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# ONE-SIDED CONSTRAINTS IN HYDRODYNAMICS

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## 1. Introduction: Systems of finite freedom

Statics of one-sided mechanical constraints may be the earliest occurrence of the concept of convexity in science: A heavy solid lies upon a fixed plane; in the elementary theory, the set of contact points between them is assumed to be finite, and the *convex hull* of these points is considered (i.e.: the smallest convex set containing these points). If the slope of the plane is less than the friction angle, a necessary and sufficient condition for equilibrium is that the vertical line drawn from the centre of gravity of the solid meets that convex hull.

Concerning *dynamics* of one-sided constraints, however, very little is given in most of the treatises on mechanics (see, however, Bouligand [1, 2] and Peres [3]). Let be  $S$  a mechanical system of finite freedom subject to a family of *one-sided constraints*, consisting of *contacts* between various solid parts of the system: these contacts may cease, while, on the contrary, the solid parts cannot overlap. Thus the kinematic effect of those constraints is expressed by inequalities involving the  $n$  generalized coordinates  $q_1, q_2, \dots, q_n$  (abbreviated in the symbol  $q$ , which denotes a "point of the configuration space") and, perhaps, the time  $t$ :

$$f_\alpha(q, t) \geq 0 \quad (\alpha \in I) \quad (1.1)$$

where  $I$  is a set of indices (here supposed to be finite). For sake of simplicity, let us suppose that the contacts are *frictionless*; that means, for every point of contact between two solid parts of  $S$ , *the reaction force is normal*. Then the description of such a reaction requires only a scalar quantity: the measure of the reaction vector along this known direction, with an orientation convention such that the measure is positive when the two solids *press* on each other. Let us denote by  $R_\alpha$  the scalar reaction associated in this way to the possible contact  $f_\alpha = 0$  in (1.1). The universally admitted laws for the one-sided reactions are

$$R_\alpha \geq 0 \quad (1.2)$$

and

$$R_\alpha f_\alpha(q, t) = 0 \quad (1.3)$$

(i.e.: as soon as a contact ceases, the corresponding reaction becomes zero).

The problem concerning such a mechanical system is put as follows: At an instant  $t_0$ , suppose given the configuration of  $S$ , i.e. the values  $q_i^0 = q_i(t_0)$  implying some contacts, namely

$$f_\alpha(q^0, t^0) = 0 \quad \text{for } \alpha \in K \subset I \quad (1.4)$$

and suppose also given the *velocity state*, i.e. the values of the derivatives  $\dot{q}_i^0 = \dot{q}_i(t_0)$  essentially *compatible* with the contacts (1.4), i.e.:

$$\left( \frac{df_\alpha}{dt} \right)_{t=t_0} = \frac{\partial f_\alpha}{\partial t} + \sum_i \frac{\partial f_\alpha}{\partial q_i} \dot{q}_i = 0 \quad \text{for } \alpha \in K. \quad (1.5)$$

To find the consequent motion, one usually proceeds by starting with the provisional hypothesis that *all the contacts in (1.4) hold during an interval*  $[t_0, t_0 + \varepsilon]$ . Thus the constraints  $f_\alpha = 0$ , for  $\alpha \in K$ , are provisionally considered as two-sided and the equations of motion written according to one of the classical methods (for instance, Lagrange's equations for the  $n$  parameters  $q_i$ , with multipliers associated to the constraints  $f_\alpha = 0$ ,  $\alpha \in K$ ); along the so described motion the values of the reactions  $R_\alpha$ ,  $\alpha \in K$  (or, equivalently, of the multipliers, if Lagrange's equations are used) are calculated: as long as the values found in that way for the  $R_\alpha$  are  $\geq 0$ , the preceding hypothesis of permanent contact is accepted and then the problem is solved. If, on the contrary, some of the calculated  $R_\alpha$  become  $< 0$  from an instant  $t_1$  onwards (may be  $t_1 = t_0$ ) the hypothesis is rejected: some contacts must cease at this instant. But, as Delassus [4] pointed out, *the contacts  $f_\alpha = 0$  which cease are not necessarily those for which the above calculated  $R_\alpha$  becomes negative.*

A very simple counter-example may be constructed without any calculation:

Two walls form an acute dihedron with horizontal edge, the slopes of the walls being such as indicated on fig. 1. A solid, homogeneous, heavy ball moves inside of the dihedron. At the initial instant  $t_0$ , contacts are assumed between the ball and both walls, without friction, and the velocity state is compatible with a possible permanence of those contacts, i.e.: the velocity vector of the center  $O$  of the ball is parallel to the edge of the dihedron or null. Under the provisional hypothesis of durable contacts, the motion of  $O$  is immediately found:  $O$  either performs a rectilinear uniform

motion parallel to the edge or, in the case of null initial velocity, remains at rest. Then the algebraic measures  $R_1$  and  $R_2$  of the reactions in the two contact points  $M_1$  and  $M_2$  between the ball and the walls are easily determined by decomposition of the weight vector  $\mathbf{P}$  of the ball along the two

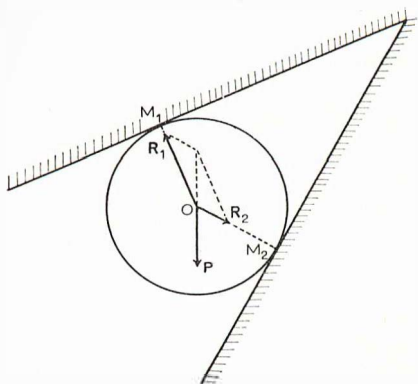


Fig. 1

radii  $OM_1$  and  $OM_2$ . Clearly, seeing the slopes of the walls in our case, *both*  $R_1$  and  $R_2$  are  $< 0$ , so that the hypothesis of durable contacts must be rejected. The actual motion is evident: the contact ceases between the ball and the upper wall, while it stays between it and the lower wall *although the precedingly calculated reaction was  $< 0$  for this latter contact*.

