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Solution Procedures For Lost Sales Base-Stock Inventory Systems with Compound Poisson Demand

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Abstract

The continuous review base-stock policy under the lost sales assumption has been extensively studied in the inventory literature. However, most of the research that deals with the optimal policy parameter determination under compound Poisson demand focuses on the stuttering Poisson case. In this paper, we extend earlier research by providing a method that calculates the best cost-driven base-stock policy under more general compound Poisson demand processes and for both the complete and the partial rejection case. This is done by developing a simple recursive formula that computes the steady-state probabilities of the number of outstanding orders for general demand and lead-time distributions. Based on this, we are able to obtain the optimal base-stock level under the complete rejection case and an approximate base-stock level under the partial rejection policy. Through a numerical investigation, we show that our approach outperforms existing methods with regard to computational requirements.

Keywords: Base-stock, Compound Poisson, Queueing system, Lost sales.

1 Introduction

Despite that several policies are proposed in the literature to deal with inventory control systems, the base-stock policy remains one of the most used for both fast and slow moving items. This might be due to the fact that such a policy can be easily implemented in practice and also allows a tractable analysis of more complex inventory systems, such as multi-echelon and multi-item systems. It is important to note that non-compound distributions such as Poisson, Normal and Gamma distributions have dominated this literature, whereas for compound distributions, the compound Poisson is largely used due to its attractive theoretical properties and empirical evidence (Babai et al. (2011), Syntetos et al. (2012)). Furthermore, the inventory literature that deals with unsatisfied demands is split into two main streams. A first stream that considers the case of backordered demands and a second one that deals with lost sales. In the latter case, two policies are often assumed: the complete rejection policy and the partial rejection policy. In the complete rejection case, a customer order is totally lost if the order exceeds the on hand inventory level,

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3 whereas in the partial rejection case, if the customer order exceeds the on-hand inventory level,
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5 the customer receives the full amount of the on-hand inventory and the rest is lost.
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8 Note that lost sales inventory models with complete or partial rejection have received far less
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10 attention from researchers than backorder models (Bijvank and Vis, 2011), especially when demand
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12 is assumed to follow a compound Poisson process. This is because lost sales inventory models are
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14 much less analytically tractable and most of the optimality results in the base-stock inventory
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16 systems have been derived under the backorder assumption.
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19 Among the considerable literature that considers the backorder case, one may cite Cheung
20
21 (1996), who analyzed the base-stock policy under a compound Poisson demand and i.i.d lead-time.
22
23 Larsen and Thorstenson (2008) compared different service level criteria for a base-stock system
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25 with compound renewal demand. More recently, Babai et al. (2011) investigated a similar model
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27 and provided a simple and fast iterative procedure for computing the optimal policy parameter for
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29 a cost-driven system.
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32 The research that focuses on the base-stock inventory system with lost sales and compound
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34 Poisson demand was pioneered by Feeney and Sherbrooke (1966), where the authors consider the
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36 cases of complete and partial rejection policies. For both policies, they derived the steady-state
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38 probabilities of the number of outstanding orders. However, as noted by Chen et al. (2011), their
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40 proposed method is inconsistent since the stationary probabilities they obtained are not the true
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42 solution of their steady-state equations. Later, Smith (1977) studied a base-stock inventory system
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44 and developed a procedure to obtain the optimal inventory level under Poisson demand. Similarly,
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46 Moinszadeh (1989) investigated a base-stock policy under partial lost sales with a simple Poisson
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48 process. More recently, Chen et al. (2011) revisited the $(S - 1, S)$ inventory system with complete
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50 and partial rejection policies and derived the exact expression of the stationary probabilities for
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52 stuttering Poisson demand. They concluded that, for a general compound Poisson demand, the
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54 stationary distribution of the number of outstanding units, which is needed to compute the optimal
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56 base-stock, is still an open question. All the above-mentioned research deals with continuous review
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58 inventory control systems where the lead-times are independent and identically distributed (i.i.d)
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60 random variables. The assumption of i.i.d lead-times means that orders may cross in time. The
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literature dealing with non orders crossing will be analyzed next. Riezebos (2006) pointed out that this assumption is realistic particularly for firms located upstream of the supply chain or those that order natural resources or replenish their stock from multiple suppliers. The non-crossing orders assumption is reasonable if a firm has one supplier with a relatively short delivery lead-time. In such case, orders should arrive in the same sequence in which they are placed. However, for a large lead-time between orders' placement, it is plausible to assume independent and identically distributed lead-times. This has been emphasized by Bijvank and Vis (2011) and Disney et al. (2016), who show a case where nearly 40% of orders cross between a supplier in Colorado and a customer in Shenzhen.

Among the literature that assumes orders crossover and closest to ours, Johansen (2005) studied a base-stock model with Erlangian lead-times and lost sales. He developed a simple procedure to compute the optimal base-stock level and showed that non-order crossover has a huge impact on the estimation of the optimal base-stock level and its associated cost. Song (1994) analyzed the effect of stochastically larger or more variable lead-times in a similar model but with backorders. She showed that a stochastically larger lead-time results in a larger optimal base stock level. However, it does not necessary lead to a higher optimal cost because the variability of the lead-time may dominate. Svoronos and Zipkin (1991) compared the impact of sequential lead-times (orders do not cross in time) versus i.i.d ones in a base-stock multi-echelon inventory system with backorders. They showed that the model with orders crossover leads to higher backorders than the one without order crossing. Moreover, even with a small variability of the lead-time, the difference between the backorder levels in the two models is still significant. Zipkin (1991) extended the work in Svoronos and Zipkin (1991) to compound Poisson demands and developed an approximate procedure to obtain the average backorders and stock levels.

Under periodic review, there has been some recent interest in investigating the base-stock policy and its optimality. Among these, van Donselaar et al. (1996) compared base-stock policies with a target service level, Erlang demand and fixed lead-time. Hill (1999) showed that the continuous-review base-stock policy with lost sales is sub-optimal. Based on Hill's finding, Johansen (2001) suggested a periodic-review base-stock inventory system that outperforms the continuous-review

base-stock policy with Poisson demand. Janakiraman and Roundy (2004) conducted a sample-path analysis and showed convexity of the expected cost. Bijvank and Johansen (2012) discussed base-stock models with compound Poisson demand and proposed an approximation procedure to determine the base-stock level. Bijvank et al. (2014) explored the optimality of the base-stock system and proved that it is asymptotically optimal as the lost-sales cost increases.

In this paper, we consider a continuous review base-stock $(S - 1, S)$ inventory system with compound Poisson demand where unfilled demands are lost. Our contribution is two-fold. First, we provide a simple recursive procedure to compute the steady-state probabilities of the number of outstanding orders for general compound Poisson demand and independent and identically distributed lead-times with a general distribution, which extends the work of Chen et al. (2011). The recursive procedure is exact under the complete rejection case and it is approximate under the partial rejection policy, except the case of stuttering Poisson demand where the steady-state probabilities are also exact. For the complete rejection policy, we model the state transitions as a Markov process and show that it is reversible, which allows to obtain a recursive procedure that computes the steady-state probabilities faster than that of Feeney and Sherbrooke (1966). Under the partial rejection case, our recursive methods extend Chen et al. (2011)'s method since it can be used as a heuristic for developing the steady-state probabilities for the more general case of a compound Poisson process. Second, we propose a method to calculate the optimal base-stock under the complete rejection case and the best base-stock level for the partial rejection policy, which outperforms the method proposed by Feeney and Sherbrooke (1966).

The remainder of the paper is organized as follows. Section 2 describes the inventory system considered and provides a method that can be used to calculate the best base-stock level. Section 3 presents the results of the numerical investigation. We end in Section 4 with conclusions and directions for further research.

2 System Analysis

We consider a single-echelon single item inventory system where the demand and the lead-time are stochastic. Demand is modeled as a compound Poisson process, i.e. the interdemand arrivals are exponentially distributed and the demand size follows an arbitrary discrete probability distribution. The stock is controlled by a continuous review base-stock policy where each replenishment order is associated with a stochastic lead-time. We assume that the lead-times are independent and identically distributed (i.i.d) random variables, which means that the outstanding orders are processed in *parallel* and order crossover may occur. We minimize the expected average cost where unfilled demands are lost. For the remainder of the paper, we use notations as detailed in Table 2. In what follows, we first assume that the lead-time is exponentially distributed. Further, we will show analytically that the stationary probabilities are insensitive to the lead-time distribution. All we need is only the mean of the lead-time distribution. In the next subsections, we analyze the cases of complete and partial rejections policies separately.

Table 1: Notations

λ	mean demand arrival rate,
μ	mean of the demand size,
f, F	pmf and cdf of the demand size,
L	mean lead-time,
S	base-stock level,
h	inventory holding cost per unit per unit of time,
b	lost sale cost per unit,
$P(j)$	steady-state probability of having j items outstanding.

