to a reinsurer.

There are several types of reinsurance contracts. The insurer can transfer its risk in a *proportional* or *linear* way, which means that he gives away a fixed percentage of its losses and of its premiums to the reinsurer: for a loss represented by a positive random variable  $X$ , he leaves  $F = aX$  with  $0 < a < 1$  to the reinsurer and supports  $X - F = (1 - a)X$ .

The insurer can also buy a *non proportional* contract, which allows him to put a ceiling on its losses since he will be exposed to  $X \wedge k$ , the reinsurer taking the remaining part  $(X - k)^+$ , with  $k \in \mathbb{R}^+$ .

A very natural question for the insurer is how to optimally transfer its risk.

Barriou and El Karoui [4] studied the optimal contract design problem using inf-convolution techniques, in particular they find again in a very simple framework a result of Borch [6], stating that the optimal risk transfer is obtained through a linear quota-share contract. The insurer has to stay exposed to a proportional part of its subscribed risk, with a coefficient equal to its relative risk aversion coefficient over the total risk aversion of the two market players. This result of optimal *linear* contracts holds under the assumption that both agents share the same risk measure, with different risk aversion coefficients. If the agents criteria are different, we prove in this paper, as in [25], that optimal contracts are *non linear* and given by a sum of Excess-of-Loss contracts (see (4.4) for a precise definition).

### *Attrition Versus Extreme Risk*

In practice, a great majority of reinsurance exchanges are done in a non proportional way. To understand this fact in view of the previous proportional optimality result, we can distinguish the risk that the insurer is facing, considering the following two components: an "attrition risk" which is essentially a high frequency and low severity risk, and the "extreme risk" that is encountered for example in natural or industrial catastrophes and which is on the contrary a low frequency and high severity risk.

The attrition risk is often considered as the heart of the insurer activity, this is the risk that he knows best and he is in a position to develop a methodology and tools to manage it. In particular, the premium that the insurer receives for the attrition risk is a major tool to manage it, as well as proportional-type reinsurance contracts if the insurer wants to reduce it.

The extreme risk is the one that the insurer really wants to reduce, this risk is considered as the heart of the reinsurer activity. The insurer and reinsurer do not have the same level of information and knowledge on the extreme risk, and this is what justifies the exchange, which is done through non proportional contracts in the majority of cases. We refer the reader to [28] and [1] (Chapter 2) for more details and practical considerations on the attritional and extreme risks.

### *The criterion choice*

A major issue for an insurance company in the design of its reinsurance policy is the choice of its risk retention, that is to say the parameter  $k$  appearing in the payoff function  $X \wedge k$ . In the literature two main criteria are used: maximizing the cumulative expected discounted dividends (see [33] for a survey) or minimizing the ruin probability.

De Finetti ([13]) and then many authors have studied the first criterion (among them [31], [23], [34], [22] and the references therein). The goal in these papers is to find, from the point of view of an insurance company, an optimal strategy concernin both the reinsurance buying process and the dividends distribution process. One of their main findings - using stochastic control techniques - is the existence of a feedback optimal strategy for reinsurance on the one hand, and a "barrier" type optimal strategy for dividends on the other hand, in which the insurance company does not pay any dividends when its wealth process exceeds a certain known level. However in practice, dividends decision are often separated from reinsurance decisions, the main issue of dividends being that it sends a strong signal to the insurance industry and to financial markets concerning the company's financial health, making the barrier strategy hardly applicable.

This paper focuses on the non proportional reinsurance optimization, using convex risk measures as a criterion. Indeed, convex risk measures constitute a flexible tool for optimization, and it is suitable to the inf-convolution operator since the inf-convolution of monetary convex risk measures is again a monetary convex risk measure (see [4]). In particular, we are able to compute the inf-convolution of Choquet integrals (that constitute a particular case of risk measures) explicitly, using *local distortions*, that we introduce (see Lemma 3.1).

## **2 Preliminaries on Risk Measures**

In this section we will give some definitions, properties and representation of convex risk measures, and then state the reinsurance optimization problem using the introduced convex risk measures as criterion.

### **2.1 Introduction**

#### **2.1.1 Axioms**

Motivated by some imperfections of traditional risk measures such as value-at-risk (which is recommended by the solvency II european regulatory requirement for insurance companies), Artzner & al. [2] and then Follmer and Schied [18] and Frittelli and Rosazza-Gianin [19] introduced the notions of coherent and convex risk measures. We will now recall their definitions, and some key properties. We state the given results for a generic space  $\mathcal{X}$  of

random variables, typically we will take  $\mathcal{X} = L^\infty(\mathbb{P})$  or  $\mathcal{X} = L^p(\mathbb{P})$  with  $p \geq 1$ , where  $(\Omega, \mathcal{F}, \mathbb{P})$  is a fixed probability space. We will skip the subscript  $\mathbb{P}$  in the definition of  $\mathcal{X}$  when there is no possible confusion.

**Definition 2.1.** *An application  $\rho : \mathcal{X} \rightarrow \mathbb{R}$  is called:*

- [MO] *Monotonic if  $X \leq Y$  a.s. implies  $\rho(X) \leq \rho(Y)$ .*
- [CI] *Cash-invariant if  $\forall c \in \mathbb{R}, \rho(X + c) = \rho(X) + c$ .*
- [CO] *Convex if  $\forall 0 \leq \lambda \leq 1, \rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ .*
- [PH] *Positively homogeneous if  $\forall \lambda \geq 0, \rho(\lambda X) = \lambda\rho(X)$ .*

*An application verifying the axioms [MO], [CI] is called a monetary risk measure. If it furthermore satisfies [CO] then it is called a convex risk measure, and if it also satisfies the axiom [PH] then we say it is a coherent risk measure.*

The cash-invariance property, also called translation invariance, gives to the value  $\rho(X)$  a monetary unit:  $\rho(X)$  may be intuitively interpreted as the minimal capital amount to add to  $X$ , and to invest in a risk free manner, to make the position  $X$  acceptable or risk free. Indeed  $\rho(X - \rho(X)) = \rho(X) - \rho(X) = 0$ .

The convexity axiom will describe the preference for diversity: it means that diversification should not increase risk.

Note that here positive random variables represent losses, then we consider *increasing* risk measures, and cash-invariant with a ”+” sign.

### 2.1.2 Dual representation

An important kind of results are the Fenchel-Legendre type dual representation theorems for risk measures. It has been given in Artzner & al. [2] for coherent risk measures and Frittelli and Rosazza-Gianin [19] for convex risk measures defined on  $L^\infty$ , and by Biagini and Frittelli [5] in the case of convex risk measures defined on  $L^p$ .

Since the dual space of  $L^\infty$  can be identified with the space  $M^{ba}$  of finitely additive set functions with finite total variation, the application of general duality theorems only gives the dual representation with respect to the subspace of finitely additive set functions normalized to 1. For the representation to hold with probability measures, we need the following additional property:

**Definition 2.2.**  *$\rho$  satisfies the Fatou property if it is lower semi-continuous with respect to the bounded pointwise convergence, i.e if for any bounded sequence  $X_n$  of  $L^\infty$  converging pointwise to  $X$ , we have*

$$\rho(X) \leq \liminf \rho(X_n).$$

**Remark 2.1.** *The Fatou property is equivalent with continuity from below (see [18]).*

We can now state the representation result:

**Theorem 2.1** ([19]). *A convex risk measure  $\rho$  on  $L^\infty$ , satisfying the Fatou property, admits the following dual representation:*

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{M}_1} \{E^{\mathbf{Q}}[X] - \alpha(\mathbf{Q})\}, \quad (2.1)$$

where  $\mathcal{M}_1$  denotes the set of all probability measures on  $\mathcal{F}$  and  $\alpha : \mathcal{M}_1 \rightarrow \mathbb{R} \cup \{+\infty\}$  is a penalty function given by:

$$\alpha(\mathbf{Q}) = \sup_{X \in L^\infty(\mathbb{P})} \{E^{\mathbf{Q}}[X] - \rho(X)\}.$$

If we interpret any probability measure  $\mathbf{Q}$  on  $(\Omega, \mathcal{F})$  as a possible model, or possible weighting of events, this representation tells us that every convex risk measure that is upper semi-continuous can be seen as a worst case expectation over a family of models minus a penalty, the penalty representing the likelihood of the given models.

In the case of risk measures  $\rho$  defined on  $L^p$ , we need the additional property of finiteness of  $\rho$  on  $L^p$ . Indeed, Biagini and Frittelli [5] proved that any convex and finite risk measure on  $L^p$  is continuous with respect to the  $L^p$  norm. Then a dual representation holds as in (2.1) in the classic  $L^p - L^q$  duality.

### 2.1.3 Examples

#### The Value-at-Risk

The Value-at-Risk is the main risk measure used in practice and it is the measure recommended by the Solvency II european regulator for insurance companies. It consists in computing an  $\alpha$ -quantile of the risk  $X$ :

$$VaR_\alpha(X) := \bar{q}_X(\alpha) = \inf\{x \in \mathbb{R} \text{ such that } \bar{F}_X(x) \leq \alpha\},$$

where  $\alpha \in [0, 1]$  and  $\bar{F}_X$  denotes the tail cumulative distribution function of  $X$ .  $\alpha$  is considered as an "acceptable" bankruptcy probability, and then  $VaR_\alpha(X)$  represents the losses that are attained only with that probability.

