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SOME TIME-FREQUENCY ASPECTS OF CONTINUOUS WAVELET DECOMPOSITIONS

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Abstract: We describe some geometric aspects of wavelet systems leading to time-frequency representations of signals. These aspects are illustrated by the use of wavelets for local frequencies characterization in signals.

I. INTRODUCTION

Since the early days of wavelet analysis, the concept of "wavelet analysis" has progressively changed. The original wavelets [1] were functions $\psi_{(b,a)}(x)$ generated from a single one $\psi(x)$ (the so-called mother wavelet) by dilations and translations.

$$\psi_{(b,a)}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \ . \tag{I.1}$$

The mother wavelet was then a function of vanishing integral, such that

$$0 \ < \ c_\psi = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} \ < \ \infty \ , \tag{I.2}$$

having moreover good localization properties in both the direct space and the Fourier space. To any function $f(x) \in H^2(I\!R)$ is associated its wavelet transform $T_f(b,a)$ defined as

$$T_f(b,a) =< f, \psi_{(b,a)} >= \frac{1}{\sqrt{a}} \int_{\mathbf{R}} f(x)\psi\left(\frac{x-b}{a}\right)^* dx \ , \tag{I.3}$$

from which $f(x)$ can be recovered thanks to the reconstruction formula

$$f(x) = \frac{1}{c_\psi} \int T_f(b,a)\psi_{(b,a)}(x) \, \frac{db}{a} \, \frac{da}{a} \ . \tag{I.4}$$

The term wavelet has now to be understood in a more general sense. The goal of wavelet analysis is mainly to provide different representations of functions (or signals), as superpositions of elementary functions. The corresponding representation is then used for different purposes, such as for instance data compression, feature extraction or pattern recognition. To this effect, it becomes necessary to be able to adapt the representation to the chosen application.

As an example, continuous wavelet decompositions can be directly generalized to the n-dimensional case as a space-scale representation , by using simply scaled and shifted copies of a

mother wavelet, as in $(I.1)$. This was very useful in a mathematical context, for the analysis of singularities for example. Another approach involves the introduction of n different wavelets, one for each direction, as done in [2] in an image processing context. One may also introduce an more precise idea of direction into the analysis, by introducing rotation degrees of freedom [3], as follows. If $\psi(\underline{x}) \in L^1(I\!R^n)$ is such that

$$0 \; < \; c_\psi = K \int_{I\!R^n} |\hat{\psi}(\underline{\xi})|^2 \frac{d\xi}{||\underline{\xi}||^n} \; < \; \infty \; , \tag{I.5}$$

where K is a finite constant [3]. Any function $f(\underline{x}) \in L^2(I\!R^n)$ can then be decomposed as

$$f(\underline{x}) = \frac{1}{c_\psi} \int T_f(\underline{b}, a, \underline{\underline{r}}) \psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x}) \, \frac{d\underline{b}}{a^n} \, \frac{da}{a} \, d\mu(\underline{\underline{r}}) \tag{I.6}$$

where

$$\psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x}) = a^{-n/2} \psi \left(\underline{\underline{r}}^{-1} \cdot \frac{\underline{x} - \underline{b}}{a} \right) \tag{I.7}$$

with $\underline{b} \in I\!R^n$, $a \in I\!R^*_+$ and $\underline{\underline{r}} \in SO(n)$, the group of all rotation matrices of $I\!R^n$, and

$$T_f(\underline{b}, a, \underline{\underline{r}}) =< f, \psi_{(\underline{b},a,\underline{\underline{r}})} > \; . \tag{I.8}$$

We describe here two aspects of continuous wavelet decompositions, directly related to time-frequency analysis and its generalisations to dimensions greater than one. The first aspect concerns the use of wavelets for the problem of characterizing local frequencies in one and two dimensional signals. The second part of this paper describes some variations around the standard construction of wavelets, leading to different time-frequency decompositions.

