Consensus and Disagreement: The Role of Quantized Behaviors in Opinion Dynamics

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CONSENSUS AND DISAGREEMENT: THE ROLE OF QUANTIZED BEHAVIOURS IN OPINION DYNAMICS*

FRANCESCA CERAGIOLI† AND PAOLO FRASCA‡

Abstract. This paper deals with continuous-time opinion dynamics that feature the interplay of continuous opinions and discrete behaviours. In our model, the opinion of one individual is only influenced by the behaviours of fellow individuals. The key technical difficulty in the study of these dynamics is that the right-hand sides of the equations are discontinuous and thus their solutions must be intended in some generalized sense: in our analysis, we consider both Carathéodory and Krasovskii solutions. We first prove existence and completeness of Carathéodory solutions from every initial condition and we highlight a pathological behavior of Carathéodory solutions, which can converge to points that are not (Carathéodory) equilibria. Notably, such points can be arbitrarily far from consensus and indeed simulations show that convergence to non-consensus configurations is common. In order to cope with these pathological attractors, we study Krasovskii solutions. We give an estimate of the asymptotic distance of all Krasovskii solutions from consensus and we prove its tightness by an example of equilibrium such that this distance is quadratic in the number of agents. This fact implies that quantization can drastically destroy consensus. However, consensus is guaranteed in some special cases, for instance when the communication among the individuals is described by either a complete or a complete bipartite graph.

Keywords. Opinion dynamics, quantized consensus, disagreement, discontinuous differential equations.

1. Introduction. A fundamental assumption in opinion dynamics is that one's opinion is attracted by others', as supported by a substantial literature [31, 33, 15, 37]. If we assume that the opinion of each individual i is described by a variable $x_i \in \mathbb{R}$, we can describe this attraction by a linear law and thus describe the evolution of the opinions by the following set of differential equations:

$$\dot{x}_i(t) = \sum_j a_{ij}[x_j(t) - x_i(t)] \qquad i \in \mathcal{I} = \{1, ..., N\} \tag{1}$$

where $a_{ij} \in \mathbb{R}_{\geq 0}$ and $a_{ij} > 0$ means that the opinion of the individual i is directly influenced by the opinion of the individual j. As long as there is at least one individual who can influence (albeit indirectly) all others, then it can be proved that *consensus* is asymptotically achieved, namely there exists $\alpha \in \mathbb{R}$ such that $x_i(t) \to \alpha$ as $t \to +\infty$ for all $i \in \mathcal{I}$. However, experience suggests that in reality consensus is not always achieved, but disagreement persists: clearly, the opinion dynamics model should include some additional feature.

In this paper, we postulate that individuals can not directly perceive the private opinions of the others, but only observe their displayed *behaviours*. Indeed, in some situations individuals may not be able to express their opinions precisely, but only through a limited number of behaviours or actions: let us think of consumers' choices, electoral votes, or stereotyped interactions in online social media. Even though we are assuming that the opinions are real-valued, behaviours are better described as elements of a finite or discrete set: hence, the behaviour of an individual shall be a suitable *quantization* of his/her opinion. If we assume to represent discrete behaviours as integers and that the relation between opinion and behavior is the same for all

*Submitted to the editors DATE.

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individuals, equations (1) can be replaced by

$$\dot{x}_i(t) = \sum_j a_{ij}[q(x_j(t)) - x_i(t)], \qquad i \in \mathcal{I}, \tag{2}$$

where we choose the quantizer map $q : \mathbb{R} \to \mathbb{Z}$ to be the "uniform" quantizer $q(s) = \lfloor s + \frac{1}{2} \rfloor$. This new model is a very simple modification of (1), which has no pretension to describe social dynamics precisely, but has the aim to make the role of quantization evident. As we shall see in this paper, the main feature that distinguishes it from (1) is that consensus is not achieved in general. From the point of view of opinion dynamics, this observation allows us to explain the persistence of disagreement as an effect of the limited number of behaviours allowed.

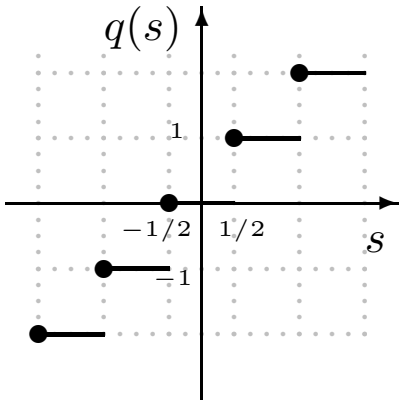

FIG. 1. *The uniform quantizer* $q : \mathbb{R} \to \mathbb{Z}$.

The aim of this paper is to study the effect of quantization of others' states in a consensus model. This is done by undertaking a rigorous mathematical analysis of (2). The key technical difficulty in the study of this model is that the right-hand sides of the equations are discontinuous and their solutions must be intended in some generalized sense. We describe our mathematical contribution as fourfold. As the *first* contribution, we prove existence and completeness of Carathéodory solutions from every initial condition. This is a relevant fact because most discontinuous systems, including well-known models of opinion dynamics, do not have complete Carathéodory solutions [13]: even proving existence for most initial conditions can be a difficult task [7, 6].

As the *second* contribution, we highlight a pathological behavior of Carathéodory solutions, which can converge to points that are not (Carathéodory) equilibria. The presence of these pathological attractors motivates us to consider also Krasovskii solutions, which have two important advantages. Firstly, Krasovskii's definition is more general than Carathéodory's, meaning that Carathéodory solutions are particular Krasovskii solutions. Hence, results that are established for Krasovskii also apply to Carathéodory. Secondly, Krasovskii solutions can not converge to points that are not (Krasovskii) equilibria, thus solving the mentioned pathological convergence.

As the *third* contribution, assuming weight-balanced interactions, we derive an estimate of the distance of Krasovskii solutions from consensus for large times and we

prove that its tightness by means of an example of equilibria. In this example, the distance from consensus is quadratically increasing in the number of agents, implying that quantization can drastically destroy consensus. Actually, not only we explicitly construct equilibria that are arbitrarily far from consensus, but also show by simulations that most graphs imply convergence to non-consensus configurations. However, consensus can be reached in some special cases. Indeed, as the *fourth* contribution, we prove convergence to consensus when communication among the individuals is described either by a complete graph or by a complete bipartite graph.

Relations with the literature. This paper relates to various bodies of work in control theory and in mathematical sociology. In the last ten years, control theorists have widely studied quantized versions of consensus algorithms. Since giving a complete overview would be impossible here, we just mention some papers whose approaches are particularly close to ours. First of all, in [10] the authors consider a discrete-time version of (2): their analysis is limited to observing that the algorithm may not converge to consensus and then abandoned. The poor perfomance of the dynamics in terms of approaching consensus explains the scarcity[1] of known results about (2). Instead, papers like [12, 22, 20, 25, 29] have considered other possible quantizations of (1): in [12, 22] all states are quantized in the right-hand side, while in [20, 25] distances between couples of states are seen through the quantizer. In these papers convergence to consensus is proved under appropriate but generally mild assumptions [38].

Surveys of the opinion dynamics literature can be found in [11, 23, 34]: all of these bring forward the need for theories that can explain both agreement and disagreement, as recognized since [28, 1]. Different explanations of persistence of disagreement have been proposed: among them, we recall in [24] the persisting influence of the initial opinions; in [27] the bounded confidence of the individuals; in [32, 2] the presence of stubborn individuals; and in [3] the occurrence of antagonistic interactions. Here, we support the claim that quantization can be a source of disagreement. The fact that others' opinion are materialized by means of discrete behaviours is a natural observation, which has been made by social scientists [24, Chapter 10] and has been addressed in several models including [36, 30, 16]. Discrete behaviours can result as consequences of limited verbalization capabilities, as postulated in [36], or represent actions taken by the individuals (CODA models), as in [30, 16]. These papers feature dynamics that may not reach consensus and that involve quantization together with other nonlinearities. In comparison, our proposed model (2) can be understood as an effort to *single out the effects of quantization only*. We consider the case of multiple quantization levels, but two-level quantization as in [16] can be easily obtained as a particular case.

Outline of the paper. In Section 2 we present some alternative forms of (2), introduce Carathéodory and Krasovskii solutions, and prove some of their basic properties. Section 3 is devoted to equilibria, providing the relevant definitions and examples. In Section 4 we prove the asymptotic estimate of the distance of solutions from consensus and show that it is sharp. In Section 5 we study the special cases of all-to-all and bipartite communication and prove that in these cases consensus is achieved. Finally, Section 6 presents some simulations on random graphs and Section 7 concludes the paper.

2. Fundamental properties of the dynamics. We begin this section by rewriting equations (2) in some alternative forms that allow us to see the model

[1] A preliminary and incomplete study of dynamics (2), where only all-to-all communication was considered, was published in the Proceedings of the European Control Conference as [14].

from different points of view.

First of all, we can think that communication among individuals is described by a weighted directed graph whose vertices are individuals. The numbers $a_{ij} \geq 0$ are the entries of the weighted adjaciency matrix A of the graph, namely $a_{ij} > 0$ if the agents i and j are linked, $a_{ij} = 0$ if they are not linked and $a_{ii} = 0$. We can then observe that

$$\begin{aligned}\dot{x}_i(t) =& \sum_{j\neq i} a_{ij}\big(q(x_j(t)) - x_j(t) + x_j(t) - x_i(t)\big)\\ =& \sum_{j\neq i} a_{ij}\big(x_j(t)) - x_i(t)\big) + \sum_{j\neq i} a_{ij}\big(q(x_j(t)) - x_j(t)\big).\end{aligned}$$

If we denote by $d_i := \sum_j a_{ij}$ the weighted degree of the i-th vertex, by $D = \mathrm{diag}(d_i)$, and by $\mathbf{q}(x) = (q(x_1), \dots, q(x_N))^\top$, then we can define the Laplacian matrix of the graph $L = D - A$ and write

$$\begin{aligned}\dot{x} =& - Lx + A(\mathbf{q}(x) - x)\\ =& - L(x - x_a\mathbf{1}) + A(\mathbf{q}(x) - x),\end{aligned} \tag{3}$$

where for the second equality we have used $\mathbf{1}^\top = (1, ..., 1)$ and $x_a(t) = \frac{1}{N}\mathbf{1}^\top x(t)$, together with the fact that $L\mathbf{1} = 0$. System (3) can be seen as the classical consensus system perturbed by others' states quantization errors. This interpretation will become useful in Section 4. Clearly, we can also write (3) as

$$\dot{x} = -Dx + A\mathbf{q}(x), \tag{4}$$

which prompts a connection with the theory of neural networks.