The measure  $\rho = VaR_\alpha$  is not convex, and then do not satisfy a dual representation, but the following Average Value-at-Risk does.

#### The Average Value-at-Risk

To overcome the non-convexity of the monetary risk measure  $VaR_\alpha$ , we consider the following Average Value-at-Risk at level  $\alpha \in (0, 1]$  of a position  $X \in \mathcal{X}$ , given by:

$$AVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \bar{q}_X(u) du.$$

The name Average Value-at-Risk is justified by the following equality, which holds if  $X$  has a continuous distribution, it says that the quantity  $AVaR_\alpha(X)$  is an average of the losses greater or equal to  $VaR_\alpha(X)$ :

$$AVaR_\alpha(X) = \mathbb{E}[X | X \geq VaR_\alpha(X)].$$

$AVaR_\alpha$  is a coherent risk measure and its dual representation is given as follows:

$$AVaR_\alpha(X) = \sup_{Q \in \mathcal{M}_1} \{E^Q(X) - \alpha(Q)\}, \quad X \in \mathcal{X},$$

where the penalty function  $\alpha$  only takes the values  $+\infty$  or  $0$ , it is given by the indicator function in the sense of convex analysis of the following set:

$$\mathcal{Q} := \{Q \in \mathcal{M}_1 \mid \frac{dQ}{d\mathbb{P}} \leq \frac{1}{\alpha}\}.$$

Note that we have:

$$\text{AVaR}_\alpha(X) = \bar{q}_X(\alpha) + \frac{1}{\alpha} \mathbb{E} [(X - \bar{q}_X(\alpha))^+].$$

#### 2.1.4 Inf-convolution

Given two convex risk measures  $\rho_1$  and  $\rho_2$  with penalty functions  $\alpha_1$  and  $\alpha_2$ , we define their inf-convolution in the following way:

$$\rho_1 \square \rho_2(X) := \inf_{F \in \mathcal{X}} \{\rho_1(X - F) + \rho_2(F)\}. \quad (2.2)$$

Barriou and El Karoui [3] proved under the assumption  $\rho_1 \square \rho_2(0) > -\infty$  that  $\rho_1 \square \rho_2$  is a finite convex risk measure. If  $\rho_1$  is continuous from below, then  $\rho_1 \square \rho_2$  is continuous from below and we have the following dual representation :

$$\rho_1 \square \rho_2(X) := \sup_{\mathbf{Q} \in \mathcal{M}_1} \{E^{\mathbf{Q}}[X] - (\alpha_1 + \alpha_2)(\mathbf{Q})\}. \quad (2.3)$$

The inf-convolution operator appears naturally in optimal risk transfer problems as we will see through the example of reinsurance optimization.

## 2.2 VaR Representation of Convex Risk Measures

**Assumption 2.1.** *We suppose in the rest of the chapter that the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  that we work with is atomless.*

This assumption induces that our probability space supports a random variable with a continuous distribution (see for instance Proposition A.27 in [18]). Its usefulness is mentioned in Example 2.1, it is also important in the proof of Proposition 4.3.

The representation theorems stated above can be reformulated making use of Choquet integrals and distortion functions, that we now introduce.

**Definition 2.3.** *Every nondecreasing function  $\psi : [0, 1] \rightarrow [0, 1]$  with  $\psi(0) = 0$  and  $\psi(1) = 1$  will be called a distortion function.*

For a given distortion function  $\psi$ , we consider a  $[0, 1]$ -valued random variable  $\Lambda_\psi$  whose law is given by:

$$\mathbb{P}(\Lambda_\psi \leq x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \psi(x), & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases} \quad (2.4)$$

We will use random variables in the form of  $\Lambda_\psi$  to give an interpretation of the Choquet integral and of convex risk measures.

**Definition 2.4.** *A set function  $c : \mathcal{F} \rightarrow [0, 1]$  is called*

1. *Normalized if  $c(\emptyset) = 0$  and  $c(\Omega) = 1$ .*



2. Monotone if  $c(A) \leq c(B)$  whenever  $A \subset B$ ,  $A, B \in \mathcal{F}$ .
3. Submodular if  $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$ ,  $\forall A, B \in \mathcal{F}$ .
4. Outer regular if for every decreasing sequence  $(A_n)$  of elements of  $\mathcal{F}$ , we have

$$c\left(\bigcap_n A_n\right) = \lim c(A_n). \quad (2.5)$$

Following [8], we call capacities normalized and monotone set functions.

**Definition 2.5.** The Choquet integral of a random variable  $X$ , with respect to a capacity  $c$  is defined by:

$$\int X dc := \int_{-\infty}^0 (c(X > x) - 1) dx + \int_0^{\infty} c(X > x) dx.$$

**Example 2.1.** Using a distortion function  $\psi$ , we can define a capacity  $c_\psi$  given by  $c_\psi(A) = \psi(\mathbb{P}(A))$ ,  $\forall A \in \mathcal{F}$ . For  $\psi(x) = x$ , the Choquet integral  $\int X dc_\psi$  is just the expectation of  $X$  under the probability measure  $\mathbb{P}$ . The function  $\psi$  is used to distort the expectation operator  $\mathbb{E}_\mathbb{P}$  into the non-linear functional  $\rho_\psi$ . It is known (see for example [18]) that when the probability space is atomless,  $\psi$  is concave if and only if  $\rho_\psi$  is a sub-linear risk measure.

If we define  $\psi(u) := \phi(\phi^{-1}(u) + \alpha)$ , with  $\alpha > 0$ , where  $\phi$  denotes the standard gaussian cumulative distribution function, we obtain the Wang transform risk measure, which is popular in insurance pricing (see Wang [36] for details on this particular Choquet integral).

**Definition 2.6.** We will say that a risk measure  $\rho$  is law invariant, if  $\rho(X) = \rho(Y)$  whenever  $X$  and  $Y$  have the same law under  $\mathbb{P}$ .

The Choquet integrals with respect to capacities constructed from distortion functions as in the previous example can be used as building blocks for any convex and law invariant risk measure which is continuous from below. This is the result of Corollary 4.72 in [18]. This means that we can use Choquet integrals instead of the usual linear expectation in the dual representation (2.1) of any convex, law invariant and continuous from below risk measure  $\rho$  as follows

$$\rho(X) = \sup_{\psi \in \mathcal{C}} \left\{ \int X dc_\psi - \gamma(\psi) \right\}, \quad (2.6)$$

where  $\mathcal{C}$  denotes the set of concave distortion functions and

$$\gamma(\psi) := \sup_{X \in L^\infty(\mathbb{P})} \left\{ \int X dc_\psi - \rho(X) \right\}.$$

In [27] and [18], a dual representation of law invariant convex risk measures is given in terms of *AVaR*. We have the same representation in terms of *VaR* risk measures, at random levels of the form  $\Lambda_\psi$ :

**Proposition 2.1.** *A convex risk measure  $\rho$  is law invariant and continuous from above if and only if*

$$\rho(X) = \sup_{\psi \in \mathcal{C}} (\mathbb{E}_{\mathbb{P}} [VaR_{\Lambda_{\psi}}(X)] - \gamma(\psi))$$

with

$$\gamma(\psi) = \sup_{X \in L^{\infty}(\mathbb{P})} (\mathbb{E}_{\mathbb{P}} [VaR_{\Lambda_{\psi}}(X)] - \rho(X)).$$

**Proof.**

We just notice that since  $\{\bar{q}_X(\Lambda_{\psi}) > x\} = \{\Lambda_{\psi} \leq \mathbb{P}(X > x)\}$ ,

$$\mathbb{P}(\bar{q}_X(\Lambda_{\psi}) > x) = \mathbb{P}(\Lambda_{\psi} \leq \mathbb{P}(X > x)) = \psi(\mathbb{P}(X > x)) = c_{\psi}(X > x).$$

Using this, we can rewrite the Choquet integral as :

$$\int X d c_{\psi} = \int_{-\infty}^0 (\mathbb{P}(\bar{q}_X(\Lambda_{\psi}) > x) - 1) dx + \int_0^{\infty} \mathbb{P}(\bar{q}_X(\Lambda_{\psi}) > x) dx = \mathbb{E}_{\mathbb{P}} [VaR_{\Lambda_{\psi}}(X)]$$

Now using the representation (2.6), we get the desired result.  $\square$

This proof enlightens in particular that the Choquet integral of a r.v.  $X$  is a non linear operator, that can be rewritten as a classic linear expectation of a quantile of  $X$ , evaluated at a random level.

**Remark 2.2.** *(Link with AVaR representations) The dual representation of law invariant convex risk measures, with AVaR as building blocks, is as follows:*

$\rho$  is convex and law invariant if and only if

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left( \int_0^1 AVaR_{\alpha}(X) \mu(d\alpha) - \gamma(\mu) \right)$$

where

$$\gamma(\mu) = \sup_{X \in L^{\infty}(\mathbb{P})} \left( \int_0^1 AVaR_{\alpha}(X) \mu(d\alpha) - \rho(X) \right)$$

and  $\mathcal{M}_1((0,1])$  denotes the set of probability measures on  $(0,1]$ .

