Quadratic Exponential Semimartingales and Application to BSDEs with jumps

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First version, May 4, 2014

Abstract

In this paper, we study a class of Quadratic Backward Stochastic Differential Equations (QBSDE in short) with jumps and unbounded terminal condition. For this propose, we introduce a new class of Quadratic semimartingale. The properties arising from this class lead us to prove existence result for the solution of a Quadratic BSDE's.

Keywords: Backward stochastic differential equation, quadratic semimartingales, unbounded terminal condition.

1 Introduction

Backward stochastic differential equations (in short BSDE's) were first introduced by Bismut in 1973 [9] as an equation for the adjoint process in the stochastic version of Pontryagin maximum principle. Pardoux and Peng [47] have generalized the existence and uniqueness result in the case when the driver is Lipschitz continuous. Later in [38] Lepeltier and San Martin exended the result of Peng-Pardoux to the linear growth case. Since then BSDE's have been widely used in stochastic control and especially in mathematical finance, as any pricing problem by replication can be written in terms of linear BSDEs, or non-linear BSDEs when portfolios constraints are taken into account as in El Karoui, Peng and Quenez [22].

When the BSDE is driven jointly by a Brownian Motion and a Poisson jump measure, Barles, Buckdahn and Pardoux [2] introduce the Lipschitz BSDE in order to give a probabilistic interpretation of viscosity solution of semilinear integral-Partial equations.

Afterwards, Becherer [7] studied bounded solution of BSDEs when the generator is lipschitz and satisfies the condition A_γ for the jump component. This condition was introduced by [**?**] to insure the comparison Theorem of the solution of the BSDE.

*Research supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon.

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Recently, Becherer Buttner and Kentia [**?**] provide a general result of existence and uniqueness for bounded solution of BSDEs that are jointly driven by a Brownian motion and a infinite activity random measure and time-inhomogeneous with non-deterministic compensator. The generator of the BSDE is lipschitz in (y, z) and doesn't need to satisfy classical global Lipschitz conditions in the jump integrand.
Another direction which has attracted many works in this area, especially in connection with applications: How to improve the existence/uniqueness of a solution under weaker conditions on the driver and also on the terminal condition.

Overview on the quadratic BSDE problem: In the case of the filtration generated by a Brownian motion, the first and important result for the Quadratic BSDEs with bounded terminal condition was obtained by Kobylanski [35]. Based on analytical point of view inspired from Boccardo, Murat and Puel [11]: From an exponential change of variable, troncation procedure and comparison Theorem, she proved a general existence of the solution by an approximation technique. The uniqueness result is also given by adding more stronger conditions on the coefficient. However, the main difficulty in this method is the strong convergence of the martingale part which is not easy to prove.
In [46], Morlais extended the method of Kobylanski in the context of exponential utility maximization problem when the model involves jumps. As a consequence, they obtain that the state process Y and the jump components U of the BSDE solution are uniformly bounded, and that the martingale component is a BMO-martingale. All those works assume that the terminal condition is bounded and they are based on the so-called Kobylanski method exept the work of Briand and hu [12]. They proved, in the continuous setting the existence of solution of Quadratic BSDEs when the terminal condition have finite exponential moment. Under strong assumptions, on the coefficient they get the uniqeness of the solution.
More recently Tevzadze [44] proposed a new different method to get the existence and uniquness of the solution of quadratic BSDE's. The method is based on a fixed point theorem but for only bounded terminal condition with small L^∞-norm.
This class of BSDE's is very useful in mathematical finance especially when we deal with exponential utilities or risk measure theory especially weather derivatives (see e.g. El Karoui and Rouge [23], Mania and Schweizer [43], Hu, Imkeller and Müller [34], Barrieu and El Karoui [5] and Becherer ([6], [7])). Actually it has been shown in [23] that in a market model with constraints on the portfolios, the indifference price is given by the resolution of a BSDE with quadratic growth coefficient. Finally let us point out that control risk-sensitive problems turn into BSDE's which fall in the same framework in El Karoui and Hamadène [24].

Main countributions: Our work was also motivated by solving a utility maximization problem of terminal wealth with exponential utility function in models involving assets with jumps. Therefore we need to consider Backward Differential Equations with jumps of the form

$$Y_t = \eta_T + \int_t^T f_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U(s,x).\widetilde{\mu}(ds, dx) \qquad (1.1)$$

where $\widetilde{\mu}$ is a martingale random measure. A solution of such BSDE associated with (f, η_T) is a triple of square integrable processes $(Y_t, Z_t, U_t)_{0\leq t\leq T}$. Our main task in this paper is to deal with quadratic BSDE's with non-bounded terminal valued and jumps. Our point of view is inspired from Barrieu and El Karoui [4] for their study in the continuous case. By adopting a forward point of view, se shall characterize first a solution of BSDE's as a quadratic Itô semimartingale Y, with a decomposition satisfying the *quadratic exponential structure condition* $q_{exp}(l, c, \delta)$, where the term *exponential* refers to the exponential feature of the jump coefficient which appears in the generator of the BSDE. More precisely, we assume that: there exists nonnegative processes constants c , δ and l such

$$-l_t - c_t|y| - \frac{1}{2}\delta|z|^2 - \frac{1}{\delta}j_t(-\delta u) \leq f(t, y, z, u) \leq l_t + c_t|y| + \frac{1}{2}\delta|z|^2 + \frac{1}{\delta}j_t(\delta u),\ a.s. \quad (1.2)$$

where $j_t(u) = \int_E (e^{u(x)} - u(x) - 1)\xi(t, x)\lambda(dx)$. The canonical structure $q_{exp}(0, 0, \delta)$ will play a essential role in the construction of the solution associated to generale $q_{exp}(l, c, \delta)$ structure condition. The simplest generator of a quadratic exponential BSDE, called the *canonical generator*, is defined as $f(t, y, z, u) = q_\delta(z, u) = \frac{\delta}{2}|z|^2 + \frac{1}{\delta}j(\delta u)$. For a given random variable ψ_T, we call *entropic process*, the process defined as $\rho_{\delta,t}(\eta_T) = \frac{1}{\delta}\ln \mathbb{E}\left[\exp(\delta\eta_T)\middle|\mathcal{F}_t\right]$ which is a solution of the canonical BSDE's associated to the coefficient q_δ and final condition ψ_T. This is a entropic dynamic risk measure which have been studied, by Barrieu and El Karoui in [5].

The backward point of view of our approach permits to relate the quadratic BSDEs to a quadratic exponential semimartingale with structure condition $q_{exp}(l, a, \delta)$, using the entropic processes. Namely, a semimartingale X with non bounded terminal condition η_T and satisfying the structure condition $q_{exp}(l, a, \delta)$, yields the following dominated inequalities $\rho_{-\delta,t}(\underline{U}_T) \leq Y_t \leq \rho_{\delta,t}(\overline{U}_T)$, where $\underline{U}_T$ and $\overline{U}_T$ are two random variable depending only on l, a, δ and η_T. The main goal in our approach is then to deduce, from this dominated inequalities, a structure properties on the martingale part and the finite variation part of X. Indeed, we obtain the canonical decomposition of an entropic quasimartingale which is a semimartingale which satisfies the entropy inequalities; as a canonical quadratic semimartingale part plus an predictable increasing process. This Doob type decomposition help us to define a general quadratic exponential semimartingale as a limit of a sequence of canonical quadratic semimartingale plus a sequence of an increasing process. Then, from the stability theorem for forward semimartingales given by Barlow and Protter [3], we prove the existence of the solution of a quadratic exponential BSDE associated to (f, η_T) for a coefficient f satisfying the structure condition $q_{exp}(l, a, \delta)$ and for non-bounded terminal condition η_T.

Finally, we have to mention that it is important to compare our approach with that used by Peng in [48, 50, 51] within the representation theorem of small g-expectation in terms of a BSDE's with coefficient g which admits a linear growth condition in z. Peng's approach is based on the notion of martingale associated to a nonlinear expectation, Monotonic limit theorem, a nonlinear Doob-Meyer's decomposition Theorem (see e.g. [49]). Moreover, Peng obtained the representation theorem for the nonlinear expectation which is dominated by a structure nonlinear expectation solution of BSDE's with coefficient given specially by $g_\mu(y, z) = \mu\big(|y| + |z|\big)$. Barrieu and El Karoui in [5] have extended this representation

theorem for a dynamic convex risk measure in terms of quadratic BSDE's with convex coefficient g which depends only in z. Our approach is an extension of the Peng's results in the more naturel framework of quadratic exponential semimartingale.

