Complex Linear-Quadratic Systems for Detection and Array Processing

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Abstract

Linear-quadratic (LQ) filters for detection and estimation are widely used in the real case. We investigate their extension in the complex case, which introduces various new questions. In particular, we calculate the optimum LQ array receiver in a non-Gaussian environment by using the deflection criterion and evaluate some of its performance.

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I. INTRODUCTION

Linear-quadratic (LQ) systems are widely used in many areas of signal processing and especially in detection problems. As an example, the optimum receiver for the detection of a normal signal in a

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noma1 noise is an LQ system. Most of the results known for LQ systems are established in the real case. However, these assumptions are too restrictive for various problems and especially in narrowband array processing. Even if the physical signals received by the sensors are real, there is a great advantage in the narrowband case to work with the complex representation, for example, by using the analytic signal (see [l, p. 2291). This especially allows definition of the complex steering vector characterizing the geometrical structure of the problem.

The most general form of a complex LQ filter calculating an output $\mathbf{y}$ in terms of a vector input $\mathbf{x}$ is

$$y(\mathbf{x}) = c + \mathbf{x}^H \mathbf{h}_1 + \mathbf{x}^T \mathbf{h}_2 + \mathbf{x}^T \mathbf{M}_1 \mathbf{x} + \mathbf{x}^H \mathbf{M}_2 \mathbf{x}^* + \mathbf{x}^T \mathbf{M}_3 \mathbf{x}, \tag{1}$$

where

c is a constant;

$\mathbf{h}_1$ is a complex vector;

$\mathbf{h}_2$ is a complex vector;

$\mathbf{M}_i$ are three complex matrices.

In this equation, $\mathbf{x}^H$ means transposition and complex conjugation. Note that because of the symmetry of the last two quadratic terms, there is no loss of generality in assuming that the matrices $\mathbf{M}_2$ and $\mathbf{M}_3$ are symmetric. The introduction of complex signals and systems yields significant changes in statistical signal processing problems. The main purpose of this correspondence is to calculate the vectors and matrices appearing (1) in such a way that y satisfies some optimality criterion introduced in the next section.

II. Statement of the Problem

The basic detection problem consists of deciding between two simple hypotheses H_0 and H_1 from an observation vector $\mathbf{x}$. When the probability distributions of $\mathbf{x}$ underH_0 and H_1 are known, the optimum procedure consists in comparing the likelihood ratio (LR) to a threshold. Our basic assumption is that we are not in this situation and that our knowledge concerning the statistical properties of $\mathbf{x}$ is much lower. If, for instance, this knowledge is limited to second-order properties of $\mathbf{x}$ under H_0 and H_1, which means that only the mean values and the covariance matrices are known, it is possible to calculate the linear filter that maximizes the output signal to noise ratio, and, in the case of nonrandom signals, this leads to the famous matched filter used in many areas of statistical signal processing (see [l, p. 5551]). The output signal-to-noise ratio is also called the *deflection* and can be defined for any filter y($\mathbf{x}$) by the expression

$$D(y) \triangleq \frac{|E_1(y) - E_0(y)|^2}{V_0(y)} \ , \tag{2}$$

where E_0 and E_1 are expectations under H_0 and H_1, respectively, and V_0 is the variance under H_0. This deflection was introduced long time ago and has been used under various assumptions, especially in the context of array processing [2]-[5]. Even if it has been essentially used in the linear case, there is no reason to limit its use to linear systems. Therefore, if the moments up to the order 4 are known, it is possible to calculate the deflection of (1) and to find the system giving its maximum value. This work extends to the complex case results obtained in [6] for the real case.

We shall outline only the principles of the method used in order to maximize the deflection of systems like (1). The first point is to note that the deflection is invariant under affine transformation, and then it is appropriate to use this property to work with LQ systems having an output with zero mean value under Ho. This is realized by subtracting the mean value. By assuming that the input vector $\mathbf{x}$ is zero-mean valued urtder H_0, the filter (1) takes the form

$$y(\mathbf{x}) = \mathbf{x}^H\mathbf{h}_1 + \mathbf{x}^T\mathbf{h}_2 + \mathbf{x}^H\mathbf{M}_1\mathbf{x} + \mathbf{x}^H\mathbf{M}_2\mathbf{x}^* + \mathbf{x}^T\mathbf{M}_1\mathbf{x} -$$

$$-\mathrm{Tr}[\mathbf{M}_1\mathbf{R} + \mathbf{M}_2\mathbf{C}^* + \mathbf{M}_3\mathbf{C}], \tag{3}$$