Remark 1 (Neural networks). *By adding a constant term $I \in \mathbb{R}^N$ in (4) we get*

$$\dot{x} = -Dx + A\mathbf{q}(x) + I, \tag{5}$$

which is the typical system used in order to describe Hopfield neural networks. In this context, individuals represent neurons and the function q is called activation function. In the basic models q is assumed to be smooth and increasing, but the literature has also considered weaker assumptions. In particular, in [21] q can be discontinuous but must be increasing and bounded, and the matrix A is assumed invertible. A major difference between our model and the one in [21] is that we have multiple equilibria (Section 3), whereas [21] assumes a single equilibrium point.

When we look at system (4), we observe that for every $\mathbf{k} \in \mathbb{Z}^N$ the vector $\mathbf{q}$ is constant on each set

$$S_{\mathbf{k}} = \{x \in \mathbb{R}^N : k_i - \frac{1}{2} \leq x_i < k_i + \frac{1}{2}, i = 1, \dots, N\}.$$

This fact makes it evident that the system is affine if restricted to each set $S_{\mathbf{k}}$. Moreover its right-hand side is discontinuous on the set $\Delta = \bigcup_{\mathbf{k}\in\mathbb{Z}^N} \partial S_{\mathbf{k}}$, where $\partial S_{\mathbf{k}}$ is the boundary of $S_{\mathbf{k}}$. In general, existence of solutions of equations with discontinuous right-hand side is not guaranteed. For this reason different types of solutions have been introduced in the literature. The notion of solution nearest to the classical one is that of Carathéodory solution. The main problem in using Carathéodory solutions is that, often, they do not exist or can not be extended to infinity. Here we prove

that Carathéodory solutions do exists and are complete. Nevertheless the discontinuity causes the strange phenomenon of solutions converging to points which are not equilibria. In order to better understand this phenomenon we also study Krasovskii solutions: such points are in fact Krasovskii equilibria.

We now formally introduce Carathéodory and Krasovskii solutions and study their basic properties. We refer the reader to [26, 5, 17] for overviews on generalized solutions of discontinuous differential equations. System (2) can be cast in the general form

$$\dot{x} = f(x), \tag{6}$$

provided $f : \mathbb{R}^N \to \mathbb{R}^N$ is the vector field given by $f_i(x) = \sum_j a_{ij}[q(x_j) - x_i]$. Let $I \subset \mathbb{R}$ be an interval of the form $(0, T)$. An absolutely continuous function $x : I \to \mathbb{R}^N$ is a *Carathéodory solution* of (6) if it satisfies (6) for almost all $t \in I$ or, equivalently, if it is a solution of the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\mathrm{d}s.$$

An absolutely continuous function $x : I \to \mathbb{R}^N$ is a Krasovskii solution of (6) if for almost every $t \in I$, it satisfies

$$\dot{x}(t) \in \mathcal{K}f(x(t)), \tag{7}$$

where

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}(\{f(y) \,:\, y \text{ such that } \|x - y\| < \delta\}).$$

We recall that in general any Carathéodory solution is also a Krasovskii solution and we observe that, for the specific dynamics (2), Krasovskii solutions coincide with Filippov solutions that are often adopted for discontinuous systems. The following result states the basic properties of the solutions of (2).

THEOREM 2.1 (Properties of solutions).

(i) (Existence) For any initial condition there exist a Carathéodory solution and a Krasovskii solution of (2).

(ii) (Boundedness) Any Carathéodory or Krasovskii solution of (2) is bounded on its domain.

(iii) (Completeness) Any Carathéodory or Krasovskii solution starting at $t_0 \in \mathbb{R}$ is defined on the set $[t_0, +\infty)$.

Proof. (i-C) First of all, we remark that the right-hand side of (2) is continuous at any point in the interior of $S_{\mathbf{k}}$ for any $\mathbf{k} \in \mathbb{Z}^N$, then local solutions with initial conditions in $\mathbb{R}^N \setminus \Delta$ do exist. Then, we consider initial conditions in Δ. For any $x_0 \in \mathbb{R}^N$ we denote by $I(x_0)$ the subset of $\{1, \dots, N\}$ of the indices i such that $x_{0i} = k_i + \frac{1}{2}$ for some $k_i \in \mathbb{Z}$ and by M the cardinality of $I(x_0)$.

We first consider initial conditions x_0 such that $M = 1$ and $I(x_0) = \{i\}$, i.e. $x_{0i} = k_i + \frac{1}{2}$ for some $k_i \in \mathbb{Z}$ and $x_{0j} \neq h + \frac{1}{2}$ for any $j \neq i$ and any $h \in \mathbb{Z}$. Let us denote

$$s_i(x) = x_i - k_i - \frac{1}{2},$$

$$S_i^+ = \{x \in \mathbb{R}^N : x_i - k_i - \frac{1}{2} \geq 0\},$$

$$S_i^- = \{x \in \mathbb{R}^N : x_i - k_i - \frac{1}{2} < 0\},$$
$$f|_{S_i^-}(x_0) = \lim_{x \in S_i^-, x \to x_0} f(x).$$

We have that $\nabla s_i(x_0) = e_i$ (the i-th vector of the canonical basis) and, if we define $a(x) = \nabla s_i(x) \cdot f(x)$, then

$$\begin{aligned} a(x_0) &= \nabla s_i(x_0) \cdot f|_{S_i^+}(x_0) = \nabla s_i(x_0) \cdot f|_{S_i^-}(x_0) \\ &= \textstyle\sum_{j \neq i} a_{ij}(q(x_{0j}) - k_i - \frac{1}{2}). \end{aligned}$$

If $a(x_0) < 0$, the vector field $f|_{S_i^-}$ points towards the half-space $s_i(x) < 0$ and then there is a solution starting at x_0 which satisfies the equations $\dot{x} = f|_{S_i^-}(x)$ and stays in S_i^- in an interval of the form $(t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. This fact is proved in [35, 8]. Analogously, if $a(x_0) \geq 0$, then there is a solution starting at x_0 which satisfies the equation $\dot{x} = f|_{S_i^+}(x)$ and stays in S_i^+ in an interval of the form $(t_0, t_0 + \epsilon)$ for some $\epsilon > 0$. Note in particular that if $a(x_0) = 0$, then the vector field $f|_{S_i^+}$ is tangent to discontinuity surface $x_i = k_i + \frac{1}{2}$, namely a hyperplane, not only at x_0 but also in a neighborhood of x_0 intersecated with the discontinuity surface. Indeed there exists a neighbourhood $N(x_0)$ of x_0 such that for all $x \in N(x_0)$ one has $q(x_j) = q(x_{0j})$ for all $j \neq i$. Moreover if $x_i = k_i + 1/2$ one gets $a(x) = a(x_0)$, and then there exists a solution starting on the hyperplane which lays on it.

We now consider initial conditions x_0 such that $1 < M \leq N$. The vector field f has 2^M limit values at x_0 corresponding to the 2^M sectors defined by the inequalities $x_i - k_i - \frac{1}{2} \geq 0$ and $x_i - k_i - \frac{1}{2} < 0$, $i \in I(x_0)$. We describe these sectors by means of vectors $B \in \{0,1\}^N$. Let $H(t) = 1$ if $t \geq 0$ and $H(t) = 0$ if $t < 0$. We define $B_i = H(x_i - k_i - \frac{1}{2})$ if $i \in I(x_0)$ and $B_i = 0$ if $i \notin I(x_0)$. Let $f_B(x_0) = \lim_{x \to x_0, x \in S_{\mathbf{k}+B}} f(x)$. In the following we denote $f_B(x_0) = f_B$. We want to prove that there exists B such that $H((f_B)_i) = B_i$ for all $i \in I(x_0)$. This means that the vector field f_B points inside the sector $S_{\mathbf{k}+B}$ at x_0.

Preliminarily, note that as $x_0 \in S_{\mathbf{k}}$, then $q(x_{0_j}) = k_j$. If $x \in S_{\mathbf{k}+B}$ from the definition of B it follows that $q(x_j) = k_j + B_j$ for all $j = 1, ..., n$. Then the ith component of f_B can be written as

$$(f_B)_i = \textstyle\sum_j a_{ij}(k_j + B_j) - d_i(k_i + \frac{1}{2}) = \sum_j a_{ij}k_j - d_i(k_i + \frac{1}{2}) + \sum_j a_{ij}B_j \tag{8}$$

Now, we start by considering the sector B^1 such that $(B^1)_i = 0$ for all $i \in I(x_0)$. If $H((f_{B^1})_i) = 0$ for all $i \in I(x_0)$, we have finished. Otherwise, there exists $i \in I(x_0)$ such that $H((f_{B^1})_i) = 1$. Assume without loss of generality that $i = 1$, i.e. $H((f_{B^1})_1) = 1$. Then for all $B \neq B^1$ we have $H((f_B)_1) = 1$. In fact if $(f_{B^1})_1 > 0$ then also $(f_B)_1 = (f_{B^1})_1 + \sum_j a_{1j}B_j > 0$.

We then examine only those B such that $B_1 = 1$. In particular the next B we consider, which we call B^2, is such that $(B^2)_1 = 1$ and $(B^2)_i = 0$ for all other $i \in I(x_0)$. If $H((f_{B^2})_i) = 0$ for all $i \in I(x_0) \setminus \{1\}$, then we have finished. Otherwise, there exists $j \in I(x_0) \setminus \{1\}$ such that $H((f_{B^1})_j) = 1$. Assume that such $j = 2$, i.e. $H((f_{B^2})_2) = 1$. Then for all $B \notin \{B^1, B^2\}$ we have $H((f_B)_2) = 1$. We can then restrict our attention to those B such that $B_1 = B_2 = 1$, and so forth. By proceeding in this way, in M step at most we find the sector B with the desired property.

As already mentioned, the meaning of the condition $H((f_B)_i) = B_i$ for all i is that the vector field f_B is directed inside $S_{\mathbf{k}+B}$ and local existence is a consequence

of the results in [35, 8]. More precisely, if f_B points strictly inside B, i.e. $(f_B)_i \neq 0$ for all i, then there is a solution that enters the sector. If instead some components of f_B are null, f_B is tangent to the boundary of $S_{\mathbf{k}+B}$, and, more precisely, to $\partial S_{\mathbf{k}+B} \cap [\cap_{i\in\tilde{I}(x_0)}\{x : x_i = k_i + 1/2\}]$, where $\tilde{I}(x_0)$ is the subset of $I(x_0)$ such that $(f_B)_i = 0$ if $i \in \tilde{I}(x_0)$.