## 2. A generalized Gauss' principle

Delassus' arguments towards a correct systematic solution were rather intricate. As in any dynamical problem, an essential stage consists in the following question: given, at an instant, the *configuration*  $q$  and the *velocity state*  $\dot{q}$ , to find the *acceleration state*  $\ddot{q}$  (i.e. the values of the  $n$  second derivatives  $\ddot{q}_i$  at the considered instant). The author has proved in [5] that an unique solution for this problem always exists and is defined by joining the conditions (1.1), (1.2), (1.3) to dynamics equations written in a classical way (for instance, Lagrange's equations). This solution possesses a *variational characterisation*, generalizing, to systems with one-sided constraints, *Gauss' variational principle* (or "principle of the least deviation").

In the classical case of systems with only two-sided frictionless constraints there is a well known formulation of Gauss' principle: given the configuration and the velocity state of such a classical system  $\mathcal{S}$  at an instant  $t$ , the consequent acceleration state is, among all the acceleration states

compatible with these data and with the constraints, that one which minimizes Appell's function

$$\mathcal{A} = \frac{1}{2} \int_{\mathcal{E}} \Gamma^2 dm - \int_{\mathcal{E}} \Gamma \cdot dF \quad (1.6)$$

( $\Gamma$  denotes the acceleration of the generic element of  $\mathcal{E}$ ;  $dm$  is the mass measure defined on  $\mathcal{E}$ , while the vectorial measure  $dF$  represents the active forces experienced by  $\mathcal{E}$ ). For the system  $S$  considered above, with generalized coordinates  $q_i (i = 1, 2, \dots, n)$ , the generic expression of  $\mathcal{A}$  for an arbitrary motion defined by  $n$  functions  $q_i(t)$  – and disregarding the subsidiary constraints (1.1) – takes the form  $G(q, \dot{q}, \ddot{q}, t)$  and this function  $G$  is *quadratic* (non homogeneous) with respect to  $\ddot{q}$  (i.e. with respect to the set of the  $n$  variables  $\ddot{q}_i$ ).

On the other hand, the one-sided constraints (1.1), together with our hypothesis (1.4) and (1.5) (configuration and velocity state compatible with the contacts  $f_\alpha = 0$  for  $\alpha \in K \subset I$ ) leads to:

$$\left( \frac{d^2 f_\alpha}{dt^2} \right)_{t=t_0} = \sum_i \frac{\partial f_\alpha}{\partial q_i} \ddot{q}_i - s_\alpha \geq 0 \quad \text{for } \alpha \in K \quad (1.7)$$

(where  $s_\alpha$  is a known function of  $q, \dot{q}, t$ ). This set of inequalities defines, in the  $\mathbf{R}^n$  space of the variable  $\ddot{q}$ , a closed convex polyhedral region  $\mathcal{C}$ . Then our generalized Gauss' principle asserts that *the solution  $\ddot{q}$  of the above formulated problem is the minimizing point in  $\mathcal{C}$ , of the preceding function  $G$* . As  $G$  is a strictly convex quadratic function relatively to  $\ddot{q}$ , this minimizing point always exists and is unique. A derivation of this extremal property is given in [5] (an alternative proof may also be founded on the theory of multipliers of H. W. Kuhn and A. W. Tucker).

A duality theorem (Moreau [6]) also gives an extremal characterization of the *reactions* corresponding to the set of one-sided constraint.

Thus, determining the acceleration state or the values of the one-sided reactions reduces to classical *quadratic programming* problems.

### 3. Unilaterality of the incompressibility constraint

Our purpose, in the present contribution, is to develop a quite similar theory for one-sided constraints in a class of mechanical system with *infinite freedom*, namely *incompressible perfect* (i.e. *non viscous*) *fluids*. In fact, the incompressibility of a perfect fluid consists of (infinitely many) frictionless constraints and these constraints are *one-sided* in the sense that

*cavities*, or bubbles, may occur inside of the flow. Similarly the contact between the fluid and the boundaries (of the containing vessel or of the immersed bodies) is one-sided too: the fluid cannot flow across the boundaries but some gaps may occur in the interspace. Those effects constitute the *cavitation* phenomenon. At the same time, the *pressure*  $p$  of the fluid, which must be considered as the *reaction* associated to the incompressibility constraint, always satisfies the *inequality*  $p \geq 0$ , or, more realistically, the inequality  $p \geq p_0$ , where  $p_0$  is the *vaporization pressure*.