where Tr means the trace, $\mathbf{R} \triangleq E_0(\mathbf{x}\mathbf{x}^H)$ is the covariance matrix of $\mathbf{x}$, an $\mathbf{C} \triangleq E_0(\mathbf{x}\mathbf{x}^T)$. These matrices define the second-order moments of $\mathbf{x}$ under H_0. They are obviously equal in the real case. The matrix $\mathbf{C}$ is often zero, and this especially appears under the assumption of circularity [7]. In all the following, we shall only consider LQ systems written as in (3). The same procedure of subtraction of the mean value can be applied to the likelihood ratio $L(\mathbf{x})$ associated with our problem. As its mean value under H_0 is obviously 1, we can introduce the displaced likelihood ratio (DLR) $L_d(\mathbf{x})$ defined by $L_d(\mathbf{x}) \triangleq L(\mathbf{x}) - 1$. Looking at (3), we note that the set of all the systems defined by this equation belong to a vector space. This space becomes a Hilbert space denoted H_{LQ} if a scalar product is introduced, and this scalar product of two filters $u(\mathbf{x})$ and $v(\mathbf{x})$ is defined by $(u, v) \triangleq E_0[u^*(\mathbf{x})u(\mathbf{x})]$. It can then be shown that the maximum value of the deflection of systems like (3) is obtained with the projection of $L_d(\mathbf{x})$ onto the Hilbert subspace H_{LQ}. It is then charactenized by the orthogonality principle (see [I, p. 3981) saying that $E_0[g^*\mathbf{x})y_{opt}(\mathbf{x})] = E_0[g^*\mathbf{x})L_d(\mathbf{x})]$ for any $g[\mathbf{x})$ belonging to H_{LQ}. However, it results from the definition of the DLR that $E_0[g^*\mathbf{x})L_d(\mathbf{x})] = E_1 g^*\mathbf{x})$, and the orthogonality relation takes the form

$$E_0[g^*(\mathbf{x})y_{opt}(\mathbf{x})] = E_1[g^*(\mathbf{x})] \; \forall g(\mathbf{x}) \in H_{LQ} \;, \tag{4}$$

which allows us to calculate $y_{opt}(\mathbf{x})$. Furthermore, the maximum value of the deflection is $D_{max} = E_0[|y_{opt}(\mathbf{x})|^2]$ Note at this step that even if the DLR has been used in the calculation, it disappears in the orthogonality equation (4). In fact, as $g(\mathbf{x})$ and $y_{opt}(\mathbf{x})$ belong H_{LQ} all the momenis appearing in (4) are of an order smaller or equal to 4. As a consequence, a statistical knowledge up to the fourth order is sufficient to obtain the optimum filter in terms of deflection.

To conclude this sectiion, it is worth discussing the uniqueness of the solution. From classical results of Hilbert spaces properties, it is clear that the projection $y_{opt}(\mathbf{x})$ defined by (4) is unique. However, as the deflection is invariant under affine transformations, all the filters such that $ay_{opt}(\mathbf{x}) + b$, $a \neq 0$, give the same deflection and then are also optimal. For simplicity, we shall only consider $y_{opt}(\mathbf{x})$ in the following. There are some technical details regarding the structure of (1) or of (3) that must be addressed. Consider in these expressions only the quadratic terms. It is clear that two different matrices $\mathbf{M}'$ and $\mathbf{M}''$ can give the same value for the quadratic form $\mathbf{x}^H\mathbf{M}\mathbf{x}^*$. This is also valid for the last term of (1), and then, the optimum system $y_{opt}(\mathbf{x})$ can take different forms. In order to ensure the uniqueness of the representation of $y_{opt}(\mathbf{x})$, it is sufficient to assume, as indicated above, that $\mathbf{M}_2$ and $\mathbf{M}_3$ are symmetric, and this assumption is introduced in all of the following. The problem is then to calculate the parameters $[\mathbf{h}_1, \mathbf{h}_2, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3]$ introduced in (3) and satisfying (4) for any $g(\mathbf{x})$ written as in (3). Applying (4) successively with

$$g(\mathbf{x}) = \mathbf{x}^H\mathbf{g}, \quad g(\mathbf{x}) = \mathbf{x}^T\mathbf{g}, \quad g(\mathbf{x}) = \mathbf{x}^H\mathbf{N}\mathbf{x} - \mathrm{Tr}[\mathbf{N}\mathbf{R}]$$

$$g(\mathbf{x}) = \mathbf{x}^H\mathbf{N}\mathbf{x}^* - \mathrm{Tr}[\mathbf{N}\mathbf{C}^*, \;\; g(\mathbf{x}) = \mathbf{x}^T\mathbf{N}\mathbf{x} - \mathrm{Tr}[\mathbf{N}\mathbf{C}]$$

where the vector $\mathbf{g}$ and the matrix $\mathbf{N}$ are arbitrary, we obtain five equations defining the optimal system. These equations are linear with respect to the unknown parameters, which, under classical general conditions, ensures the uniqueness of the solution.