In fact we observe that there exists a solution issuing from x_0 which lies on the set $\partial S_{\mathbf{k}+B} \cap [\cap_{i\in\tilde{I}(x_0)}\{x : x_i = k_i + 1/2\}]$. Let us consider $x \in \partial S_{\mathbf{k}+B} \cap [\cap_{i\in\tilde{I}(x_0)}\{x : x_i = k_i + 1/2\}]$ and let us compute $(f_B(x))_i$.

$$\begin{aligned}(f_B(x))_i &= \sum_j a_{ij}q(x_j) - d_i x_i \\ &= \sum_{j\notin I(x_0)} a_{ij}q(x_j) + \sum_{j\in I(x_0)\setminus\tilde{I}(x_0)} a_{ij}q(x_j) + \sum_{j\in\tilde{I}(x_0)} a_{ij}q(x_j) - d_i\left(k_i + \frac{1}{2}\right).\end{aligned}$$

Note that if $j \in \tilde{I}(x_0)$ then $q(x_j) = q((x_0)_j) = k_j + B_j$ as $x \in \partial S_{\mathbf{k}+B} \cap [\cap_{i\in\tilde{I}(x_0)}\{x : x_i = k_i + 1/2\}]$. If $j \notin I(x_0)$ then also $q(x_j) = q((x_0)_j) = k_j + B_j$, provided that x is sufficiently near x_0 so that the quantity $x_j - k_j - 1/2$ does not change sign. Finally, if $j \in I(x_0)\setminus\tilde{I}(x_0)$ then $x_j = k_j + 1/2$ and $(f_B(x_0))_j > 0$ as $H((f_B))_j = B_j$. Let $N_B(x_0) = \{x \in \partial S_{\mathbf{k}+B} \cap [\cap_{i\in\tilde{I}(x_0)}\{x : x_i = k_i + 1/2\}] : x_j > k_j + 1/2\}$. In a neighborhood of x_0 intersecated with $N_B(x_0)$ also $q(x_j)$ remains constant and so does $(f_B(x))_i$ for all $i \in \tilde{I}(x_0)$. This means that the vector field $f_B(x)$ is tangent to the set $N_B(x_0)$ in a neighbourhood of x_0 and thus there exists a solution issuing from x_0 and lying on this set.

We conclude that there exist a solution φ of (2), $t_0 \in \mathbb{R}$ and $\epsilon > 0$ such that $\varphi(t_0) = x_0$ and $\dot{\varphi}(t) = f_B(\varphi(t))$ for almost every $t \in (t_0, t_0 + \epsilon)$.

(i-K) Since the function q is locally bounded, then the Krasovskii set-valued function $\mathcal{K}f(x)$ is upper semi-continuous with compact and convex values and local existence of Krasovskii solutions is a consequence of the existence theorem in [4], page 98, Theorem 3.

(ii-C) Let $x(t)$ be a Carathéodory solution and let m be any index in $\mathcal{I}$ such that $x_m(t) = \min\{x_i(t), i \in \mathcal{I}\}$. For $i \in \mathcal{I}$, $x_i(t) \geq x_m(t)$. Let $q(x_m(t)) = q_m(t)$. If $x_m(t) \in \left[q_m(t) - \frac{1}{2}, q_m(t)\right]$, one has $q(x_i(t)) \geq x_m(t)$ for all $i \in \mathcal{I}$ and then $\dot{x}_m(t) = \sum_j a_{mj}[q(x_j(t)) - x_m(t)] \geq 0$. When $x_m(t) \in \left(q_m(t), q_m(t) + \frac{1}{2}\right)$, instead, $x_m(t)$ may be decreasing as there may be other indices i such that $q(x_i(t)) = q_m(t)$. Nevertheless as already remarked $\dot{x}_m(t) \geq 0$ when $x_m(t) = q_m(t)$ and then $x_m(t)$ is lower bounded by $\min\{x_m(0), q_m(0)\}$. Analogously, if M is any index such that $x_M(t) = \max\{x_i(t), i \in \mathcal{I}\}$, $q(x_M(t)) = q_M(0)$, we get that $x_M(t)$ is upper bounded by $\max\{x_M(0), q_M(0)\}$, and the function $q_M(t)$ is nonincreasing.

(ii-K) Let now $x(t)$ be a Krasovskii solution and m, M, x_m, x_M, q_m, q_M be defined as in (ii-C). We have to distinguish three cases. If $x_m(t) \in \left(q_m(t), q_m(t) + \frac{1}{2}\right)$, then (as for Carathéodory solutions) $\dot{x}_m(t)$ may be negative, but $x_m(t)$ remains lower bounded by $\min\{x_m(0), q_m(0)\}$. If $x_m(t) \in \left(q_m(t) - \frac{1}{2}, q_m(t)\right]$, then (as for Carathéodory solutions) $\dot{x}_m(t) \geq 0$. The remaining case when $x_m(t) = q_m(t) - \frac{1}{2}$ is more delicate and specific to Krasovskii solutions. Indeed, there can exist an index ℓ such that $x_\ell(t) = x_m(t)$ but $q(y) = q_m(t) - 1$ for some points y in the neighborhood of x_ℓ, which makes the (set-valued) right-hand side include negative values. In such a case, $x_m(t)$ would be allowed to decrease, but this fact would in turn lead to the situation of the first case. In conclusion, $x_m(t)$ is lower bounded by $q_m(0) - 1$ and similarly $x_M(t)$ is upper bounded by $q_M(0) + 1$.

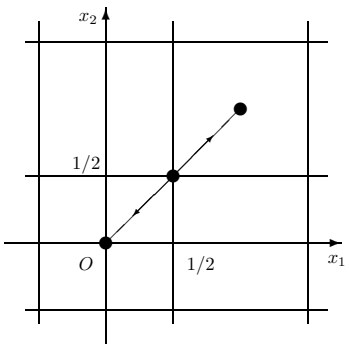

FIG. 2. *A pair of distinct trajectories issuing from the point* $(1/2, 1/2)$*, as discussed in Example 1.*

(iii-C-K) Carathéodory solutions can be continued up to $+\infty$ thanks to local existence and boundedness of solutions (the proof is perfectly analogous to the classical one). Completeness of Krasovskii solutions can either be deduced as for Carathéodory solutions or obtained as a consequence of Lemma 5.1 page 53 in [19]. ∎

The following example shows that in general uniqueness of solutions is not guaranteed.

EXAMPLE 1 (Multiple Carathéodory solutions). *Consider the dynamics*

$$\begin{aligned}\dot{x}_1 =& q(x_2) - x_1\\ \dot{x}_2 =& q(x_1) - x_2\end{aligned}$$

and let $f(x_1, x_2) = (q(x_2) - x_1, q(x_1) - x_2)^\top$*. Note that the vector field* f *restricted to the line* $x_2 = x_1$ *is parallel to this line. Then if we take initial conditions on this line, there exist Carathéodory solutions which lie it. In particular if* $(x_1(0), x_2(0))^\top = (1/2, 1/2)^\top$*, there are two solutions issuing from this point which correspond to the limit values of* f *when restricted to the two sets* $S_{(0,0)} = \{x \in \mathbb{R}^2 : -\frac{1}{2} \leq x_i < +\frac{1}{2}, i = 1, 2\}$ *and* $S_{(1,1)} = \{x \in \mathbb{R}^2 : \frac{1}{2} \leq x_i < \frac{3}{2}, i = 1, 2\}$*. These solutions converge to* $(0, 0)^\top$ *and* $(1, 1)^\top$*, respectively. Their trajectories are the line segments joining the initial condition with the points* $(0, 0)^\top$ *and* $(1, 1)^\top$*; see Figure 2.*

Many consensus-seeking dynamics enjoy a monotonicity property, such that the smallest (largest) component is nondecreasing (nonincreasing). This fact is *not* true for system (2). Nevertheless in the next proposition we prove monotonicity of maximum and minimum quantization level in the case of Carathéodory solutions. This monotonicity of the extremal quantization levels does *not* hold for Krasovskii solutions, as can be seen from Example 4 in the next section.

PROPOSITION 2.2 (Monotonicity and limit of minimum and maximum quantization level for Carathéodory solutions). *Let* x *be a Carathéodory solution and define* $x_m(t) = \min\{x_i(t), i \in \mathcal{I}\}$*,* $q_m(t) = q(x_m(t))$*,* $x_M(t) = \max\{x_i(t), i \in \mathcal{I}\}$ *and* $q_M(t) = q(x_M(t))$*. Then, function* q_m *is nondecreasing, function* q_M *is nonincreasing and they both are definitively constant.*

Proof. We have shown in the proof of Theorem 2.1 (iiC) that, for Carathéodory solutions, $x_m(t) \geq \min\{x_m(0), q_m(0)\}$ for all $t \geq 0$. This means that $x_m(t)$ can not leave the quantization interval $\left[q_m(0) - \frac{1}{2}, q_m(0) + \frac{1}{2}\right)$ from below but only from above. Consequently, the smallest level $q_m(t)$ is nondecreasing. Since it is bounded

and take values in $\mathbb{Z}$, it follows that it is definitively constant, i.e. for any Carathéodory solution $x(t)$ of (2) there exist $T^* \in \mathbb{R}$ and $q_m^* \in \mathbb{Z}$ such that for any $t \geq T^*$ $\min\{q(x_i(t)), i = 1, \ldots, N\} = q_m^*$. Analogous considerations can be repeated for $q_M(t)$. ∎

We observe that the function q takes values in $\mathbb{Z}$, and this means that we a priori admit infinite types of behaviours. Nevertheless a consequence of the previous remark is that the number of quantization levels actually assumed in the evolution of the system is finite, once the initial condition is fixed. In particular, if the initial values of all the components are in the interval $(0, 1)$, the quantizer only takes on the values 0 and 1 and opinions remain in $(0, 1)$: in this way we get the case of binary behaviours.

3. Equilibria and lack of consensus. Generally speaking, a point x^* is said to be an equilibrium of (6) if the function $x(t) = x^*$ is a solution of (6). Since we have different types of solution, we distinguish between different types of equilibria.