The paper is structured as follows: in a second section, we give a model and preliminary notation. In the third section, we define the quadratic exponential semimartingale and we study the entropic quadratic exponential semimartingale. In particular, we give the characterization of an entropic quasimartingale and its Doob decomposition. Then, a stability results of this class of semimartingale are given in the fourth section. The fifth section is dedicated to give application of the quadratic exponential semimartingale to prove existence result for a class of QBSDE's associated to (f, η_T) where the coefficient f satisfies the **(SC)** structure condition and for non-bounded terminal condition η_T.

2 Model and Preliminaries

2.1 Notations and Setting

We start with a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with finite horizon time $T < +\infty$ and a filtration $\mathbb{F} = \left(\mathcal{F}_t\right)_{t\in[0,T]}$ satisfying the usual conditions of right continuity and completness such that we can take all semimartingales to have right continuous paths with left limits. For simplicity, we assume $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Without losing any generality we shall work with a random measure to characterize the jumps of any quasi-left continuous process X.
We define a left continuous random measure μ on the measurable space $(E, \mathcal{E})$ with compensator ν.
The measure $\mathbb{P} \otimes \nu$ is define on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}) = (\Omega \times [0,T] \times E, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{E})$ by

$$\mathbb{P} \otimes \nu(\widetilde{B}) = \mathbb{E}\left[\int_{[0,T]\times E} \mathbf{1}_{\widetilde{B}}(\omega, t, e)\nu(\omega, dt, de)\right], \quad \widetilde{B} \in \widetilde{\mathcal{F}}.$$

Let $\mathcal{P}$ denote the predicatble σ-field on $\Omega \times [0,T]$ and define $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$. For any $\widetilde{\mathcal{P}}$-measurable function f with values in $\mathbb{R}$, we have

$$f \star \mu_t := \int_0^t \int_E y(w, s, x)\mu(w, ds, dx), \ y \star \nu_t = \int_0^t \int_E y(w, s, x)\nu(w, ds, dx).$$

The random measure $\tilde{\mu}$ is defined as the compensated random measure of μ such that

$$\tilde{\mu}(\omega, dt, dx) = \mu(\omega, dt, dx) - \nu(dt, dx)$$

Let denote by $\mathcal{G}_{loc}(\mu)$, the set of $\widetilde{\mathcal{P}}$- measurable functions U with values in $\mathbb{R}$ such that

$$|U|^2.\nu_t < \infty, \quad a.s.$$

In addition, if $|H|.\nu_t < +\infty$ a.s, $H.\widetilde{\star}\tilde{\mu}$ is a local martingale.

Weak predictable representation property

we assume the following representation theorem for any local martingale M:

$$M = M_0 + M^c + M^d.$$

where M^c is the continuous part of the martingale, M^d is the discontinuous part defined as $M^d = U.(\mu - \nu)$. for any $U \in \mathcal{G}_{loc}(\mu)$.

Now we introduce the following spaces

$\mathcal{M}_0^p$ is the set of martingale M such that $M_0 = 0$ and $\mathbb{E}\left[\sup_{t\leq T} |M_t|^p\right] < +\infty$.

$\mathcal{D}_{\exp}$ is the set of local semimartingales X such that $\exp(X) \in \mathcal{D}$ where $\mathcal{D}$ is the class of optionnal processes for which the absolute value is dominated by a uniformly integrable martingale.

$\mathcal{U}_{\exp}$ is the set of local martingales M such that $\mathcal{E}(M)$ is uniformly integrable.

3 Quadratic exponential semimartingales

In all our work, we shall consider the class of quasi-left continuous semimartingales X with canonical decomposition $X = X_0 - V + M$, with V is a continuous predictable process with finite total variation $|V|$, M is a càdlàg local martingale satisfying the decomposition $M = M^c + M^d$ with M^c is the continuous part of the martingale M and $M^d = U.\widetilde{\mu}$ is the purely discontinuous part .
The quadratic exponential semimartingales are the generalization of the quadratic semimartingales in jump diffusion models. In fact the extra term in " exponential" comes from jumps and lead us to generalize the results given by [4].

Definition 3.1. *The process X is a local quadratic exponential special semimartingale if there exists two positive continuous increasing processes Λ and C and a positive constant δ such that the processes $\delta M^c + (e^{\delta U} - 1).\widetilde{\mu}$, $-\delta M^c + (e^{-\delta U} - 1).\widetilde{\mu}$ are still local martingales and the the finite variation of X satisfies the structure condition $\mathcal{Q}(\Lambda, C, \delta)$:*

$$-\frac{\delta}{2}d\langle M^c\rangle_t - \frac{1}{\delta}d\Lambda_t - |X_t|.dC_t - \frac{1}{\delta}dj_t(-\delta\Delta M^d) \ll dV_t \ll \frac{\delta}{2}d\langle M^c\rangle_t + d\Lambda_t + |X_t|.dC_t + \frac{1}{\delta}dj_t(\delta\Delta M^d)$$

Here, the process $j(\gamma\Delta M^d)$ is the compensator of the increasing process $A_t^\gamma :== \sum_{s\leq t}(e^{\gamma\Delta M_s^d} - \gamma\Delta M_s^d - 1) < +\infty$ a.s. $dB_t \ll dA_t$ stands for $A - B$ is an increasing process.

Remark 3.1. *(About the dual predictable compensator)*
• *Before studying the properties of this class of local semimartingale, let first remark that for all $\gamma \in \{-\delta, \delta\}$, the increasing càdlàg process $j(\delta\Delta M)$ is continuous applying Chap IV T[40] Dellacherie[14]. Moreover using representation theorem of the discontinuous martingale $M^d = U.\widetilde{\mu}$, then:*

$$j_t(\delta\Delta M^d) = (e^{\delta U} - \delta U - 1).\nu_t.$$

• *Let remark that for* $a = e^{\Delta U} - 1$, $b = e^{-\Delta U} - 1$ *since* $-2ab \leq a^2 + b^2$, *we find*

$$2[(e^{\delta U} - \delta U - 1) + (e^{-\delta U} + \delta U - 1)] \leq |e^{\delta U} - 1|^2 + |e^{-\delta U} - 1|^2$$

Since by assumption the processes $\delta M^c + (e^{\delta U} - 1)\widetilde{\mu}$*and* $-\delta M^c + (e^{-\delta U} - 1)\widetilde{\mu}$ *are local martingales, the processes* $|e^{\delta U} - 1|^2.\nu_t$ *a.s and* $|e^{-\delta U} - 1|^2.\nu_t < +\infty$ *a.s, therefore the predictable compensator* $j(\gamma \Delta M^d)$ *of* A^γ *is well defined for* $\gamma \in \{-\delta, \delta\}$.

To understand better the class of local quadratic exponential semimartingales and theirs properties, we divide the class in three classes:

– *The canonical class*: The finite variation part of X satisfies:

$$V_t = \frac{1}{2}\langle M^c \rangle_t + j_t(\delta M^d) \text{ or } V_t = -\frac{1}{2}\langle M^c \rangle_t - j_t(\Delta M^d)$$

– *The class* $\mathcal{Q}(0, 0, 1)$, where the finite variation part of X satisfies:

$$-j_t(-\Delta M^d) - \frac{1}{2}\langle M^c \rangle_t \ll V_t \ll \frac{1}{2}\langle M^c \rangle_t + j_t(\Delta M^d)$$

– *The general class* $\mathcal{Q}(\Lambda, C, \delta))$, where the finite variation part of X satisfies:

$$-\frac{\delta}{2}\langle M^c \rangle_t - \frac{1}{\delta}\Lambda_t - |X| * C_t - \frac{1}{\delta} j_t(-\delta \Delta M^d) \ll V_t \ll \frac{\delta}{2}\langle M^c \rangle_t + \frac{1}{\delta}\Lambda_t + |X| * C_t + \frac{1}{\delta} j_t(\delta \Delta M^d)$$

3.1 The canonical class

3.1.1 The exponential of Doléans-Dade

We describe the relation between the exponential transform of the first class of local quadratic exponential semimartingale and the exponential of Doléans-Dade . Let first recall that for any càdlàg local semimartingale X, the exponential of Doléans-Dade Z of X solving the EDS $dZ_t = Z_{t^-} dX_t$, $Z_0 = 1$ is given by:

$$Z_t = \mathcal{E}(X)_t = \exp(X_t - \langle X^c \rangle_t) \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}, \quad t \geq 0. \tag{3.3}$$

This formula is given by the Itô's formula for discontinuous processes a local martingale M with $(\Delta M > -1)$ is in fact a positive local martingale and there is a relation between exponential of a canonical quadratic exponential semimartingale and Doléans-Dade of some local martingale.