These two representations with VaR and AVaR risk measures, evaluated at random levels, lead to no contradiction since

$$\int_0^1 AVaR_{\alpha}(X) \mu(d\alpha) = \int X d c_{\psi} = \mathbb{E}_{\mathbb{P}} [VaR_{\Lambda_{\psi}}(X)]$$

where  $\psi$  is the distortion function defined by  $\frac{d\psi}{dt} = \int_t^1 \frac{1}{s} \mu(ds)$ , with  $\mu$  a given probability measure on  $(0,1]$ .

### 2.3 Optimal risk transfer and inf-convolution

Following the framework of [25] and [3], we consider the problem of optimal risk sharing between an insurer and a reinsurer in a principal-agent framework.

The exposure of the insurer (the agent) is modeled by a positive random variable  $X$ . Both the insurer and the reinsurer assess their risk using an increasing law invariant convex monetary risk measure (resp.  $\rho_1$  and  $\rho_2$ ). For a given loss level  $X$ , the insurer will take in charge  $X - F$  and transfer to the reinsurer a quantity  $F$ , and for this he will pay a premium  $\pi(F)$ .

The insurer (the agent) minimizes his risk under the constraint that a transaction takes place, he solves :

$$\inf_{F, \pi} \{\rho_1(X - F + \pi)\} \text{ under the constraint } \rho_2(F - \pi) \leq \rho_2(0) = 0 \quad (2.7)$$

Binding this last constraint and using the cash-additivity property for  $\rho_2$  gives the optimal price  $\pi = \rho_2(F) - \rho_R(0) = \rho_R(F)$ . This is an indifference pricing rule for the reinsurer, that is to say the price at which he is indifferent (from a risk perspective) between entering and not entering into the transaction.

Replacing  $\pi = \rho_2(F)$  in (2.7) and using the cash-additivity property of  $\rho_1$ , the insurer program becomes equivalent to the following one:

$$\inf_F \{\rho_A(X - F) + \rho_R(F)\} =: \rho_A \square \rho_R(X)$$

We are left with the inf-convolution of  $\rho_1$  and  $\rho_2$ , problem for which we give some explicit solutions in Theorem 4.1 and in Section 5.

### 3 Comonotonic Risk Measures as Choquet Integrals

We start this section with some definitions and properties related to the important notion of comonotonicity.

Let us begin with the definition of comonotonicity for random variables:

**Definition 3.1.** *Two random variables  $X$  and  $Y$  on  $(\Omega, \mathcal{F})$  are called comonotone if*

$$\forall (\omega, \omega') \in \Omega^2, (X(\omega) - X(\omega')) (Y(\omega) - Y(\omega')) \geq 0$$

Otherwise said,  $X$  and  $Y$  are comonotone if we cannot use one variable as a hedge against the other. Denneberg [15] proved that two variables  $X$  and  $Y$  are comonotone if there exist a third random variable  $Z$  such that  $X$  and  $Y$  can be written as nondecreasing functions of  $Z$ .

The variables  $\xi_1$  and  $\xi_2$  corresponding to what is payable by the insurer and the reinsurer in most of common reinsurance contracts correspond to comonotone random variables. For instance, if the total loss is represented by a positive random variable  $X$ ,  $\xi_1 = \alpha X$  and  $\xi_2 = (1 - \alpha)X$  are comonotone. This is also true for  $\xi_1 = X \wedge k$  and  $\xi_2 = (X - k)^+$ .

The risk evaluation corresponding to an aggregate position  $X + Y$  of comonotone variables  $X$  and  $Y$  is then just the sum of the individual risks:

**Definition 3.2.** *A risk measure  $\rho$  is called comonotonic if*

$$\rho(X + Y) = \rho(X) + \rho(Y), \text{ whenever } X, Y \text{ are comonotone random variables.}$$

#### 3.1 Representation results on $L^p$

In [18], it is proved that a monetary risk measure  $\rho$  on  $L^\infty(\mathbb{P})$  is comonotonic if and only if there exists a capacity  $c$  such that

$$\rho(X) = \int X dc, \quad (3.1)$$

and in this case,  $c$  is given by  $c(A) = \rho(\mathbf{1}_A)$ .

We can extend the representation (3.1) to risk measures defined on  $L^p$  spaces, but we will need the additional requirement that  $\rho$  is convex. Indeed, we will need the continuity of the Choquet integral on  $L^p$  to prove the representation on  $L^p$ , and it will be given by the following extended Namioka-Klee theorem proved in [5] for convex functionals:

**Theorem 3.1** (Biagini - Frittelli [5]). *Any finite, convex and monotone increasing risk measure  $\rho : L^p \rightarrow \mathbb{R}$  is continuous on  $L^p$ .*

Using this continuity result, we can state a representation on  $L^p$ , which is a direct consequence of Theorem 4.82 in [18]. Note that Kervarec [26], in the 3rd chapter of her PhD dissertation, also proves a continuity result for coherent risk measures defined on a space analogous to  $L^p$  in a situation with model uncertainty, with a non dominated family of probability measures.

We can now state the representation on  $L^p$ .

**Theorem 3.2.** *Let  $\rho$  be a convex and comonotonic monetary risk measure on  $L^p$ , and let  $C(X) := \int X dc$ , where  $c$  is a capacity given by  $c(A) = \rho(\mathbf{1}_A)$ . If  $\rho$  and  $C$  are finite on  $L^p$ , then*

$$\rho(X) = C(X) \text{ for any r.v } X \text{ in } L^p. \quad (3.2)$$

**Proof.**

Of course Theorem 3.1 applies and  $\rho$  is  $L^p$ -continuous.

First note that since  $\rho$  is comonotonic and since the pair  $(X, X)$  is comonotone, then  $\rho(2X) = 2\rho(X)$ , by induction  $\rho(nX) = n\rho(X)$  for  $n \in \mathbb{N}^*$ , and this easily implies that  $\rho(qX) = q\rho(X)$  for any  $q \in \mathbb{Q}^+$ . We can use the  $L^p$ -continuity of  $\rho$  to deduce that  $\rho(\lambda X) = \lambda\rho(X)$  for any  $\lambda \in \mathbb{R}^+$ . We have obtained that  $\rho$  is positively homogeneous, in particular  $\rho(0) = 0$ , and if we define  $c(A) := \rho(\mathbf{1}_A)$  for any  $A$  in  $\mathcal{F}$ , then  $c$  is a capacity (monotone and normalized), which is also submodular by convexity of  $\rho$ .

We can define the Choquet integral  $C(X) := \int X dc$ .  $C$  is a convex (since  $c$  is submodular) and comonotonic monetary risk measure, so using the finiteness of  $C$  on  $L^p$  and Theorem 3.1, we obtain the continuity of  $C$  on  $L^p$ .

Now, by Theorem 4.82 in [18], we know that  $C$  and  $\rho$  coincide on  $L^\infty$ . By the continuity of  $\rho$  and  $C$  on  $L^p$  and density of  $L^\infty$  in  $L^p$ , we have the desired result.

□

### 3.2 A quantile weighting result for general Choquet integrals

We denote by  $\rho_\psi$  the Choquet integral with respect to the capacity  $c_\psi$  constructed from a distortion function  $\psi$ . Recall that  $\mathcal{X} = L^\infty$  or  $\mathcal{X} = L^p$ .

**Assumption 3.1.** *For any interval  $[a, b]$  such that  $x \rightarrow \mathbb{P}(X > x)$  is constant on  $[a, b]$ , the function  $x \rightarrow c(X > x)$  is also constant on  $[a, b]$ . We will say that  $\bar{c}_X$  inherits the constant intervals of  $\bar{F}_X$ .*

**Remark 3.1.** *If  $Q$  is a probability measure, then  $x \rightarrow Q(X > x)$  inherits the constant intervals of  $\bar{F}_X$  if and only if the law of  $X$  under  $Q$  is absolutely continuous with respect to the law of  $X$  under  $\mathbb{P}$ .*

**Lemma 3.1.** *Let  $c$  be a capacity on  $\mathcal{F}$ . Let  $X$  be a random variable in  $\mathcal{X}$  such that  $\bar{c}_X$  inherits the constant intervals of  $\bar{F}_X$ , then there exists a distortion function  $\psi_X$  such that the Choquet integral with respect to  $c$  coincides with  $\rho_{\psi_X}$  at  $X$  :*

$$\int X dc = \int X d c_{\psi_X}.$$

Furthermore, if the capacity  $c$  is outer regular then  $\psi_X$  is right-continuous.

**Proof.** Recall that  $\bar{q}_X(u) = \inf\{x \in \mathbb{R} \mid \bar{F}_X(x) \leq u\}$ . In particular  $\bar{q}_X(\bar{F}_X(x)) \leq x$ .

Step 1: We first prove that for any real number  $x$ ,

$$c\{X > x\} = c\{X > \bar{q}_X(\bar{F}_X(x))\}.$$

For any  $x$  such that  $\bar{F}_X(x)$  is a point of continuity of  $\bar{q}_X$ , we have  $\bar{q}_X(\bar{F}_X(x)) = x$ , so the claim is obvious in that case. Then let  $x$  be such that  $\bar{F}_X(x)$  is a discontinuity point for the function  $\bar{q}_X$ .