We call x^* a *Carathéodory equilibrium* of (6) if the function $x(t) = x^*$ is a Carathédodory solution of (6), and we call x^* a *Krasovskii equilibrium* of (6) if the function $x(t) = x^*$ is a Krasovskii solution of (6). Carathéodory equilibria are found by looking for solutions of the equation $f(x^*) = 0$, whereas Krasovskii equilibria are points x^* such that $0 \in \mathcal{K}f(x^*)$. If we denote by E_C the set of Carathéodory equilibria of (6) and by E_K the set of Krasovskii equilibria, it is evident that $E_C \subset E_K$. As an example of a Krasovskii equilibrium that is not a Carathéodory equilibrium we can take the point $(1/2, 1/2)^\top$ in Example 1. Actually points of the form $(h+1/2, ..., h+1/2)^\top$ are always Krasovskii equilibria of (2).

Let us call *consensus point* a point $\overline{x} \in \mathbb{R}^N$ such that $\overline{x}_i = \overline{x}_j$ for all $i, j = 1, ..., N$. Clearly, integer consensus points like $x^* = k\mathbf{1}$ with $k \in \mathbb{Z}$ are Carathéodory equilibria of (2) because $f(k\mathbf{1}) = 0$. Such a point $k\mathbf{1}$ belongs to the interior of $S_{k\mathbf{1}}$ and, consequently, is locally asymptotically stable. However, Carathéodory equilibria are not necessarily consensus points and may belong to the discontinuity set.

Example 2 (Non-consensus Carathéodory equilibrium). *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= q(x_3) - x_1 \\
\dot{x}_2 &= q(x_3) - x_2 \\
\dot{x}_3 &= q(x_1) + q(x_2) + q(x_4) - 3x_3 \\
\dot{x}_4 &= q(x_3) + q(x_5) - 2x_4 \\
\dot{x}_5 &= q(x_4) - x_5
\end{aligned}$$

The point $\bar{x} = (0, 0, 1/3, 1/2, 1)^\top$ is a Carathéodory equilibrium point which lies on the boundary of $S_{(0,0,0,1,1)}$ as $\bar{x}_4 = 1/2$. The intersection of a neighborhood of $\bar{x}$ with set $S_{(0,0,0,1,1)}$ is contained in the attraction region of $\bar{x}$, but $\bar{x}$ is not Lyapunov stable. Indeed, we recall the an equilibrium point x^* is said to be Lyapunov stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all initial conditions x_0 with $\|x_0 - x^*\| < \delta$ all corresponding (Carathéodory) solutions $x(t; x_0)$ are such that $\|x(t; x_0) - x^*\| < \epsilon$ for all $t \geq 0$. In our case $\bar{x}$ is not Lyapunov stable as for small values of $\alpha > 0$ solutions starting from points of the form $\bar{x} = (0, 0, 1/3, 1/2 - \alpha, 1)^\top$ follow the vector field $(-x_1, -x_2, -3x_3, 1 - 2x_4, -x_5)^\top$ towards the point $(0, 0, 0, 1/2, 0)^\top$.

This example also shows that solutions *can converge to points that are not consensus points.*

For smooth systems, if a (classical) solution converges to a point, then such point is a (classical) equilibrium. This property does not hold true for Carathéodory solutions of systems with discontinuous right-hand side, and, in particular, in the case of systems of the form (4). This motivates the following definition of extended equilibrium. Let $\mathbf{k} \in \mathbb{Z}$ and $f_{\mathbf{k}}$ the vector field whose components are

$$(f_{\mathbf{k}})_i(x) = \sum_{j \neq i} a_{ij}(k_j - x_i).$$

Clearly $f_{\mathbf{k}}$ coincides with f on the set $S_{\mathbf{k}}$. We call *extended equilibrium* of (2) a point $x^* \in \mathbb{R}^N$ such that there exists $\mathbf{k} \in \mathbb{Z}^N$ such that $f_{\mathbf{k}}(x^*) = 0$ and $x^* \in \overline{S_{\mathbf{k}}}$. We denote by E_e the set of extended equilibria of (2). It is evident that $E_C \subset E_e \subset E_K$. The following example shows that these inclusions are strict in general.

EXAMPLE 3 (4-path graph: equilibria). *Let us consider (2) on a 4-node line graph:*

$$\begin{aligned} \dot{x}_1 &= q(x_2) - x_1 \\ \dot{x}_2 &= q(x_1) + q(x_3) - 2x_2 \\ \dot{x}_3 &= q(x_2) + q(x_4) - 2x_3 \\ \dot{x}_4 &= q(x_3) - x_4 \end{aligned} \tag{9}$$

Then,

- *The point $x^A = (0, \frac{1}{2}, \frac{1}{2}, 1)^\top$ is an extended equilibrium, because $x^A \in \overline{S_{(0,0,1,1)}}$ and $f_{(0,0,1,1)}(x^A) = 0$. However, x^A can not be a Carathéodory equilibrium, because $q(x_2^A) = 1 \neq x_1^A = 0$.*
- *The point $x^B = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$ is a Krasovskii equilibrium, as it can be computed that $0 \in \mathcal{K}f(x^B)$. However, x^B can not be an extended equilibrium, because $x_1^B = \frac{1}{2}$ can not be equal to any quantizer value, so that the first component of the vector field can not be zero in any neighbourhood of the point.*

This example shows that extended equilibria include points, like x^A, that are not consensus points. Interestingly, there exist Carathéodory solutions that asymptotically converge to x^A: it is enough to take the solution issuing from an initial condition in $S_{(0,0,1,1)}$. In spite of being attractive, x^A is *not* a Carathéodory equilibrium and actually Carathéodory solutions originating from x^A converge to $(1,1,1,1)^\top$. This pathological behaviour might appear surprising, but is allowed by the discontinuity of the vector field.

In the previous example one can find a Krasovskii solution that is not a Carathéodory solution: it slides on the discontinuity set and connects two equilibria.

EXAMPLE 4 (4-path graph: Krasovskii trajectories). *Let us consider again the dynamics (9) on the 4-node path graph. Consider the parametrized segment $x_a = (a, \frac{1}{2}, \frac{1}{2}, 1-a)^\top$ for $a \in [0, \frac{1}{2}]$, which interpolates between the Krasovskii equilibria $x_0 = x^A$ and $x_{\frac{1}{2}} = x^B$. For every $a \in (0, \frac{1}{2})$ it holds true that*

$$\mathcal{K}f(x_a) = \overline{\text{co}}\,\{(1-a, 0, 1, a)^\top, (-a, 0, 0, a)^\top, (-a, -1, 0, a-1)^\top, (1-a, -1, 1, a-1)^\top\}.$$

Then, there is a Krasovskii solution $\varphi(t)$ such that

- *$\varphi(0) = x_B = (1/2, 1/2, 1/2, 1/2)$;*
- *for any $t > 0$, there exists $a \in (0, 1/2)$ such that $\varphi(t) = x_a$;*

- *if* $\varphi(t) = x_a$*, then* $\dot\varphi(t) = (-a, 0, 0, a)$*;*
- $\lim_{t\to+\infty} \varphi(t) = x_A = (0, 1/2, 1/2, 1)$.

Note that $\varphi(t)$ *can not be a Carathéodory solution as* $f(x_a) = (1-a, 0, 1, a)$.

This example also shows that the minimum quantizer level $q_m(t)$ assumed along Krasovskii solutions may be decreasing. In fact, if we compute $q_m(t)$ for the Krasovskii solution showed above, then $q_m(0) = 1$ whereas $q_m(t) = 0$ if $t > 0$. This fact should be contrasted with the monotonicity of Carathéodory solutions established in Proposition 2.2.

3.1. Path graph: equilibria. We now show, by means of an example, that equilibria can be significantly far from consensus: in the case the communication graph is a path, such distance can be proportional to the square of number of individuals.

If the graph is a path, equations (2) read

$$\begin{aligned} \dot x_1 =& q(x_2) - x_1 \\ \dot x_i =& q(x_{i-1}) + q(x_{i+1}) - 2x_i,\ i = 2, \dots, N-1 \\ \dot x_N =& q(x_{N-1}) - x_N \end{aligned} \tag{10}$$

PROPOSITION 3.1. *If* x^* *is an extended equilibrium, then*

$$\max\left\{|x_j^* - x_i^*| \,:\, i, j = 1, \dots, N\right\} \le \frac{(N-2)^2}{4}.$$

Furthermore, there exists an extended equilibrium $\bar x^*$ *such that*

$$\bar x_N^* - \bar x_1^* = \begin{cases} \frac{(N-2)^2}{4} & \text{if } N \text{ is even} \\ \frac{(N-1)(N-3)}{4} & \text{if } N \text{ is odd.} \end{cases}$$

Proof. Let us characterize extended equilibria of (10). Let $\mathbf{k} \in \mathbb{Z}^N$. If x^* is such that $f_{\mathbf{k}}(x^*) = 0$ then

$$\begin{aligned} x_1^* =& k_2 \\ x_i^* =& \frac{k_{i-1} + k_{i+1}}{2},\ i = 2, \dots, N-1 \\ x_N^* =& k_{N-1}. \end{aligned}$$

Then, $x^* \in \overline{S}_{\mathbf{k}}$ if and only if the following inequalities are satisfied

$$\begin{aligned} k_1 - \frac12 \le& k_2 \le k_1 + \frac12 \\ k_i - \frac12 \le& \frac{k_{i-1} + k_{i+1}}{2} \le k_i + \frac12,\ i \in \{2, \dots, N\} \\ k_N - \frac12 \le& k_{N-1} \le k_N + \frac12 \end{aligned}$$

Since $k_i \in \mathbb{Z}$, from the first inequality it follows that $k_1 = k_2$ and from the last one it follows that $k_N = k_{N-1}$. Moreover the inequalities with $i = 2, \dots, N-1$ can be written in the form

$$k_i - k_{i-1} - 1 \le k_{i+1} - k_i \le k_i - k_{i-1} + 1$$

that implies

$$|k_{i+1} - k_i| - |k_i - k_{i-1}| \le |(k_{i+1} - k_i) - (k_i - k_{i-1})| \le 1$$

i.e.

$$|k_{i+1}-k_i| \le |k_i-k_{i-1}|+1 \text{ and } |k_i-k_{i-1}| \ge |k_{i+1}-k_i|-1.$$