Proposition 3.1. *(Doléans Dade martingale and canonical quadratic semimartingale). Let* $\bar{M} = \bar{M}^c + \bar{U}.\widetilde{\mu}$ *and* $\underline{M} = \underline{M}^c + \underline{U}.\widetilde{\mu}$ *two càdlàg local martingales such that* $\bar{M}^c + (e^{\bar{U}} - 1).\widetilde{\mu}$ *and* $-\underline{M}^c + (e^{-\underline{U}} - 1).\widetilde{\mu}$ *are still càdlàg local martingales. Let define the canonical local quadratic exponential semimartingale:*

$$r(\bar{M}_t) = r(\bar{M}_0) + \bar{M}_t - \frac{1}{2}\langle \bar{M}^c \rangle_t - (e^{\bar{U}} - \bar{U} - 1).\nu_t,$$
$$\underline{r}(\underline{M}_t) = \underline{r}(\underline{M}_0) + \underline{M}_t + \frac{1}{2}\langle \underline{M}^c \rangle_t + (e^{-\underline{U}} + \underline{U} - 1).\nu_t$$

then we find the following processes:

$$\exp[r(\bar{M})-r(\bar{M}_0)] = \mathcal{E}\left(\bar{M}^c + (e^{\bar{U}}-1).\widetilde{\mu}\right) \text{ and } \exp[-\underline{r}(\underline{M})+\underline{r}(\underline{M}_0)] = \mathcal{E}\left(-\underline{M}^c + (e^{-\underline{U}}-1).\widetilde{\mu}\right)$$

are positive local martingales.

Proof. We apply the Doléans-Dade exponential formula (3.3) with $\bar{X} = \bar{M}^c + (e^{\bar{U}}-1).\widetilde{\mu}$ and $\underline{X} = -\underline{M}^c + (e^{-\underline{U}}-1).\widetilde{\mu}$. and we find the expected results. □

Definition 3.2. *($\mathcal{Q}$- local martingale) A local semimartingale X is a $\mathcal{Q}$-local martingale if* $\exp(X)$ *is a positive local martingale.*

The canonical local quadratic exponential semimartingales $\bar{r}(\bar{M})$ and $-\underline{r}(\underline{M})$ defined above are $\mathcal{Q}$- local martingales.

3.1.2 The entropic risk measure

The canonical local quadratic exponential semimartingales $\bar{r}(\bar{M})$ and $\underline{r}(\underline{M})$ are $\mathcal{Q}$- local martingales. In fact the uniform integrability of this class of semimartingales can be obtained through the local martingales $\bar{M}$ and $\underline{M}$.

$$\mathbb{E}\left[\exp\{\frac{1}{2}\langle M^c\rangle_\tau + ((1+U)\ln(1+U)-U).\nu_\tau\}\right] < +\infty. \tag{3.4}$$

where $\tau = \inf\{t \geq 0, \mathcal{E}(M) = 0\}$. This condition is sufficient and not necessary, another sufficient condition for a local semimartingale X to belong to $\mathcal{D}_{exp}$ is satisfying if there exists a positive uniformly integrable martingale M such that $\exp(X) \leq M$. In particular theses sufficient conditions are satisfying for the dynamic entropic risk measure (see Barrieu and El Karoui for more details[5]).

Proposition 3.2. *Let consider the fixed horizon time $T > 0$ and $\psi_T \in \mathcal{F}_T$ such that* $\exp(|\psi_T|) \in L^1$ *and consider the two dynamic risk measures:*

$$\bar{\rho}_t(\psi_T) = \ln\left[\mathbb{E}\left(\exp(\psi_T)|\mathcal{F}_t\right)\right], \text{ and } \underline{\rho}_t(\psi_T) = -\ln\left[\mathbb{E}\left(\exp(-\psi_T)|\mathcal{F}_t\right)\right]$$

There exists local martingales $\bar{M} = \bar{M}^c + \bar{U}.\widetilde{\mu}$ and $\underline{M} = \underline{M}^c + \underline{U}.\widetilde{\mu}$ such that:

$$-d\bar{\rho}_t(\psi_T) = -d\bar{M}_t + \frac{1}{2}d\langle\bar{M}^c\rangle_t + \int_E (e^{\bar{U}(s,x)} - \bar{U}(s,x) - 1).\nu(dt,dx), \quad \bar{\rho}_T(\psi_T) = \psi_T$$

$$-d\underline{\rho}_t(\psi_T) = -d\underline{M}_t - \frac{1}{2}d\langle\underline{M}^c\rangle_t - \int_E (e^{-\underline{U}(s,x)} + \underline{U}(s,x) - 1).\nu(dt,dx), \quad \underline{\rho}_T(\psi_T) = \psi_T$$

Moreover the local martingales $\bar{M}^c + (e^{\bar{U}}-1).\widetilde{\mu}$ and $-\underline{M}^c + (e^{-\underline{U}}-1).\widetilde{\mu}$ belong to $\mathcal{U}_{\exp}$. The dynamic risk measures $\bar{\rho}(\psi_T)$ and $\underline{\rho}(\psi_T)$ are uniformly integrable canonical quadratic exponential semimartingales.

Proof. We have $\exp(\bar{\rho}_t(\psi_T)) = \mathbb{E}\left[\exp(\psi_T)|\mathcal{F}_t\right]$ which is a positive uniform integrable martingale since $\exp(|\psi_T|) \in L^1$ then there exists a martingale $\bar{X} \in \mathcal{U}_{exp}$ satisfying $\Delta\bar{X} > -1$ such that $\exp(\bar{\rho}_t(\psi_T)) = \mathcal{E}(\bar{X}_t)$.
Using martingale representation Theorem there exists a continuous martingale M^c and a

process U satisfying $(e^U-1)\in\mathcal{G}_{loc}(\mu)$ such that $\bar{X}=\bar{X}_0+\bar{M}^c+(e^{\bar{U}}-1).\widetilde{\mu}$. Therefore we find $\exp(\bar{\rho}_t(\psi_T))=\mathcal{E}(\bar{X}_0+\bar{M}^c_t+(e^{\bar{U}}-1).\widetilde{\mu}_t)=\exp(\bar{r}(\bar{M}_t))$.
We use the same arguments to prove that there exists a martingale $\underline{X}=\underline{X}_0-\underline{M}^c+(e^{-\underline{U}}-1).\widetilde{\mu}\in\mathcal{U}_{\exp}$ such that $\exp(-\underline{\rho}_t(\psi_T))=\mathcal{E}(\underline{X}_0-\underline{M}^c_t+(e^{-\underline{U}}-1).\widetilde{\mu}_t)=\exp(-\underline{r}(\underline{M}_t))$. □

We adopt a forward and backward points of view to describe the canonical local quadratic exponential semimartingales class. In the forward point of view, we give condition of some martingales using Doléans-Dade exponential formula to find that for any canonical local quadratic exponential semimartingale X, $\exp(X)$ or $\exp(-X)$ is a local martingale. In the backward point of view, we fix a terminal condition $X_T\in\mathcal{F}_T$ such that $\exp(|X_T|)\in L^1$, then we can prove that some dynamic entropic risk measures of ψ_T belongs to canonical quadratic exponential semimartingale class.
In this point of view, we do not make assumption on the martingale part of the canonical semimartingale to satisfy the Lepingle and Mémin condition (3.4) since the exponential condition on the terminal condition is sufficient to find uniform integrability condition.