III. Application to Narrowband Array Process

Consider a narrowband array processor such that the N components of the observation vector $\mathbf{x}$ are constructed from the complex envelope of the signals processed by the N sensors of the array. Under H_0, (noise only) $\mathbf{x} = \mathbf{b}$, vector describing the noise received by the sensors. Under H_1 (signal plus noise), we have $\mathbf{x} = m\mathbf{s} + \mathbf{b}$, where $\mathbf{s}$ is the complex steering vector of the signal, and m is its complex amplitude. The first- and second-order moments of $\mathbf{x}$ are assumed to be known. Under H_0 , $\mathbf{x}$ has a zero mean value, and its second-order properties are defined by the matrices $\mathbf{R}$ and $\mathbf{C}$ introduced above. Under H_1, we have

$$\mathbf{m}_1 = \mu\mathbf{s}, \; \mathbf{R}_{1-0} \stackrel{\triangle}{=} E_1(\mathbf{x}\mathbf{x}^H) - E_0(\mathbf{x}\mathbf{x}^H) = \pi_s\mathbf{s}\mathbf{s}^H,$$

$$\mathbf{C}_{1-0} \stackrel{\triangle}{=} E_1(\mathbf{x}\mathbf{x}^T) - E_0(\mathbf{x}\mathbf{x}^T) = \gamma_s\mathbf{s}\mathbf{s}^H, \tag{5}$$

where $\mu \triangleq E[m]$, $\pi_s \triangleq E[|m|^2]$, and $\gamma_s \triangleq E[m^2]$ In particular, if the signal is random and with a zero mean value, $\mu = 0$ and π_s is proportional to the input power. For the nonrandom signal, the expectations appearing in the previous relations are simply canceled. Under these assumptions, the problem is to find the optimal second-order LQ filter that maximizes the deflection.

As the components x_i of $\mathbf{x}$ are under H_0 related to the complex envelopes of a stationary narrowband signal, it is shown in [7] that if the indices i_k. are integers taken in $[l.2, ..., N]$, then $E_0[x_{i_1}, x_{i_2}, ..., x_{i_m}] = 0$ and $E_0[x^*_{i_1}, x_{i_2}, ..., x_{i_n}] = 0$ for any m and n. As a result, we deduce that $\mathbf{C}$ and the third-order moment of $\mathbf{x}$ under H_0 are zero. Inserting this result into the five equations indicated at the end of the previous section and defining thc optimal complex LQ filter, we find the quantities $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ defining the optimal second-order LQ array receiver. After some algebra, which is presented more in detail in [9], we obtain that the vectors defining the linear part of (1) satisfy $\mathbf{h}_1 = \mathbf{h}_2 = \mu\mathbf{R}^{-1}\mathbf{s}$ and then are equal to zero if the signal is random with zero mean value ($\mu = 0$) but are proportional to the spatial matched filter $\mathbf{x}\mathbf{R}^{-1}\mathbf{s}$ and its conjugate for nonrandom signals. Similarly, it is easy to deduce from the assumptions introduced that the optimal matrix $\mathbf{M}_1$ is Hermitian. Finally, it appears that $\mathbf{M}_2 = \mathbf{M}_3^*$. In particular, for the stationary narrowband random signal case, we find that $\gamma_s = 0$, and then, the optimal symmetric matrices $\mathbf{M}_2$ and $\mathbf{M}_3$ are zero. Thus, for the random signal case, the optimal second-order LQ array receiver is purely quadratic and can be written as $y_{opt} = \mathbf{x}^H\mathbf{M}_1\mathbf{x} - \mathrm{Tr}[\mathbf{R}\mathbf{M}_1]$, whereas it remains a widely linear part [8] for nonrandom signals. Applying the expression previously given for the maximum value of the deflection to LQ systems yields

$$D_{max} = 2|\mu|^2\mathbf{s}^H\mathbf{R}^{-1}\mathbf{s} + \pi_s\mathbf{s}^H\mathbf{M}_1\mathbf{s} + \gamma_s\mathbf{s}^T\mathbf{M_3}\mathbf{s} + \gamma_s^*\mathbf{s}^H\mathbf{M}_3^*\mathbf{s}^* \tag{6}$$

The first term of the right-hand side of this expression, which is equal to zero for the random signal case, is a result of the widely linear part of the LQ array receiver, whereas the other terms, which are noted d_q, are a result of the purely quadratic pait. It is easy to see that $d_q > 0$, and we deduce that in order to detect a given nonrandom signal in a given environment and for a given number of sensors, it is always better, in terms of deflection maximization, to use a LQ array receiver than a widely linear one.