We have  $\bar{q}_X(\bar{F}_X(x)) < x$ , and  $\bar{F}_X$  is constant on the interval  $[\bar{q}_X(\bar{F}_X(x)), x]$ , then by Assumption 3.1,  $\bar{c}_X$  is also constant on this interval which is the desired claim.

Step 2: For a given random variable  $X$  and a capacity  $c$ , we define

$$\psi_X(u) := c(X > \bar{q}_X(u)). \quad (3.3)$$

$\psi_X$  is a distortion function since  $c$  is monotone and normalized, and using step 1 above, we have

$$\psi_X(\mathbb{P}(X > x)) = c(X > \bar{q}_X(\mathbb{P}(X > x))) = c(X > x).$$

And hence,

$$\int_{-\infty}^0 (c(X > x) - 1) dx + \int_0^{\infty} c(X > x) dx = \int_{-\infty}^0 (c_{\psi_X}(X > x) - 1) dx + \int_0^{\infty} c_{\psi_X}(X > x) dx,$$

which is the desired result.

The proof that the outer regularity of  $c$  implies the right continuity of  $\psi_X$  is as follows: Since  $\bar{q}_X$  is right-continuous and non increasing, for every non increasing sequence  $(u_n)$  converging to  $u$ ,  $\bar{q}_X(u_n)$  goes to  $\bar{q}_X(u)$  and  $\{X > \bar{q}_X(u_{n+1})\} \subset \{X > \bar{q}_X(u_n)\}$ . Then the application of the property (2.5) with  $A_n = \{X > \bar{q}_X(u_n)\}$  gives the right continuity of  $\psi_X$ .

□

In [27], Kusuoka provided a representation of *law invariant* risk measures as an integral of weighted quantiles. More precisely, he proved that  $\rho$  is a monetary risk measure which is law invariant and comonotonic if and only if there exists a distortion function  $\psi$  such that for every  $X$  in  $L^\infty(\mathbb{P})$ ,

$$\rho(X) = \int_0^1 \bar{q}_X(u) \psi'(u) du. \quad (3.4)$$

We can still write a quantile representation for a monetary risk measure which is comonotonic but not necessarily law invariant:

**Proposition 3.1.** *Let  $\rho$  be a monetary risk measure satisfying the Fatou property of Definition 2.2, which is comonotonic, and let  $X$  be in  $L^\infty$ . If  $x \rightarrow \rho(\mathbf{1}_{\{X>x\}})$  inherits the constant intervals of  $\bar{F}_X$ , then there exists a distortion function  $\psi_X$  such that*

$$\rho(X) = \int_0^1 \bar{q}_X(u) \psi'_X(u) du. \quad (3.5)$$

**Proof.**

Since  $\rho$  is comonotonic, there exists a capacity  $c$  such that  $\rho(X) = \int X dc$ . The Fatou property of  $\rho$  is equivalent with its continuity from above ( Lemma 4.20 in [18]), and this implies that  $c(A) = \rho(\mathbf{1}_A)$  is outer regular. At the point  $X$ ,  $\rho$  coincides by Lemma 3.1 with a right-continuous  $\psi_X$ -distortion risk measure, and we re-write it as follows:

$$\int_{-\infty}^0 (c_{\psi_X}(X > x) - 1) dx + \int_0^\infty c_{\psi_X}(X > x) dx = \mathbb{E}_{\mathbb{P}} [\bar{q}_X(\Lambda_{\psi_X})]$$

and the last quantity, by definition of  $\Lambda_{\psi_X}$ , is equal to  $\int_0^1 \bar{q}_X(u) \psi'_X(u) du$ , which ends the proof. □

**Remark 3.2.** *The same proof shows that the result is true for risk measures defined on  $L^p$ , provided that  $C(X) = \int X dc$  is finite, where  $c(A) := \rho(\mathbf{1}_A)$ .*

### 3.3 An extension to subadditive comonotonic functionals

The two following results provide representation of subadditive comonotonic risk measures on  $L^\infty$  and  $L^p$  as Choquet integrals.

**Proposition 3.2.**  *$\rho$  is an increasing, comonotonic and cash sub-additive functional on  $L^\infty(\mathbb{P})$ , if and only if there exist a set function  $c$  such that for each  $X$  in  $L^\infty(\mathbb{P})$ ,  $\rho(X) = \int_\Omega X dc$ , where for  $A \in \mathcal{F}$ ,  $c$  is defined by  $c(A) = \nu.\tilde{c}(A)$  and  $\tilde{c}$  is a capacity on  $\mathcal{F}$ . In that case  $\nu = \rho(1) = c(\Omega)$ .*

**Proof.**

First suppose that  $\rho$  is an increasing, comonotonic and cash sub-additive risk measure. Using the monotony and cash sub-additivity of  $\rho$ , we have that  $\rho$  is continuous with respect to the supremum norm since

$$X \leq Y + \|X - Y\|_\infty \Rightarrow \rho(X) \leq \rho(Y + \|X - Y\|_\infty) \leq \rho(Y) + \|X - Y\|_\infty,$$

Then  $|\rho(X) - \rho(Y)| \leq \|X - Y\|_\infty$ . Lipschitz continuity and comonotonicity imply that  $\rho$  is positively homogeneous (see lemma 4.77 in [18]). Then  $\rho(0) = 0$ , and by comonotonicity and positive homogeneity,

$$\rho(X + m) = \rho(X) + \rho(m) = \rho(X) + m\rho(1)$$

and  $\rho(1) \leq 1$  by cash sub-additivity. So  $\rho$  is linear cash-subadditive. Set  $\nu := \rho(1)$  and define  $\tilde{\rho}(X) := \rho(\frac{1}{\nu}X)$ , then  $\tilde{\rho}$  is increasing, comonotonic, and cash additive, using (3.1), we know that there exist a capacity  $\tilde{c}$  such that  $\tilde{\rho}(X) = \int X d\tilde{c}$ . Setting  $c(A) = \nu.\tilde{c}(A)$ , we obtain  $\rho(X) = \int X dc$ .

Now suppose that  $\rho$  is the Choquet integral on  $L^\infty(\mathbb{P})$  with respect to a monotone set function  $c$  such that  $c(\Omega) \leq 1$ , then it is proved in [15] that  $\rho$  is increasing and comonotonic. Furthermore,  $\rho(X + m) = \int X dc + mc(\Omega) \leq \rho(X) + m$ .

□

Using the same arguments as in the proofs of Theorem 3.2 and Proposition 3.2, we can show the following:

**Corollary 3.1.** *Let  $\rho$  be an increasing, convex, comonotonic and cash sub-additive functional on  $L^p$ . Let  $C(X) := \int X dc$  where  $c$  is the capacity given by  $c(A) := \rho(\mathbf{1}_A)$ . Assume that  $\rho$  and  $C$  are finite on  $L^p$ , then*

$$\rho(X) = \int X dc, \quad \forall X \in L^p.$$

In view of the above results of sections 3.1 and 3.3, the inf-convolution of comonotonic risk measures reduces to the inf-convolution of Choquet integrals, the study of which will require some preliminary results that we enunciate and prove below.

## 4 Inf-Convolution of Choquet Integrals

As shown in the following proposition, the Kusuoka-type quantile representation (3.5) is stable under inf-convolution. This extends a result in [24] to non necessarily law invariant measures.

**Proposition 4.1.** *Let  $\rho_1$  and  $\rho_2$  be convex comonotonic risk measures satisfying the Fatou property and let  $X \in \mathcal{X}$ . Assume that  $\rho_1 \square \rho_2(0) > -\infty$  and the functions  $x \rightarrow \rho_i(\mathbf{1}_{\{X > x\}})$  inherit the constant intervals of  $\bar{F}_X$ .*

*If  $\mathcal{X} = L^p$ , we also assume that  $\rho_i$  and its associated comonotonic risk measure is finite on  $L^p$ ,  $i = 1, 2$ .*

*Then there exist two distortion functions  $g_1^X$  and  $g_2^X$  such that*

$$\rho_1 \square \rho_2(X) = \int \bar{q}_X(u) (g_1^X \wedge g_2^X)'(u) du \quad (4.1)$$

where  $(g_1^X \wedge g_2^X)(x) := \min(g_1^X(x), g_2^X(x))$ .

**Proof.** Since  $\rho_1$  and  $\rho_2$  are convex and comonotonic, using either (3.1) or (3.2) we have the existence of two submodular capacities  $c_1$  and  $c_2$  such that  $\rho_1, \rho_2$  are the associated Choquet integrals. Let  $c_{1,2}(A) := \min(c_1(A), c_2(A))$ ,  $\forall A \in \mathcal{F}$ . The dual representation of  $\rho_1$  and  $\rho_2$  is given by:

$$\rho_i(X) = \sup_{Q \in \mathcal{M}_{1,f}} \{E_Q(X) - \alpha_i(Q)\} \quad (4.2)$$

with  $\alpha_i(Q) = 0$  if  $Q \in \mathcal{C}_i$  and  $+\infty$  if not.