The problem of foreseeing whether the cavitation does appear, from a given non-cavitating state of the fluid, may be handled by the same logical plan as in the case of finite freedom systems sketched above: First, using standard hydrodynamical methods, the pressure  $p$  is calculated at every point of the fluid (together with the acceleration of the fluid) *under the provisional hypothesis of a non cavitating flow*. If that  $p$  is everywhere  $\geq p_0$  the hypothesis is accepted. If, on the contrary, this function takes values strictly less than  $p_0$  in some points, one concludes that cavitation appears; however, the “intuitive” assertion that *cavitation appears exactly at those points is not correct in generality*. In other words, the possibility of cavitation *globally* modifies the determination of the pressure function in the fluid domain.

Such a discrepancy between intuitive inference and a precise analysis of the hydrodynamical phenomenon was already pointed out by Riabouchinsky [7, 8, 9] and Demtchenko [10, 11]. These authors, working in bidimensional hydrodynamics, used the representation of the flow by complex variables and analytic functions; they did not always succeed in solving the problem without any ambiguity. Actually we are going to show in the following pages, that, taking account of the unilaterality of the incompressibility constraint, the acceleration field and the pressure field of the fluid are thoroughly determined. We will give two extremal characterizations (generalized Gauss’ principle and its dual) for these two fields.

In order to require of the reader nothing but a very superficial knowledge of hydrodynamics, we shall restrict ourselves to the *one-dimensional case of a flow in a narrow pipe*.

We hope that readers not concerned with such mechanical problems will, however, be interested in the presentation of an *infinite-dimensional quadratic programming device* which leads to very simple effective computation (e.g. by graphical treatment).

#### **4. The hydrodynamical problem for a narrow pipe of constant cross section**

Let us consider a perfect (i.e. non viscous) incompressible fluid in laminar



flow through a rigid pipe. In order to avoid some formal complications we restrict ourselves to the special case where the cross section  $S$  of the pipe is constant. The general case of a section  $S$  varying along the pipe may be reduced to the same computational device by changes of variable and of unknown functions. Moreover, we suppose that the cross dimensions of the pipe are small with respect to its length and to the radius of curvature of a mean curve representing the pipe. Then, the approximation of *unidimensional hydrodynamics* holds: the *velocity*  $v$  of the fluid, as well as its *pressure*  $p$ , depend on the time  $t$  and on only one space variable  $x$ , the *curvilinear abscissa* along the pipe (or, more precisely, along the mean curve of it).

The fluid undergoes some continuously distributed exterior forces (usually: gravity forces) at a vectorial rate  $\mathbf{F}$  per unit of volume: the vector  $\mathbf{F}$  may be a function of the variable  $x$  and we denote by  $F(x)$  its component along the oriented tangent of the mean curve. Let

$$U(x) = \int F(x) dx. \quad (4.1)$$

In the usual case of gravity forces, calling  $h(x)$  the height of the point  $x$  of the mean curve above a horizontal reference plane, we have

$$U(x) = -\rho gh(x) \quad (4.2)$$

( $\rho$ : constant fluid density,  $g$ : gravity intensity).

The *acceleration* of the fluid at the point  $x$  is

$$\gamma = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \quad (4.3)$$

and we write the equation of the *unidimensional hydrodynamics*

$$\rho \gamma = - \frac{\partial p}{\partial x} + F. \quad (4.4)$$

The problem under consideration refers to a *fluid column* with the following data.

*Initial configuration:* At the initial instant 0, the fluid column fills, without cavitation, the part of the pipe lying between the curvilinear abscissae  $x = a$  and  $x = b > a$ .

*Initial velocities:* The initial distribution of velocities is assumed to be compatible with non-cavitation; that means the conservation of the flow along the pipe, therefore

$$v(x, 0) = v_0 \quad (4.5)$$

(given, independent of  $x$ ).

*Boundary conditions:* At one of the two extremities of the fluid column, namely for  $x = a$ , the pressure takes a given value

$$p(a) = p_a \quad (4.6)$$

while the other extremity  $x = b$  is *confined* by a “piston” moving with the exact velocity  $v_0$  given in (4.5) (as the initial state of velocity is assumed to be compatible with non-cavitation) and with a given *acceleration*

$$\frac{d^2x}{dt^2} = \Gamma. \quad (4.7)$$

Evidently, some other combinations of boundary conditions could have been imposed.

## 5. Non-cavitation assumption

If it is *a priori* assumed that cavitation does not occur during an interval of time  $(0, \epsilon)$  then the conservation of the flow along the pipe implies for  $\gamma$  a value independant of  $x$ , necessarily equal to the value  $\Gamma$  given in (4.7). Then (4.4) is integrated as

$$-\rho \Gamma x - p(x) + U(x) = C \text{ (constant)} \quad (5.1)$$

and the constant  $C$  is determined by setting  $x = a$  and using (4.6).

If, for every  $x \in [a, b]$  the function  $p(x)$  defined by (5.1) is greater than or equal to the *vaporization pressure*  $p_0$ , the hypothesis of a non-cavitating flow is accepted. If, on the contrary, the preceding expression of  $p(x)$  takes values strictly less than  $p_0$  on a region of the fluid column, one must draw the inference of *inceptive cavitation*. Then the computation must be carried out in a quite different way.