Let us now introduce the assumption of fourth-order Gaussian noise, which means that that all the moments up to the order 4 are those of a Gaussian noise, but no assumption is introduced for the other moments that are completely unknown. By applying the formula giving the fourth-order moment of circular complex Gaussian random variables [l], we obtain, after some simple algebra

$$\mathbf{M}_1 \triangleq \mathbf{M}_1^G = \pi_s\mathbf{R}^{-1}\mathbf{s}\mathbf{s}^H\mathbf{R} \triangleq \pi_s\mathbf{h}_s\mathbf{h}_s^H \tag{7}$$

$$\mathbf{M}_2 \triangleq \mathbf{M}_2^G = (1/2)\gamma_s\mathbf{R}^{-1}\mathbf{s}\mathbf{s}^T\mathbf{R}^{-T} = (1/2)\gamma_s\mathbf{h}_s\mathbf{h}_s^T. \tag{8}$$

and $\mathbf{M}_3 \stackrel{\triangle}{=} \mathbf{M}_3^G = \mathbf{M}_3^{G*}$. All these results show that the optimal LQ array receiver in a complex circular fourth-order Gaussian noise can be directly deduced from the spatial matched filter $\mathbf{h}_s$ defined in (8) both for the random and nonrandom signal. The fact that the linear matched filter can appear in a quadratic structure was already shown by a completely different approach in [2].

Finally, the corresponding maximum deflection is then given by

$$D_G = 2|\mu|^2 \mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} + (\pi_s^2 + |\gamma_s|^2)(\mathbf{s}^H \mathbf{R}^{-1} \mathbf{s})^2. \tag{9}$$

IV. Optimal Performance in a Spatially Fourth-Order White Noise

To evaluate more in detail the effect of the non-Gaussian character of the noise on the structure and the performance of the optimal second-order LQ array receiver, we now apply the results of the previous section to the particular case of a stationary narrowband spatially fourth-order white noise.

In most of the papers dealing with array processing, it is assumed that in a quiescent environment, the noise is spatially white. This means that the components x_i of the observation vector $\mathbf{x}$ arc under H_0 independent and identically distributed (i.i.d.) complex random variables.

In the stationary case, the only nonzero moments up to the fourth order are

$$E[x_i^* x_j = \eta_2 \delta(i,j)$$

$$E[x_i^* x_j^* x_k x_l] = (\eta_4 - 2\eta_2^2)\delta(i,j,k,l) + \eta_2^2[\delta(i,k)\delta(j,l) + \delta(i,l)\delta(j,k)], \tag{10}$$

where the symbols $\delta(.)$ are extensions of the Kronecker Delta symbols and are equal to one if all the indices are equal and to zero in the other cases. We shall say that a noise is a complex fourth-order white noise if its first moments are given by (lo), and no particular assumption is introduced on the higher order moments. It is, in some sense, an extension of the concept of complex second-order white noise, where only the first equation of (10) is introduced.

By using our previous results, we can find the optimal second-order LQ array receiver for the detection of a signal defined by $\mu, \pi_s, \gamma_s, \mathbf{s}$ in a stationary narrowband complex spatially fourthorder white noise defined by η_2 and η_4. For the liner part, we obtain $\mathbf{h}_1 = \mathbf{h}_2^* = (\mu/\eta_2)\mathbf{s}$. On the other hand, the quadratic part is characterized by the matrices

$$\mathbf{M}_1 = \mathbf{M}_1^G - \frac{\pi_s}{\eta_2^2}\frac{\beta-2}{\beta-1}\mathbf{\Lambda}_m$$

$$\mathbf{M}_2 = \mathbf{M}_2^G - \frac{\gamma_s}{2\eta_2^2}\frac{\beta-2}{\beta}\mathbf{\Lambda}_p. \tag{11}$$