2.1 Analysis of Complete Rejection Policy

In this subsection, we start by assuming the lost sales case with a complete rejection policy and exponential lead-time distribution. Under a Poisson demand process, the number of outstanding orders forms an $M/M/S/S$ queue with S servers and a finite capacity S (Tijms, 1994; Johansen, 2013). This queueing system is also called Erlang's loss model. The transition rates for this queue are shown in Figure 1. It is well known (Tijms, 1994; Johansen, 2013) that the steady-state

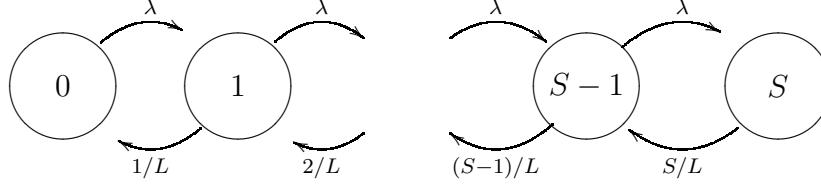


Figure 1: Markov chain of the number of outstanding orders

probabilities of having j outstanding orders are given by:

$$P(j) = \frac{(\lambda L)^j / j!}{\sum_{k=0}^S (\lambda L)^k / k!}, j = 0, 1, \dots, S \quad (1)$$

Equation (1) is known as the Erlang B-formula. If the demand follows a compound Poisson process where inter-demand arrivals are exponentially distributed with mean $1/\lambda$ and the demand size follows an arbitrary discrete probability distribution with pmf f , then the number of outstanding orders forms a multi-class queueing system with batch arrival and batch service time. In fact, we observe that outstanding orders can be classified into S -classes based on customer demand sizes. The base-stock level S can be interpreted as the number of servers shared by S -classes. An order of class i arrives according to a Poisson process of rate $\lambda f(i)$ and requires i servers simultaneously for a holding time with mean L . In order to characterize the performance of the base-stock system, one must determine the behavior of the aforementioned multi-class queue in the steady-state. To this end, let n_i denote the number of class i outstanding orders. Customers are admitted as long as the sum of the demand sizes of all outstanding orders does not exceed S or, equivalently, if there are still free servers. The state of the system is represented by the vector $\mathbf{n} = (n_1, n_2, \dots, n_S)$. The set of all possible states is denoted by:

$$\Omega := \left\{ \mathbf{n} = (n_1, n_2, \dots, n_S) \in \mathbb{N}^S \mid \sum_{i=1}^S i n_i \leq S \right\}. \quad (2)$$

Define

$$\Omega(j) := \left\{ \mathbf{n} = (n_1, n_2, \dots, n_j) \in \mathbb{N}^j \mid \sum_{i=1}^j n_i = j, j \leq S \right\}, \quad (3)$$

where \mathbb{N} is the set of non-negative integers. Figure 2 illustrates a fragment of the transition rates for a three-class queue where $\mathbf{n} = (n_1, n_2, n_3) \in \Omega$, $f(1) + f(2) + f(3) = 1$ and $S = 3$.

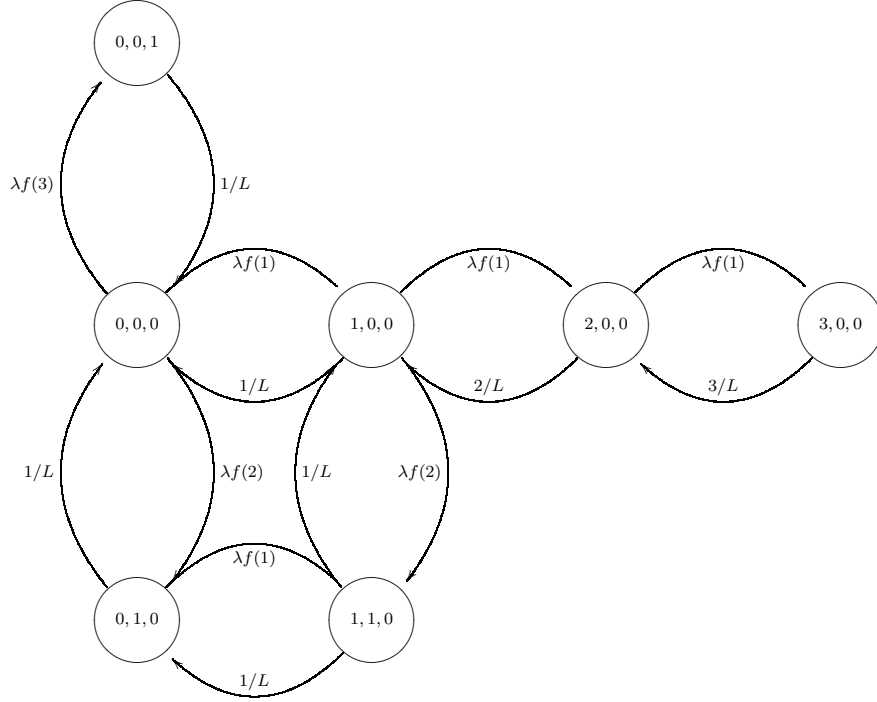


Figure 2: Sample transition rates for a three-class queue where $\mathbf{n} = (n_1, n_2, n_3) \in \Omega$, $f(1) + f(2) + f(3) = 1$ and $S = 3$.

Let $n_i(t)$ be the number of the class i outstanding orders at time t . In addition, let the vector $\{\mathbf{n}(t) = (n_1(t), n_2(t), \dots, n_S(t)), t \geq 0\}$ denote the random process of the number of outstanding orders of different classes at time t . The state space of $\mathbf{n}(t)$ is given by Ω . Since the inter-arrival and holding times are exponentially distributed, $\{\mathbf{n}(t), t \geq 0\}$ is a continuous time Markov chain. The state of this Markov chain changes either due to the arrival of an order of class i at the rate $\lambda f(i)$ or due to a departure at the rate n_i/L . The process $\{\mathbf{n}(t), t \geq 0\}$ is irreducible. In addition, since it is defined over a finite state space, it is positive recurrent, which implies that a stationary distribution exists.

Let $P(j) = \sum_{\mathbf{n} \in \Omega(j)} \pi(\mathbf{n})$, $0 \leq j \leq S$ be the steady-state probability to have j outstanding items in the system. The following proposition provides the steady-state probability of having j outstanding items in the $(S-1, S)$ base-stock system.

Proposition 1. *Under the complete rejection policy, the steady-state probability $P(j)$, $0 \leq j \leq S$ is given by:*

$$P(j) = \left(\sum_{\mathbf{n} \in \Omega} \prod_{i=1}^S \frac{(f(i)\lambda L)^{n_i}}{n_i!} \right)^{-1} \sum_{\mathbf{n} \in \Omega(j)} \prod_{i=1}^S \frac{(f(i)\lambda L)^{n_i}}{n_i!}. \quad (4)$$

Proof. See Appendix A. ■

By showing reversibility of the process $\mathbf{n}(\mathbf{t})$, Proposition 1 extends earlier investigations that assume either a Poisson demand (Smith, 1977) or a stuttering Poisson demand (Chen et al., 2011) and it shows that the steady-state probabilities of the number of outstanding items has a product-form. Note that the steady-state probability $P(j)$ is exactly the same as the steady-state probability of having j busy servers in the multi-class queue mentioned above. From Expression (4), the steady-state probability $P(j)$, $0 \leq j \leq S$ only depends on the mean service time L . A simple property follows immediately.

In the following, we relax the exponential lead-time assumption.

Corollary 1. *If the lead-time follows a general nonnegative and continuous distribution, then the steady-state probability $P(j)$, $0 \leq j \leq S$ is insensitive to the lead-time distribution and depends only on its mean.*

Proof. See Appendix B ■

Corollary 1 implies that the assumption of the exponential lead-time distribution can be relaxed. Next, we provide a recursive algorithm to simplify the calculation of the steady-state probabilities. In fact, because the cardinality of the state space Ω is high, it is not straightforward to compute $P(j)$, $0 \leq j \leq S$ in its present form, even for small values of S . To illustrate this, assume that all demand sizes are equal to 1. Then, the cardinality of Ω is the same as the total number of

ways to place $j = 1, 2, \dots, S$ indistinguishable balls into S distinguishable urns. This is equivalent to find the total number of solutions that satisfy: $n_1 + n_2 + \dots + n_S = j$, where n_i represents the number of balls in the i -th urn. For a given number j , Feller (1957) showed that the cardinality of placing j indistinguishable balls into S distinguishable urns is given by $\binom{S+j-1}{S-1}$. Thus, the cardinality of Ω is $|\Omega| = \sum_{j=0}^S \binom{S+j-1}{S-1}$. Hence, enumerating the terms in Expression (4) might be very cumbersome, especially if S is high. In order to overcome this computational problem, Corollary 2 provides a one-dimensional recursion independent of the order-classes.