II. WAVELETS AND ONE-DIMENSIONAL TIME-FREQUENCY SIGNAL ANALYSIS

Wavelet analysis has been shown in a signal analysis context to provide a very precise local version of Fourier analysis. Indeed, by Plancherel's formula, one has

$$T_f(b, a) = \frac{\sqrt{a}}{2\pi} \int_{\mathbf{R}} \hat{f}(\xi) e^{i\xi.b} \hat{\psi}(a\xi)^* \, d\xi \; . \tag{II.1}$$

Then, assuming that $\psi(x)$ is "well localized" around $x = 0$, and that $\hat{\psi}(\xi)$ is "well localized" around $\xi = \omega_0 \neq 0$ (because of the admissibility condition), $T_f(b, a)$ measures a position-frequency content of $f(x)$ in a neighborhood of the point $(b, \omega_0/a)$ in the position frequency plane. Moreover, because of the isometry property of the wavelet transform

$$\frac{1}{c_\psi} \int_{I\!R \times I\!R^*_+} |T_f(b, a)|^2 \frac{db}{a} \frac{da}{a} = ||f||^2 \; , \tag{II.2}$$

the squared modulus $|T_f(b, a)|^2$ of the wavelet transform can be interpreted as an energy density in the half plane.

In the context of one-dimensional signal analysis, the signals to be analyzed are in general real-valued functions, and are then completely determined by their positive frequency content.

It is then sufficient to work in the $H^2(I\!R)$-context, and then quite convenient to use progressive (or analytic) wavelets, i.e. wavelets $\psi(x)$ such that $\hat{\psi}(\xi) = 0$ for negative values of ξ.

Let us illustrate the time-frequency aspects of wavelet analysis with the following simple examples. The wavelet transform of

$$f(x) = \cos(\omega x)$$

is obviously given by

$$T_f(b,a) = \frac{\sqrt{a}}{2}\left[e^{i\omega b}\hat{\psi}(a\omega)^* + e^{-i\omega b}\hat{\psi}(-a\omega)^*\right] \ .$$

If the wavelet $\psi(x)$ is such that $\hat{\psi}(\xi) \neq 0$ for negative ξs, $|T_f(b,a)|$ is an oscillatory function, which does not yield a satisfactory description of the time-frequency energy distribution of $f(x)$. Such a phenomenon disappears if $\psi(x)$ is a progressive wavelet. The main consequence of the progressivity is the absence of beats, which gives a perfect localization of the energy in the position-scale half-plane, around the horizontal line $a = \omega_0/\omega$.

In the more realistic situation where the amplitude of the plane wave depends on time, it is possible to characterize such a dependence under suitable assumptions. Indeed, let now

$$f(x) = A(x)\cos(\omega x) \qquad (II.3)$$

and let $\psi(x)$ be a progressive wavelet. Then it follows directly from Taylor's formula that

$$T_f(b,a) = \frac{\sqrt{a}}{2}A(b)e^{i\omega b}\,\hat{\psi}(a\omega)^* + r(b,a) \ , \qquad (II.4)$$

where $r(b,a)$ is bounded as

$$|r(b,a)| \leq K.a^{-1}.Sup\,\{|A'(x)|\}$$

for some constant K (to get such an estimate, the wavelet $\psi(x)$ must decay enough at infinity so that $\int |x||\psi(x)|\,dx < \infty$). Clearly if $a|A'/A| << 1$, the remainder can be neglected and $A(x)$ and ω can be directly obtained from $T_f(b,a)$. The continuous wavelet transform thus yields a direct access to the frequency and local amplitude of the signal. This is the simplest example of the use of wavelet analysis as a local Fourier analysis.

A little bit more complex is the case where the frequency of the analyzed signal is allowed to change with x. Before entering the subject, we first need to describe the notion of instantaneous frequency. To analyze a signal in terms of local amplitude and frequency, it is convenient to write it in the form

$$f(x) = a(x)\cos(\varphi(x)) \qquad (II.5)$$

and to define an instantaneous frequency $\nu_i(x)$ by $\nu_i(x) = \varphi'(x)/2\pi$. Such a presentation is far from unique, but this problem can be solved by the introduction of the so-called analytic signal $Z_f(x)$ of $f(x)$, defined by its Fourier transform as $\widehat{Z_f}(\xi) = 2H(\xi)\hat{f}(\xi)$ where $H(\xi)$ is the Heaviside step function. $Z_f(x)$ can clearly be written in a unique way as