From the left inequality we deduce that

$$|k_i-k_{i-1}| \le i-2,\ i\in\{2,\dots,N\}$$

and from the right inequality

$$|k_i-k_{i-1}| \le N-i,\ i\in\{2,\dots,N\}.$$

These two inequalities imply that

$$|k_i-k_{i-1}| \le \min\{i-2, N-i\},\ i\in\{2,\dots,N\},$$

that can be written as

$$|k_i-k_{i-1}| \le i-2,\ i\in\{2,\dots,N\}, i\le \frac{N+2}{2}$$

and

$$|k_i-k_{i-1}| \le N-i,\ i\in\{2,\dots,N\}, i> \frac{N+2}{2}.$$

Assuming without loss of generality that $j\ge i$, we can deduce that

$$|k_j-k_i| \le |k_j-k_{j-1}|+|k_{j-1}-k_{j-2}|+\ldots+|k_{i+1}-k_i| \le |k_N-k_{N-1}|+\ldots+|k_2-k_1|$$

and also

$$|x_j-x_i| \le |k_N-k_{N-1}|+\ldots+|k_2-k_1|$$

Let us distinguish the cases N is even and N is odd. If N is even, $N=2n, n\in\mathbb{N}$, and $\frac{N+2}{2}=n+1$. We have

$$\begin{aligned}
&|k_N-k_{N-1}|+|k_{N-1}-k_{N-2}|+\ldots+|k_{n+1}-k_n|+\ldots+|k_2-k_1|\\
&\le \big(0+1+2+\ldots+(2n-n-2)\big)+\big((n+1-2)+(n-2)+\ldots+1+0\big)\\
&\le \frac{(n-2)(n-1)}{2}+\frac{(n-1)n}{2}=(n-1)^2=\frac{(N-2)^2}{4}.
\end{aligned}$$

If N is odd, $N=2m+1, m\in\mathbb{N}$, and $\frac{N+2}{2}=m+\frac{3}{2}$. We have

$$\begin{aligned}
&|k_N-k_{N-1}|+|k_{N-1}-k_{N-2}|+\ldots|k_{n+1}-k_n|+\ldots.+|k_2-k_1|\\
&\le \big(0+1+2+\ldots+(2m+1-m-2)\big)+\big(m+1-2)+(m-2)+\ldots+1+0\big)\\
&\le 2\frac{(m-1)m}{2}=(m-1)m=\frac{(N-1)(N-3)}{4}.
\end{aligned}$$

The first statement then follows from the fact $(N-1)(N-3)\le(N-2)^2$.

In order to prove the second statement, if we select $\mathbf{k}$ such that $k_1=0$ and

$$k_i-k_{i-1}=\begin{cases} i-2,\ i\in\{2,\dots,N\}, i\le\frac{N+2}{2}\\ N-i,\ i\in\{2,\dots,N\}, i>\frac{N+2}{2},\end{cases}$$

we do obtain an extended equilibrium $\bar{x}^*$ that achieves the above bounds. □

Point $\bar{x}^*$ in the previous proposition shows that the system has extended equilibria that, provided N is large enough, are arbitrarily far from consensus. We shall come back to this point in Example 6. Figure 3 shows convergence to a non-consensus state.

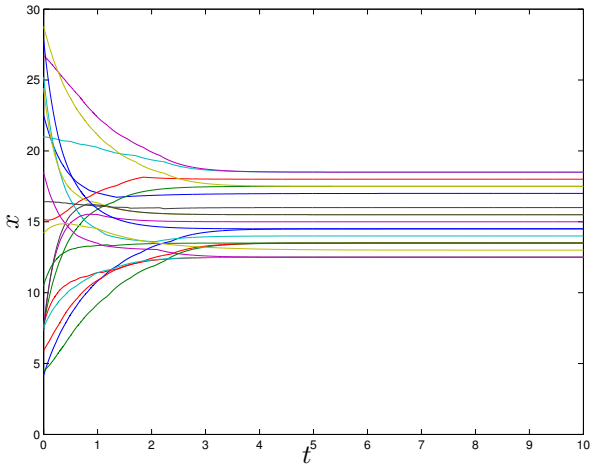

FIG. 3. *Simulation via explicit Euler method of* (10) *for* $N = 20$ *from random initial conditions in* $[0, 30]$*, showing convergence to a non-consensus state.*

4. Quantization as disturbance. A classical approach to obtain (conservative) results about quantized systems is to see the quantized dynamics as the perturbation of a "nominal" system, where the perturbation is due to the quantization error $q(x) - x$. This idea is also useful for our problem. In this section we get an asymptotic estimate of the distance of solutions from consensus. In order to prove it we need to assume the graph to be weight balanced. Below, we recall a few concepts of graph theory needed to make use of this assumption.

We have already defined $d_i = \sum_{j=1}^N a_{ij}$. Let $d^i = \sum_{k=1}^N a_{ki}$. The (directed weighted) graph with adjacency matrix A is said to be *weight-balanced* if $d_i = d^i$ for all $i = 1, ..., N$. Note that $L\mathbf{1} = 0$ and $\mathbf{1}^\top L = 0$ if and only if the graph is weight-balanced. Given an edge (i, j), we shall refer to i and to j as the tail and the head of the edge, respectively. A path is an ordered list of edges such that the head of each edge is equal to the tail of the following one. The graph is said to be

1. *strongly connected* if for any i, j there is an path from i to j;
2. *connected* if there exists one node j such that for any i there is path from i to j;
3. *weakly connected* if for each pair of nodes i, j, one can construct a path which connects i and j by possibly reverting the direction of some edges.

These three notions of connectedness coincide when the graph is *symmetric*, that is when $a_{ij} > 0$ if and only if $a_{ji} > 0$. On the contrary, weakly connected weight-balanced graphs are strongly connected [18, Proposition 2]. We recall the following result, which can be derived from [9, Theorem 1.37] and [18, Formula (1) and Section 2.2].

LEMMA 4.1. *Let* L *be the Laplacian matrix of a weight-balanced and weakly connected graph. Then:*

(i) The matrix $\mathrm{Sym}(L) = \frac{L+L^\top}{2}$ *is positive semi-definite.*

(ii) Denoted by λ_* *the smallest non-zero eigenvalue of* $\mathrm{Sym}(L)$*,*

$$x^\top L x \geq \lambda_* \|x - x_a \mathbf{1}\|^2 ,$$

for all $x \in \mathbb{R}^N$*.*

THEOREM 4.2 (Convergence to a set). *Assume that the graph with adjacency matrix A is weight balanced and weakly connected. If $x(t)$ is any Carathéodory or Krasovskii solution of (2) and*

$$M = \left\{ x \in \mathbb{R}^N : \inf_{\alpha \in \mathbb{R}} \|x - \alpha \mathbf{1}\| \leq \frac{||A||}{\lambda_*} \frac{\sqrt{N}}{2} \right\},$$

then $\operatorname{dist}(x(t), M) \to 0$ *as* $t \to +\infty$.

Proof. We prove the statement for Krasovskii solutions, as Carathéodory solutions are a special case. Let $y(t) = x(t) - x_a(t)\mathbf{1}$. Then $\dot{y}(t) = \dot{x}(t) - \dot{x}_a(t)\mathbf{1}$. Consider the function $V(y) = \frac{1}{2} y^\top y$. We have that

$$\begin{aligned}
\nabla V(y) \cdot \dot{y} =& y^\top \dot{y}(t) \\
=& (x - x_a \mathbf{1})^\top [\dot{x} - \dot{x}_a \mathbf{1}] \\
=& (x - x_a \mathbf{1})^\top \dot{x} - x^\top \dot{x}_a \mathbf{1} + x_a \mathbf{1}^\top \dot{x}_a \mathbf{1} \\
=& (x - x_a \mathbf{1})^\top \dot{x} - \dot{x}_a x^\top \mathbf{1} + \dot{x}_a N x_a \\
=& (x - x_a \mathbf{1})^\top \dot{x} - \dot{x}_a N x_a + \dot{x}_a N x_a \\
=& (x - x_a \mathbf{1})^\top \dot{x}.
\end{aligned}$$

From (3) we get

$$\dot{x} \in -L(x - x_a \mathbf{1}) + A\mathcal{K}(\mathbf{q}(x) - x) \subseteq -L(x - x_a \mathbf{1}) + A(\mathcal{K}\mathbf{q}(x) - x).$$

For any $v \in \mathcal{K}\mathbf{q}(x) - x$, it holds $\|v\| \leq \frac{\sqrt{N}}{2}$. Then, if $v \in \mathcal{K}\mathbf{q}(x) - x$ is such that $\dot{y} = -L(x - x_a \mathbf{1}) + Av$, we have

$$\begin{aligned}
\nabla V(y) \cdot \dot{y} =& (x - x_a \mathbf{1})^\top [-L(x - x_a \mathbf{1}) + Av] \\
=& -(x - x_a \mathbf{1})^\top L(x - x_a \mathbf{1}) + (x - x_a \mathbf{1})^\top Av \\
=& -(x - x_a \mathbf{1})^\top \operatorname{Sym}(L)(x - x_a \mathbf{1}) + (x - x_a \mathbf{1})^\top Av \\
\leq& -\lambda_* \|x - x_a \mathbf{1}\|^2 + \|x - x_a \mathbf{1}\| \|A\| \frac{\sqrt{N}}{2} \\
\leq& \|x - x_a \mathbf{1}\| \left[-\lambda_* \|x - x_a \mathbf{1}\| + \|A\| \frac{\sqrt{N}}{2} \right].
\end{aligned}$$

We conclude that $\operatorname{dist}(x(t), M) \to 0$ as $t \to +\infty$, because otherwise V would decrease unboundedly along solutions, which is forbidden by V being nonnegative. ∎

Note that the infimum $\inf_{\alpha \in \mathbb{R}} \|x - \alpha \mathbf{1}\|$ is in fact achieved for $\alpha = \frac{1}{N} \sum_{i=1}^N x_i$. Then we can further elaborate that

$$M = \left\{ x \in \mathbb{R}^N : \frac{1}{\sqrt{N}} \|x - \frac{1}{N} \sum_{i=1}^N x_i \mathbf{1}\| \leq \frac{||A||}{2\lambda_*} \right\}.$$

In order to better understand the interest of this overapproximation of the limit set, we specialize it to two families of graphs with uniform weights: complete and path graphs with $a_{ij} = 1$ if i and j are connected and $a_{ij} = 0$ if they are not.