$\mathcal{C}_i$  is the core of  $c_i$  defined by  $\mathcal{C}_i = \{Q \in \mathcal{M}_{1,f} | Q(A) \leq c_i(A), \forall A \in \mathcal{F}\}$  and  $\mathcal{M}_{1,f}$  denotes the set of finitely additive set functions. Now theorem 3.6 in [3] says that  $\rho_1 \square \rho_2$  is a convex risk measure with a penalty function given by  $\tilde{\alpha}(Q) = \alpha_1(Q) + \alpha_2(Q)$ , which equals 0 if  $Q \in \tilde{\mathcal{C}} = \mathcal{C}_1 \cap \mathcal{C}_2 = \{Q \in \mathcal{M}_{1,f} | Q(A) \leq \min(c_1(A), c_2(A)), \forall A \in \mathcal{F}\}$  and  $+\infty$  if not.  $\tilde{\alpha}(Q)$  is the core of  $c_{1,2}$ .  $c_{1,2}$  is obviously a capacity, and using theorem 4.88 in [18],  $c_{1,2}$  is a

submodular capacity (here the Assumption 2.1 of atomless probability space is important) and we have

$$\rho_1 \square \rho_2(X) = \int X dc_{1,2}. \quad (4.3)$$

Now since  $c_{1,2}(X > x)$  inherits the constant intervals of  $\bar{F}_X(x)$  (since  $c_1$  and  $c_2$  do), we can use Lemma 3.1 to provide the existence of a distortion function  $\tilde{g}$  such that  $\int X dc_{1,2}$  coincides with the Choquet integral with respect to the capacity  $\tilde{g}(\mathbb{P}(\cdot))$  at  $X$ . We have

$$\tilde{g}(u) = c_{1,2}(X > \bar{q}_X(u)) = c_1(X > \bar{q}_X(u)) \wedge c_2(X > \bar{q}_X(u)) = \min(g_1^X(u), g_2^X(u)),$$

with  $g_i^X(u) = c_i(X > \bar{q}_X(u))$ ,  $i = 1, 2$ . This implies, as in the demonstration of Proposition 3.1, that  $\rho_1 \square \rho_2(X) = \int \bar{q}_X(u)(g_1^X \wedge g_2^X)'(u) du$  (this is where the Fatou property is needed).

□

**Remark 4.1.** *The formula (4.3) in the previous proof tells us that the inf-convolution  $\rho_1 \square \rho_2(X)$  is again a Choquet integral with respect to the capacity*

$$c_{1,2}(A) = \min(c_1(A), c_2(A)).$$

*In the particular case where  $c_1$  and  $c_2$  are constructed with distortion functions  $\psi_1$  and  $\psi_2$  as in Example 2.1, then  $c_{1,2}$  is also a distorted probability with respect to the function  $\psi_{1,2}(u) := \min(\psi_1(u), \psi_2(u))$ .*

*If at a given point  $u^*$  in  $[0, 1]$ , we have  $\psi_1(u^*) < \psi_2(u^*)$ , then we can say that at the given quantile level  $u^*$ , agent 1 is less risk averse than agent 2, since he uses a less severe distortion of the a-priori probability  $\mathbb{P}$  at this point  $u^*$ . Now the form of  $\psi_{1,2}$  suggests that the inf-convolution splits the total risk  $X$  in many parts, which are assigned to the agent being the less risk averse on the corresponding quantile intervals (see Theorem 4.1 and Section 5 for more details).*

Using this result, we are able to solve explicitly the inf-convolution of convex Choquet integrals.

Recall that the core  $\mathcal{C}_i$  of the capacity  $c_i$  is defined by

$$\mathcal{C}_i = \{Q \in \mathcal{M}_{1,f} \mid Q(A) \leq c_i(A), \forall A \in \mathcal{F}\}$$

where  $\mathcal{M}_{1,f}$  denotes the set of finitely additive set functions on  $\Omega$ .

In order to correctly formulate our next theorem, we need the following definitions.

**Definition 4.1.** *We will say that a subset  $B$  of  $[0, 1]$  is densely ordered if for any  $x, y$  in  $B$  with  $x < y$ , there exists  $z$  in  $B$  such that  $x < z < y$ .*

For instance the set  $\mathbb{Q} \cap [0, 1]$  is numerable and densely ordered.

**Definition 4.2.** *Let  $f$  be a real valued function defined on  $[0, 1]$ . We define the changing sign set of  $f$  by those  $u$  in  $(0, 1)$  such that for every sequence  $(u_n)$  increasing towards  $u$  and every sequence  $(v_n)$  decreasing towards  $u$ , there is an integer  $N$ , such that  $\forall n \geq N$ ,  $\text{sgn}[f(u_n)] = -\text{sgn}[f(v_n)]$ , where as usual,  $\text{sgn}(x) = -1$  if  $x < 0$  and  $\text{sgn}(x) = 1$  if  $x > 0$ .*

The two previous definitions are used to formulate the following assumption on two distortion functions.



**Assumption 4.1.** Let  $\psi_1$  and  $\psi_2$  be two distortion functions and denote  $f := \psi_1 - \psi_2$ . The set of changing sign points of  $f$  contains no densely ordered subset.

The economic interpretation of the previous assumption is as follows: it means that  $\psi_1 - \psi_2$  does not change sign too often. Recall from the previous remark that these functions represent uncertainty weighting, if  $\psi_1(u) \geq \psi_2(u)$  for a certain point  $u$ , an economic agent using  $\psi_1$  as a distortion function assigns more uncertainty to the point  $u$  than an agent 2 using  $\psi_2$ . Now when the sign of  $\psi_1 - \psi_2$  changes, it means that the order of uncertainty weighting between agent 1 and agent 2 changes. If it changes very often in the sense that Assumption 4.1 is not satisfied, this means that the agents relative agreement on the probability weighting changes very often.

A sufficient condition for Assumption 4.1 to hold is that the distortions  $\psi_1$  and  $\psi_2$  are continuous and have a finite number of crossing points. This will be the case in all the examples we consider.

**Theorem 4.1.** Let  $\rho_1$  and  $\rho_2$  be two comonotonic risk measures satisfying the assumptions of Proposition 4.3. Suppose also that the associated local distortions  $g_1^X$  and  $g_2^X$  are continuous and satisfy Assumption 4.1.

Then there exists a possibly infinite increasing sequence  $\{k_n, n \leq N\}$  of real numbers corresponding to quantile values of  $X$ , such that

$$\rho_1 \square \rho_2(X) = \rho_1(X - Y^*) + \rho_2(Y^*)$$

where  $Y^*$  is given by:

$$Y^* = \sum_{p=0}^N (X - k_{2p})^+ - (X - k_{2p+1})^+. \quad (4.4)$$

**Remark 4.2.** If the sequence  $\{k_n, n \geq 0\}$  only takes two non zero values  $k_1$  and  $k_2$ , then the first agent (the insurer in that case) takes in charge  $X \wedge k_1 + (X - k_2)^+$  and agent two (the reinsurer) is exposed to  $(X - k_1)^+ - (X - k_2)^+$ . This is exactly the definition of the Excess-of-Loss contract, denoted "A-XS-B", that we find in a large majority of catastrophe reinsurance market exchanges (here  $B = k_1$  and  $A = k_2 - k_1$ ).

**Remark 4.3.** This result means that the inf-convolution of law invariant and comonotonic risk measures is given by a generalization of the Excess-of-Loss contract, with more threshold values. The domain  $\mathbb{R}^+$  of attainable losses is divided in "ranges", and each range is alternatively at the charge of one of the two agents.

The case where  $N$  is infinite is of course impossible in practice, but it gives an additional conceptual insight to the optimal structure of the risk transfer.

**Proof.**

We start by writing the representation (4.1) for the inf-convolution and divide it in two parts. For  $X \in \mathcal{X}$ , let  $A := \{x \in [0, 1] : g_1^X(x) \leq g_2^X(x)\}$  (we will write  $g_1$  and  $g_2$  instead of  $g_1^X$  and  $g_2^X$  for convenience),

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \int_0^1 \bar{q}_X(u) (g_1 \wedge g_2)'(u) du \\ &= \int_0^1 \bar{q}_X(u) g_1'(u) \mathbf{1}_A(u) du + \int_0^1 \bar{q}_X(u) g_2'(u) \mathbf{1}_{A^c}(u) du. \end{aligned}$$

Since  $g_1$  and  $g_2$  satisfy Assumption 4.1 and are continuous, the following non increasing sequence  $\{u_n, n \geq 1\}$  is well defined: it corresponds to the points of equality of the functions  $g_1$  and  $g_2$  defined as follows :

$$u_0 = 1; u_{n+1} = \sup\{x < u_n \text{ such that } g_1(x) = g_2(x) \text{ and } g_1(x^-) \neq g_2(x^-)\}.$$

For each  $n$  we define  $k_n := \bar{q}_X(u_n)$ .