3.2 The second class: $\mathcal{Q}(0,0,1)$

3.2.1 The exponential transform

In the first part, we use the Doléans-Dade formula to explain how the canonical local quadratic exponential semimartingale can be represented using an exponential transform. The same technics can be developped for $\mathcal{Q}(0,0,1)$- local semimartingale using the multiplicative decomposition Theorem studied by Meyer and Yoeurp [45] which stands that for any càdlàg positive local submartingale Z there exists an predictable increasing process A ($A_0=0$) and a local martingale M ($\Delta M>-1, M_0=0$) such that:

$$Z_t=Z_0\exp(A_t).\mathcal{E}(M_t),\quad t\geq 0.$$

Theorem 3.1. *Let X a làdlàg process, X is a $\mathcal{Q}(0,0,1)$-local semimartingale if and only if $\exp(X)$ and $\exp(-X)$ are local submartingales. In both cases, X is a càdlàg process.*

Proof. Let consider a $\mathcal{Q}(0,0,1)$- local semimartingale X with canonical decomposition $X=X_0-V+M$ where V is the finite variation part of X (continuous) and M is a local martingale, then there exists $U\in\mathcal{G}_{loc}(\mu)$ such that $M=M^c+U.\widetilde{\mu}$. Applying Itô' s formula to $\bar{Z}=\exp(X)$, we find

$$d\bar{Z}_t=\bar{Z}_{t^-}\left[dM^c_t+\int_E(e^{U(t,x)}-1).\widetilde{\mu}(dt,dx)-dV_t+\frac{1}{2}d\langle M^c\rangle_t+\int_E(e^{U(t,x)}-U(t,x)-1)\nu(dt,dx)\right]$$

Since X is a $\mathcal{Q}(0,0,1)$- semimartingale then $A=-V+\frac{1}{2}\langle M^c\rangle+(e^U-U-1).\nu$ is an increasing continuous predictable process. Therefore the process $Z=\exp(X)$ is a positive local submartingale and satisfies the following Meyer and Yoeurp multiplicative decomposition:

$$\exp(X_t-X_0)=\exp(A_t)\mathcal{E}(M^c_t+(e^U-1).\widetilde{\mu}_t),\quad t\geq 0.$$

We use the same arguments to prove that $\exp(-X)$ is a local positive submartingale.
Let now assume that $\exp(X)$ and $\exp(-X)$ are local submartingales where X is a làdlàg process. Using Meyer and Yoeurp multiplicative decomposition, there exist local martingales

$\bar{M}, \underline{M}$ and increasing predictable processes $\bar{A}, \underline{A}$ such that $\exp(X_t - X_0) = \exp(\bar{A}_t)\mathcal{E}(\bar{M}_t)$ and $\exp(-X_t + X_0) = \exp(\underline{A}_t)\mathcal{E}(\underline{M}_t)$. Using the representation martingale Theorem, there exist $\bar{U}, \underline{U} \in \mathcal{G}_{loc}(\mu)$ and continuous local martingales $\bar{M}^c, \underline{M}^c$ such that $\bar{M} = \bar{M}^c + (e^{\bar{U}} - 1).\widetilde{\mu}$ and $\underline{M} = \underline{M}^c + (e^{\underline{U}} - 1).\widetilde{\mu}$. Hence we find $\exp(X - X_0) = \exp(\bar{A})\exp(r(\bar{M})$ and $\exp(-X + X_0) = \exp(\underline{A})\exp(r(\underline{M})$ and we get

$$X_t - X_0 = \bar{A}_t + \bar{M}_t - \frac{1}{2}\langle \bar{M}^c \rangle - (e^{\bar{U}} - \bar{U} - 1).\nu_t \text{ and } -X_t + X_0 = \underline{A}_t + \underline{M}_t - \frac{1}{2}\langle \underline{M}^c \rangle - (e^{\underline{U}} - \underline{U} - 1).\widetilde{\nu}_t.$$

Using the uniqueness of the representation of the semimartingale X, we deduce, $\underline{M} = -\bar{M}$, then we find $\bar{A}_t + \underline{A}_t = \langle \bar{M}^c \rangle_t + (e^{\bar{U}} - \bar{U} - 1).\nu_t + (e^{-\bar{U}} + \bar{U} - 1).\nu_t$. The process $\bar{A}$ and $\underline{A}$ are continuous, moreover from Radon Nikodym's Theorem, there exists a predictable process with $0 \leq \alpha_t \leq 2$ such that $d\bar{A}_t = \frac{\alpha_t}{2} d\left[\langle \bar{M}^c \rangle_t + (e^{\bar{U}} - \bar{U} - 1).\nu_t + (e^{-\bar{U}} + \bar{U} - 1).\nu_t\right]$. Therefore the process X satisfies the dynamics $dX_t = dM_t - dV_t$ where:

$$dV_t = \frac{(1 - \alpha_t)}{2} d\langle M^c \rangle_t + \frac{(2 - \alpha_t)}{2} d\left[(e^{\bar{U}} - \bar{U} - 1).\nu_t\right] - \frac{\alpha_t}{2} d\left[(e^{-\bar{U}} + \bar{U} - 1).\nu_t\right]$$

Since $0 \leq \alpha_t \leq 2$, the local semimartingale X satisfies the structure condition $\mathcal{Q}(0, 0, 1)$. Moreover the finite variation part V of X is a predictable continuous process. We deduce X is $\mathcal{Q}(0, 0, 1)$- local semimartingale and that all jumps of X come from the local martingale part which is càdlàg process. □

Definition 3.3. *Let consider a local semimartingale X, if $\exp(X)$ is a local submartingale then X is called $\mathcal{Q}$- local submartingale.*

From Theorem 3.1, any $\mathcal{Q}(0, 0, 1)$- local semimartingale is a $\mathcal{Q}$- local submartingale and the reverse holds true.

3.2.2 The entropic submartingales

We are interested to find uniform integrability condition for $\mathcal{Q}(0, 0, 1)$- local semimartingales. Since $\mathcal{Q}(0, 0, 1)$- local semimartingales are $\mathcal{Q}$- local submartingales, we use the same technics developped for standard local submartingales. We recall that to prove $X \in \mathcal{D}_{exp}$, it is sufficient to prove there exists a positive martingale $L \in \mathcal{D}$ such that $\exp(X) \leq L$.
To construct the positiive martingale L, let first give some useful definitions.

Definition 3.4. *A process $X \in \mathcal{D}_{exp}$ is called an entropic submartingale if for any stopping times $\sigma \leq \tau$:*

$$X_\sigma \leq \bar{\rho}_\sigma(X_\tau), \quad \sigma \leq \tau.$$

where $\bar{\rho}$ stands for the usual entropic risk measure defined above. In the same point of view, X is called a entropic supermartingale if $-X$ is an entropic submartingale. If X and $-X$ are entropic submartingales, X is called entropic quasi-martingale.

Theorem 3.2. *Let $T > 0$ the fixed horizon time and consider a semimartingale $X = X_0 - V + M^c + U.\widetilde{\mu}$ such that $\exp(|X_T|) \in L^1$ then X is a $\mathcal{Q}(0, 0, 1)$-semimartingale $\in \mathcal{D}_{exp}$ if and only if X and $-X$ are entropic submartingales.Moreover, in all cases the martingales $M^c + (e^U - 1).\widetilde{\mu}$ and $-M^c + (e^{-U} - 1).\widetilde{\mu}$ belong to $\mathcal{U}_{exp}$.*

Proof. Let consider a $\mathcal{Q}(0,0,1)$-semimartingale $X = X_0 + M^c + U.\widetilde{\mu} - V \in \mathcal{D}_{exp}$ such that $\exp(|X_T|) \in L^1$. Since X is $\mathcal{Q}$- submartingale we find:

$$\exp(X_t) \leq \mathbb{E}\left[\exp(X_T)|\mathcal{F}_t\right] \in \mathcal{D} \text{ and } \exp(-X_t) \leq \mathbb{E}\left[\exp(-X_T)|\mathcal{F}_t\right] \in \mathcal{D}$$

and for any stopping times: $\sigma \leq \tau \leq T$:

$$X_\sigma \leq \ln\left(\mathbb{E}\left[\exp(X_\tau)|\mathcal{F}_\sigma\right]\right) = \bar{\rho}_\sigma(X_\tau) \text{ and } -X_\sigma \leq \ln\left(\mathbb{E}\left[\exp(-X_\tau)|\mathcal{F}_\sigma\right]\right) = \bar{\rho}_\sigma(-X_\tau).$$

then X and $-X$ are entropic submartingales. Let prove the reverse, assume X and $-X$ are entropic submartingales then for any stopping times $\sigma \leq \tau$, $\exp(X_\sigma)) \leq \mathbb{E}\left[\exp(X_\tau)|\mathcal{F}_\sigma\right]$ and $\exp(-X_\sigma)) \leq \mathbb{E}\left[\exp(-X_\tau)|\mathcal{F}_\sigma\right]$, then X is a uniformly integrable $\mathcal{Q}$-submartingale and from Theorem 3.1, X is a $\mathcal{Q}(0,0,1)$-semimartingale. Since X and $-X$ belong to $\mathcal{D}_{exp}$ then for a fixed horizon time T, $\exp(X_T)$ and $\exp(-X_T)$ belong to L^1 which lead to conclude $\exp(|X_T|) \in L^1$.
Moreover since X and $-X$ are $\mathcal{Q}$-submartingales using Meyer-Yoeurp multiplicative decomposition Theorem, there exist increasing processes $\bar{A}$ and $\underline{A}$ ($\bar{A}_0 = 0$ and $\underline{A}_0 = 0$)) such that:

$$\exp(X_t - X_0) = \exp(\bar{A}_t)\mathcal{E}(M_t^c + (e^U - 1).\widetilde{\mu}_t) \text{ and } \exp(-X_t + X_0) = \exp(\underline{A}_t)\mathcal{E}(-M_t^c + (e^{-U} - 1).\widetilde{\mu}_t).$$

Therefore we deduce that $\mathcal{E}(M_t^c + (e^U - 1).\widetilde{\mu}) \leq \exp(X_t - X_0)$ and $\mathcal{E}(-M_t^c + (e^{-U} - 1).\widetilde{\mu}) \leq \exp(-X_t + X_0)$. Since $|X - X_0| \in \mathcal{D}_{exp}$, we conclude the martingales $M^c + (e^U - 1).\widetilde{\mu}$ and $-M^c + (e^{-U} - 1).\widetilde{\mu} \in \mathcal{U}_{exp}$. □

To conclude this part, we can make some links with the sublinear g-expectation of Peng [52] since if we define the g-expectation of X by $\mathbb{E}^g(X)$, we can define the submartingale under the g-expectation. Therefore, we deduce that if X is $\mathcal{Q}(0,0,1)$-semimartingale such that $|X| \in \mathcal{D}_{exp}$, X and $-X$ are submartingales under $\mathbb{E}^g = \ln\left[\mathbb{E}(\exp)\right]$.