EXAMPLE 5 (Complete graph). *On complete graphs, we have*

$$\dot{x}_i = \sum_{j\neq i} \big(q(x_j) - x_i\big) \quad \textit{for all } i,$$

while $||A|| = N-1$ *and* $\lambda_* = N$*, so that asymptotically*

$$\frac{1}{\sqrt{N}}\|x - \frac{1}{N}\sum_{i=1}^{N} x_i \mathbf{1}\| \leq \frac{1}{2}.$$

This bound implies that the limit points are close to consensus. In fact, we shall prove below (Theorem 5.3) that in this case limit points are precisely consensus points.

EXAMPLE 6 (Path graph). *On path graphs with* N *nodes,* $\lambda_* = 1-\cos(\frac{\pi}{N})$ *and* $||A|| \leq 2$*, because it always holds by Gershgorin's disk lemma that* $||A|| \leq d_{\max}$*. Then, an overapproximation of set* M *can be defined by*

$$\begin{aligned}\frac{1}{\sqrt{N}}\|x - \frac{1}{N}\sum_{i=1}^{N} x_i \mathbf{1}\| &\leq \frac{2}{2(1-\cos\frac{\pi}{N})} \\ &= \frac{1}{\frac{\pi^2}{2N^2} - \frac{\pi^4}{4N^4} + o(\frac{1}{N^4})} \\ &= \frac{2}{\pi^2}\frac{N^2}{1 - \frac{\pi^2}{2N^2} + o(\frac{1}{N^2})} = \frac{2}{\pi^2}N^2 + O(1) \quad \textit{as } N\to\infty.\end{aligned}$$

Let us now consider the following equilibrium $x^* \in E_e$ *for odd* $N = 2m+1$*, which was constructed in the proof of Proposition 3.1: the point* x^* *is symmetric with respect to the median value* x^*_{m+1} *and such that*

$$\begin{aligned} x^*_1 &= 0 \\ x^*_i &= \frac{i^2 - 3i + 3}{2} \quad \textit{if } 2 \leq i \leq m \\ x^*_{m+1} &= \frac{m(m-1)}{2}.\end{aligned}$$

For instance, for $N = 9$ *we have* $\mathbf{k} = (0,0,1,3,6,9,11,12,12)^\top$ *and correspondingly* $x^* = (0, \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, 6, \frac{17}{2}, \frac{21}{2}, \frac{23}{2}, 12)^\top$*. Consequently,* $x^*_a = x^*_{m+1}$ *and*

$$\begin{aligned}\|x^* - x^*_a\|^2 =& 2\sum_{i=1}^{m}(x^*_i - x^*_{m+1})^2 \\ =& 2\sum_{i=1}^{m}\left(\frac{i^2-3i+3}{2} - \frac{m(m-1)}{2}\right)^2 \\ =& \frac{1}{2}\sum_{i=1}^{m}\big(i^4 - 6i^3 - (2m^2-2m-15)i^2 + 6(m^2-m-3)i \\ &\qquad + m^4 - 2m^3 - 5m^2 + 6m + 9\big).\end{aligned}$$

By recalling that for all $a \in \mathbb{N}$*, it holds* $\sum_{i=1}^{n} i^a = \frac{1}{a+1}n^{a+1} + o(n^{a+1})$ *as* $n\to\infty$ *and identifying the highest order terms in the above expression, we finally observe that*

$$\frac{1}{\sqrt{N}}\|x^* - x^*_a\| = \frac{1}{\sqrt{120}}\ N^2 + o(N^2) \quad \textit{as } N\to\infty.$$

This example shows that the result in Proposition 4.2 can not be significantly improved in general, as the estimate on the limit set is asymptotically tight for large networks in the sense of the Euclidean distance from the consensus. Nevertheless, stronger results can be obtained for specific topologies, as we do in the following section.

5. Convergence to consensus on special graphs. The previous sections emphasize the fact that in general we can not expect consensus for a system of the form (2). Nevertheless we can prove that consensus is achieved when the communication graph has some particular form, namely if it is complete or complete bipartite, and weights are uniform.

5.1. Complete graph. System (2) in the case of the complete graph with uniform weights reads

$$\dot{x} = f(x) \tag{11}$$

where $f : \mathbb{R}^N \to \mathbb{R}^N$ is given by $f_i(x) = \sum_{j\neq i}[q(x_j) - x_i]$. We define

$$\begin{aligned}\hat{\mathbf{q}}(x) &= \Big(\sum_{j\neq 1} q(x_j), \dots, \sum_{j\neq N} q(x_j)\Big)^\top \\ &= \Big(\sum_{j=1}^{N} q(x_j) - q(x_1), \dots, \sum_{j=1}^{N} q(x_j) - q(x_N)\Big)^\top,\end{aligned}$$

and we observe that for every $\mathbf{k} \in \mathbb{Z}^N$ the function $\hat{\mathbf{q}}$ is constant on each set $S_{\mathbf{k}}$. The discontinuous vector field can be written as

$$f(x) = \hat{\mathbf{q}}(x) - (N-1)x. \tag{12}$$

This form makes evident that in each set $S_{\mathbf{k}}$ trajectories are line segments.

We now prove that the set E_e of extended equilibria reduces to quantized consensus points which actually are Carathéodory equilibria as they belong to the interior of some $S_{\mathbf{k}}$.

PROPOSITION 5.1 (Extended equilibria for complete graph). *The sets of Carathéodory and extended equilibria of* (11) *coincide and are equal to*

$$E_C = E_e = \{x \in \mathbb{Z}^N : \ \exists h \in \mathbb{Z} \text{ such that } \ x_i = h \ \forall i = 1, \dots, N\}.$$

Proof. Clearly, any point of E_e is a Carathéodory equilibrium of (11). On the other hand x^* is an extended equilibrium if there exists $\mathbf{k} \in \mathbb{Z}^N$ such that $f_{\mathbf{k}}(x^*) = 0$ and x^* belongs to $\overline{S_{\mathbf{k}}}$. If $K = \sum_{j=1}^{N} k_j$, x^* is a partial equilibrium if for all $i = 1, \dots, N$

$$x_i^* = \frac{K - k_i}{N-1}$$

and

$$k_i - \frac{1}{2} \leq x_i^* \leq k_i + \frac{1}{2}$$

By multiplying these bounds by $N-1$, we get

$$(N-1)\left(k_i - \frac{1}{2}\right) \leq K - k_i \leq (N-1)\left(k_i + \frac{1}{2}\right)$$

that implies, for all $i = 1, \dots, N$,

$$\frac{K}{N} - \frac{N-1}{2N} \leq k_i \leq \frac{K}{N} + \frac{N-1}{2N}.$$

Since $k_i \in \mathbb{Z}$ interval $\left[\frac{K}{N} - \frac{N-1}{2N}, \frac{K}{N} + \frac{N-1}{2N}\right]$, and these bounds imply that $k_1 = \ldots = k_n$. We conclude that if x^* is an extended equilibrium then $x^* \in S_{(h,\dots,h)}$ for some $h \in \mathbb{Z}$. In this case $f(x^*)_i = (N-1)h - (N-1)x_i^*$ for all $i = 1, \dots, N$, and finally $x_i^* = h$ for all $i = 1, \dots, N$.

We remark that also for (11) there may be Krasovskii equilibria which are not extended equilibria: see, for instance, the initial condition in Example 1 or, more generally, any point $x^* = (k + \frac{1}{2})\mathbf{1}$ with integer k. Actually, our convergence result below will imply that all Krasovskii equilibria are consensus points. Before it, we provide the reader with three statements that describe qualitative properties of the solutions.

REMARK 2 (Local stability). *Note that extended equilibria of (11) are Lyapunov stable and locally asymptotically stable. In fact for any equilibrium x^* there exists a neighborhood where the vector field can be written as $f(x) = -(N-1)(x - x^*)$.*

REMARK 3 (Finite-time exit for Carathéodory solutions). *From the proof of Proposition 5.1 it follows that in each $S_{\mathbf{k}}$ with $\mathbf{k} \neq (h, \dots, h)^\top$ for any $h \in \mathbb{Z}$, trajectories are line segments that are generated by Carathéodory solutions corresponding to a vector field whose equilibrium is out of $S_{\mathbf{k}}$. This consideration implies that all solutions to (2) escape from every set $S_{\mathbf{k}}$ with $\mathbf{k} \neq (h, \dots, h)^\top$ in finite time. On the other hand, once inside some $S_{(h,h,\dots,h)}$, the equilibrium is reached asymptotically.*

LEMMA 5.2 (Invariance properties). *Let $x(t)$ be a Carathéodory or Krasovskii solution of (11) and let $s_{ij}(x) = x_i - x_j$ and $S_{ij} = \{x \in \mathbb{R}^N : s_{ij}(x) = 0\}$.*

(A) (Invariant manifolds) If $x(t_0) \in S_{ij}$, then $x(t) \in S_{ij}$ for all $t \geq t_0$.

(B) (Order preservation) If $x_i(t_0) \leq x_j(t_0)$ for some $t_0 \in \mathbb{R}$ then $x_i(t) \leq x_j(t)$ for all $t \geq t_0$.

Proof. We prove the statements for Krasovskii solutions. (A) Let $x(t)$ be a Krasovskii solution such that $x(t_0) = x_0 \in S_{ij}$. We prove that $x(t)$ can not leave S_{ij}. If $x_0 \in S_{ij} \backslash \Delta$, $f(x_0)$ is parallel to S_{ij} in fact $\nabla s_{ij}(x_0) = e_i - e_j$ and

$$\begin{aligned}\nabla s_{ij}(x_0) \cdot f(x_0) =& f_i(x_0) - f_j(x_0) \\ =& \sum_{h \neq i} [q(x_{0h}) - x_{0i}] - \sum_{h \neq j} [q(x_{0h}) - x_{0j}] \\ =& q(x_{0j}) - q(x_{0i}) - (N-1)(x_{0i} - x_{0j}) = 0.\end{aligned}$$

Let now $x_0 \in S_{ij} \cap \Delta$, i.e. $x_{0h} = k_h + \frac{1}{2}$ for $h \in \mathcal{I}' \subset \mathcal{I}$ and some $k_h \in \mathbb{Z}$. We first consider the case $i, j \notin \mathcal{I}'$. In this case for any limit value $f^l(x_0)$ of $f(x_0)$ at x_0 one has

$$\nabla s_{ij}(x_0) \cdot f^l(x_0) = f_i^l(x_0) - f_j^l(x_0) = q(x_{0j}) - q(x_{0i}) - (N-1)(x_{0i} - x_{0j}) = 0.$$

meaning that all $f^l(x_0)$ are parallel to S_{ij}.