Corollary 2. *The steady-state probabilities $P(j), 0 \leq j \leq S$ can be computed recursively as:*

Step 1. Set $p(0) = 1$.

Step 2. Compute $p(n+1) = \frac{\lambda L}{n+1} \sum_{k=0}^n (n-k+1)f(n-k+1)p(k), n = 0, 2, \dots, S-1$.

Step 3. Set $P(n) \leftarrow p(n) / \sum_{k=0}^S p(k), n = 0, 1, \dots, S$.

Proof. See Appendix C. ■

It is interesting to see from Corollary 2 that Step 2 also computes the probability distribution of the total demand during the replenishment lead-time, which is a compound Poisson process (see Equation (5) in Adelson (1966)). Therefore, the stationary probabilities derived in Feeney and Sherbrooke (1966) are exact under a complete rejection policy. Recall that their method to derive the steady-state probabilities is inconsistent since the stationary probabilities they obtained are not the true solution to their steady-state equations. Thus, our recursion provides an efficient and numerically stable method even for a large S compared to the convolution method of Feeney and Sherbrooke (1966). The computational effort of the convolution algorithm of Feeney and Sherbrooke (1966) is $O(S^3)$ and less efficient than our recursion, which requires $O(S^2)$ time. In fact, to calculate the unnormalized probability in Step 2, $O(S)$ arithmetic operations are performed. Since the unnormalized probability should be obtained for $n = 1, \dots, S$, the overall effort of Step 2, and of the algorithm as a whole, is $O(S^2)$. In the convolution algorithm, one needs $n = 1, \dots, S$ arithmetic operations to compute the n -fold convolution of the size distribution with itself. In addition, $n = 1, \dots, S$ other operations are necessary to convolve the Poisson process

and the convolved distribution of the demand size. The computational effort is therefore $O(S^2)$ to obtain the compound Poisson demand distribution. Since the calculation is repeated S times to obtain the unnormalized probability, the overall effort is $O(S^3)$. In Section 3, through numerical investigation, we show examples of the computational times of both methods.

2.2 Analysis of Partial Rejection Policy

A partial rejection occurs when a fraction of the demand is immediately satisfied and the remainder of it is lost if no item is on stock. Assume that the lead-time follows a general distribution, then, using the same reasoning as for the case of the complete rejection policy, the system can be seen as a multi-class queuing network with a state-dependent arrival rate and S queues, where the i -th queue accepts only class i . The i -th queue in isolation behaves like an $M/G/\infty$. Orders of class i arrive according to a Poisson process of rate $\lambda f(i)$ and requires simultaneously i servers for a holding time *i.i.d* with cdf G and mean L . The state space of the system, denoted by Ω , is given by (2). If the system is in state $\mathbf{n} \in \Omega(j)$, then the state-dependent arrival rate of this network can be written as:

$$\lambda_i(\mathbf{n}) = \begin{cases} \lambda f(i) & \text{if } \mathbf{n} \in \Omega(j), j < S, \\ \lambda(1 - F(i - 1)) & \text{if } \mathbf{n} \in \Omega(S), \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

According to Theorem 3.14 of Kelly (1979), the above network is quasi-reversible and has stationary probabilities of a product-form if and only if we can define a function ψ such that:

$$\lambda_i(\mathbf{n}) = \frac{\psi(\mathbf{n})}{\psi(\mathbf{n} - \mathbf{e}_i)} \lambda f(i). \quad (6)$$

Then the stationary probabilities are given by

$$P(j) = \left(\sum_{\mathbf{n} \in \Omega} \prod_{i=1}^S \frac{(f(i)\lambda L)^{n_i}}{n_i!} \right)^{-1} \sum_{\mathbf{n} \in \Omega(j)} \psi(\mathbf{n}) \prod_{i=1}^S \frac{(f(i)\lambda L)^{n_i}}{n_i!}. \quad (7)$$

The function ψ expresses the dependence upon the number of customers of each class and guarantees that the balance equations hold. ψ can also be interpreted as the weight of any direct path for the state 0 to the state $j > 0$, defined as the product of $\lambda_i(\mathbf{n})$, $\mathbf{n} \in \Omega(j)$. The weight $\psi(\mathbf{n})$ is path independent. We refer the reader to Kelly (1979) and Serfozo (1999) who give details on how the function ψ can be defined in a state-dependent service rate queuing network. In the context of inventory control systems, Smeitink (1990) used the results of Kelly (1979) to obtain an explicit expression of the function ψ and proved that the base-stock system with partial backorders is insensitive to the lead-time distribution. Proposition 2 provides a condition under which quasi-reversibility is preserved.

Proposition 2. *For the partial rejection policy, the process $\{\mathbf{n}(t), t \geq 0\}$ is quasi-reversible and has steady-state probabilities $P(j)$ given by (7) if and only if the demand size follows a geometric distribution.*

Proof. See Appendix D. ■

Proposition 2 states that the quasi-reversibility of the above network does not hold if the demand size distribution is not geometric, since for $\mathbf{n} \in \Omega(S)$, ψ cannot be explicitly determined. Intuitively speaking, the function ψ should guarantee that an arrival of an order of class $i + 1, \dots$ can always be “converted” into class i if it cannot be completely satisfied. Although the steady-state probabilities $P(j), 0 \leq j \leq S$ can be computed by (7), enumerating the terms in (7) may not be feasible. However, Corollary 3 allows us to compute $P(j), 0 \leq j \leq S$ in a one-dimensional recursion.

Corollary 3. *The steady-state probabilities $P(j)$ can be computed recursively as:*

Step 1. Set $p(0) = 1$.

Step 2. Compute $p(n+1) = \frac{\lambda L}{n+1} \sum_{k=0}^n (n-k+1)f(n-k+1)p(k), n = 0, 2, \dots, S-2$.

Step 3. Compute $p(S) = \frac{\lambda L}{S} \sum_{k=1}^S k(1-F(k-1))p(S-k)$.

Step 4. Set $P(n) \leftarrow p(n) / \sum_{k=0}^S p(k), n = 0, 1, \dots, S$.

Proof. See Appendix E. ■

The base-stock model with a general compound Poisson demand process and a partial rejection policy cannot be solved in closed-form, because the function ψ may depend on other parameters of the system, such as the class of the order. Therefore, we propose an approximate solution to compute the steady-state probabilities under a general compound Poisson demand process. Given that a general compound Poisson process can always be approximated by a stuttering Poisson process that preserves the first two moments of the original demand process, we may find a function ψ using the approximate stuttering Poisson process. Consequently, Corollary 3 computes approximate steady-state probabilities for a general compound Poisson process using the original pmf f and cdf F of the demand size distribution while it provides the exact steady-state probabilities under geometric demand distribution. We will show through a numerical study that the approximation produced by Corollary 3 is very accurate.

2.3 Expected Total Cost

The expected average cost of the system under the complete or partial rejection policies can be expressed by using the fraction of the demand lost under both policies. Let $B(S)$ denote this fraction for a given base-stock S and $E(O)$ be the expected number of outstanding items. From Little's Theorem, one can write:

$$E(O) = \lambda\mu(1 - B(S))L \implies B(S) = 1 - \frac{E(O)}{\lambda L\mu}, \text{ where } E(O) = \sum_{n=0}^S nP(n). \quad (8)$$

The expected inventory level is given by $S - \lambda L\mu(1 - B(S))$ and the expected number of lost sales is $\lambda\mu B(S)$. The expected average cost of the $(S - 1, S)$ base-stock policy is therefore:

$$Z(S) = h(S - \lambda L\mu) + (b + hL)\lambda\mu B(S). \quad (9)$$

Note that the expression of the expected total cost becomes the one given by Smith (1977) for a simple Poisson arrival process with $\mu = 1$. Lemma 1 will help to compute the best base-stock

level.

Lemma 1. *For the base-stock policy with compound Poisson demand, the following statements hold:*

- i) $Z(S)$ is convex under the partial rejection policy.*
- ii) $Z(S)$ is not necessarily convex under the complete rejection policy.*
- iii) If $Z(S)$ is convex, then the optimal base stock level, denoted by S^* , satisfies the following inequality:*

$$B(S^* + 1) \leq \frac{h}{\lambda(b + hL)\lambda\mu} \leq B(S^*). \quad (10)$$

Proof. See Appendix F. ■

By Lemma 1, a simple procedure to find the best base-stock value is as follows:

Step 1. Set $S = 0$.