## 6. The problem for cavitating flows

Now we take into account the possibility of cavitation. Let us consider the part of the fluid lying, at the instant 0, between the curvilinear abscissae  $x_1$  and  $x_2 > x_1$  of the pipe. Since the fluid is incompressible and, at this instant, without cavitation, the volume of pipe occupied by this fluid part cannot decrease afterwards. The time derivative of this volume, namely

$$[v(x_2) - v(x_1)]S$$

is zero by (4.5) so that the second derivative, namely

$$[\gamma(x_2) - \gamma(x_1)]S$$



is non-negative. That means that  $\gamma(x)$  is a *non-decreasing function of  $x$*  along the pipe.

Every region where  $\partial\gamma/\partial x > 0$  is a region of inception cavitation: there the pressure  $p$  equals the vaporization pressure  $p_0$ , so that, for every  $x$

$$(p - p_0) \frac{\partial\gamma}{\partial x} = 0. \quad (6.1)$$

By putting

$$p - p_0 = q(x) \quad (6.2)$$

(note that  $p_0$  may be a function of  $x$  if, for instance, the temperature is not the same at every point of the fluid column) that gives, for every  $x$

$$q \frac{\partial\gamma}{\partial x} = 0 \quad (6.3)$$

with

$$q \geq 0 \quad (6.4)$$

$$\frac{\partial\gamma}{\partial x} \geq 0. \quad (6.5)$$

Besides, the hydrodynamical equation (4.4) holds for every  $x$  and leads to

$$\rho\gamma = - \frac{\partial q}{\partial x} + F - \frac{\partial p_0}{\partial x}. \quad (6.6)$$

Finally we have to write the boundary condition (4.6) under the form

$$q(a) = p_a - p_0(a) = q_a \text{ (known positive quantity)} \quad (6.7)$$

and a boundary condition referring to the "moving piston": the fluid cannot pass through the piston, but a gap may occur between them. Since the initial configuration of the system, as well as the initial velocity distribution, are assumed to be non-cavitationnal (i.e.: there is no initial gap and the piston has the same velocity  $v_0$  as the fluid), the equation (4.7) yields

$$\Gamma - \gamma(b) \geq 0. \quad (6.8)$$

If this expression is strictly positive, there is an inception cavitation on the surface of the piston, and in that case  $p - p_0 = 0$  for  $x = b$ . So we have

$$q(b)[\Gamma - \gamma(b)] = 0. \quad (6.9)$$

We are going to show that the conditions (6.3) to (6.9) determine one and only one solution for the functions  $p(x)$  (or  $q$ ) and  $\gamma(x)$ .

Let us first consider the new unknown function

$$W(x) = U(x) - p(x) = U(x) - p_0(x) - q(x). \quad (6.10)$$

By (6.6), the conditions (6.3), (6.4) and (6.5) respectively become, for every  $x \in [a, b]$ ,

$$(U - p_0 - W) \frac{\partial^2 W}{\partial x^2} = 0, \quad (6.11)$$

$$W \leq U - p_0 \quad (6.12)$$

$$\frac{\partial^2 W}{\partial x^2} \geq 0. \quad (6.13)$$

The boundary condition (6.7) becomes

$$W(a) = U(a) - p_a \quad (6.14)$$

while (6.8) and (6.9) lead to

$$\left. \begin{aligned} \frac{\partial W}{\partial x} &\leq \rho \Gamma \\ (U - p_0 - W) \left\{ \rho \Gamma - \frac{\partial W}{\partial x} \right\} &= 0, \end{aligned} \right\} \text{ for } x = b. \quad (6.15)$$

$$(6.16)$$

## 7. Graphical solution

Let us draw the graph [G] of the known function  $U(x) - p_0(x)$ . By (6.12) the graph [W] of the unknown function  $W(x)$  lies everywhere below the curve [G]; by (6.13) the function  $W(x)$  is *convex* and (6.11) shows that the graph [W] is rectilinear in any part where it differs from the known curve [G]. Furthermore, (6.14) defines the starting point A (for  $x = a$ ) of the curve [W]. The behaviour of this curve at its second extremity B (for  $x = b$ ) is described as follows: by (6.15), the slope of [W] at this point does not exceed the known value  $\rho \Gamma$ ; and, by (6.16), this slope can differ from  $\rho \Gamma$  only if the curve [W] has the same extremity as [G] for  $x = b$ .

Thus, the curve [W] consists of some arcs of the known curve [G] and of some straight lines tangent to this curve, which are easy to trace with a great precision (figs. 2 and 3). It may be said that [W] *has the shape of a thread going round [G] from below and stretched between the point A and a point infinitely distant to the right, in the direction of slope  $\rho \Gamma$* .

The values of  $x$  for which the curve [W] differs from the curve [G] lead to strictly positive values for  $q = p - p_0$  and to nullity of  $\partial \gamma / \partial x$ : cavitation