3.3 General class: $\mathcal{Q}(\delta, \Lambda, C)$

3.3.1 The exponential transform

We use some exponential transformations for general $\mathcal{Q}(\Lambda, C, \delta)$ local quadratic exponential semimartingale such that the new tansformed process belong to the class $\mathcal{Q}(0,0,1)$. Therefore, we can apply the same methodology using in the previous sections to find general results for $\mathcal{Q}(\Lambda, C, \delta)$ local semimartingales.

Proposition 3.3. *Let consider a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale $X = X_0 - V + M^c + U.\widetilde{\mu}$ then*

1. *For any $\lambda \neq 0$, the process λX is a $\mathcal{Q}(\Lambda, C, \frac{\delta}{|\lambda|})$-local semimartingale and a $\mathcal{Q}(\lambda\Lambda, C, \delta)$-local semimartingale when $\lambda > 1$.*

2. *Let define the two transformations:*

$$Y^{\Lambda,C}(X) = X + \Lambda + |X| * C \text{ and } \bar{Y}^{\Lambda,C}(|X|) = e^C|X| + e^C * \Lambda.$$

then the two processes $Y^{\Lambda,C}(\delta X)$ and $\bar{Y}^{\Lambda,C}(|\delta X|)$ are $\mathcal{Q}$-local submartingales.

3. *Exponential transformation: Let $U^{\Lambda,C}(X)$ the transformation*

$$U_t^{\Lambda,C}(e^X) = e^{X_t} + \int_0^t e^{X_s} d\Lambda_s + \int_0^t e^{X_s} |X_s| dC_s.$$

then $U^{\Lambda,C}(e^{\delta X})$ is a positive local submartingale.

Proof. 1. Let consider a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale $X = X_0 - V + M^c + M^d$ (where $M^d = U.\widetilde{\mu}$) and consider $\lambda \neq 0$, hence $\lambda X = \lambda X_0 - \lambda V + \lambda M^c + \lambda M^d$ and λX satisfies the condition

$$\begin{cases} -|\lambda|\frac{\delta}{2} d\langle M^c\rangle_t - \frac{|\lambda|}{\delta} d\Lambda_t - |\lambda X_t|.dC_t - |\lambda|\frac{1}{\delta} dj_t[-\delta \text{ sign } (\lambda)\Delta M_t^d] \ll \lambda dV_t, \\ \lambda dV_t \ll |\lambda|\frac{\delta}{2} d\langle M^c\rangle_t + \frac{|\lambda|}{\delta} d\Lambda_t + |\lambda X_t|.dC_t + |\lambda|\frac{1}{\delta} dj_t[\delta \text{sign}(\lambda)\Delta M_t^d] \end{cases}$$

Since $j(\delta\Delta M^d) = j[\frac{\delta}{\lambda}(\lambda\Delta M^d]$ then we find

$$\begin{cases} -\frac{\delta}{|\lambda|}\frac{1}{2} d\langle \lambda M^c\rangle_t - \frac{|\lambda|}{\delta} d\Lambda_t - |\lambda X_t|.dC_t - \frac{|\lambda|}{\delta} dj_t[-\frac{\delta}{|\lambda|}(\lambda\Delta M_t^d)] \ll \lambda dV_t, \\ \lambda dV_t \ll \frac{\delta}{|\lambda|}\frac{1}{2} d\langle \lambda M^c\rangle_t + \frac{|\lambda|}{\delta} d\Lambda_t + |\lambda X_t|.dC_t + \frac{|\lambda|}{\delta} dj_t[\frac{\delta}{|\lambda|}(\lambda\Delta M_t^d)]. \end{cases}$$

then λX is a $\mathcal{Q}(\Lambda, C, \frac{\delta}{|\lambda|})$-local semimartingale. Moreover for $\lambda > 1$:

$$\frac{\delta}{|\lambda|}\frac{1}{2} d\langle \lambda M^c\rangle_t \ll \frac{\delta}{2} d\langle M^c\rangle_t \text{ and } \frac{|\lambda|}{\delta} j_t[\frac{\delta}{|\lambda|}(\lambda\Delta M_t^d)] \ll \frac{1}{\delta} j_t[\delta(\lambda\Delta M^d)]$$

For more details on this inequality see Lemma 6.3 in Appendix .
We find that for $\lambda > 1$, λX is a $\mathcal{Q}(|\lambda|\Lambda, C, \delta)$-semimartingale.

2. Let consider the $Y^{\Lambda,C}(\delta X) = \delta X_0 + \widetilde{M}_t - \widetilde{V}_t$, where $\widetilde{M}$ is the local martingale part given by $\widetilde{M} = \delta M^c + \delta M^d$ and $\widetilde{V}$ the finite variation part given by $\widetilde{V} = \delta V - \Lambda - |\delta X| * C$. Since X is $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale we have $d\widetilde{V}_t \ll dj_t(\delta\Delta M^d) + \frac{\delta^2}{2} d\langle M^c\rangle_t$. We conclude $d\widetilde{V}_t \ll dj_t(\Delta\widetilde{M}^d) + \frac{1}{2} d\langle \widetilde{M}^c\rangle_t$ and the process A defined by $dA_t = -d\widetilde{V}_t + dj_t(\Delta\widetilde{M}^d) + \frac{1}{2} d\langle \widetilde{M}^c\rangle_t$ is an increasing process and $Y^{\Lambda,C}(\delta X) = \delta X_0 + \widetilde{M} - \frac{1}{2}\langle \widetilde{M}^c\rangle - j(\Delta\widetilde{M}^d) + A$ then we conclude $\exp(Y^{\Lambda,C}(\delta X))$ is a local submartingale then it is $\mathcal{Q}$-local submartingale.
Let prove now that the process $\bar{Y}^{\Lambda,C}(|X|)$ belong to the $\mathcal{Q}(0,0,1)$-class. Applying the Meyer-Itô's formula, we find the decomposition:

$$de^{C_t}|X_t| = e^{C_t}\left[|X_t| dC_t - \text{sign}(X_{t^-}) dV_t + dL_t^X + d\left[(|X_- + U| - |X_-|).\nu_t\right] + d\bar{M}_t\right]$$

where $d\bar{M}_t = \text{sign}(X_{t^-}) dM_t^c + d\left[(|X_- + U| - |X_-|).\widetilde{\mu}_t\right]$ and L^X stands for the local time of X at 0. Therefore the decomposition of the semimartingale $\bar{Y}^{\Lambda,C}([\delta X|)$ satisfies $d\bar{Y}^{\Lambda,C}(|X|) = -d\widetilde{V}_t + d\widetilde{M}_t$ where $\widetilde{M} = \delta\bar{M}$ and

$$d\widetilde{V}_t = -e^{C_t}\left[|\delta X_t| dC_t + d\Lambda_t - \delta\text{sign}(X_{t^-}) dV_t + dL_t^{\delta X} + d\left[\delta(|X_- + U| - |X_-|).\nu_t\right]\right].$$

Since the process X is a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale, the process A defined by $dA_t = \delta(|X_t|dC_t + \frac{1}{\delta}d\Lambda_t - \text{sign}(X_{t^-})dV_t + \frac{\delta}{2}d\langle M^c\rangle_t + \frac{1}{\delta}dj_t[\text{sign}(X_{t^-})]\delta|\Delta M^d|) + \frac{1}{\delta}dL_t^{\delta X})$ is an increasing process. Therefore we get:

$$-d\widetilde{V}_t = e^{C_t}\left[-\frac{\delta^2}{2}d\langle M^c\rangle_t - dj_t[\delta\text{sign}(X_{t^-})\Delta M_t] + d\left(\delta(|X_- + U| - |X_-|).\nu_t\right).\right]$$