We can write the set  $A$  as a numerable union of disjoint intervals  $A = \bigcup_{n \geq 0} [u_{n+1}, u_n]$ , where  $u_0 = 1, \lim_{n \rightarrow +\infty} u_n = 0$  and possibly there are integers  $k \in \mathbb{N}$  such that  $g_1(x) = g_2(x)$  for any  $x \in [u_{k+1}, u_k]$ . The non increasing sequence  $\{u_n, n \geq 1\}$  corresponds to points of equality of the functions  $g_1$  and  $g_2$ . We can rewrite the inf-convolution

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \sum_{p \geq 1} \int_0^1 \bar{q}_X(u) \mathbf{1}_{(u_{2p+1} \leq u \leq u_{2p})} g_1'(u) du \\ &\quad + \sum_{p \geq 1} \int_0^1 \bar{q}_X(u) \mathbf{1}_{(u_{2p+2} \leq u \leq u_{2p+1})} g_1'(u) du. \end{aligned} \quad (4.5)$$

Then we remark that

$$\begin{aligned} \bar{q}_X(u) \mathbf{1}_{(u_{i+1} \leq u \leq u_i)} &= (\bar{q}_X(u) - \bar{q}_X(u_i))^+ - (\bar{q}_X(u) - \bar{q}_X(u_{i+1}))^+ \\ &\quad - \bar{q}_X(u_{i+1}) \mathbf{1}_{\{u \leq u_{i+1}\}} + \bar{q}_X(u_i) \mathbf{1}_{\{u \leq u_i\}}, \end{aligned}$$

We define the increasing sequence  $(k_i)$  by  $k_i := \bar{q}_X(u_i)$ , and the random variables  $Z := (X - k_i)^+ - (X - k_{i+1})^+$  and  $Z_i = \mathbf{1}_{\{X > k_i\}}$ , then we have

$$\bar{q}_X(u) \mathbf{1}_{(u_{i+1} \leq u \leq u_i)} = \bar{q}_Z(u) - k_{i+1} \bar{q}_{Z_{i+1}}(u) + k_i \bar{q}_{Z_i}(u).$$

We can plug this equality into (4.5) to obtain

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \rho_1 \left( \sum_{p \geq 1} (X - k_{2p})^+ - (X - k_{2p+1})^+ \right) \\ &\quad + \rho_2 \left( \sum_{p \geq 1} (X - k_{2p+1})^+ - (X - k_{2p+2})^+ \right) \\ &\quad - \sum_{p \geq 1} \left( k_{2p+1} \rho_1(X > k_{2p+1}) - k_{2p} \rho_1(X > k_{2p}) \right) \\ &\quad - \sum_{p \geq 1} \left( k_{2p+2} \rho_2(X > k_{2p+2}) - k_{2p+1} \rho_2(X > k_{2p+1}) \right) \end{aligned}$$

Where we denoted  $\rho(X > k) := \rho(\mathbf{1}_{\{X > k\}})$ .

Finally, by noticing that for any  $i$ ,  $\rho_1(\mathbf{1}_{\{X > k_i\}}) = g_1(u_i) = g_2(u_i) = \rho_2(\mathbf{1}_{\{X > k_i\}})$ , we obtain the desired claim.  $\square$

The last proof shows in particular that the points  $k_i$  defining the risk transfer are such that  $\rho_1(X > k_i) = \rho_2(X > k_i)$ . This is the counterpart on  $\mathbb{R}$  of the equality at the points  $u_i$  belonging to  $[0, 1]$  of the local distortion functions  $g_1^X$  and  $g_2^X$ .

In Figure 1 below, we represented two possible functions  $g_1^X$  and  $g_2^X$  crossing twice on  $(0, 1)$ , and thus generating a layer as the optimal structure for the risk sharing. Assume that an insurer uses the local distortion at  $X$  corresponding to the red curve, and that a reinsurer uses the one corresponding to the green curve.

In the interval  $(w, 1)$ , the reinsurer uses a more severe probability distortion, and thus all the risk is assumed by the insurer in that region, since  $(w, 1)$  corresponds to losses  $X$  belonging to  $(0, k_1)$ , with  $k_1 = \bar{q}_X(w)$ . The same thing happens for the interval  $(0, v)$  corresponding to losses belonging to  $(k_2, +\infty)$ , with of course  $k_2 = \bar{q}_X(v)$ . Finally, the reinsurer assumes all the losses belonging to  $(k_1, k_2)$ , since he has a less severe distortion on the interval  $(v, w)$ .

With the notations of the last proof,  $k_1$  and  $k_2$  are the only two finite points such that we have the "agreement"  $\rho_1(X > k_i) = \rho_2(X > k_i)$ .

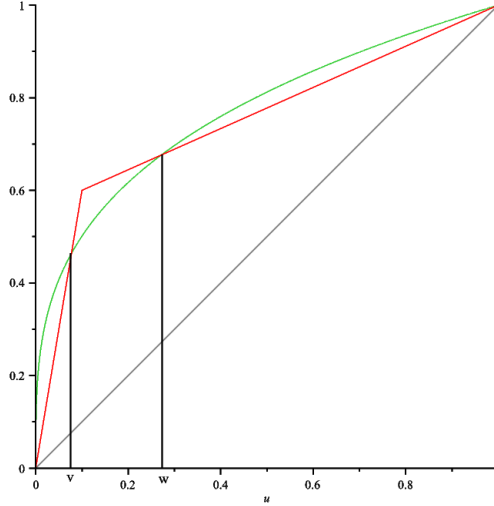


Figure 1: Example of configuration leading to a layer as optimal risk transfer

## 5 Examples

### 5.1 AVaR Examples

1. Let  $\alpha$  and  $\beta$  be in  $(0, 1)$  and consider  $\rho_1 = AVaR_\alpha$  and  $\rho_2 = AVaR_\beta$ . Then we have

$$(AVaR_\alpha \square AVaR_\beta)(X) = AVaR_{\alpha \wedge \beta}(X), \quad \forall X \in L^\infty.$$

Indeed,  $AVaR_\alpha$  is the Choquet integral with respect to the distorted probability  $c(A) = \psi_\alpha(\mathbb{P}(A))$  where

$$\psi_\alpha(u) = \frac{u}{\alpha} \wedge 1.$$

It is easy to see that if  $\beta < \alpha$ , then  $\psi_\alpha(u) \leq \psi_\beta(u)$  for any  $u$  in  $[0, 1]$ . This means that the distortion  $\psi_\beta$  is always more severe than the distortion  $\psi_\alpha$ , and from Theorem 4.1 we know that the agent using the less severe distortion assumes all the risk.

2. A classic measure used in reinsurance as a risk measure or as a pricing functional is the *Proportional Hazard* transform, or PH-transform [35]. It is defined by the following Choquet integral

$$\rho_{PH}(X) = \int X dc, \quad \text{with } c(X > x) = \mathbb{P}(X > x)^r, \quad 0 < r < 1.$$

Using Theorem 4.1, and the fact that the functions  $\psi_\alpha(u) = \frac{u}{\alpha} \wedge 1$  and  $\psi_{PH}(u) = u^r$  only coincide at the point  $u^* = \alpha^{\frac{1}{1-r}}$ , we obtain that

$$\rho_{PH} \square AVaR_\alpha(X) = \rho_{PH}(X \wedge k^*) + AVaR_\alpha((X - k^*)^+),$$

with  $k^* = \bar{q}_X(\alpha^{\frac{1}{1-r}})$ .

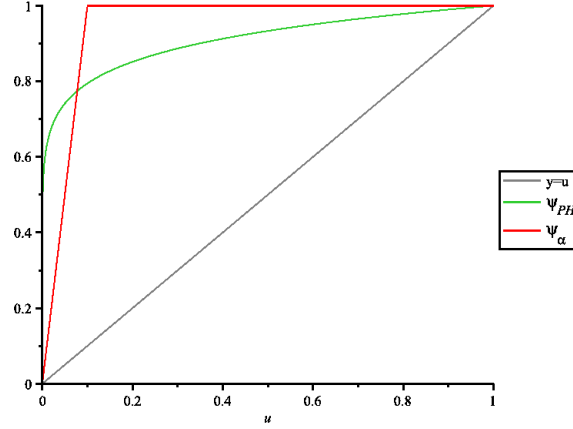


Figure 2: PH-transform and  $AVaR_\alpha$  distortions

We can take for example  $\alpha = r = \frac{1}{10}$ , in that case it leads to the distortion functions represented in Figure 2 and  $u^* = \alpha^{\frac{1}{1-r}} = 0.0774$ . If  $X$  has a beta distribution with density  $f_X(x) = \frac{x^{v-1}(1-x)^{w-1}}{\beta(v,w)}$ ,  $v = 4$  and  $w = 8$ , then  $k^* = 0.53$ .

Assume that an insurance company is facing a peril corresponding to a random destruction rate which has a  $\beta(4, 8)$  distribution on  $[0, 1]$ . If the total insurable value is given by  $M > 0$ , then an insurer using the PH-transform Choquet integral will cede every losses above the value  $0.53M$  to a reinsurer using the  $AVaR_\alpha$  as risk measure.

## 5.2 Epsilon-contaminated capacities

Let  $\nu$  be the capacity given as the *Epsilon-contamination* of a given probability measure  $Q$ , which means that  $\nu$  is the submodular capacity defined by

$$\nu(A) := (1 - \epsilon)Q(A) + \epsilon, \quad 0 < \epsilon < 1 \quad \text{for } A \neq \{\emptyset\} \quad \text{and } \nu(\{\emptyset\}) = 0$$

Let  $X$  be a bounded random variable, denote  $X^* := \sup X(\omega)$  and assume that  $X^* > 0$ . Applying the definition of the Choquet integral of  $X$  with respect to  $\nu$ , we get

$$\begin{aligned} \int X d\nu &= \int_{-\infty}^0 (\nu(X > x) - 1) dx + \int_0^{X^*} \nu(X > x) dx + \int_{X^*}^{+\infty} \nu(X > x) dx \\ &= \int_{-\infty}^0 (1 - \epsilon)(Q(X > x) - 1) dx + \int_0^{X^*} [(1 - \epsilon)Q(X > x) + \epsilon] dx \\ &= (1 - \epsilon)E^Q[X] + \epsilon X^*. \end{aligned}$$

In this case the Choquet integral is affected by the supremum of  $X$ .