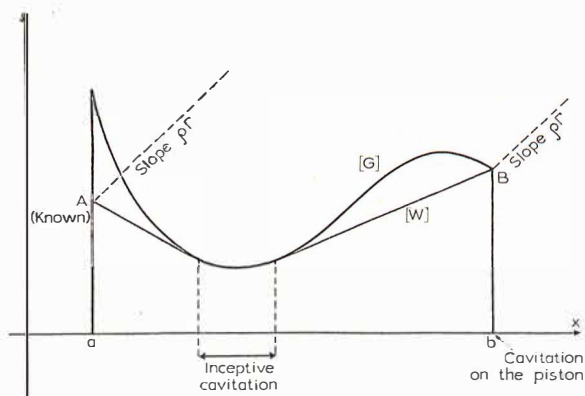


Fig. 2

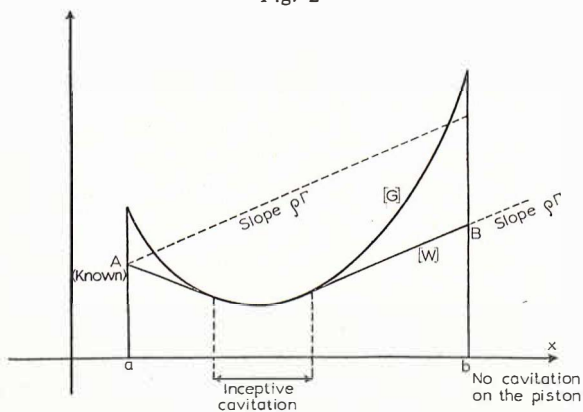


Fig. 3

does not appear on the corresponding parts of the pipe. On the contrary, there is certainly an inceptive cavitation at the points where  $\partial\gamma/\partial x > 0$ . Similarly, if the slope of  $[W]$  at the point B is strictly less than  $\rho\Gamma$ , then a gap appears between the fluid and the piston.

When, as in fig. 4, the given pressure  $p_a$  is large enough and the given acceleration  $\Gamma$  not too large, the straight line drawn from A with the slope  $\rho\Gamma$  lies everywhere below the curve  $[G]$ . In that case this straight line represents the solution  $W(x)$ , and corresponds to the calculation of section 5: Cavitation does not appear.

When, on the contrary, the preceding straight line cuts the curve  $[G]$ , the values of  $x$  for which this straight line lies below  $[G]$  are precisely those ones for which the calculation of section 5 would have given positive values

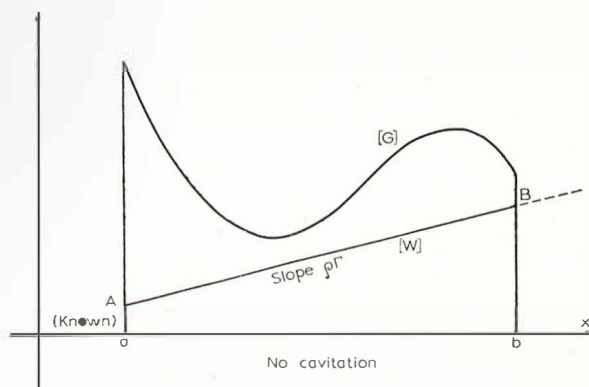


Fig. 4

of  $q = p - p_0$ . Clearly, from figs. 2 or 3, these intervals for  $x$  are strictly smaller than the intervals corresponding to the straight parts of the actual curve  $[W]$ . That means that, in some points where the calculation of section 5, would have given  $p - p_0 < 0$ , cavitation actually does not appear. Such a discrepancy between intuitive inference and the actual behaviour of the fluid is exactly analogous to the case presented in section 1, for systems of finite freedom.

## 8. Extremal principle for the acceleration field

The graphical solution of section 7 already implies an extremal characterization of the solution: *clearly the function  $W$  is the largest convex function less than the given function  $U - p_0$  and satisfying the boundary conditions (6.14), (6.15), (6.16).*

In two- or three-dimensional hydrodynamics a similar property holds, but is far less easy to deal with, since convexity is then replaced by *subharmonicity*. Hence, it is useful to have some other extremal characterizations. We first state once more a generalization of Gauss' extremal principle, concerning the *acceleration field*  $\gamma$ .

For brevity, let us put

$$F - \frac{\partial p_0}{\partial x} = f(x) \quad (8.1)$$

so that (6.6) is rewritten as

$$\rho\gamma + \frac{\partial q}{\partial x} = f \quad (a \leq x \leq b). \quad (8.2)$$

Let us denote by  $\gamma_1(x)$ ,  $q_1(x)$  the solution of our problem, that is functions  $\gamma$  and  $q$  fulfilling the conditions (6.3) to (6.9). We state:

*$\mathcal{A}$ -PRINCIPLE: Among all the acceleration fields  $\gamma(x)$  satisfying the kinematic conditions (6.5) and (6.8), the solution  $\gamma_1$  strictly minimizes the functional (generalized Appell's function):*

$$\mathcal{A}(\gamma) = \frac{1}{2} \int_a^b \rho \gamma^2 dx - \int_a^b f \gamma dx - \gamma(a) q_a. \quad (8.3)$$

*Proof:* Owing to the fact that  $\gamma_1$  and  $q_1$  fulfil (8.2) and (6.7),  $\mathcal{A}(\gamma)$  becomes

$$\mathcal{A}(\gamma) = \frac{1}{2} \int_a^b \rho \gamma^2 dx - \int_a^b \left( \rho \gamma_1 + \frac{\partial q_1}{\partial x} \right) \gamma dx - \gamma(a) q_1(a)$$

or, integrating by parts,

$$\mathcal{A}(\gamma) = \frac{1}{2} \int_a^b \rho (\gamma^2 - 2\gamma \gamma_1) dx + \int_a^b q_1 \frac{\partial \gamma}{\partial x} dx - \gamma(b) q_1(b).$$