Then we consider the case $x_{0i} = x_{0j} = k + \frac{1}{2}$ for some $k \in \mathbb{Z}$. We consider four limit values $f^{(i)}(x_0)$, $f^{(ii)}(x_0)$, $f^{(iii)}(x)$, $f^{(iv)}(x)$ of vector the field $f(x_0)$, depending on the fact that x_0 is approached from the following sectors:

(i) $x_i > k + \frac{1}{2}$, $x_j < k + \frac{1}{2}$

(ii) $x_i > k + \frac{1}{2}$, $x_j > k + \frac{1}{2}$
(iii) $x_i < k + \frac{1}{2}$, $x_j > k + \frac{1}{2}$
(iv) $x_i < k + \frac{1}{2}$, $x_j < k + \frac{1}{2}$.

Note that

$$\nabla s_{ij}(x_0) \cdot f^{(ii)}(x_0) = \nabla s_{ij}(x_0) \cdot f^{(iv)}(x_0) = 0,$$

meaning that $f^{(ii)}(x_0)$ and $f^{(iv)}(x_0)$ are parallel to S_{ij} at x_0. In sector (i) one has $s_{ij}(x) = x_i - x_j > 0$ and

$$\nabla s_{ij}(x_0) \cdot f^{(i)}(x_0) = -1.$$

This means that $f^{(i)}(x_0)$ points towards S_{ij} and the solution can not move towards the sector (i). In fact assume by contradiction that there exists $T > 0$ such that the solution $x(t)$ belongs to the sector (i) when $t \in (0, T)$. The fact that $x(t)$ belongs to the sector (i) means that $x_i(t) - x_j(t) > 0$ when $t \in (0, T)$. On the other hand $x_i(t) - x_j(t) = s_{ij}(t)$ and $\frac{d}{dt} s_{ij}(x(t)) = \nabla s_{ij}(x(t)) \cdot \dot{x}(t) = \nabla s_{ij}(x(t)) \cdot f^{(i)}(x(t)) < 0$ meaning that $s_{ij}(x(t))$ is decreasing in $(0, T)$ so that $x_i(t) - x_j(t) < x_i(0) - x_j(0) = 0$, contradiction. Analogously in (iii) one has $s_{ij}(x) = x_i - x_j < 0$ and $\nabla s_{ij}(x_0) \cdot f^{(iii)}(x_0) = 1$. This means that $f^{(iii)}(x_0)$ points towards S_{ij} and the solution cannot move towards sector (iii).

(B) This is an immediate consequence of the previous statement. Assume first that $x_i(t_0) < x_j(t_0)$ and $x_i(t^*) > x_j(t^*)$ for some $t^* > t_0$. Then there exist $\bar{t} \in (t_0, t^*)$ such that $x_i(\bar{t}) = x_j(\bar{t})$ and $t^* > t_0$ such that $x_i(t^*) > x_j(t^*)$, which contradicts (A). In the case $x_i(t_0) = x_j(t_0)$ and $x_i(t^*) > x_j(t^*)$ for some $t^* > t_0$ the contradiction with (A) is immediate. ∎

We now state the main result of this section: any Carathéodory or Krasovskii solution converges to a consensus point. The convergence is illustrated in Figure 4.

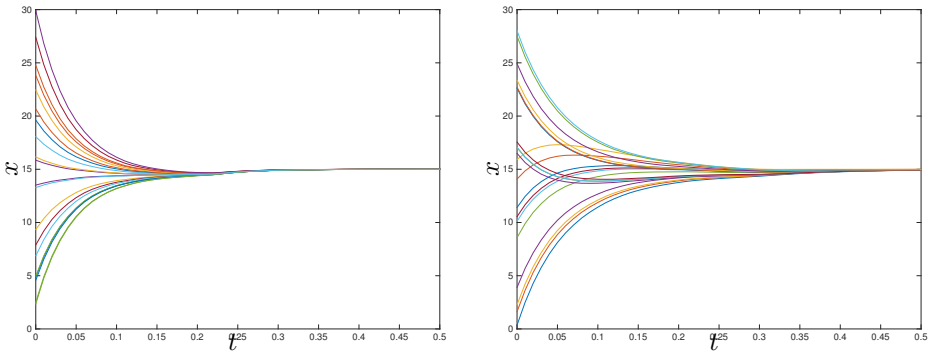

FIG. 4. *Simulations of* (11) *and* (14) *(respectively, left and right plot) for* $N = 20$ *from random initial conditions in* $[0, 30]$*, obtained via the explicit Euler method with step size* 10^{-2}*. Both simulations show convergence of the states to the integer consensus value* $h = 15$*. System* (11) *also exhibits order preservation as per Lemma 5.2.*

THEOREM 5.3 (Convergence–Complete graph). *Any Carathéodory or Krasovskii solution* $x(t)$ *of* (11) *converges to a consensus point. Furthermore, if* $x(t)$ *is Carathéodory, then the limit point is necessarily of the form* $(h, \dots, h)^\top$ *with integer* h*. If instead* $x(t)$ *is Krasovskii, then the limit may be of the form* $(h + \frac{1}{2}, \dots, h + \frac{1}{2})^\top$*.*

Proof. As in the proof of Lemma 5.2 we prove convergence for the more general case of Krasovskii solutions. Let $x(t)$ be any Krasovskii solution of (11). Thanks to

the previous lemma, we can assume without loss of generality that $x_1(t_0) \leq \cdots \leq x_N(t_0)$. Indeed, by order preservation we have $x_1(t) \leq \ldots \leq x_N(t)$ for all $t \geq t_0$ and $x_N(t) - x_1(t) \geq 0$ for all $t \geq t_0$. We prove that $x_N(t) - x_1(t) \to 0$ as $t \to +\infty$. If there exists T such that $x_N(T) = x_1(T)$, then, thanks to Lemma 5.2, $x_N(t) - x_1(t) = 0$ with $t \geq T$. If instead $x_N(t) > x_1(t)$ for all $t \geq t_0$, then we can recall that $\dot{x}(t) \in -Dx(t) + A\mathcal{K}\mathbf{q}(x(t))$, where $a_{ij} = 1$ if $i \neq j$ and $a_{ij} = 0$ if $i = j$, and there exists a measurable function $v(t) \in \mathcal{K}\mathbf{q}(x(t))$ such that $\dot{x}(t) = -Dx(t) + Av(t)$. We can then write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x_N(t) - x_1(t)) &= \sum_{j \neq N} v_j(t) - (N-1)x_N(t) - \sum_{j \neq 1} v_j(t) + (N-1)x_1(t) \\ &= -(N-1)(x_N(t) - x_1(t)) - (v_N(t) - v_1(t))\end{aligned}$$

As $x_N(t) > x_1(t)$ we have that $v_N(t) \geq v_1(t)$, i.e. $v_N(t) - v_1(t) \geq 0$ and then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x_N(t) - x_1(t)) &= -(N-1)(x_N(t) - x_1(t)) - (v_N(t) - v_1(t)) \\ &\leq -(N-1)(x_N(t) - x_1(t)),\end{aligned} \tag{13}$$

which implies that $x_N(t) - x_1(t) \to 0$ as $t \to +\infty$.

We now distinguish two cases. If $x(t)$ is in the interior of $S_{(h,\ldots,h)}$ for some t and some $h \in \mathbb{Z}$, then

$$\dot{x}(t) = f(x(t)) = -(N-1)(x(t) - (h, \ldots, h)^\top)$$

and $x(t) \to (h, \ldots, h)^\top$ as $t \to +\infty$. Otherwise, the only way for $x(t)$ to converge to consensus without entering the interior of any $S_{(m,\ldots,m)}$ for any $m \in \mathbb{Z}$ is that $x(t)$ converges to $(h + \frac{1}{2}, \ldots, h + \frac{1}{2})^\top$ for some $h \in \mathbb{Z}$. In this case, Remark 3 implies that $x(t)$ actually reaches $(h + \frac{1}{2}, \ldots, h + \frac{1}{2})^\top$ in finite time. Afterwards, if $x(t)$ is a Carathéodory solution, then it cannot stay there (because of Proposition 5.1), nor it can leave the consensus manifold (because of the differential inequality (13)): thus, it necessarily converges to $(h, \ldots, h)$ or to $(h+1, \ldots, h+1)$. If instead $x(t)$ is a Krasovskii solution, it may stay at $(h + \frac{1}{2}, \ldots, h + \frac{1}{2})$, which is Krasovskii equilibrium. ∎

5.2. Complete bipartite graph. In the case the graph is complete bipartite, equations (2) read

$$\dot{x}_i = \sum_{h \in \mathcal{Q}} [q(x_h) - x_i] \qquad \dot{x}_h = \sum_{i \in \mathcal{P}} [q(x_i) - x_h] \tag{14}$$

where $\mathcal{P} = \{1, \ldots, p\}$ and $\mathcal{Q} = \{p+1, \ldots, N\}$.

THEOREM 5.4 (Convergence–Complete bipartite). *Any Carathéodory or Krasovskii solution to (14) converges to a consensus point.*

Proof. Let $x(t)$ be any Krasovskii solution of (14). First of all we prove that $x_i(t) - x_j(t) \to 0$ as $t \to +\infty$ for $i, j \in \mathcal{P}$ and $x_h(t) - x_k(t) \to 0$ as $t \to +\infty$ for $h, k \in \mathcal{Q}$. We recall that $x(t)$ is such that

$$\dot{x}(t) \in -Dx(t) + \mathcal{K}(A\mathbf{q}(x(t)))$$

and that there exists a measurable function $v(t)$ such that $\dot{x}(t) = v(t) - Dx(t)$ with $v(t) \in \mathcal{K}(A\mathbf{q}(x(t)))$. In the case of the bipartite graph $d_{ii} = N - p$, $d_{hh} = p$, $(A\mathbf{q}(x))_i = \sum_{h \in \mathcal{Q}} q(x_h)$, $(A\mathbf{q}(x))_h = \sum_{i \in \mathcal{P}} q(x_i)$ with $i = 1, \ldots, p$ and $h = p+1, \ldots, N$.