$$Z_f(x) = A(x)e^{i\phi(x)} \ , \qquad (II.6)$$

where $A(x)$ is a positive real-valued function, and $\phi(x)$ takes values in say $[0, 2\pi[$. (A, ϕ) is called the canonical pair of $f(x)$, and the expression

$$f(x) = A(x)\cos(\phi(x)) \tag{II.7}$$

is the canonical presentation of $f(x)$. It is easily checked that the instantaneous frequency coincides with the usual frequency in the case of plane waves.

How to use wavelet analysis for the characterization of the canonical pair $(A(x), \phi(x))$ of a signal $f(x)$? To carry on such an analysis, it is necessary to make the same assumptions as before, namely to assume that $A(x)$ is slowly varying compared with the oscillatory term $\cos(\phi(x))$. Then, assuming that $\psi(x) = A_\psi(x)\exp(i\phi_\psi(x))$ is progressive and also satisfies the same conditions, the wavelet coefficient $T_f(b, a)$ takes the form of an oscillatory integral. Using classical approximation techniques for such integrals, it can be shown [4] that $T_f(b, a)$ is locally maximum around a curve $\mathcal{R}$ in the (b, a) half-plane, called the ridge of the wavelet transform, of equation

$$a = a_r(b) = \frac{\phi'_\psi(0)}{\phi'(b)} \tag{II.8}$$

Moreover, the restriction of the wavelet transform to its ridge (the so-called *skeleton of the transform*) has a very simple expression:

$$T_f(b, a_r(b)) = Corr(b)Z_f(b) + r(b) \tag{II.9}$$

where $Corr(b)$ is a corrective factor, completely determined by the wavelet $\psi(x)$ and the ridge $a_r(b)$, and $r(b)$ is some remainder, whose size can be estimated by standard techniques. Then the knowledge of the ridge of the wavelet transform is a sufficient information to characterize $A(x)$ and $\nu_i(x)$. It turns out that a fast algorithm can be built to numerically extract the ridge from the wavelet transform . The main idea of the algorithm is that the ridge is the place in the time-scale half plane where the local frequency of the wavelet transform (i.e. some derivative of the argument of $T_f(b, a)$ with respect to b, up to a factor 2π) coincides with the instantaneous frequency of the wavelet $\psi_{(b,a)}(x)$. It can then be reached for instance by a fixed-point algorithm [4].

Of course, when the analyzed function is simply of the form $(II.5)$, the use of wavelet analysis is not necessary. One can directly compute the analytic signal $Z_f(x)$ and then deduce the canonical pair $(A(x), \phi(x))$. However, if the signal to be analyzed is now a linear combination of components of the form $(II.5)$, one is clearly interested in the individual canonical pairs of the components, and the global analytic signal cannot give such an answer. Now, since wavelet transform is a linear transform, the wavelet transform of the composite signal is the sum of the wavelet transforms of the components, that possess their own ridges. Then, if the ridges are not too close from each other in the (b, a) half-plane (in which case the approximation of oscillatory integrals fails), one can associate a skeleton to every ridge, and then compute the canonical pairs of all components. We refer to [4] and [5] for a more detailed presentation of that method, and for numerical illustrations.

III. WAVELETS AND TWO-DIMENSIONAL SPACE-FREQUENCY SIGNAL ANALYSIS

Wavelets are in general used in the context of two-dimensional signal analysis as a position-scale tool. Indeed, they are often used to detect abrupt changes and singularities in images [2]. It turns out that the method of "ridge and skeleton" previously described can be generalized to the two-dimensional setting, giving rise to a method for the detection and characterization of local frequencies in two-dimensional signals and images.