That also holds for  $\gamma = \gamma_1$ , so that

$$\mathcal{A}(\gamma_1) = -\frac{1}{2} \int_a^b \rho \gamma_1^2 dx + \int_a^b q_1 \frac{\partial \gamma_1}{\partial x} dx - \gamma_1(b) q_1(b)$$

or, since  $\gamma_1$  and  $q_1$  fulfil (6.3) and (6.9),

$$\mathcal{A}(\gamma_1) = -\frac{1}{2} \int_a^b \rho \gamma_1^2 dx - \Gamma q_1(b).$$

Subtracting, we find

$$\mathcal{A}(\gamma) - \mathcal{A}(\gamma_1) = \frac{1}{2} \int_a^b \rho (\gamma - \gamma_1)^2 dx + \int_a^b q_1 \frac{\partial \gamma}{\partial x} dx + [\Gamma - \gamma(b)] q_1(b).$$

Here, the first integral is non-negative; the second and third terms are also non-negative by virtue of (6.4), (6.5) and (6.8). That proves the minimizing property of  $\gamma_1$ ; and the minimization is *strict* since the nullity of  $\mathcal{A}(\gamma) - \mathcal{A}(\gamma_1)$  implies nullity of all the three non-negative terms in the right member, in particular

$$\frac{1}{2} \int_a^b \rho (\gamma - \gamma_1)^2 dx = 0.$$

By continuity of the non-negative function  $(\gamma - \gamma_1)^2$  we have  $\gamma = \gamma_1$ .

Let us now remark that, in the function space  $\mathcal{C}_1[a, b]$  of once differentiable functions on the interval  $[a, b]$  the set of the  $\gamma$  fulfilling (6.5) and (6.8) is a *convex set*  $C$ . On the other hand, the functional  $\mathcal{B}$  is a *quadratic numerical function* defined on the linear space  $\mathcal{C}_1[a, b]$ . So the problem of determining  $\gamma_1$ , graphically solved in section 7, turns out to be an *infinite dimensional quadratic programming problem*.

## 9. Extremal principle for the pressure field

With the same notations as in the preceding section, we state:

**$\mathcal{B}$ -PRINCIPLE:** *Among all the differentiable functions  $q(x)$  satisfying the sthenic conditions (i.e. condition referring to forces) (6.4) and (6.7), the solution  $q_1$  strictly minimizes the functional*

$$\mathcal{B}(q) = \frac{1}{2} \int_a^b \left( \frac{\partial q}{\partial x} \right)^2 dx - \int_a^b f \frac{\partial q}{\partial x} dx + \rho \Gamma q(b). \quad (9.1)$$

*Proof:* Owing to the fact that  $\gamma_1$  and  $q_1$  fulfil (8.2),  $\mathcal{B}(q)$  turns into

$$\mathcal{B}(q) = \frac{1}{2} \int_a^b \left( \frac{\partial q}{\partial x} \right)^2 dx - \int_a^b \left( \rho \gamma_1 + \frac{\partial q_1}{\partial x} \right) \frac{\partial q}{\partial x} dx + \rho \Gamma q(b)$$

or, integrating by parts,

$$\begin{aligned} \mathcal{B}(q) = \frac{1}{2} \int_a^b \left[ \left( \frac{\partial q}{\partial x} \right)^2 - 2 \frac{\partial q_1}{\partial x} \frac{\partial q}{\partial x} \right] dx + \int_a^b \rho q \frac{\partial \gamma_1}{\partial x} dx + \\ + \rho [\Gamma - \gamma_1(b)] q(b) + \rho q_a \gamma_1(a). \end{aligned}$$

That also holds for  $q = q_1$ , so that

$$\mathcal{B}(q_1) = -\frac{1}{2} \int_a^b \left( \frac{\partial q_1}{\partial x} \right)^2 dx + \int_a^b \rho q_1 \frac{\partial \gamma_1}{\partial x} dx + \rho [\Gamma - \gamma_1(b)] q_1(b) + \rho q_a \gamma_1(a).$$

The second and the third terms are zero, since  $q_1$  fulfils (6.3) and (6.9); thus, subtracting, we have

$$\mathcal{B}(q) - \mathcal{B}(q_1) = \frac{1}{2} \int_a^b \left( \frac{\partial q}{\partial x} - \frac{\partial q_1}{\partial x} \right)^2 dx + \int_a^b \rho q \frac{\partial \gamma_1}{\partial x} dx + \rho [\Gamma - \gamma_1(b)] q(b).$$

Here, the first integral is non-negative; so is the second one by virtue of (6.4) for  $q$  and (6.5) for  $\gamma_1$ ; so is the last term by virtue of (6.8) for  $\gamma_1$ , and (6.4) for  $q$ . That proves the minimizing property of  $q_1$ ; and the minimization is *strict* since the nullity of  $\mathcal{B}(q) - \mathcal{B}(q_1)$  implies nullity of all the three



non-negative terms in the right member, in particular of

$$\frac{1}{2} \int_a^b \left( \frac{\partial q}{\partial x} - \frac{\partial q_1}{\partial x} \right)^2 dx.$$

By continuity of the integrand, that leads to

$$\frac{\partial}{\partial x} (q - q_1) = 0 \quad \text{for every } x \in [a, b]$$

and thus to  $q = q_1$  since  $q(a) = q_1(a) = q_a$ .