Step 2. Check whether (10) in Lemma 1 holds. If yes stop. Otherwise, go to Step 3.

Step 3. Increment S by 1 and go to Step 2.

3 Numerical Results

The advantages of the proposed procedure arise particularly in the spare parts case, which is characterized by intermittent demand. According to Johnston et al. (2003), such items may generate over 40% of the sales and account for 60% of the total stock value. Hence, their inventories should be carefully controlled. Lengu et al. (2014) showed that the demand of spare parts can be modeled with a compound Poisson process. The authors distinguished between four compound Poisson demand processes, namely Poisson-Poisson, Poisson-Geometric, Poisson-Logarithmic-series and Poisson-Negative Binomial distributions, which, indeed, provide a good fit for most of the 15,000 SKUs they considered. Based on these aforementioned demand processes, we intend to assess the performance of our proposed procedures and Feeney-Sherbrooke methods. Since our proposed algorithm and the convolution method of Feeney and Sherbrooke (1966) are exact under the com-

plete rejection case, we only compare the computational effort required to find the steady-state probabilities. Under the partial rejection case, both methods are approximate, therefore, we assess the performance of both methods in terms of cost error from the total optimal cost obtained by simulation. Note that, in the case of a Poisson-Geometric demand, our proposed procedure and the one of Chen et al. (2011) can be used interchangeably to compute the optimal base-stock level. We omit the case where the demand follows a Poisson-Negative Binomial distribution since the results we found are similar to those obtained with the Poisson-Geometric and Poisson-Logarithmic-series cases. The probability mass function for the demand size distribution is described as:

$$f(i) = \begin{cases} \frac{e^\mu \mu^{i-1}}{(i-1)!} & \text{for Shifted Poisson with mean } \mu + 1, i \geq 1. \\ \frac{-\theta^i}{i \log(1-\theta)}, 0 < \theta < 1 & \text{for Logarithmic-series with mean } \frac{-\theta}{(1-\theta)\log(1-\theta)}, i \geq 1. \end{cases} \quad (11)$$

Since the Feeney-Sherbrooke approximation uses a convolution algorithm to compute the steady-state probabilities, one needs to express the sum of n independent random variables, each of which has the same Poisson/Logarithmic-series distribution. This is given by:

$$\begin{cases} e^{n\mu} (n\mu)^{i-n} / (i-n)! & \text{for shifted Poisson demand sizes, } n = 1, 2, \dots, \text{ and } i = n, n+1, \dots \\ \frac{n! \theta^i |s(i, n)|}{i! (-\log(1-\theta))^n} & \text{for Logarithmic-series demand sizes, } n = 1, 2, \dots, \text{ and } i = n, n+1, \dots \end{cases} \quad (12)$$

where $|s(i, n)|$ is the unsigned Stirling number of the first kind (see Johnson et al. (2005) for the convolution of Logarithmic-series random variables).

Computational performance under complete rejection policy The experiments are coded in Matlab 2015a and run on a machine with an Intel Core i7 processor of 2.6 GHz and 16 GB of RAM. Table 2 reports the CPU time needed to compute the steady-state probabilities as a function of λ in seconds, the demand size distribution and the inventory level S . We also report the optimal value of the inventory level denoted by S^* for $h = 1$ and $b = 20$. The value of S^* is found by means of a grid search starting from $S = 0$. In Table 2, we show that, under the logarithmic demand size distribution and when $\lambda = 0.5$, our algorithm requires, in most of

the cases, less than one second to compute the steady-state probabilities for a given S , whereas Feeney and Sherbrooke (1966)'s method needs more than 20 seconds. Under the shifted Poisson demand size process and $\lambda = 0.5$, both methods take less than one second to compute the steady-states probabilities. As λ increases, the optimal S^* increases and so does the CPU time. Under the logarithmic demand size distribution, the convolution method of Feeney and Sherbrooke (1966) becomes very time consuming when S exceeds 100 (going up to 2,500 seconds), while our method is still providing the steady-state probabilities in less than one second. For a shifted Poisson demand size distribution, the proposed procedure uses approximately half of the CPU time required when running Feeney and Sherbrooke (1966)'s method. The CPU time under logarithmic and Poisson demand size are so different when using Feeney and Sherbrooke (1966)'s method because the logarithmic distribution involves the calculation of the Stirling numbers $|s(i, n)|$, $n!$ and $i!$, which require more running time than $n!$ in the shifted Poisson demand size process. This shows the efficiency of our algorithm since the saving in CPU time is significant compared to that of Feeney and Sherbrooke (1966), especially for high values of λ .

Performance under partial rejection policy The optimal base-stock level is computed using a simulation model built in *Arena Rockwell*. The simulation model runs for only one replication length of 1 million time units, using the batch means method (Law, 2014), which is high enough to obtain the system operating costs (stock level and lost sales) with an interval of confidence of 99%. The simulation starts with S items on-hand and runs for different scenarios designed in the *Process Analyzer* tool of Arena. Let S^a , S^f and S^* be the best values of the base-stock level that minimize the total cost. S^a , S^f are computed by Lemma 1 where $B(S)$ is found using the steady-state probabilities in Corollary 3 and in Feeney and Sherbrooke (1966) respectively, while S^* is obtained from the simulation model. The fixed parameter settings we used are $L = 7$, $h = 1$ and $b = 10$. Please note that a constant lead-time numerical value $L = 7$ is used due to insensitivity property shown in Corollary 1. The results are reported in Tables 3 and 4. We notice that our procedure performs well for all parameter settings. It often provides the optimal base-stock values, regardless of the demand characteristics. One can observe that in only few cases

Table 2: CPU time (in seconds) for the steady-state probabilities under the complete rejection policy

λ	S	Logarithmic demand size distribution with parameter θ						Shifted Poisson demand size process with mean $\mu + 1$					
		Our algorithm			Feeney-Sherbrooke algorithm			Our algorithm			Feeney-Sherbrooke algorithm		
		$\theta = 0.8$	$\theta = 0.9$	$\theta = 0.99$	$\theta = 0.8$	$\theta = 0.9$	$\theta = 0.99$	$\mu = 2$	$\mu = 4$	$\mu = 8$	$\mu = 2$	$\mu = 4$	$\mu = 8$
0.5	100	0.02	0.04	0.03	46.10	27.10	27.00	0.39	0.43	0.45	0.79	0.91	0.84
	200	0.11	0.06	0.06	193.57	144.23	104.45	1.51	1.47	1.49	3.09	3.01	3.00
	300	0.22	0.21	0.14	349.28	274.45	231.44	3.34	3.27	3.33	9.10	6.76	6.73
	400	0.24	0.24	0.24	407.24	403.55	409.47	5.89	5.79	5.89	12.22	11.99	11.89
	500	0.37	0.37	0.37	738.01	632.17	639.02	9.27	9.03	9.14	19.97	18.68	18.54
	600	0.53	0.53	0.53	933.94	926.86	928.78	13.09	13.09	13.15	27.26	26.87	26.63
	700	0.73	0.73	0.72	1550.73	1310.27	1269.93	17.82	17.74	17.85	36.89	36.50	36.26
	800	1.13	0.95	0.94	1788.70	1666.37	1671.75	23.13	23.17	23.21	47.95	47.66	47.10
	900	1.21	1.22	1.18	2148.34	2105.73	2108.64	29.26	29.29	29.28	60.64	60.04	59.49
	1000	1.56	1.49	1.49	2756.98	2596.79	2726.17	36.06	35.94	36.10	74.61	73.93	73.51
5		$S^* = 14$	$S^* = 22$	$S^* = 106$				$S^* = 18$	$S^* = 30$	$S^* = 53$			
	100	0.03	0.02	0.02	30.07	25.95	26.42	0.45	0.43	0.44	0.96	0.92	0.91
	200	0.06	0.06	0.06	124.74	101.29	100.46	1.47	1.47	1.48	3.17	3.14	3.09
	300	0.16	0.13	0.13	258.27	223.53	223.23	3.28	3.30	3.30	7.01	6.96	6.85
	400	0.24	0.23	0.24	473.20	397.39	398.18	5.81	5.81	5.84	12.31	12.21	12.08
	500	0.39	0.36	0.37	667.63	623.78	625.86	9.06	9.06	9.09	19.13	19.04	18.80
	600	0.53	0.53	0.52	980.56	903.90	903.03	13.00	13.02	13.08	27.46	27.18	26.85
	700	0.72	0.71	0.71	1535.00	1228.93	1231.28	17.67	17.72	17.74	37.18	36.86	36.37
	800	0.93	0.93	0.93	1610.19	1613.79	1612.68	23.03	23.15	23.12	48.42	48.01	47.37
	900	1.18	1.17	1.17	2044.54	2044.76	2047.84	29.10	29.18	29.18	61.04	61.66	59.79
10	1000	1.45	1.46	1.44	2529.38	2536.78	2763.02	35.85	38.56	35.94	75.11	79.73	73.58
		$S^* = 113$	$S^* = 183$	$S^* = 1081$				$S^* = 131$	$S^* = 217$	$S^* = 390$			
	100	0.04	0.02	0.03	26.00	26.56	26.22	0.44	0.44	0.44	0.94	0.92	0.93
	200	0.06	0.06	0.06	101.55	101.12	101.29	1.47	1.46	1.50	3.23	3.19	3.20
	300	0.13	0.13	0.13	223.18	224.75	225.14	3.28	3.27	3.31	7.15	7.05	7.03
	400	0.24	0.23	0.23	396.98	400.89	398.71	5.81	5.78	5.89	12.57	12.44	12.37
	500	0.37	0.36	0.36	624.88	629.70	626.37	9.04	9.05	9.14	19.49	19.24	19.18
	600	0.54	0.52	0.53	901.39	907.96	904.66	13.01	13.00	13.11	27.81	27.54	27.38
	700	0.73	0.71	0.71	1322.57	1236.70	1241.64	17.67	17.64	17.82	37.65	37.29	37.08
	800	0.95	0.93	0.93	1619.62	1626.62	2046.99	23.06	23.02	23.24	49.03	48.46	48.13
	900	1.20	1.17	1.28	2049.48	2060.06	2054.98	29.22	29.08	29.29	61.62	61.07	60.65
	1000	1.48	1.45	1.45	2534.84	2548.80	2551.18	35.92	35.84	36.09	75.89	75.08	74.66
		$S^* = 212$	$S^* = 341$	$S^* = 1996$				$S^* = 247$	$S^* = 411$	$S^* = 737$			