The starting point of the method is the same as before, except that there is no standard notion of analytic signal available in more than one dimension (a possible definition is proposed in [6]). We then assume that the (real-valued) two-dimensional signal can be locally modeled as

$$f(\underline{x}) = A(\underline{x}) \cos(\phi(\underline{x})) \ , \tag{III.1}$$

where the local amplitude and phase are such that

$$\left| \frac{1}{A(\underline{x})} \ \nabla_{\underline{x}} A(\underline{x}) \right| \ << \ \nabla_{\underline{x}} \phi(\underline{x}) \ . \tag{III.2}$$

Let $\psi(x) \in L^2(I\!R^2)$ be an admissible wavelet. Set $Z_f^{\pm}(\underline{x}) = A(\underline{x}) \exp[\pm i\phi(\underline{x})]$. Then the two-dimensional wavelet transform (with rotations) of $f(\underline{x})$

$$T_f(\underline{b}, a, \theta) = \frac{1}{a} \int_{I\!R^2} f(\underline{x}) \ \psi \left(\underline{\underline{r}}_{-\theta} \cdot \frac{\underline{x} - \underline{b}}{a} \right)^* d\underline{x} \tag{III.3}$$

can be approximated as

$$T_f(\underline{b}, a, \theta) = \frac{Z_f^+(\underline{x}_0^+) \, \psi \left(\underline{\underline{r}}_{-\theta} \cdot \frac{\underline{x}_0^+ - \underline{b}}{a} \right)^*}{Corr^+(\underline{b}, a, \theta)} + \frac{Z_f^-(\underline{x}_0^-) \, \psi \left(\underline{\underline{r}}_{-\theta} \cdot \frac{\underline{x}_0^- - \underline{b}}{a} \right)^*}{Corr^-(\underline{b}, a, \theta)} + r(\underline{b}, a, \theta) \tag{III.4}$$

where $\underline{x}_0^{\pm} = \underline{x}_0^{\pm}(\underline{b}, a, \theta)$are the solutions (assumed to be unique for simplicity) of the stationarity equations

$$\nabla_{\underline{x}} \phi(\underline{x}_0^{\pm}) = \pm \frac{1}{a} \underline{\underline{r}}_{\theta} \cdot \nabla_{\underline{x}} \phi_{\psi} \left(\underline{\underline{r}}_{-\theta} \cdot \frac{\underline{x}_0^{\pm} - \underline{b}}{a} \right) \tag{III.5}$$

$Corr^{\pm}(\underline{b}, a, \theta)$ are corrective factors whose expression are completely explicit, and $r(\underline{b}, a, \theta)$ is a remainder whose size depends on the derivatives of $A(\underline{x})$ and $\Phi(\underline{x})$ (see [7] for more details).

Assume now that the wavelet $\psi(\underline{x})$ is "concentrated" around the origin in the direct space, and around some frequency $\underline{k}_0$ in the Fourier space. Then $\psi_{(\underline{b},a,\theta)}(\underline{x})$ is concentrated around the point $(\underline{b}, \underline{\underline{r}}_{\theta} \cdot \underline{k}_0/a)$ in the phase space. Due to such localization properties, $T_f(\underline{b}, a, \theta)$ is locally maximum around the sets (the ridges of the transform)

$$\mathcal{R}^{\pm} = \left\{ (\underline{b}, a, \theta) \in I\!R^2 \times I\!R_+^* \times [0, 2\pi[\ / \ \underline{x}_0^{\pm}(\underline{b}, a, \theta) = \underline{b} \right\} \tag{III.6}$$

corresponding to the two components of the signal $Z_f^{\pm}(\underline{x})$. Otherwise stated

$$\begin{aligned} \left(\nabla_{\underline{x}} \phi(\underline{x}_0^{\pm}) \right)_{\underline{x}_0^{\pm} = \underline{b}} &= \pm \ \frac{1}{a} \underline{\underline{r}}_{\theta} \left(\nabla_{\underline{x}} \, \phi_{\psi} \big(\underline{\underline{r}}_{-\theta} \cdot \frac{\underline{x}_0^{\pm} - \underline{b}}{a} \big) \right)_{\underline{x}_0^{\pm} = \underline{b}} \\ &= \ \pm \frac{1}{a} \underline{\underline{r}}_{\theta} \, \nabla_{\underline{x}} \, \phi_{\psi}(0) \end{aligned} \tag{III.7}$$

For simplicity, we will focus only on the values of a and θ corresponding to a given ridge, say $\mathcal{R}^+$. The ridge then appears as a 2-dimensional surface $\{a_r(\underline{b}), \theta_r(\underline{b})\}$ in the four-dimensional parameter space. $(k_0/a_r(\underline{b}), \theta_r(\underline{b}))$ are the polar coordinates of a wavevector located at the point $\underline{b}$, i.e. a local wavevector. But due to the existence of two components, implying the existence of two ridges, there is still an undeterminacy of π in the angular coordinate, i.e. one only has a local wavedirector (unoriented wavevector).