From Lemma 6.3 (see Appendix for details), for any $k \geq 1$, $j(k\Delta M) \geq kj(\Delta M)$, therefore since C is an increasing process with the initial condition $C_0 = 0$, we get $j_s[\delta e^{C_s}\text{sign}(X_{s^-})\Delta M_s] - e^{C_s}j_s[\delta\text{sign}(X_{s^-})\Delta M_s] \geq 0$. Moreover for any $s \geq 0$, $\frac{\delta^2}{2}\langle e^{C_s}M^c\rangle_s - \frac{\delta^2}{2}e^{C_s}\langle M^c\rangle_s \geq 0$, then we obtain:

$$-d\widetilde{V}_t = -\frac{1}{2}d\langle e^{C_t}\delta\text{sign}(X_{t^-})M^c\rangle_s - dj_t[\delta e^{C_t}\text{sign}(X_{t^-})\Delta M_t] + d\left(\delta(|X_- + U| - |X_-|).\nu_t\right) + d\bar{A}_t$$

where $\bar{A}$ is an increasing process. Finally we get:

$$\begin{aligned} d\bar{Y}_t^{\Lambda,C}(|X|) = {} & e^{C_t}\delta\text{sign}(X_{t^-})dM_t^c + \int_E e^{C_t}\delta(|X_{t^-} + U(t,x)| - |X_{t^-}|)\widetilde{\mu}(dt,dx) \\ & - \frac{1}{2}d\langle e^{C_t}\delta\text{sign}(X_{t^-})M^c\rangle_t - j_t[\delta e^{C_t}(|X_{t^-} + \Delta M_t| - |X_{t^-}|)] + d\widetilde{A}_t \end{aligned}$$

where

$$\widetilde{A}_t = \bar{A}_t + \int_0^t\int_E \left[\exp\left(e^{C_s}\delta(|X_{s^-} + U(s,x)| - |X_{s^-}|)\right) - \exp\left(e^{C_s}\text{sign}(\delta X_{s^-})U(s,x)\right)\right]\nu(ds,dx)$$

Since $|y+u| - |y| \geq \text{sign}(y)u$ we deduce $\widetilde{A}$ is increasing then we get:

$$\bar{Y}^{\Lambda,C}(|X|) = |\delta X_0| + \widetilde{M} - \frac{1}{2}\langle\widetilde{M}\rangle - j(\Delta\widetilde{M}) + \widetilde{A}.$$

Therefore, $\exp(\bar{X}^{\Lambda,C})$ is a local submartingale then it is $\mathcal{Q}$-local submartingale.

3. Let apply Itô's formula to find the decomposition of $U^{\Lambda,C}(e^{\delta X})$:

$$dU_t^{\Lambda,C}(e^{\delta X}) = e^{\delta X_{t^-}}\left[\delta dM_t^c + d[(e^{\delta U} - 1).\widetilde{\mu}_t] - \delta dV_t + \frac{\delta^2}{2}d\langle M^c\rangle_t + dj_t(\delta\Delta M_t) + |\delta X_t|dC_t\right].$$

Since X is $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale then the process A defined by $dA_t = -dV_t + \frac{\delta}{2}d\langle M^c\rangle_t + \frac{1}{\delta}dj_t(\delta\Delta M_t) + |\delta X_t|dC_t$ is an increasing process, we deduce the process $U^{\Lambda,C}(e^{\delta X})$ is a positive local submartingale. □

Theorem 3.3. *Let X a làdlàg optionnal process X. X is a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale if and only if* $\exp\left[Y^{\Lambda,C}(\delta X)\right]$ *and* $\exp\left[Y^{\Lambda,C}(-\delta X)\right]$ *are submartingales or equivently if the processes $U^{\Lambda,C}(e^{\delta X})$ and $U^{\Lambda,C}(e^{-\delta X})$ are local submartingales. In all cases; X is a càdlàg process.*

Proof. Let consider a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale X, using Proposition 3.3-2, we prove the process $\exp(Y^{\Lambda,C}(\delta X))$ is a local submartingale. The same arguments lead us to conclude also that $\exp(Y^{\Lambda,C}(-\delta X))$ is a local submartingale since $-X$ as the same structure condition as X.

Let now consider that the both processes $\exp(Y^{\Lambda,C}(\delta X))$ and $\exp(Y^{\Lambda,C}(-\delta X))$ are positive submartingales then we can apply the Yoeurp-Meyer decomposition as Theorem 3.1 and conclude there exists continuous local martingales $\bar{M}^c, \underline{M}^c$, increasing processes $\bar{A}, \underline{A}$ and $\bar{U}, \underline{U} \in \mathcal{G}_{loc}(\mu)$ such that

$$\exp[Y_t^{\Lambda,C}(\delta X)] = \exp(\delta X_0)\exp(\bar{M}_t - \frac{1}{2}\langle \bar{M}^c\rangle_t - (e^{\bar{U}} - \bar{U} - 1).\nu_t + \bar{A}_t)$$
$$\exp[Y_t^{\Lambda,C}(-\delta X)] = \exp(-\delta X_0)\exp(\underline{M}_t - \frac{1}{2}\langle \underline{M}^c\rangle_t - (e^{\underline{U}} - \underline{U} - 1).\nu_t + \underline{A}_t)$$

then we find $\delta X_t + \Lambda_t + |X_t| * C_t = \delta X_0 + \bar{M}_t - \frac{1}{2}\langle \bar{M}^c\rangle_t - (e^{\bar{U}} - \bar{U} - 1).\nu_t + \bar{A}_t$ and $-\delta X_t + \Lambda_t + |X_t| * C_t = -\delta X_0 + \underline{M}_t - \frac{1}{2}\langle \underline{M}^c\rangle_t - (e^{\underline{U}} - \underline{U} - 1).\nu_t + \underline{A}_t$. Therefore $\bar{M} = -\underline{M}$ from uniqueness of the decomposition, moreover $\bar{A}_t + \underline{A}_t = \langle M^c\rangle + (e^{\bar{U}} - \bar{U} - 1)\nu_t + (e^{-\bar{U}} + \bar{U} - 1)\nu_t + 2\Lambda_t + 2|X_t| * C_t$. We deduce the both processes $\bar{A}$ and $\underline{A}$ are continuous and from Radon Nikodym Theorem, there exists a predictable process $0 \leq \alpha \leq 2$ such that $dA_t = \frac{\alpha_t}{2}\left[\langle M^c\rangle + (e^{\bar{U}} - \bar{U} - 1)\nu_t + (e^{-\bar{U}} + \bar{U} - 1)\nu_t + 2\Lambda_t + 2|X_t| * C_t\right]$ then we find the decomposition of $X = X_0 - V + \widetilde{M}$ where:

$$d\widetilde{V}_t = \frac{\delta}{2}(1-\alpha_t)d\langle \widetilde{M}^c\rangle_t + \frac{(2-\alpha_t)}{2}\frac{1}{\delta}dj_t(\delta\Delta\widetilde{M}_t) + \frac{(2-\alpha_t)}{2}\frac{1}{\delta}d\Lambda_t + \frac{(2-\alpha_t)}{2}|X_t|dC_t - \frac{\alpha_t}{2}dj_t(-\delta\Delta\widetilde{M}_t)$$

Since the predictable process $0 \leq \alpha \leq 2$, we find:

$$\begin{cases} -\frac{\delta}{2}d\langle \widetilde{M}^c\rangle_t - \frac{1}{\delta}d\Lambda_t - |X_t|.dC_t - \frac{1}{\delta}dj_t[-\delta\Delta\widetilde{M}_t^d)] \ll dV_t, \\ dV_t \ll \frac{\delta}{2}d\langle \widetilde{M}^c\rangle_t + \frac{1}{\delta}d\Lambda_t + |X_t|.dC_t + \frac{1}{\delta}dj_t[\delta\Delta\widetilde{M}_t^d)]. \end{cases}$$

then X is a $\mathcal{Q}(\Lambda, C, \delta)$- local semimartingale. equivalently, we can use the same arguments for the positive local submartingale $U^{\Lambda,C}(e^{\delta X})$ and $U^{\Lambda,C}(e^{\delta X})$ to find that the process X is a $\mathcal{Q}(\Lambda, C, \delta)$-local semimartingale. Moreover since the finite variation part of $\widetilde{V}$ is continuous, jumps come from the local martingale part. Hence, the process X is a càdlàg local semimartingale. □

In all the rest of the paper, since from a multiplicative transformation (see Proposition 3.3), we can transform the general class $\mathcal{Q}(\Lambda, C, \delta)$ to the class $\mathcal{Q}(\Lambda, C, 1)$. We can give all results in the class $\mathcal{Q}(\Lambda, C) := \mathcal{Q}(\Lambda, C, 1)$ without losing any generality.