In the same way as in the preceding section, determining  $q_1$  may be regarded as an *infinite dimensional quadratic programming problem*, for the set  $D$  of functions  $q$  fulfilling (6.4) and (6.7) is *convex* in some linear space of functions, while the functional  $\mathcal{B}$  is *quadratic*. This will be made precise in the next section.

## 10. Geometrical formulation

In order to deal only with elements of the function space  $E = \mathcal{C}[a, b]$  of numerical functions which are continuous on  $[a, b]$  we now introduce some new notations.

We put:

$$\rho\gamma = X \in \mathcal{C}[a, b] \quad (10.1)$$

$$\frac{\partial q}{\partial x} = Y \in \mathcal{C}[a, b]. \quad (10.2)$$

The sthenic conditions (6.4) and (6.7) for  $q$  are rewritten as one condition for  $Y$

$$q_a + \int_a^x Y(\xi) d\xi \geq 0 \quad \text{for every } x \in [a, b] \quad (10.3)$$

which defines a *convex subset*  $Q$  of  $\mathcal{C}[a, b]$ .

The kinematic conditions (6.5) and (6.8) for  $\gamma$  become

$$\left. \begin{array}{l} X \text{ differentiable} \\ \frac{\partial X}{\partial x} \geq 0 \end{array} \right\} \quad (10.4)$$

$$X(b) \leq \rho\Gamma \quad (10.5)$$

which also defines a *convex subset*  $P$  of  $\mathcal{C}[a, b]$ .

We equip the linear space  $\mathcal{C}[a, b]$  with a euclidean (or “prehilbertian”) *metric* by associating to any pair  $Z_1, Z_2$  of elements of  $\mathcal{C}[a, b]$  the *scalar product*

$$(Z_1|Z_2) = \int_a^b \frac{1}{\rho} Z_1(x) Z_2(x) dx$$

which gives rise to the *norm*

$$\|Z\| = \sqrt{(Z|Z)}.$$

Using these notations and the relation (10.1), the functional  $\mathcal{A}$  may be expressed as

$$\mathcal{A}(\gamma) = \frac{1}{2} \|X\|^2 - (f|X) - \frac{1}{\rho} q_a X(a). \quad (10.6)$$

Similarly, using (10.2), if  $q$  fulfils (6.7), we have

$$q(b) = q_a + \int_a^b Y(x) dx$$

so that

$$\begin{aligned} \frac{1}{\rho} \mathcal{B}(q) &= \frac{1}{2} \|Y\|^2 - (f|Y) + \Gamma \left[ q_a + \int_a^b Y(x) dx \right] \\ &= \frac{1}{2} \|Y\|^2 - (f - \rho \Gamma|Y) + \Gamma q_a \\ &= \frac{1}{2} \|Y - (f - \rho \Gamma)\|^2 + \frac{1}{2} \|f - \rho \Gamma\|^2 + \Gamma q_a. \end{aligned} \quad (10.7)$$

Then, the  $\mathcal{B}$ -principle of section 9 takes an interesting form:

*The solution  $Y_1 = \partial q_1 / \partial x$  of the problem is the nearest point in the convex set  $Q$  defined by (10.3) from the known point  $f - \rho \Gamma$ . By the way it must be remarked that this known element  $f - \rho \Gamma$  is the very value which would be found for  $Y = \partial q / \partial x$  under the non-cavitation assumption of section 5.*

The existence of this nearest point, for every  $f \in \mathcal{C}[a, b]$  results from the “graphical” solution explained in section 7. The fact that  $Q$  is a *closed convex set* in  $\mathcal{C}[a, b] = E$  (relatively to the topology of the euclidean norm  $\| \cdot \|$ ) is not sufficient to ensure this existence, for  $E$  is not *complete* (it is not a Hilbert space).

The expression (10.6) of the functional  $\mathcal{A}$  does not lead to so simple an interpretation concerning the  $\mathcal{A}$ -principle. Indeed, the last term in (10.6) is linear with regard to  $X$ , as is the last term of (10.7) with regard to  $Y$ , but  $-X(a)$  is not continuous for the topology of the norm  $\| \cdot \|$  (it is only lower semi-continuous on the set  $P$ ) so that it cannot be expressed as a scalar product of  $X$  by a fixed element of  $E$ .

## 11. Duality between the two principles

Extremal characterizations of  $X_1 = \rho\gamma_1$  and  $Y_1 = \partial q_1/\partial x$  in  $E$  may be enunciated using the "prox" formalism which we have introduced in former publications, in an abstract setting (cf. Moreau [6, 12]). Let be  $\varphi$  a function everywhere defined on the space  $E$  taking values in  $]-\infty, +\infty]$  (possibly the value  $+\infty$ ) and *convex*. Let be  $z$  an element of  $E$ . The numerical function

$$u \rightarrow \frac{1}{2}\|u-z\|^2 + \varphi(u) \quad (11.1)$$

attains its infimum in at most one point; we denote by  $\text{prox}_\varphi z$  this point, if it exists (in the case where  $E$  is a Hilbert space, the usual sufficient condition of existence for  $\text{prox}_\varphi z$  is lower semi-continuity of the convex numerical function  $\varphi$ ).