Like in the one-dimensional case, it can be shown that the ridges are the places where the local wavevector of the wavelet transform (i.e. the gradient of the phase of $T_f(\underline{b}, a, \theta)$) coincides with that of the wavelet $\psi_{(\underline{b},a,\theta)}(\underline{x})$. It is then possible to specify a fixed-point algorithm that follows continuously a given ridge.

Again, the restriction of the wavelet transform to the associated ridge possesses a remarkable property: it gives direct access to the signal itself. Indeed, we have the following two-dimensional skeleton:

$$T_f(\underline{b}, a_r(\underline{b}), \theta_r(\underline{b})) = Corr(\underline{b}) Z_f(\underline{b}) + r(\underline{b}) \qquad (III.8)$$

where $Corr(\underline{b})$ is a corrective factor, completely determined by the wavelet $\psi(\underline{x})$ and the ridge $\{a_r(\underline{b}), \theta_r(\underline{b})\}$, and $r(\underline{b})$ is some remainder, whose size can be estimated by standard techniques.

To illustrate the method, we have chosen a simple example, namely $f(\underline{x}) = e^{-||\underline{x}||^2} \cos(\underline{k}_1 \cdot \underline{x}) + \cos(\underline{k}_2 \cdot \underline{x})$, with $\underline{k}_1 \perp \underline{k}_2$ (see fig. 1). The two corresponding ridges are represented as local directors in fig. 2. From each ridge it is possible to reconstruct the corresponding component from the associated skeleton, as shown in fig. 3.