By analogy with Dow and Werlang [], we define the uncertainty aversion of a capacity  $\nu$  at  $A \in \mathcal{F}$  by

$$e(\nu, A) := \nu(A) + \nu(A^c) - 1.$$

It is easy to check that an Epsilon contamination  $\nu$  of  $Q$  is a constant uncertainty aversion capacity, i.e  $e(\nu, A) = \epsilon$ , for any  $A \neq \{\emptyset\}$ ,  $A \neq \Omega$ .

If two agents are using respectively an  $\epsilon_1$  and  $\epsilon_2$ -contaminated capacities  $\nu_1$  and  $\nu_2$ , we will say that agent one is more uncertainty averse than agent two if  $\epsilon_1 > \epsilon_2$ .

**Remark 5.1.** *We give here slightly different definitions than Carlier, Dana and Shahidi [] or Dow and Werlang []. This is due to the facts that we consider submodular instead of supermodular capacities and that a positive random variable  $X$  represents losses for us, instead of gains, so a risk averse agent seeks for instance to increase rather than decrease the expectation of  $X$  with respect to the a priori probability measure.*

**Remark 5.2.** *A capacity  $\nu$ , given as an Epsilon-contamination of a probability measure  $Q$  is not a distortion of the probability  $Q$  in the sense of Example 2.1. In particular, if we define a capacity  $\tilde{\nu}$  by*

$$\tilde{\nu}(A) := \psi(Q(A)), \quad \text{with } \psi \text{ given by } \psi(0) = 0 \text{ and } \psi(u) = (1 - \epsilon)u + \epsilon, \quad 0 < u \leq 1,$$

then  $\nu \neq \tilde{\nu}$ . Indeed, for a  $Q$ -negligible set  $B \neq \{\emptyset\}$ , we have

$$\tilde{\nu}(B) = 0 \quad \text{and} \quad \nu(B) = \epsilon > 0.$$

We will nonetheless write the Choquet integral with respect to  $\nu$  using local distortion functions.

Since  $\int X d\nu$  is affected by  $\sup X(\omega)$ , we only consider bounded random variables  $X$ , but the simple uniform distribution on  $[0, 1]$  already provides an interesting example.

Recall that our a priori given probability space is  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will write  $\nu(X > x)$  as  $\psi'_X(\mathbb{P}(X > x))$  for some local distortion function  $\psi'_X$ .

We assume that agent one uses a Choquet integral with respect to the capacity  $\nu_1$ , given as an  $\epsilon_1$ -contamination of a probability measure  $Q$ , such that  $Q \ll \mathbb{P}$ . Agent two uses a Choquet integral with respect to  $\nu_2$ , the  $\epsilon_2$ -contamination of  $\mathbb{P}$ . Let us take  $\epsilon_1 < \epsilon_2$ . In other word we have

$$\begin{aligned} \nu_1(A) &= (1 - \epsilon_1)Q(A) + \epsilon_1, \quad \text{with } Q \ll \mathbb{P} \quad \text{for } A \neq \{\emptyset\} \quad \text{and } \nu_1(\{\emptyset\}) = 0, \\ \nu_2(A) &= (1 - \epsilon_2)\mathbb{P}(A) + \epsilon_2 \quad \text{for } A \neq \{\emptyset\} \quad \text{and } \nu_2(\{\emptyset\}) = 0. \end{aligned}$$

Notice that  $\nu_1$  and  $\nu_2$  are two different examples of submodular capacities which are not given by a distortion of the probability measure  $\mathbb{P}$ .

The following lemma gives an example of two different optimal risk transfers, with two random variables having the same law.

**Lemma 5.1.** *Let  $Z$  and  $Z'$  be two independent and uniformly distributed random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $[0, 1]$ . Let  $\theta > 0$  be a given parameter and define  $Q$  as the  $\theta$ -Esscher transform of  $\mathbb{P}$ , i.e  $\frac{dQ}{d\mathbb{P}} = \frac{e^{\theta Z}}{\mathbb{E}\mathbb{P}[e^{\theta Z}]}$ . We have*

$$\begin{aligned} \int Z d\nu_1 &= \int_0^1 \psi_Z^{\nu_1}[\mathbb{P}(X > x)]dx \quad \text{where} \quad \psi_Z^{\nu_1}(u) = \epsilon_1 + (1 - \epsilon_1) \left( \frac{e^\theta - e^{(1-u)\theta}}{e^\theta - 1} \right), \\ \int Z d\nu_2 &= \int_0^1 \psi_Z^{\nu_2}[\mathbb{P}(X > x)]dx \quad \text{where} \quad \psi_Z^{\nu_2}(u) = \epsilon_2 + (1 - \epsilon_2)u, \\ \int Z' d\nu_1 &= \int_0^1 \psi_{Z'}^{\nu_1}[\mathbb{P}(X > x)]dx \quad \text{where} \quad \psi_{Z'}^{\nu_1}(u) = \epsilon_1 + (1 - \epsilon_1)u, \\ \int Z' d\nu_2 &= \int_0^1 \psi_{Z'}^{\nu_2}[\mathbb{P}(X > x)]dx \quad \text{where} \quad \psi_{Z'}^{\nu_2}(u) = \psi_Z^{\nu_2}(u) = \epsilon_2 + (1 - \epsilon_2)u. \end{aligned}$$

As a consequence, we have the following optimal risk transfer structures:

$$\rho_1 \square \rho_2(Z) = \rho_1\left((X - k^*)^+\right) + \rho_2(X \wedge k^*)$$

with  $k^* = \bar{q}_X^{\mathbb{P}}(u^*)$ , where  $u^* \in (0, 1)$  is the solution of the equation  $\psi_Z^{\nu_1}(u) = \psi_Z^{\nu_2}(u)$ . Furthermore, we have

$$\rho_1 \square \rho_2(Z') = \rho_1(X),$$

where we denoted by  $\rho_1$  and  $\rho_2$  the Choquet integrals with respect to  $\nu_1$  and  $\nu_2$ .

**Proof.** The functions  $\nu_1(Z > x)$  and  $\nu_2(Z > x)$  for  $x \in (0, 1)$  inherit the constant intervals of  $\mathbb{P}(Z > x)$  since  $\mathbb{P}(Z > x)$  is never constant on  $(0, 1)$ , so there is nothing to check, and this is also true for  $Z'$  instead of  $Z$ .

Then Lemma 3.1 gives that

$$\int Z d\nu_i = \int_0^1 \psi_Z^{\nu_i}[\mathbb{P}(X > x)] dx, \quad \text{with } \psi_Z^{\nu_i}(u) = \nu_i(Z > \bar{q}_Z^{\mathbb{P}}(u)), \quad i = 1, 2.$$

We calculate for instance, for  $0 < u \leq 1$

$$\nu_1(Z > \bar{q}_Z^{\mathbb{P}}(u)) = (1 - \epsilon_1)Q(Z > \bar{q}_Z^{\mathbb{P}}(u)) + \epsilon_1 \quad \text{and} \quad \psi_Z^{\nu_1}(0) = 0.$$

Moreover

$$Q(Z > \bar{q}_Z^{\mathbb{P}}(u)) = \mathbb{E}^{\mathbb{P}} \left[ \frac{e^{\theta Z}}{\mathbb{E}^{\mathbb{P}}[e^{\theta Z}]} \mathbf{1}_{\{Z > \bar{q}_Z^{\mathbb{P}}(u)\}} \right] = \frac{e^{\theta} - e^{(1-u)\theta}}{e^{\theta} - 1},$$

since  $\bar{q}_Z^{\mathbb{P}}(u) = 1 - u$ . This proves the first claim of the Lemma. The functions  $\psi_Z^{\nu_2}$ ,  $\psi_{Z'}^{\nu_1}$ ,  $\psi_{Z'}^{\nu_2}$  are computed similarly, and using the fact that  $Z$  is independent of  $Z'$ . The different optimal risk transfer structures are direct applications of Theorem 4.1. □

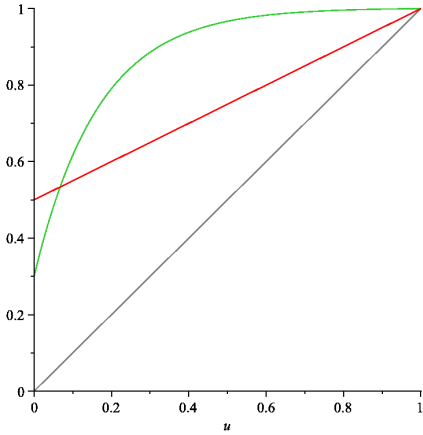


Figure 3: The functions  $\psi_Z^{\nu_1}$  (green curve) and  $\psi_Z^{\nu_2}$  (red curve)

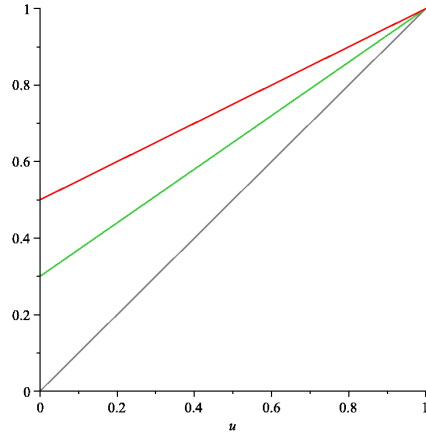


Figure 4: The functions  $\psi_{Z'}^{\nu_1}$  (green curve) and  $\psi_{Z'}^{\nu_2}$  (red curve)

In Figures 3 and 4 above, we represented the local distortions  $\psi_Z^{\nu_i}$  and  $\psi_{Z'}^{\nu_i}$ ,  $i = 1, 2$ , with  $(\epsilon_1, \epsilon_2, \theta) = (0.3, 0.5, 6)$ .