Here, we define  $\varphi$  by

$$\varphi(u) = \begin{cases} +\infty & \text{if } u \notin P \\ -\frac{1}{\rho} q_a u(a) & \text{if } u \in P \end{cases} \quad (11.2)$$

and we define another convex function  $\psi$  by

$$\psi(u) = \begin{cases} +\infty & \text{if } u \notin Q \\ (\rho\Gamma|u) + \Gamma q_a & \text{if } u \in Q. \end{cases} \quad (11.3)$$

Then the  $\mathcal{A}$ -principle characterizes the solution  $X_1$  as

$$X_1 = \text{prox}_\varphi f \quad (11.4)$$

and the  $\mathcal{B}$ -principle characterizes  $Y_1$  as

$$Y_1 = \text{prox}_\psi f. \quad (11.5)$$

Actually, these two principles turn out to be equivalent by means of a *duality theory* developed, in abstract Hilbert space context, in [12].

We first state:

*The two functions  $\varphi$  and  $\psi$  are superdual, i.e.: for every  $X$  and  $Y$  in  $E$  the following inequality holds:*

$$\varphi(X) + \psi(Y) - (X|Y) \geq 0. \quad (11.6)$$

*Proof:* If  $X \notin P$  or  $Y \notin Q$  the inequality is trivial since the left member takes the value  $+\infty$ . Let us now suppose  $X \in P$  and  $Y \in Q$ ; the left member is written, with the  $q$  function associated to  $Y$  by (10.2), as

$$\begin{aligned}
-\frac{1}{\rho} q_a X(a) + \Gamma q(b) - \int_a^b \frac{1}{\rho} X(x) \dot{Y}(x) dx &= \\
&= -\frac{1}{\rho} q_a X(a) + \Gamma q(b) - \frac{1}{\rho} [qX]_a^b + \frac{1}{\rho} \int_a^b q \frac{\partial X}{\partial x} dx \\
&= \frac{1}{\rho} [\rho \Gamma - X(b)] q(b) + \frac{1}{\rho} \int_a^b q \frac{\partial X}{\partial x} dx
\end{aligned}$$

which is  $\geq 0$  by virtue of (6.4), (6.7), (10.4), (10.5).

The two points  $X \in E$  and  $Y \in E$  are said to be conjugate relative to the pair of superdual function  $\varphi, \psi$  if equality holds in (11.6).

From the above expression for the left member of (11.6) it results that this equality is *equivalent* to the joint conditions (6.3), (6.4), (6.7), (6.9), (10.4), (10.5). By rewriting (6.6) in terms of  $X, Y, f$ , our fundamental problem turns into the following one:

To find  $X$  and  $Y$  in  $E$  such that

$$X + Y = f \quad (11.7)$$

$$\varphi(X) + \psi(Y) - (X|Y) = 0. \quad (11.8)$$

Arguments from [12] prove that these two conditions are equivalent to (11.4) and (11.5). Actually, this *duality-decomposition* theory attains its full efficacy only after *completing*  $E$  into the Hilbert space  $L^2[a, b]$  and extending  $\varphi$  and  $\psi$  as a pair of *dual functions* on this Hilbert space. Such a completion is a routine procedure in functional analysis but we think it is beyond the scope of the present book. Hilbert space techniques become a useful tool when dealing with the more difficult problems of two- or three-dimensional hydrodynamics.

Here we have a nonlinear generalization of the classical idea of decomposition of a Hilbert space into the sum of two orthogonal complementary subspaces. As a matter of fact, the case of "bilateral" incompressibility studied in section 5 may be reduced to such linear decomposition into complementary subspaces.

In the finite freedom case sketched in section 2, the duality-decomposition theory may be carried out without topological difficulties, the corresponding space  $E$  being finite-dimensional.

## 12. Conclusion

Here, we have presented the simplest and most schematical case of calculating inception cavitation in hydrodynamics. Our aim was to develop

a non-classical example of an infinite dimensional quadratic programming device. At the same time, we have given a physical introduction to our theory of dual functions in Hilbert spaces, published elsewhere ([12], where references can also be found concerning the theory of dual convex functions on more general topological linear spaces).

From a mechanical standpoint, we must prevent any misuse of the present theory by insisting on the various hypotheses used. First, *viscosity* and *turbulence* are neglected: that will be fairly often justifiable; in particular, at least, when the initial state of the fluid is rest ( $v_0 = 0$ ). On the other hand, in the unidimensional frame where we have worked, we disregarded whether the inceptive cavitation was *diffuse in the fluid* (growing bubbles) or whether it consisted in *getting loose from the pipe boundary*; in the last case the longitudinal dilatation of the fluid  $\partial\gamma/\partial x > 0$  results from narrowing down the fluid vein. In both cases it seems that *capillary action* on the interfaces liquid-vapor or liquid-boundary may play some part as well as the statistical distribution of vaporization germs. Those elements may only be neglected in fairly large scale phenomena. There is another sort of limitation to consider: in studying the transient phenomena which take place in the large penstocks of hydroelectric power stations, one cannot generally neglect water compressibility nor elasticity of the pipe boundaries (cf. Escande [13, 14]).

It must be noted, anyway, that Riabouchinsky [7, 8, 9], using quite different mathematical methods, for bidimensional cases, but starting with the same physical hypothesis, has obtained a good experimental confirmation.

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