Fixed parameters: $h = 1, b = 20$.

the proposed procedure gives $S^a = S^* \pm 1$. However, even with this low error, the percentage difference from the optimal total cost is negligible.

If we use the Feeney-Sherbrooke approximation, the model exhibits a large error for all considered cases. Its overall performance is strongly affected by the variability of both Poisson inter-arrival and demand size distributions as depicted in Figures 3 and 4. The maximum error observed is roughly 17% when $\lambda = 5$ and the demand size distribution is a logarithmic-series with the parameter $\theta = 0.6$. Tables 3 and 4 indicate that the percentage error from the simulated cost increases as λ or the parameters (μ or θ) of the demand size distribution increase, while, for our method, the effect of the demand parameters is negligible.

We also conducted a sensitivity analysis of the total cost with respect to the values of the holding cost h , the lost sales cost b and the mean lead-time L . For the base case, we set $L = 7$, $h = 1$ and $b = 10$. We then vary each of these parameters from -90% to 90% in steps of 10% while keeping the other parameters unchanged. For all the tested cases, the arrival rate of orders is $\lambda = 0.8$ and the mean demand size is 4 (i.e. a parameter $\theta = 0.903$ in the case of Logarithmic-series demand size distribution and a parameter $\mu = 4$ in the case of shifted Poisson demand size). The percentage errors of Feeney-Sherbrooke's cost from the optimal total cost are plotted in Figures 5 and 6 for shifted Poisson demand and Logarithmic-series size distributions respectively. The results show that the performance of Feeney-Sherbrooke's cost is sensitive to the cost parameters and particularly to the mean lead-time L since the error can go up to 10% for high values of L . The results of our proposed procedure are not reported here since the percentage error from the optimal total cost is less than 0.04% in all cases, which shows that the performance of our proposed procedure is insensitive to the parameters h , b and L .

Table 3: Performance of the considered procedures when the demand size follows a shifted Poisson process with mean $\mu + 1$.

λ	μ	Best base-stock			Total cost using simulation			Error from optimal simulation cost in%	
		S^a	S^J	S^*	$Z(S^a)$	$Z(S^J)$	$Z(S^*)$	Our method	Feeney-Sherbrooke's method
0.5	2	12	15	12	8.92	9.26	8.92	0.0	3.8
	3	16	20	16	11.79	12.22	11.79	0.0	3.7
	4	21	25	21	14.61	15.14	14.61	0.0	3.7
	5	25	30	25	17.41	18.05	17.41	0.0	3.7
	6	29	35	29	20.20	20.94	20.20	0.0	3.7
	7	33	40	33	22.98	23.82	22.98	0.0	3.7
	8	37	45	37	25.76	26.70	25.76	0.0	3.6
	9	42	50	41	28.54	29.58	28.54	0.0	3.6
	10	46	55	46	31.31	32.45	31.31	0.0	3.6
0.6	2	15	18	15	9.95	10.35	9.95	0.0	4.1
	3	20	24	20	13.13	13.65	13.13	0.0	4.0
	4	25	30	25	16.27	16.92	16.27	0.0	4.0
	5	30	36	30	19.38	20.16	19.38	0.0	4.0
	6	35	42	35	22.48	23.39	22.48	0.0	4.0
	7	40	48	40	25.58	26.61	25.58	0.0	4.0
	8	45	54	45	28.67	29.82	28.67	0.0	4.0
	9	50	60	50	31.75	33.03	31.75	0.0	4.0
	10	55	66	55	34.83	36.24	34.83	0.0	4.0
0.7	2	17	21	17	10.87	11.36	10.87	0.0	4.5
	3	23	28	23	14.34	14.99	14.34	0.0	4.5
	4	29	35	29	17.76	18.57	17.76	0.0	4.6
	5	35	42	35	21.15	22.12	21.15	0.0	4.6
	6	41	49	41	24.53	25.67	24.53	0.0	4.6
	7	47	56	47	27.90	29.20	27.90	0.0	4.6
	8	53	63	53	31.27	32.73	31.27	0.0	4.6
	9	59	70	59	34.63	36.25	34.63	0.0	4.7
	10	64	77	65	37.99	39.77	37.99	0.0	4.7
0.8	2	20	24	20	11.74	12.33	11.74	0.0	5.0
	3	27	32	27	15.49	16.27	15.49	0.0	5.0
	4	33	40	33	19.18	20.15	19.18	0.0	5.1
	5	40	48	40	22.84	24.01	22.84	0.0	5.1
	6	47	56	47	26.49	27.86	26.49	0.0	5.2
	7	54	64	54	30.13	31.69	30.13	0.0	5.2
	8	60	72	60	33.76	35.52	33.76	0.0	5.2
	9	67	80	67	37.38	39.34	37.38	0.0	5.2
	10	74	88	74	41.00	43.16	41.00	0.0	5.3
0.9	2	22	27	22	12.56	13.27	12.56	0.0	5.6
	3	30	36	30	16.56	17.50	16.56	0.0	5.7
	4	38	45	38	20.51	21.69	20.51	0.0	5.8
	5	45	54	45	24.42	25.85	24.42	0.0	5.8
	6	53	63	53	28.32	29.99	28.32	0.0	5.9
	7	60	72	60	32.21	34.11	32.21	0.0	5.9
	8	68	81	68	36.09	38.23	36.09	0.0	5.9
	9	75	90	75	39.96	42.35	39.96	0.0	6.0
	10	83	99	83	43.83	46.47	43.83	0.0	6.0
1	2	25	30	25	13.33	14.17	13.33	0.0	6.3
	3	33	40	33	17.57	18.70	17.57	0.0	6.4
	4	42	50	42	21.76	23.17	21.76	0.0	6.5
	5	50	60	50	25.91	27.62	25.91	0.0	6.6
	6	58	70	58	30.04	32.04	30.04	0.0	6.6
	7	67	80	67	34.16	36.46	34.16	0.0	6.7
	8	75	90	75	38.28	40.87	38.28	0.0	6.8
	9	83	100	84	42.39	45.27	42.39	0.0	6.8
	10	92	110	92	46.48	49.67	46.48	0.0	6.8
2	2	49	58	49	19.51	21.45	19.51	0.0	10.0
	3	65	77	65	25.68	28.17	25.68	0.0	9.7
	4	82	97	82	31.78	35.27	31.78	0.0	11.0
	5	98	116	98	37.83	41.90	37.83	0.0	10.8
	6	114	135	115	43.84	48.49	43.84	0.0	10.6
	7	131	154	131	49.83	55.07	49.83	0.0	10.5
	8	147	173	147	55.82	61.64	55.82	0.0	10.4
	9	163	193	163	61.80	68.66	61.80	0.0	11.1
	10	180	212	180	67.77	75.21	67.77	0.0	11.0
5	2	118	136	118	31.74	36.74	31.74	0.0	15.7
	3	157	182	158	41.75	48.97	41.74	0.0	17.3
	4	197	227	197	51.62	60.56	51.62	0.0	17.3
	5	236	272	236	61.41	72.09	61.41	0.0	17.4
	6	275	317	276	71.16	83.59	71.15	0.0	17.5
	7	314	361	315	80.87	94.51	80.87	0.0	16.9
	8	353	406	354	90.57	105.99	90.57	0.0	17.0
	9	393	451	393	100.23	117.46	100.23	0.0	17.2
	10	432	496	433	109.91	128.92	109.91	0.0	17.3
Fixed parameters: $L = 7, h = 1, b = 10$.									