We clearly see in Figure 4 that for the r.v.  $Z'$ , there is no risk transfer, agent one will assume all the risk, since he is less risk averse than agent two. Whereas Figure 3 shows that there is a unique agreement point  $k^*$  such that  $\rho_1(X > k^*) = \rho_2(X > k^*)$ , and  $k^*$  is approximately given by  $\bar{q}_Z^{\mathbb{P}}(0.067) = 0.932$ .

### 5.3 Volatility uncertainty

Computing the Choquet integral of a r.v  $X$  with respect to a submodular capacity is a way to inflate the  $\mathbb{P}$ -expectation of  $X$ , and hence to take into account the model uncertainty we have relatively to the probability measure  $\mathbb{P}$ .

In the recent literature on model uncertainty, we can find important contributions to the particular case of volatility uncertainty ([29], [14] or [32]). These papers all use probabilities of a certain form that we now describe.

Let  $\Omega = \mathcal{C}(\mathbb{R}^+, \mathbb{R}^d)$  be the canonical space of  $\mathbb{R}^d$ -valued continuous paths defined on  $\mathbb{R}^+$ , equipped with its Borel  $\sigma$ -field,  $B$  the canonical process on  $\Omega$  and  $\mathbb{P}_0$  the Wiener measure. Let  $\alpha : \mathbb{R}^+ \rightarrow \mathbb{S}_d^{>0}$  be a deterministic function such that  $\int_0^1 |\alpha_t| dt < +\infty$ , where  $\mathbb{S}_d^{>0}$  denotes the space of all  $d \times d$  real valued positive definite matrices.

We define the probability measure  $\mathbb{P}^\alpha$  on  $\Omega$  by

$$\mathbb{P}^\alpha = \mathbb{P}_0 \circ (X^\alpha)^{-1}, \quad \text{where} \quad X_t^\alpha := \int_0^t \alpha^{1/2}(s) dB_s, \quad t \in [0, 1], \quad \mathbb{P}_0 - a.s.$$

which means that  $\mathbb{P}^\alpha$  is the law under  $\mathbb{P}_0$  of the process  $X^\alpha$ . Then a key observation is that the  $\mathbb{P}^\alpha$ -distribution of  $B$  is equal to the  $\mathbb{P}_0$ -distribution of  $X^\alpha$ . In particular,  $B$  has a  $\mathbb{P}^\alpha$ -quadratic variation which is absolutely continuous with respect to the Lebesgue measure  $dt$  with density  $\alpha$ . Then for different functions  $\alpha$ , the measures  $\mathbb{P}^\alpha$  offer a way to model the quadratic variation uncertainty, or volatility uncertainty, for the canonical process  $B$ .

Assume that two agents use the same distortion function  $\psi$ , but disagree on the volatility process. More precisely, let  $c_1$  and  $c_2$  be two submodular capacities on  $\mathcal{F}$  given by

$$c_i(A) := \psi[\mathbb{P}^{\alpha_i}(A)], \quad i = 1, 2, \quad A \in \mathcal{F},$$

where  $\psi$  is a given distortion function, and  $\alpha_1, \alpha_2$  are two different volatility processes. Let us consider the corresponding Choquet integrals evaluated at  $B_1$ :

$$\rho_i(B_1) = \int_{-\infty}^0 [\psi(\mathbb{P}^{\alpha_i}(B_1 > x)) - 1] dx + \int_0^{+\infty} \psi(\mathbb{P}^{\alpha_i}(B_1 > x)) dx, \quad i = 1, 2.$$

Since the probabilities are different we cannot use this form to find an optimal risk transfer between agent 1 and agent two. Notice that the measures  $\mathbb{P}_0, \mathbb{P}^{\alpha_1}$  and  $\mathbb{P}^{\alpha_2}$  are mutually singular. But we can use local distortion functions, and do as if  $\mathbb{P}_0$  is a reference measure.

**Lemma 5.2.** *We have*

$$\rho_1 \square \rho_2(B_1) = \rho_1(-B_1^-) + \rho_2(B_1^+).$$

where, as usual,  $X^- = \max(-X, 0)$  and  $X^+ = \max(X, 0)$ .

**Proof.**

We can use Lemma 3.1, and write everything in terms of  $\mathbb{P}_0(B_1 > x)$  as follows.

Since  $B_1$  under  $\mathbb{P}_0$  is a standard gaussian r.v, the function  $x \rightarrow \mathbb{P}_0(B_1 > x)$  has no constant intervals. By Lemma 3.1, there exist local distortion functions  $\psi_{B_1}^i$  such that

$$\psi(\mathbb{P}^{\alpha_i}(B_1 > x)) = \psi_{B_1}^i(\mathbb{P}_0(B_1 > x)), \quad x \in \mathbb{R} \quad i = 1, 2.$$

We have for example for agent one ( $i = 1$ )

$$\psi_{B_1}^1(u) = c_1(B_1 > \bar{q}(u)) = \psi[\mathbb{P}^{\alpha_1}(B_1 > x)],$$

where we denoted  $\bar{q}(u) := \bar{q}_{B_1}^{\mathbb{P}_0}(u)$  for simplicity. Moreover, using the fact that  $B$  is a  $\mathbb{P}_0$ -Brownian motion, and the Dubins-Schwarz theorem, we get

$$\mathbb{P}^{\alpha_1}(B_1 > x) = \mathbb{P}_0(X_1^{\alpha_1} > x) = \mathbb{P}_0\left(\int_0^1 \alpha_1^{1/2}(s) dB_s > x\right) = \bar{F}\left(\frac{\bar{q}(u)}{\|\alpha_1^{1/2}\|_{L^2}}\right),$$

where  $\bar{F}$  is the tail distribution function of the standard gaussian law. So we found that

$$\psi_{B_1}^1(u) = \psi\left[\bar{F}\left(\frac{\bar{q}(u)}{\|\alpha_1^{1/2}\|_{L^2}}\right)\right], \quad \text{and of course} \quad \psi_{B_1}^2(u) = \psi\left[\bar{F}\left(\frac{\bar{q}(u)}{\|\alpha_2^{1/2}\|_{L^2}}\right)\right].$$

The functions  $\psi_{B_1}^1$  and  $\psi_{B_1}^2$  are equal on  $(0, 1)$  only at  $u^* = 1/2$ . Then using Theorem 4.3, we get

$$\rho_1 \square \rho_2(B_1) = \rho_1(B_1 \wedge k^*) + \rho_2((X - k^*)^+),$$

with  $k^* = \bar{q}(u^*) = 0$ . This ends the proof. □

In Figure 5 below, we represented these two local distortions, in the particular case where  $\psi(u) = \sqrt{u}$ , and the functions  $\alpha_1$  and  $\alpha_2$  are constants respectively equal to 4 and 9.

Notice that the Choquet integrals  $\rho_1$  and  $\rho_2$  are convex, but the local distortions  $\psi_{B_1}^1$  and  $\psi_{B_1}^2$  are not concave. This cannot happen for classic distorted probabilities since the Choquet integral with respect to a distorted probability is convex if and only if the distortion is concave, provided the probability space is atomless.

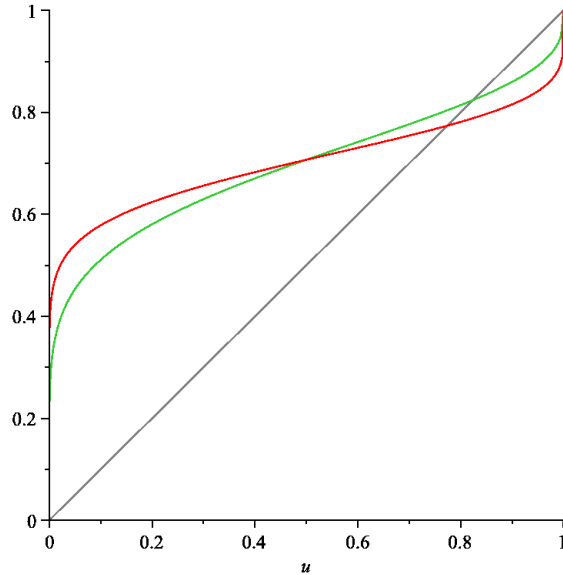


Figure 5: The functions  $\psi_{B_1}^1$  (red curve) and  $\psi_{B_1}^2$  (green curve)

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