3.3.2 Uniform Integrable $\mathcal{Q}(\Lambda, C)$- semimartingales

We use the entropic submartingales to characterize the integrability condition for $\mathcal{Q}(0, 0, 1)$-class. Given an fixed horizon time, we find in this part sufficient condition on the terminal condition to have uniform integrability of general local quadratic exponential semimartingales. First, let give some generalization of entropic submartingales for general $\mathcal{Q}(\Lambda, C)$-semimartingales.

Theorem 3.4. *let X be a càdlàg process and T a fixed horizon time.*

1. *Assuming,* $\exp(|X_T|) \in L^1$*, the process* X *is a* $\mathcal{Q}(\Lambda, C)$*-semimartingale which belongs to* $\mathcal{D}_{exp}$ *if and only if for any stopping times* $\sigma \leq \tau \leq T$*:*

$$X_\sigma \leq \rho_\sigma(X_\tau + \Lambda_{\sigma,\tau} + |X| * C_{\sigma,\tau}) \text{ and } -X_\sigma \leq \rho_\sigma(-X_\tau + \Lambda_{\sigma,\tau} + |X| * C_{\sigma,\tau}). \quad (3.5)$$

2. *Assuming* $U_T^{\Lambda,C}(e^{|X|}) \in L^1$*, the process* X *is a* $\mathcal{Q}(\Lambda, C)$*-semimartingale which belongs to* $\mathcal{D}_{exp}$ *if and only if for any stopping times* $\sigma \leq \tau \leq T$*:*

$$X_\sigma \leq \rho_\sigma(X_\tau + \Lambda_{\sigma,\tau} + |X| * C_{\sigma,\tau}) \text{ and } -X_\sigma \leq \rho_\sigma(-X_\tau + \Lambda_{\sigma,\tau} + |X| * C_{\sigma,\tau}).$$

Proof. 1. Let X a $\mathcal{Q}(\Lambda, C)$-semimartingales which belongs to the class $\mathcal{D}_{exp}$. From Theeorem 3.3, $\exp(Y^{\Lambda,C}(X))$ and $\exp(Y^{\Lambda,C}(-X))$ are submartingales which belong to the class $\mathcal{D}$. Therefore for any stopping times $\sigma \leq \tau \leq T$:

$$\exp(Y_\sigma^{\Lambda,C}(X) \leq \mathbb{E}\left[\exp(Y_\tau^{\Lambda,C}(X)|\mathcal{F}_\sigma\right] \text{ and } \exp(Y_\sigma^{\Lambda,C}(-X) \leq \mathbb{E}\left[\exp(Y_\tau^{\Lambda,C}(-X)|\mathcal{F}_\sigma\right]$$

then the $\mathcal{Q}(\Lambda, C)$ semimartingale X satisfies the entropy inequalities (3.5). Let assume the inequalities (3.5) are satified then we conclude $\exp(Y^{\Lambda,C}(X))$ and $\exp(Y^{\Lambda,C}(-X))$ are submartingales which belong to the class $\mathcal{D}$ then from Theorem 3.3, X is a $\mathcal{Q}(\Lambda, C)$ semimartingales which belong to the class $\mathcal{D}_{exp}$.

2. We use the same arguments with the positive submartingales $U^{\Lambda,C}(e^X)$ and $U^{\Lambda,C}(e^{-X})$. □

The Theorem 3.4 gives sufficient integrable condition for $\mathcal{Q}(\Lambda, C)$-semimartingale X such that it belongs to the class $\mathcal{D}_{exp}$. We can find another condition using the transformation $\bar{Y}^{\Lambda,C}(|X|)$ since it is a $\mathcal{Q}$-submartingale. Therefore, using the same arguments as assertions in Theorem 3.4, we find $\bar{Y}^{\Lambda,C}(|X_t|) \leq \bar{\rho}_t[\exp(\bar{Y}^{\Lambda,C}(|X_T|)]$ which is equivalent to the condition given by [4] in the continuous case (see Hypotehsis 2.8 [4]):

$$|X_t| \leq \rho_t\left[e^{C_{t,T}}|Y_T| + \int_t^T e^{C_{t,s}} d\Lambda_s\right], \quad t \leq T. \quad (3.6)$$

This assumption is a necessary and sufficient condition for the process $\bar{Y}^{\Lambda,C}(|X|)$ to be in class $\mathcal{D}_{exp}$ (the proof is given in Lemma 2.9 of [4]). In the same way, assertions in Proposition 2.10 of [4] still hold since the authors give the result in the general case (without using the continuity of processes). Moreover using the same $LLogL$ Doob-inequality, we can find the same sufficient condition on the terminal value $\bar{Y}^{\Lambda,C}(|X|)$ such that $|X| \in \mathcal{D}_{exp}$.

Proposition 3.4. *Let consider an fixed horizon time* $T > 0$ *and let* L *be a positive submartingale such that* $\max L_T := \max_{t\in[0,T]} L_t \in (1, +\infty)$*. For any* $m > 0$*, let* u_m *the convex function defined on* $\mathbb{R}^+$ *defined by* $u_m(x) = x - m - m\ln(x)$ *and* $u(x) := u_1(x)$*, the following assertions are satisfied:*

1. *Using the Doléans Dade representation of positive martingale* L*,* $L_t = \mathcal{E}(M_t^c + (e^U - 1).\widetilde{\mu})$*,* $t \leq T$*, we find:*

$$H^{ent} := \mathbb{E}[L_T \ln(L_T)] = \mathbb{E}\left[L_T\left(\frac{1}{2}\langle M^c\rangle_T + (Ue^U - e^U + 1).\nu_T\right)\right].$$

2. *The following sharp inequality holds true:*

$$u(\mathbb{E}(\max L_T)) \leq \mathbb{E}\left(L_T \ln(L_T)\right).$$

Moreover, if L is a positive $\mathcal{D}$-submartingale, the previous inequality becomes:

$$u_m(\mathbb{E}(\max L_T)) - u_m(L_0) \leq \mathbb{E}[L_T \ln(L_T)] - \mathbb{E}(L_T) \ln\left[\mathbb{E}(L_T)\right].$$

where $m = \mathbb{E}(L_T)$.

Proof. :

1. To prove the assertion, let us first prove that the equality

$$\mathbb{E}(\max L_T) - 1 = \mathbb{E}\left[L_T \ln(\max L_T)\right]$$

holds true in our case. From Dellacherie [16] p.375, $\max L_t(\omega) = L_t(\omega)$ for every jump time t or every increasing of right of $s \longrightarrow \max L_s(\omega)$. Therefore $L = \max L$ on the right support of $d\max L$. Therefore we find $\max L_t = 1 + \int_0^t d\max L_s = \int_0^t \frac{L_s}{\max L_s} d\max L_s$ then $\mathbb{E}(\max L_T) - 1 = \mathbb{E}\left[L_T \ln(\max L_T)\right]$. holds true. From this equality, it is sufficient that $\max L_T \in L^1$ to find $L_T \ln(L_T) \in L^1$. Let assume, $\max L_T \in L^1$ and let define the stopping times T_K such that the positive local martingale $L_t = \mathcal{E}(M_t + (e^U - 1)\widetilde{\mu}_t) \leq K$. The stopping times T_K is increasing and goes to infinity with K. Let define the process $N^{\mathbb{Q}} = M^c - \langle M^c \rangle + U.(\widetilde{\mu} - (e^U - 1).\nu)$. is a martingale with respect to $\mathbb{Q} = L_T\mathbb{P}$ and we get:

$$\mathbb{E}\left[L_T\left(\frac{1}{2}\langle M^c\rangle_T + (e^U - U - 1).\nu_T\right)\right] = \lim_K \mathbb{E}\left[L_T\left(\frac{1}{2}\langle M^c\rangle_{T\wedge T_K} + (e^U - U - 1).\nu_{T\wedge T_K}\right)\right]$$
$$= \lim_K \mathbb{E}\left[L_{T\wedge T_K}\left(\frac{1}{2}\langle M^c\rangle_{T\wedge T_K} + (e^U - U - 1).\nu_{T\wedge T_K}\right)\right]$$

Since $\mathbb{E}(L_{T\wedge T_K} N^{\mathbb{Q}}_{T\wedge T_K}) = 0$, we find:

$$\mathbb{E}\left[L_{T\wedge T_K} \ln(L_{T\wedge T_K})\right] = \mathbb{E}\left[L_{T\wedge T_K}\left(\frac{1}{2}\langle M^c\rangle_{T\wedge T_K} + U(e^U - 1).\nu_{T\wedge T_K} + (e^U - U - 1).\nu_{T\wedge T_K}\right)\right]$$

We have $\mathbb{E}\left[L_{T\wedge T_K} \ln(L_{T\wedge T_K})\right] \leq \mathbb{E}\left[L_{T\wedge T_K} \ln(\max L_{T\wedge T_K})\right] \leq \mathbb{E}\left[\max L_T\right] - 1 \leq +\infty$, then we get the result by taking the limit when K goes to infinity.