Table 4: Performance of the considered procedures when the demand size follows a Logarithmic-series distribution with parameter θ .

λ	θ	Best base-stock			Total cost using simulation			Error from optimal simulation cost in %	
		S^a	S^f	S^*	$Z(S^a)$	$Z(S^f)$	$Z(S^*)$	Our method	Feeney-Sherbrooke's method
0.5	0.2	5	5	5	3.18	3.18	3.18	0.0	0.0
	0.4	5	6	5	3.84	3.91	3.84	0.0	1.8
	0.6	6	7	6	5.10	5.14	5.10	0.0	0.8
	0.8	9	11	9	8.39	8.60	8.39	0.0	2.5
	0.9	13	15	13	14.13	14.26	14.13	0.0	0.9
	0.95	19	23	19	24.18	24.42	24.18	0.0	1.0
0.6	0.2	6	6	6	3.54	3.54	3.54	0.0	0.0
	0.4	6	7	6	4.30	4.34	4.30	0.0	0.9
	0.6	8	9	8	5.73	5.85	5.73	0.0	2.1
	0.8	11	13	11	9.47	9.65	9.47	0.0	1.8
	0.9	16	20	16	16.05	16.39	16.05	0.0	2.1
	0.95	24	30	25	27.62	28.02	27.62	0.0	1.4
0.7	0.2	6	7	6	3.87	3.87	3.87	0.0	0.2
	0.4	7	9	7	4.72	4.93	4.72	0.0	4.4
	0.6	9	11	9	6.28	6.53	6.28	0.0	3.9
	0.8	13	16	13	10.46	10.77	10.46	0.0	3.0
	0.9	20	24	20	17.79	18.21	17.79	0.0	2.4
	0.95	30	36	30	30.72	31.17	30.72	0.0	1.5
0.8	0.2	7	8	7	4.17	4.19	4.17	0.0	0.5
	0.4	9	10	9	5.11	5.27	5.11	0.0	3.2
	0.6	10	13	10	6.82	7.20	6.82	0.0	5.5
	0.8	15	19	15	11.37	11.86	11.37	0.0	4.3
	0.9	23	28	23	19.42	19.94	19.42	0.0	2.7
	0.95	35	43	35	33.65	34.30	33.65	0.0	1.9
0.9	0.2	8	9	8	4.45	4.48	4.45	0.0	0.7
	0.4	10	11	10	5.46	5.60	5.46	0.0	2.7
	0.6	12	14	12	7.31	7.58	7.31	0.0	3.7
	0.8	17	21	17	12.22	12.67	12.22	0.0	3.7
	0.9	26	32	26	20.96	21.58	20.96	0.0	3.0
	0.95	41	50	41	36.41	37.30	36.41	0.0	2.4
1	0.2	9	10	9	4.73	4.77	4.73	0.0	1.0
	0.4	11	13	11	5.81	6.21	5.81	0.0	6.8
	0.6	13	16	13	7.78	8.22	7.78	0.0	5.6
	0.8	19	24	19	13.06	13.72	13.06	0.0	5.0
	0.9	30	36	30	22.43	23.16	22.43	0.0	3.2
	0.95	46	57	46	39.04	40.18	39.04	0.0	2.9
2	0.2	18	21	18	6.94	7.47	6.94	0.0	7.6
	0.4	21	25	21	8.57	9.36	8.57	0.0	9.2
	0.6	26	32	26	11.61	12.78	11.61	0.0	10.1
	0.8	40	48	40	19.72	21.16	19.72	0.0	7.3
	0.9	62	76	62	34.13	36.51	34.13	0.0	7.0
	0.95	100	122	100	60.09	63.53	60.09	0.0	5.7
5	0.2	44	50	44	11.29	12.94	11.29	0.0	14.6
	0.4	51	59	51	14.01	16.10	14.01	0.0	14.9
	0.6	65	76	65	19.06	22.36	19.06	0.0	17.3
	0.8	99	117	98	32.78	37.67	32.77	0.0	14.9
	0.9	155	186	155	57.40	64.84	57.40	0.0	13.0
	0.95	253	304	252	102.05	113.65	102.05	0.0	11.4
Fixed parameters: $L = 7, h = 1, b = 10$.									

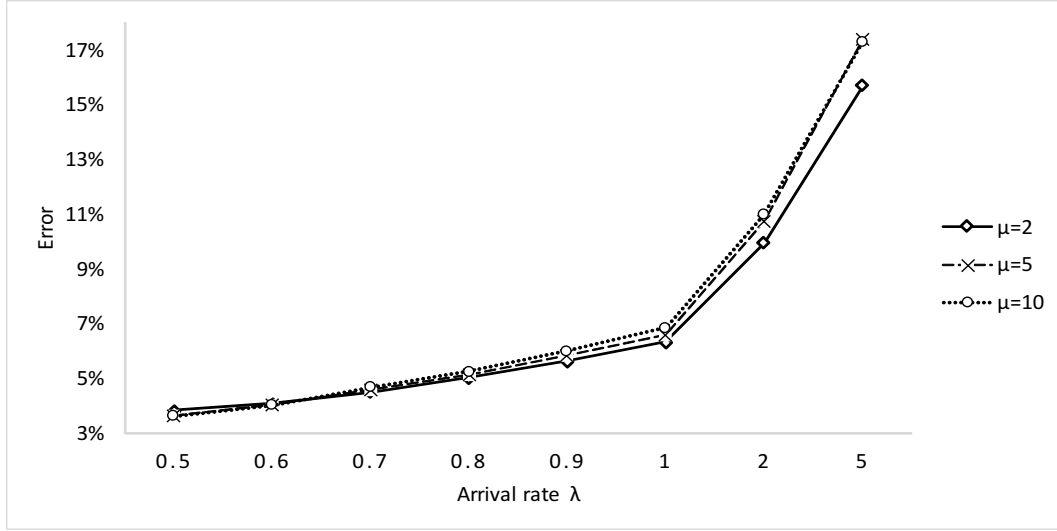


Figure 3: Cost error of using Feeney-Sherbrooke's method from the optimal total cost for a shifted Poisson demand sizes with mean $\mu + 1$.

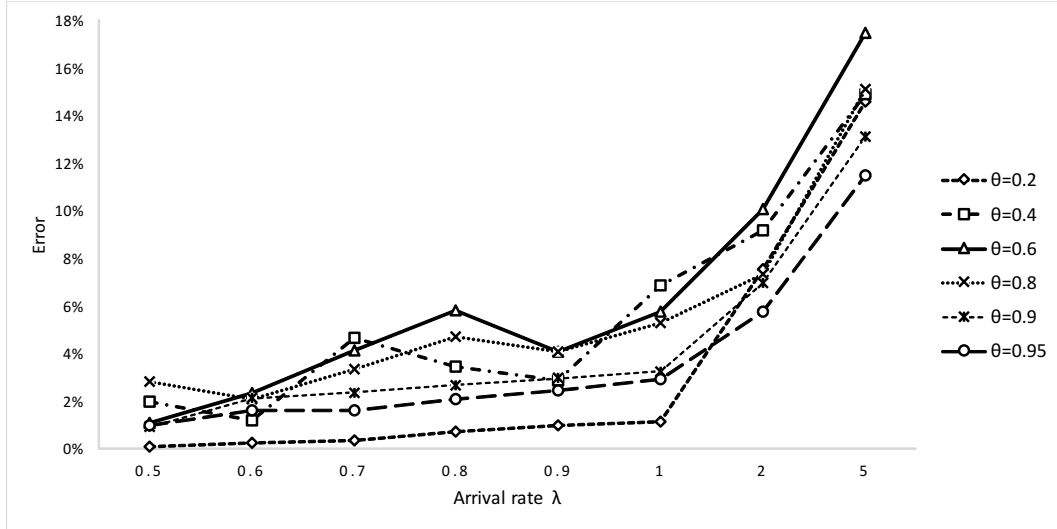


Figure 4: Cost error of using Feeney-Sherbrooke's method from the optimal total cost for a logarithmic-series demand sizes with parameter θ .