In these equations, the quantity β defined by η_4/η_2^2 is the kurtosis of the noise and is equal to 2 for complex circular Gaussian noise. Furthermore, the matrices $\mathbf{\Lambda}$ are deduced from the components s_i of $\mathbf{s}$ by

$$\mathbf{\Lambda}_m \stackrel{\triangle}{=} \mathrm{diag}[|s_1|^2, |s_2|^2, ..., |s_N|^2]$$

$$\mathbf{\Lambda}_p \stackrel{\triangle}{=} \mathrm{diag}[s_1^2, s_2^2, ..., s_N^2]. \tag{12}$$

Finally, the optimal matrices in the fourth-order Gaussian case are $\mathbf{M}_1^G = (\pi_s/\eta_2^2)\mathbf{s}\mathbf{s}^H$, and $\mathbf{M}_2^G = (2\pi_s/\eta_2^2)\mathbf{s}\mathbf{s}^T$ These expressions show that the nondiagonal elements of $\mathbf{M}_1$ and $\mathbf{M}_2$ are the same as in the fourth-order Gaussian case ($\beta = 2$). In other words, the moment η_4 only appears for the computation of the diagonal elements of $\mathbf{M}_1$ and $\mathbf{M}_2$ which are the only ones that are affected by the non-Gaussian character of the noise. Furthermore, we see that when the noise is not Gaussian, the optimal matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are not dyadic, and the optimal second-order LQ array receiver cannot be implemented with the output of a linear processor alone as in the fourth-order Gaussian case. Finally, remember that $\mathbf{M}_3 = \mathbf{M}_2^*$

The performance of the optimal array receiver is characterized by the value of its maximum deflection, which is defined by (6). Introducing the assumption $|s_1|^2 = 1$ (omnidirectional sensors), we deduce from the previous expressions that

$$D_{max}(\beta) = D_G - \frac{N}{\eta_2^2}(\beta - 2)\left[\frac{\pi_s^2}{\beta - 1} + \frac{|\gamma_s|^2}{\beta}\right], \tag{13}$$

where D_G, corresponding to (9), is the maximum deflection in the fourth-order Gaussian case ($\beta = 2$), given by

$$D_G = \frac{2N|\mu|^2}{\eta_2} + \left(\frac{N}{\eta_2}\right)^2 (\pi_s^2 + |\gamma_s|^2). \tag{14}$$

Note that for the random signal case, the expression (13) becomes

$$D_{max}(\beta) = D_G\left[1 - \frac{\beta - 2}{N(\beta - 1)}\right] \tag{15}$$

Let us now discuss these last expressions. Note that the term D_G of (13) is a function of the signal input power and the number of sensors and only depends on the second-order moment η_2 of the noise. The last term of (13) is, in reality, the most interesting. It not only depends on the previously mentioned parameters, but it is also the only term where the fourth-order moments of the noise appear. We deduce from the Schwarz inequality that $\beta \geq 1$, and it is clear that the last term of (13) is equal to zero in the fourth-order Gaussian case, whereas it can be quite large when β tends to 1. The case of$\beta = 1$ corresponds to a situation of singular detection, which has been discussed in [6] and [10] for real processes. Furthermore, we can easily verify that $D_{max} > D_G$ for $\beta < 2$ (lower-Gaussian case) and $D_{max} < D_G$ for $\beta > 2$

(upper-Gaussian case). Furthermore, it results from (13) and (14) that D_{max} is a decreasing function of β. As β increases, i.e., as the noise becomes strongly upper-Gaussian, D_{max} monotonically decreases from infinity obtained for $\beta = 1$ to the value

$$D_{max}(\infty) = D_G - \frac{N}{\eta_2^2}(\pi_s^2 + |\gamma_s|^2) \tag{16}$$

obtained when β becomes infinite. Note from (13) and (14) that for large values of the number N of the sensors, the asymptotic value given by (16) is approximately equal to D_G, which means that D_{max} becomes almost independent of β. Finally, note that D_{max} is an increasing function of N and of the input signal-to-noise ratio π_s/η_2.

V. Conclusion

The problem of optimum detection using a complex LQ filter has been investigated in the non-Gaussian case using a deflection criterion. This optimal filter is the solution of a linear system of equations, and its calculation requires the statistical knowledge of the noise up to the fourth order. It has been computed both for random and nonrandom signals. The performance has been analyzed in the presence of spatially fourth-order white noise. In the non-Gaussian case, the optimal LQ array receiver cannot be implemented with only the output of a linear array receiver. Moreover, the gain in deflection due to the adaptation of the LQ array receiver to the fourth-order statistics of the noise has been computed for the particular case of a random signal and a spatial fourth-order white noise. It has been shown that this gain may become very large for strongly lower or upper Gaussian noise.

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