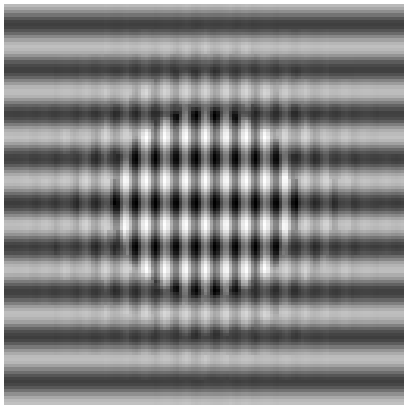

Fig. 1: Analyzed image

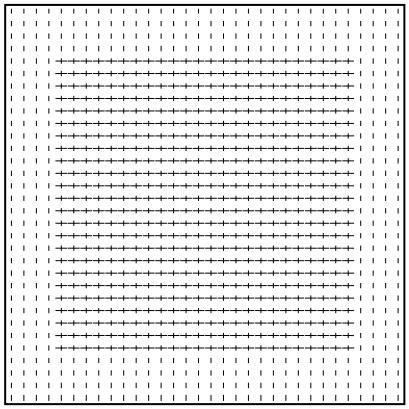

Fig. 2: Ridge of the wavelet transform

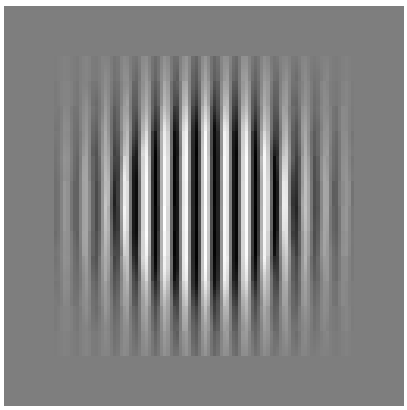

Fig. 3a: First component

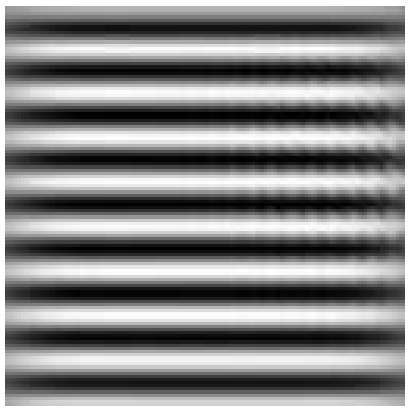

Fig. 3b: Second component

IV. WAVELETS AND PHASE SPACE

Let us now turn to the geometric description of continuous wavelet systems built on the phase space, focusing on two simple examples. In the two examples described above, the key point was to use a wavelet transform mapping the L^2 space of the considered space ($I\!R$ or $I\!R^2$) into the L^2 space of the associated space-frequency space (or phase space). This is not the case of all wavelet transforms.

Consider for example R. Murenzi's construction of n-dimensional wavelets [3], as described in $(I.5-8)$. The associated wavelet transform maps $L^2(I\!R^n)$ into $L^2(I\!R^n \times I\!R_+^* \times SO(n))$, i.e. the L^2 space of a space of dimension $n+1+n(n-1)/2 = n(n+1)/2+1$, bigger than the phase space for $n \geq 2$. This means that all the parameters are not necessary for a space-frequency analysis.

Consider as an example the simplest case, namely the case $n = 3$. It is well known that the rotations of $I\!R^3$ can be parametrized by 3 Euler angles ϕ, θ and ρ, as (see e.g. [8])

$$\underline{\underline{r}} = \underline{\underline{r}}_{12}(\phi) \cdot \underline{\underline{r}}_{23}(\theta) \cdot \underline{\underline{r}}_{12}(\rho) \qquad (IV.1)$$

(where $\underline{\underline{r}}_{ij}(\phi)$ is the rotation matrix $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ in the Oij plane). Assuming that the wavelet $\hat{\psi}(\underline{\xi})$ is "concentrated" around some frequency $\underline{k}_0$ in the Fourier space, $\widehat{\psi_{(\underline{b},a,\underline{\underline{r}})}}(\underline{\xi})$ is concentrated around $\underline{\underline{r}} \cdot \underline{k}_0/a$. If the coordinate system is chosen in such a way that $\underline{k}_0$ is parallel to the third axis and then invariant by $\underline{\underline{r}}_{12}(\rho)$, then $\widehat{\psi_{(\underline{b},a,\underline{\underline{r}})}}(\underline{\xi})$ is concentrated around

$$\xi = \frac{1}{a}\, \underline{\underline{r}}_{12}(\phi) \cdot \underline{\underline{r}}_{23}(\theta) \cdot \underline{k}_0 = \frac{1}{a}\, \underline{\underline{r}}_0(\theta,\phi) \cdot \underline{k}_0 \ . \qquad (IV.2)$$

In other words, $(k_0/a, \phi, \theta)$ are the spherical coordinates of a local wavevector naturally associated with $\psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x})$. Given a signal $f(\underline{x}) \in L^2(I\!R^3)$ to analyse, $T_f(\underline{b}, a, \underline{\underline{r}})$ can then be interpreted as measuring the phase space content of $f(\underline{x})$ in the neighborhood of the point $(\underline{b}; k_0/a, \phi, \theta)$. This shows that a reduced wavelet transform is sufficient for space-frequency analysis. From such a reduced wavelet transform, one can then derive a 3-dimensional "ridge and skeleton" method.

The next question is the completeness of such a reduced wavelet representation. A simple calculation shows that the wavelet coefficients $T_f(\underline{b}, a, \underline{\underline{r}}_0(\theta,\phi))$ represent a sufficient information to characterize a three-dimensional signal $f(\underline{x})$. The corresponding reconstruction formula reads

$$f(\underline{x}) = \frac{1}{k_\psi} \int T_f(\underline{b}, a, \underline{\underline{r}}_0(\theta,\phi) \psi_{(\underline{b},a,\underline{\underline{r}}_0(\theta,\phi))}(\underline{x})\, \frac{d\underline{b}}{a^n}\, \frac{da}{a}\, \sin\theta\, d\theta\, d\phi \ , \qquad (IV.3)$$

where

$$k_\psi = \int_{I\!R^n} |\hat{\psi}(\underline{\xi})|^2 \frac{d\underline{\xi}}{||\underline{\xi}||^n} \ . \qquad (IV.4)$$