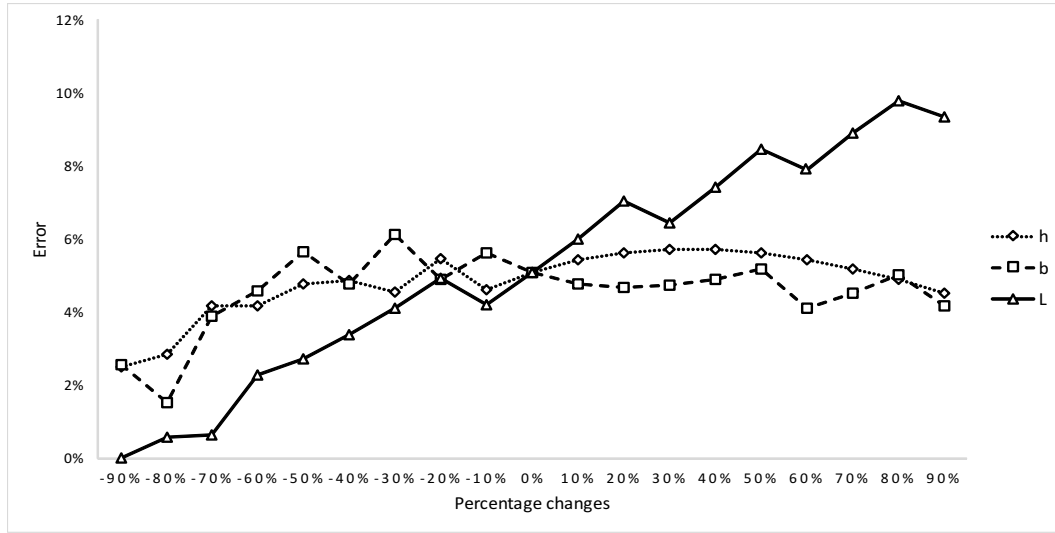


Figure 5: Sensitivity of Feeney-Sherbrooke's cost to optimal total cost for a shifted Poisson demand size.

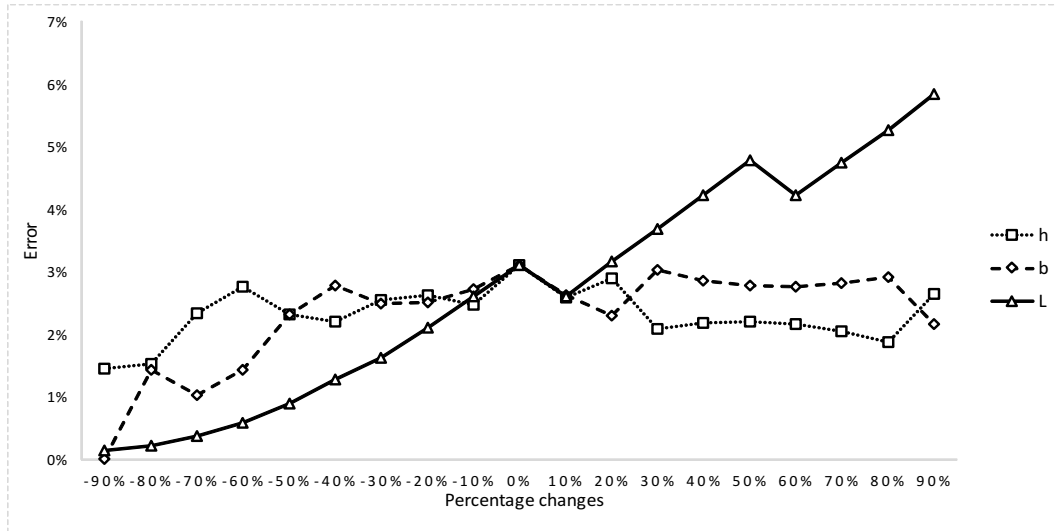


Figure 6: Sensitivity of Feeney-Sherbrooke's cost to the optimal total cost for a logarithmic-series demand size.

4 Conclusion

We derived simple recursion procedures for computing the steady-state probabilities of the number of outstanding orders for $(S - 1, S)$ base-stock systems with lost sales. The recursions work for both complete and partial rejection policies. For a general compound Poisson demand process with a discrete probability distribution of the demand sizes, our recursion procedures allow to compute the optimal and the best (approximate) base-stock level under complete and partial rejection policies respectively. The recursions we propose can also be used for continuous demand sizes through discretization that preserves the moments of the original distribution. Our numerical results reveal that the proposed procedure with the partial rejection case provides the optimal base-stock level in most of the instances considered. The cost error resulting from the optimal policy is close to zero. We also show that the convolution algorithm of Feeney-Sherbrooke yields a poor approximation of the optimal base-stock policy since the cost error can go up to 17% from the optimal cost.

Our method can be extended to multi-echelon inventory systems where each echelon uses a base-stock policy. For such models, most of the earlier research assumes Poisson demand processes. Future research can also include the case of base-stock systems with multiple locations and emergency/lateral transshipment, i.e. instead of losing excess demand, it can be satisfied either by an emergency order or by using a lateral transshipment.

Acknowledgements

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Appendix A: Proof of Proposition 1

Proof. To obtain the steady-state probabilities $P(j), 0 \leq j \leq S$, consider the case where $S = \infty$. Then, the state space of $\mathbf{n}(\mathbf{t})$ is the set of all positive integers, which implies that there is no

interaction between the different classes. The individual components of $\mathbf{n}(\mathbf{t})$ change independently of each other and they are continuous time Markov chains. Without loss of generality, assume that we have only S classes. The system can be seen as a network of S infinite server queues where the i -th queue accepts only class i . The i -th queue in isolation behaves as $M/M/\infty$, which is a reversible queue. The steady-state probabilities, denoted by $\pi_\infty(n_i)$, can therefore be written as: $\pi_\infty(n_i) = \frac{(\lambda L f(i))^{n_i} e^{-\lambda L f(i)}}{n_i!}$. Since the process $\mathbf{n}(\mathbf{t})$ forms S independent $M/M/\infty$ queues and its components change independently of each other, it is a reversible process. We can write the steady-state probabilities of the whole system with $S = \infty$ as a product-form:

$$\pi_\infty(\mathbf{n}) = \prod_{i=1}^S \frac{(\lambda L f(i))^{n_i} e^{-\lambda L f(i)}}{n_i!}, \quad (13)$$

where S in (13) is the number of classes. Now, the state space of the original multi-class queue $\mathbf{n}(\mathbf{t})$ is the same as the state space of the multi-class queue with $S = \infty$, except that it is truncated to Ω . Hence, we obtain the steady-state probabilities of the original multi-class queue by applying Corollary 1.10 of Kelly (1979). This Corollary states that if a reversible Markov process, with a given state space (say Θ) and an equilibrium probability, is truncated to a subset $\Omega \subset \Theta$, then the resulting Markov process is reversible and has the same equilibrium distribution as with state space Θ but restricted to Ω . It follows that the steady-state probabilities $\pi(\mathbf{n})$, $\mathbf{n} \in \Omega$ of the original system are given by:

$$\pi(\mathbf{n}) = \frac{1}{G} \prod_{i=1}^S \frac{(\lambda L f(i))^{n_i}}{n_i!}, \text{ where } G = \sum_{\mathbf{n} \in \Omega} \prod_{i=1}^S \frac{(f(i) \lambda L)^{n_i}}{n_i!}. \quad (14)$$

If we are interested in the total number of busy servers, then the steady-state probability of having j busy servers is written as:

$$P(j) = \sum_{\mathbf{n} \in \Omega(j)} \pi(\mathbf{n}) = \left(\sum_{\mathbf{n} \in \Omega} \prod_{i=1}^S \frac{(f(i) \lambda L)^{n_i}}{n_i!} \right)^{-1} \sum_{\mathbf{n} \in \Omega(j)} \prod_{i=1}^S \frac{(f(i) \lambda L)^{n_i}}{n_i!}. \quad (15)$$