We denote by $v_i(t)$ the i-th component of $v(t)$ and prove that $v_i(t) = v_j(t)$ for $i, j \in \mathcal{P}$. Clearly if $x_h(t) \neq k + 1/2$, $k \in \mathbb{Z}$, for all $h \in \mathcal{Q}$, then $v_i(t) = \sum_{h\in\mathcal{Q}} q(x_h(t))$ for all $i \in \mathcal{P}$. If $x_h(t) = k_h + 1/2$, for some $k_h \in \mathbb{Z}$ and for some $h \in \mathcal{Q}' \subset \mathcal{Q}$, there exist $2^{|\mathcal{Q}'|} = M$ limit values $Q^1, \dots, Q^M$ for $\mathbf{q}(x)$ at $x(t)$, and $v(t) \in \mathcal{K}(A\mathbf{q}(x(t))$ can be written $v(t) = \alpha_1 AQ^1 + \dots \alpha_M AQ^M$, with $\alpha_1, \dots, \alpha_M \in [0, 1]$ and $\alpha_1 + \dots + \alpha_M = 1$. As $(A\mathbf{q}(x))_i = \sum_{h\in\mathcal{Q}} q(x_h)$ for any $i = 1, \dots, p$ we get that $v_i(t) = v_j(t)$ for any $i, j \in \mathcal{P}$. Analogously, one has $v_h(t) = v_k(t)$ for $h, k \in \mathcal{Q}$.

For any $i, j \in \mathcal{P}$ we thus get

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_i(t) - x_j(t)) = v_i(t) - px_i(t) - v_j(t) + px_j(t) = -p(x_i(t) - x_j(t))$$

which implies that $x_i(t) - x_j(t) \to 0$ as $t \to +\infty$. Moreover if there exists τ such that $x_i(\tau) = x_j(\tau)$ for $i, j \in \mathcal{P}$, then $x_i(t) = x_j(t)$ for all $t \geq \tau$. Analogously one has $x_h(t) - x_k(t) \to 0$ as $t \to +\infty$ for any $h, k \in \mathcal{Q}$ and if there exists τ' such that $x_h(\tau') = x_k(\tau')$ for $i, j \in \mathcal{P}$, then $x_h(t) = x_k(t)$ for all $t \geq \tau'$. The fact that $x_i(t) - x_j(t) \to 0$ and $x_h(t) - x_k(t) \to 0$ as $t \to +\infty$ also implies that either there exists T such that $x_i(t) \leq x_h(t)$ for all $i \in \mathcal{P}$, $h \in \mathcal{Q}$ and for all $t \geq T$ or $x_i(t) - x_h(t) \to 0$ for all $i \in \mathcal{P}$, $h \in \mathcal{Q}$.

Let $m(t) \in \{1, \dots, N\}$ be such that $x_{m(t)}(t) = \min\{x_i(t), i = 1, \dots, N\}$ and $M(t) \in \{1, \dots, N\}$ be such that $x_{M(t)}(t) = \max\{x_i(t), i = 1, \dots, N\}$. We will simply denote $x_{m(t)}(t) = x_m(t)$ and $x_{M(t)}(t) = x_M(t)$. Thanks to the previous remark we can assume that $m(t) \in \mathcal{P}$ and $M(t) \in \mathcal{Q}$ definitively. We first assume that there exist $T' \in \mathbb{R}$ and $q \in \mathbb{Z}$ such that $x_m(T'), x_M(T') \in (q - 1/2, q + 1/2)$. In this case, from (14) we get that $x_m(t), x_M(t) \in (q - 1/2, q + 1/2)$ for all $t \geq T'$ and $x_i(t) \to q$ for all $i \in \mathcal{P}$, $x_h(t) \to q$ for all $h \in \mathcal{Q}$.

We then prove that there exist T and $q \in \mathbb{Z}$ such that either $x_m(T), x_M(T) \in (q - 1/2, q + 1/2)$ or $x_m(t) = x_M(t) = q + 1/2$ for all $t \geq T$.

Note that as far as $x_m(t) \in [q - 1/2, q + 1/2)$ and $x_M(t) > q + 1/2$, for almost all t one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_M(t) - x_m(t)) \in \mathcal{K}\Big(\sum_{i\in\mathcal{P}}[q(x_i) - x_M] - \sum_{h\in\mathcal{Q}}[q(x_h) - x_m]\Big)$$

and if $v \in \mathcal{K}(\sum_{i\in\mathcal{P}}[q(x_i) - x_M] - \sum_{h\in\mathcal{Q}}[q(x_h) - x_m])$, then $v \leq -1$. This fact implies that $x_M(t) - x_m(t)$ decreases until one of the following cases is reached:

- $q - 1/2 < x_m(t') < q + 1/2,\ x_M(t') = q + 1/2$,
- $x_m(t') = q - 1/2,\ x_M(t') = q + 1/2$,
- $x_m(t') = x_M(t') = q + 1/2$.

In the first case, $q(x_i(t')) = q$. Since $q - (q + 1/2) = -1/2 < 0$, we get that there exists $t'' > t$ such that $x_M(t)$ decreases in (t', t'') and $x_m(T), x_M(T) \in (q-1/2, q+1/2)$ at some time $T \in (t', t'')$.

The second case is analogous to the first one with $x_m(t)$ increasing and $x_M(t)$ decreasing in a right neighbourhood (t', t'') of t', so that $x_m(T), x_M(T) \in (q-1/2, q+1/2)$ at some time $T \in (t', t'')$.

Finally in the third case it is sufficient to recall that if $x_i(t') = x_j(t')$ and $x_h(t') = x_k(t')$ then $x_i(t) = x_j(t)$ and $x_h(t) = x_k(t)$ for all $t \geq t'$. This implies that $x_i(t), x_h(t) \to q$ or $x_i(t), x_h(t) \to q + 1$ or $x_i(t) = x_h(t) = q + 1/2$ for all $i \in \mathcal{P}$, $h \in \mathcal{Q}$.

By exhausting the cases, we have proved the claim and consequently completed the proof. ∎

6. Simulations on general graphs. We now present some simulations in order to illustrate and explore the convergence properties of (2) on general graphs. The simulations show that convergence to non-consensus equilibria is very common on non-structured graphs: two examples are shown in Figures 5 and 6.

We have constructed our numerical solutions, issuing from random initial conditions, by applying the explicit Euler method with a fixed step. Numerical solutions constructed in this way are approximations of certain Krasovskii solutions, which include attractive Krasovskii solutions sliding on the discontinuity set. However, we are unable to observe in the simulations any chattering phenomenon that would indicate the presence of solutions with this feature. It thus remains an open problem determining whether there exists any non-Carathéodory non-constant solutions with non-negligible basin of attraction. Instead, our numerical solutions cannot approximate any unstable Krasovskii solutions sliding on the discontinuity set. Nevertheless, we know that there exist some, because an example of those has been constructed analytically in Example 3.

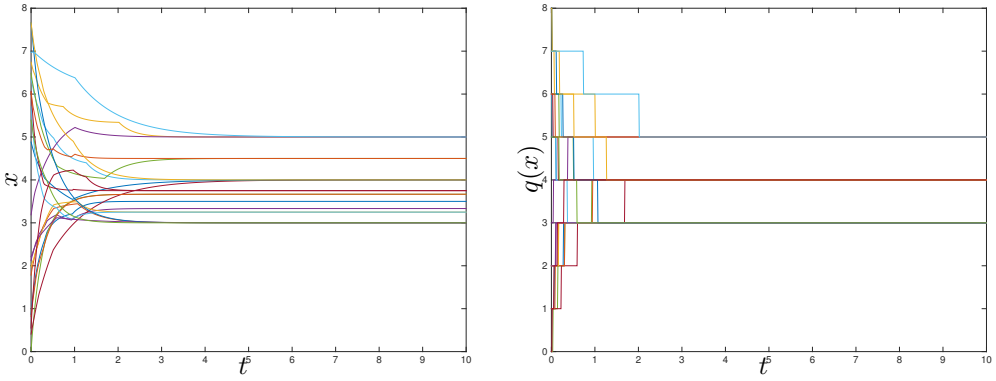

FIG. 5. *Evolution of x and $q(x(t))$ on a connected graph with $N = 20$ generated as an instance of random geometric graph with connectivity radius* 0.2. *The size of Euler method step is* 0.01.

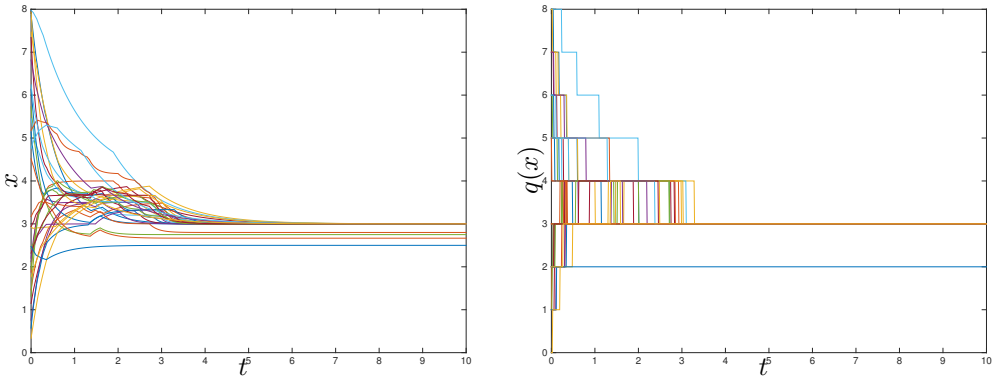

FIG. 6. *Evolution of x and $q(x(t))$ on a connected directed graph with $N = 20$, randomly generated by connecting each ordered pair of nodes with probability* 0.1. *The size of Euler method step is* 0.01.

7. Conclusion. In this article we have addressed the problem of opinions formation in a group where opinions are continuous variables that are communicated to other individuals by means of discrete variables, which may represent behaviours, choices, or verbal expressions. We have shown that in general we can not expect consensus and that unexpected equilibria may appear. We think that these features are relevant from the point of view of opinion dynamics, as a good model should be able to justify both agreement and disagreement. Our positive results consist in an asymptotic estimate of the distance from consensus depending on the communication graph and in the proof of consensus for some special cases, when the communication graph is either complete or complete bipartite. Two main questions remain open: proving (or disproving) that solutions converge to equilibria, and finding necessary and sufficient conditions for consensus.

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