This can be generalized to arbitrary dimensions, using the n-dimensional Euler angles. Let $\psi(\underline{x}) \in L^1(I\!R^n)$ be such that $\hat{\psi}(\underline{\xi})$ is "concentrated" around some frequency $\underline{k}_0$ in the Fourier space. Any n-dimensional rotation matrix is characterized by $n(n-1)/2$ Euler angles, and can be expressed as a product

$$\underline{\underline{r}} = \underline{\underline{r}}_0 \cdot \underline{\underline{s}} \ , \qquad (IV.5)$$

where $\underline{\underline{s}} \in SO(n-1)$, the group of all rotation matrices that leave $\underline{k}_0$ invariant, and $\underline{\underline{r}}_0$ is a product of $n(n-1)/2-(n-1)(n-2)/2 = n-1$ rotation matrices $\underline{\underline{r}}_{ij}(\theta)$, characterized by $n-1$ Euler angles (see [8], [3] for more details). It follows from the same arguments as above that the frequency localization of $\psi_{(\underline{b},a,\underline{\underline{r}})}(\underline{x})$ is completely determined by a and $\underline{\underline{r}}_0$. The corresponding reduced wavelet transform is sufficient to characterize the phase space content of n-dimensional signals. Moreover, like in the three-dimensional case, it is also sufficient to characterize functions of n variables, in the sense that a reconstruction formula is available.

The last example we consider is also motivated by signal analysis problems. Let us focus for simplicity on the one-dimensional case. For some problems (in speech processing or vision for example) it is useful to enlarge the class of analysis functions under consideration. Indeed the usual wavelets, as described in $(I.1-4)$, are functions of constant relative bandwidth (i.e. the width of the Fourier transform $\widehat{\psi_{(b,a)}}$ is proportional to the corresponding frequency ω_0/a where $\psi_{(b,a)}$ is centered). As shown in [9] in a speech analysis context, such a description of the frequency space is not always optimal, in the sense that it is not adapted to the structure of speech signal. To vary the relative bandwidth as a function of the frequency (and/or the time), one can introduce an additional set of transformations, namely modulations [9], [10]. In other words, one can try to combine wavelet analysis and windowed Fourier analysis, by considering the following family of functions

$$\psi_{(b,\omega,a)}(x) = \frac{1}{\sqrt{a}} e^{i\omega(x-b)} \psi\left(\frac{x-b}{a}\right) \ . \qquad (IV.6)$$

If $\hat{\psi}(\xi)$ is localized around the origin in the Fourier space for instance, and has a bandwidth B, then $\widehat{\psi_{(b,\omega,a)}}(\xi)$ is localized around $\xi = \omega$ and has a bandwidth equal to B/a. However, such a family of functions cannot be directly used to provide a simple analysis-reconstruction scheme as before, because the corresponding admissibility constant c_ψ would in general diverge. This is due to the fact that the set of parameters (b, ω, a) is bigger than the phase space of $I\!R$, the extra dimension being noncompact and then of infinite volume. Nevertheless, this problem can be avoided by introducing a relation between a and ω for example. Setting

$$a = \beta(\omega) \qquad (IV.7)$$

(β is called the scale function) reduces the parameter space to a two-dimensional one. It can then be shown in particular that the choice

$$\beta_{\sigma,\tau}(\omega) = \frac{1}{\sigma\omega + \tau} \quad \tau \neq 0 \qquad (IV.8)$$

directly leads to a simple analysis-reconstruction scheme [10], in the sense that as soon as $\psi \in L^1(I\!R) \cap L^2(I\!R)$, any $f \in L^2(I\!R)$ can be decomposed as

$$f(x) = \frac{1}{2\pi||\psi||^2} \int < f, \psi_{(b,\omega,\beta_{\sigma,\tau}(\omega))} > \psi_{(b,\omega,\beta_{\sigma,\tau}(\omega))}(x)\, db\, d\omega \ . \qquad (IV.9)$$

Such a family of β functions is actually not rich enough to include functions of practical interest; nevertheless, the construction can be slightly modified to allow more general scale functions (see [11]). A similar analysis has been carried out in the n-dimensional case in [12].