■

Appendix B: Proof of Corollary 1

Proof. Let $U(t) = (\mathbf{n}(t), x) = \{(n_1(t), \dots, n_S(t), x_{i,k}), 1 \leq i \leq S, 1 \leq k \leq n_i\}$, where $x_{i,k}$ is the time elapsed prior to t since the k th order of class i has arrived. $U(t)$ is a Generalized Semi-Markov Process (GSMP). One can approximate the general lead-time distribution G , in weak convergence sense, by a mixture of Erlang distributions that have the same mean as the original distribution. The mixture of Erlang distributions can be written as

$$G_n(t) = \sum_{k=1}^{\infty} q_k E_{n,k}(t), \quad (16)$$

where $n \geq 1$, $q_k = G(\frac{k}{n}) - G(\frac{k-1}{n})$, $\sum_{k=1}^{\infty} q_k = 1$ and $E_{n,k}$ is an Erlang distribution with k stages and mean k/n . By Theorem 5.5.1 of Tijms (2003), we have $\lim_{n \rightarrow \infty} G_n(t) = G(t), t \geq 0$. Define by $U_{n,k}(t)$ a sequence of state processes with the approximated service-time distribution $E_{n,k}$ that keeps track of all orders in the service stage with mean k/n for each n and k . $U_{n,k}(t)$ is a continuous time Markov process that has unique steady-state probabilities. In addition, as shown by Burman et al. (1984), since the detailed balance equation is satisfied by the steady-state probabilities of the exponential lead-time distribution case, the steady-state probabilities of the process $U_{n,k}(t)$ are also given by Equation (4), where L is replaced by k/n . Finally, by Theorem 3 of Whitt (1980), the steady-state probabilities of the process $\lim_{\substack{k \rightarrow \infty \\ n \rightarrow \infty}} U_{n,k}(t)$ imply that the process $U(t)$ has a stationary distribution given by Equation (4) in a weak convergence sense. Another proof for the insensitivity to the service time distribution can be found in Burman et al. (1984) and in Bonald and Proutière (2002) for the circuit-switching network context. ■

Appendix C: Proof of Corollary 2

Proof. From Corollary 1, we assume without loss of generality that the lead-time follows an exponential distribution with mean L . If we let $\pi(\mathbf{n})$ denote the stationary probabilities that $\mathbf{n} \in \Omega$ orders are outstanding in the system, then, since the process $\{\mathbf{n}(t), t \geq 0\}$ is reversible, the detailed

balance equation holds and can be written as:

$$\frac{n_i}{L}\pi(\mathbf{n}) = \lambda f(i)\pi(\mathbf{n} - e_i) \quad \text{for } \mathbf{n} \in \Omega(j), j \leq S, i \leq S, \quad (17)$$

where e_i denotes a vector with the i -th element being 1 and the other elements being 0.

Summing up the above equation over all states $\mathbf{n} \in \Omega(j)$, we obtain:

$$\begin{aligned} \sum_{\mathbf{n} \in \Omega(j)} n_i \pi(\mathbf{n}) &= \sum_{\mathbf{n} \in \Omega(j)} \lambda L f(i) \pi(\mathbf{n} - e_i) = \lambda L f(i) \sum_{\mathbf{n} \in \Omega(j)} \pi(\mathbf{n} - e_i) = \lambda L f(i) \sum_{\mathbf{n} \in \Omega(j-i)} \pi(\mathbf{n}) \\ &= \lambda L f(i) P(j-i), 1 \leq i \leq j \leq S. \end{aligned} \quad (18)$$

Multiplying both sides of (18) by i and summing from 1 to j , we get:

$$\sum_{i=1}^j i \lambda L f(i) P(j-i) = \sum_{i=1}^j \sum_{\mathbf{n} \in \Omega(j)} i n_i \pi(\mathbf{n}) = \sum_{\mathbf{n} \in \Omega(j)} \left(\sum_{i=1}^j i n_i \right) \pi(\mathbf{n}) = j P(j). \quad (19)$$

The result of Corollary 2 follows from Equation (19). The recursion derived in (19) is analogous to that derived by Kaufman (1981) for circuit-switched networks and that by Ross (1995) for the stochastic knapsack problem. ■

Appendix D: Proof of Proposition 2

Proof. Assume that the function ψ exists for any distribution of the demand size. Therefore, the process $\{\mathbf{n}(t), t \geq 0\}$ is quasi-reversible. From Equations (5) and (6), we have:

$$\begin{cases} \frac{\psi(\mathbf{n})}{\psi(\mathbf{n} - e_i)} \lambda f(i) = \lambda f(i) & \text{for } \mathbf{n} \in \Omega(j), i < S, j < S, \\ \frac{\psi(\mathbf{n})}{\psi(\mathbf{n} - e_i)} \lambda f(i) = \lambda(1 - F(i-1)) & \text{for } \mathbf{n} \in \Omega(S), i \leq S, \end{cases} \quad (20)$$

which means that:

$$\begin{cases} \psi(\mathbf{n}) = 1 & \text{for } \mathbf{n} \in \Omega(j), i < S, j < S, \\ \psi(\mathbf{n}) = \frac{1 - F(i-1)}{f(i)} & \text{for } \mathbf{n} \in \Omega(S), i \leq S. \end{cases} \quad (21)$$

Relation (21) holds for any type of demand size distribution and all order classes. Therefore, for an order of class 1, we have $\psi(\mathbf{n}) = \frac{1 - F(0)}{f(1)} = \frac{1}{f(1)}$, $\mathbf{n} \in \Omega(S)$. But $\psi(\mathbf{n}) = \frac{1 - F(i-1)}{f(i)}$, $\mathbf{n} \in \Omega(S)$ for an order of class $i > 1$, which means that the distribution of the demand size should be characterized by $\frac{1}{f(1)} = \frac{1 - F(i-1)}{f(i)}$, for all $i \geq 2$. The only distribution for which such a relation holds is the geometric one. In fact, the pmf of the geometric distribution is $f(i) = (1 - \theta)\theta^{i-1}$, $0 < \theta < 1$ and we have $1 - F(i-1) = \theta^{i-1} = \frac{f(i)}{1 - \theta}$, which means that $\psi(\mathbf{n}) = \frac{1}{1 - \theta}$, for $\mathbf{n} \in \Omega(S)$. Since the process $\{\mathbf{n}(t), t \geq 0\}$ is quasi-reversible, from Kelly (1979) (section 3.5), the steady-state probability $P(j)$ is given by (7). ■

Appendix E: Proof of Corollary 3

Proof. The quasi-reversibility allows us to write the balance equations as follows:

$$\lambda f(i)\pi(\mathbf{n} - e_i) = \frac{n_i}{L}\pi(\mathbf{n}) \text{ for } \mathbf{n} \in \Omega(j), i < S, j < S, \quad (22)$$

$$\begin{aligned} \frac{\psi(\mathbf{n})}{\psi(\mathbf{n} - e_i)} \lambda f(i)\pi(\mathbf{n} - e_i) &= \psi(\mathbf{n}) \lambda f(i)\pi(\mathbf{n} - e_i) = \lambda(1 - F(i-1))\pi(\mathbf{n} - e_i), \\ &= \frac{n_i}{L}\pi(\mathbf{n}) \text{ for } \mathbf{n} \in \Omega(S), i \leq S. \end{aligned} \quad (23)$$

Using the above balance equations and by a reasoning similar to the one we introduced with the complete rejection policy, Corollary 3 follows. ■

Appendix F: Proof of Lemma 1

Proof. Recall that the number of outstanding items in a base-stock policy with lost sales and compound Poisson demand is equivalent to the number of busy servers of an $M^X/G/S/S$ queue with batch arrival. Statement (i) holds by Theorem 3 of Wolff and Wang (2002), where the arbitrary arrival process is replaced by a compound Poisson process. For statement (ii), the proof is by counterexample. First assume that the total cost is convex. The fraction of the demand lost $B(S)$ is also convex. This implies that, for a given S , we have $B(S-1) - B(S) > 0$ and $B(S-1) - B(S) > B(S) - B(S+1)$. Suppose that all orders have equal demand sizes. For example, the demand

size is equal to 5. We fix S to 4. Then we should have $B(3) - B(4) > B(4) - B(5)$, however, $B(3) = B(4) = 1 > B(5) = B(6) = B(7) = B(8) = B(9)$, which means that the relation $B(S - 1) - B(S) > B(S) - B(S + 1)$ does not hold. Finally, for statement (iii), the convexity of the expected total cost allows us to write the following:

$$Z(S^*) < Z(S) \Rightarrow h(S^* - S) \leq (b + hL)\lambda\mu [B(S) - B(S^*)] \text{ for all } S = 0, 1, \dots \quad (24)$$

and

$$\frac{h}{\lambda(b + hL)\lambda\mu} \leq \sum_{i=S}^{S^*-1} \frac{B(i) - B(i + 1)}{S^* - S} \text{ if, } S^* > S, \quad (25)$$

$$\frac{h}{\lambda(b + hL)\lambda\mu} \geq \sum_{i=S^*}^{S-1} \frac{B(i) - B(i + 1)}{S - S^*} \text{ if, } S > S^*. \quad (26)$$

The convexity of $B(S)$ implies $B(S - 1) - B(S) > 0$ and $B(S - 1) - B(S) > B(S) - B(S + 1)$. With (25) and (26), one can show that the optimal base-stock level S^* satisfies:

$$B(S^* + 1) \leq \frac{h}{\lambda(b + hL)\lambda\mu} \leq B(S^*). \quad (27)$$

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