Again in the second example, one obtains a wavelet-type representation by restricting the parameter set to the phase space associated with the original space. This is not a coincidence,

and the explanation of this fact comes from the group theoretical interpretation of wavelets derived in [13]. Indeed, in all the cases we have been considering since the beginning of this paper, the wavelets are constructed from the action of a group G on $L^2(I\!R^n)$ (the affine group in one dimension, the affine group extended by rotations, the affine group extended by modulations, ...):

$$\pi : G \to \mathcal{U}(L^2(I\!R^n)) \qquad (IV.10)$$

where $\mathcal{U}(L^2(I\!R^n))$ stands for the unitary operators on $L^2(I\!R^n)$ In all these cases, it can be shown (see [12] for instance) that there is a quotient space G/H of G (i.e. a set of equivalence classes of elements of G) naturally associated with the action of G on $L^2(I\!R^n)$. For example, in the case of Murenzi's wavelets, the H subgroup is the group of $(n-1)$-dimensional rotation matrices. This quotient space, known as the phase space by the specialists of geometric quantization, is of dimension $2n$, and is exactly the parameter space used for the wavelet transform or the reduced wavelet transform we described. This shows that for all such groups, there is a naturally associated wavelet transform well suited for space-frequency analysis:

$$T : L^2(I\!R^n) \to L^2(G/H) \ . \qquad (IV.11)$$

The wavelet systems associated with quotients of groups was also described in [14]. This language also allows to clarify the role played by the group G. The group fixes the geometry of the phase space that is associated with the considered wavelet analysis. It is also the symmetry (or covariance) group of the wavelet representation. Indeed, there is also a natural action λ of G on $L^2(G/H)$, coming from the group multiplication law of G.

$$\lambda : G \to \mathcal{U}(L^2(G/H)) \ . \qquad (IV.12)$$

The covariance is thus expressed as follows (for simplicity we restrict to the case where the phase space is the group G itself, like in the case of one-dimensional wavelets. In general, the full covariance is lost, and there only remains a twisted covariance, i.e. a covariance modulo the action of H [12]).

$$\lambda \circ T = T \circ \pi \ . \qquad (IV.13)$$

In other words: the space-frequency wavelet transform of $\pi(g) \cdot .f$ of a $L^2(I\!R^n)$ function $f(x)$ acted on by an element g of the covariance group G is equal to $\lambda(g) \cdot T \cdot f$, the wavelet transform T_f of $f(x)$, acted on by g. For example, considering the case of image analysis by two-dimensional Murenzi's wavelets, one would say that the wavelet transform of a rotated, scaled and shifted copy of an image is equal to a rotated, scaled and shifted copy of the wavelet transform of the original image. This would not be the case of the windowed Fourier transform of the image, since there is no natural action of rotations on the underlying group, the two-dimensional Weyl-Heisenberg group (generated by space and frequency translations).

CONCLUSIONS:

We have described in this paper two aspects of space-frequency wavelet analysis, namely two examples of application to the problem of local frequencies estimation, and some underlying geometrical description. We have in particular seen that given a wavelet transform taking its values in the L^2 space of the phase space, it can be used for instance to build "ridge and skeleton" algorithms for characterizing amplitude and frequency modulated components in signals, under some special assumptions on the modulation laws.

Nevertheless, a given space-frequency representation may be particularly accurate for a given class of signals, and less accurate for other classes. This opens the problem of selecting the most accurate representation for a given signal and a given problem. Such a criterion was proposed in the context of orthonormal bases of wavelet packets (see e.g. [15] for a review), for signal compression problems. However, it is very unlikely that the same criterion can be used for different problems (such as for example modeling signals as superpositions of amplitude and frequency modulated components). We now have a large number of available space-frequency representations of signals, and being able to select the best one for a given signal remains a very appealing difficult problem to solve.

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