2. The proof is done in [4], since authors used the first assertion to prove the result.

□

Let X be a $\mathcal{Q}(\Lambda, C)$-semimartingale, applying the result of Proposition 3.4 to the positive submartingale $\exp(\bar{Y}^{\Lambda,C}(|X|))$, we conclude if $\mathbb{E}\left(\bar{Y}^{\Lambda,C}_T(|X|)\exp[\bar{Y}^{\Lambda,C}_T(|X|)]\right) \in L^1$ then we have $\max \mathbb{E}\left(\bar{Y}^{\Lambda,C}_T(|X|)\exp[\bar{Y}^{\Lambda,C}_T(|X|)]\right) \in L^1$ and the inequality (3.6) is satisfied, therefore $\bar{Y}^{\Lambda,C}(|X|)$ belongs to class $\mathcal{D}_{exp}$. To conclude this part, let recall the definition of the class of $\mathcal{Q}(\Lambda, C)$-semimartingales which belong to $\mathcal{D}_{exp}$ given by [4].

Definition 3.5. *Let η_T be a $\mathcal{F}_T$-random variable such that*

$$\exp[\gamma \bar{Y}_T^{\Lambda,C}(|\eta_T|)] = \exp[\gamma(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s)]$$

belongs to L^1, for all $\gamma > 0$. We define a class of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ of $\mathcal{Q}(\Lambda, C)$-semimartingales X such that

$$|X_t| \leq \bar{\rho}_t \left[e^{C_{t,T}}|\eta_T| + \int_t^T e^{C_{t,s}} d\Lambda_s \right], \quad a.s.$$

4 Quadratic-exponential variation and stability result

4.1 A priori estimates

We now focus on the estimate of the martingale part of a semimartingale $X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$. The estimates of the discontinuous martingales part allow us to conclude the predictable projection $j(\gamma \Delta M^d)$, $\gamma \in \{-1, 1\}$ is well defined when the semimartingale X lives in a suitable space.

Proposition 4.5. *Let consider a semimartingale $X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$ which follows the decomposition $X = X_0 - V + M^c + M^d$, where there exists a process $U \in \mathcal{G}_{loc}(\mu)$ such that $M^d = U.\widetilde{\mu}$ then for any $p \geq 1$ the matingales, $\bar{M} = M^c + (e^U - 1).\widetilde{\mu}$ and $\underline{M} = -M^c + (e^{-U} - 1).\widetilde{\mu}$ belong to $\mathcal{M}_0^p$.*

Moreover if for any stopping times $\sigma \leq T$ there exists a constant $c > 0$ such that

$$\mathbb{E}\left[\exp(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s) | \mathcal{F}_\sigma \right] \leq c,$$

then the processes $\bar{M}$ and $\underline{M}$ are BMO *martingales.*

Proof. As already seen, in the previous section, the martingale part of the canonical exponential quadratic semimartingale given by $\bar{M}^c + (e^U - 1).\tilde{\mu}$ belongs to $\mathcal{U}_{\exp}$ which is not verified in this general class. However, we can control $\bar{M}^c + (e^U - 1).\tilde{\mu}$ in $\mathcal{M}_0^p$.

1. Let $X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$, from Proposition 3.3, $Y^{\Lambda,C}(X) = X + \Lambda + |X| * C$ and $Y^{\Lambda,C}(-X)$ are $\mathcal{Q}$-local submartingale. Moreover let recall the process $\bar{Y}^{\Lambda,C}(X) = e^C.|X| + e^C * \Lambda$ satisfies $Y^{\Lambda,C}(X) \leq \bar{Y}^{\Lambda,C}(X)$ and $Y^{\Lambda,C}(-X) \leq \bar{Y}^{\Lambda,C}(X)$,
   therefore since $X \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$, for any $p \geq 1$, we find

$$\exp(p|Y_t^{\Lambda,C}(|X|)|) \leq \exp[p\bar{Y}_t^{\Lambda,C}(X)] \leq \mathbb{E}\left[\exp[p(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s)] | \mathcal{F}_t \right] \quad (4.7)$$

   We conclude that

$$\mathbb{E}\left[\sup_{t \leq T} \exp(p|Y_t^{\Lambda,C}(|X|)|) \right] < +\infty. \quad (4.8)$$

   From the submartingale property of $\exp(Y^{\Lambda,C}(X))$ and $\exp(Y^{\Lambda,C}(-X))$, from Yoeurp-Meyer decomposition, there exist increasing processes $\bar{A}$ and $\underline{A}$ such that:

$$\bar{K}_t := \exp(Y_t^{\Lambda,C}(X)) = \exp(X_0)\mathcal{E}(\bar{M}_t)\exp(\bar{A}_t)$$
$$\underline{K}_t := \exp(Y_t^{\Lambda,C}(-X)) = \exp(-X_0)\mathcal{E}(\underline{M}_t)\exp(\underline{A}_t)$$

Since $\bar{A}$ and $\underline{A}$ are increasing, from (4.8) we conclude $\bar{Z} := \mathcal{E}(\bar{M})$ and $\underline{Z} := \mathcal{E}(\underline{M})$ are uniformly integrable. Hence, $\bar{M}$ and $\underline{M} \in \mathcal{U}_{\exp}$.
Moreover $\bar{Z}$ and $\underline{Z}$ belong to $\mathcal{M}^p$, for any $p \geq 1$. Using intergration by part formula we find $d\bar{K}_t = \bar{K}_{t^-}\left[d\bar{A}_t + d\bar{M}_t\right]$ and $d\underline{K}_t = \underline{K}_{t^-}\left[d\underline{A}_t + d\underline{M}_t\right]$, that leads to $d[\bar{K}]_t = \bar{K}^2_{t^-} d[\bar{M}]_t$ and $d[\underline{K}]_t = \underline{K}^2_{t^-} d[\underline{M}]_t$. Therefore we find for any stopping times $\sigma \leq T$, $[\bar{M}]_{\sigma,T} = \int_\sigma^T \frac{d[\bar{K}]_t}{\bar{K}^2_{t^-}}$ and $[\underline{M}]_{\sigma,T} = \int_\sigma^T \frac{d[\underline{K}]_t}{\underline{K}^2_{t^-}}$, then we obtain

$$[\bar{M}]_{\sigma,T} \leq \sup_{\sigma\leq t\leq T}\left(\frac{1}{\bar{K}^2_t}\right) \times [\bar{K}]_{\sigma,T} \quad \text{and} \quad [\underline{M}]_{\sigma,T} \leq \sup_{\sigma\leq t\leq T}\left(\frac{1}{\underline{K}^2_t}\right) \times [\underline{K}]_{\sigma,T} \tag{4.9}$$

However, using Itô's decomposition of the submartingales $\bar{K}^2$ and $\underline{K}^2$, we find a priori estilmates of $[\bar{K}]_T$ and $[\underline{K}]_T$

$$\begin{aligned} d\bar{K}^2_t &= 2\bar{K}_{t^-} d\bar{K}_t + d[\bar{K}]_t = 2\bar{K}^2_{t^-}[d\bar{M}_t + d\bar{A}_t] + d[\bar{K}]_t \\ d\underline{K}^2_t &= 2\underline{K}_{t^-} d\underline{K}_t + d[\underline{K}]_t = 2\underline{K}^2_{t^-}[d\underline{M}_t + d\underline{A}_t] + d[\underline{K}]_t \end{aligned}$$

Therefore for any stopping times $\sigma \leq T$, we find

$$\mathbb{E}\left[[\bar{K}]_{\sigma,T}|\mathcal{F}_\sigma\right] \leq \mathbb{E}\left[\bar{K}^2_T|\mathcal{F}_\sigma\right] \quad \text{and} \quad \mathbb{E}\left[[\underline{K}]_{\sigma,T}|\mathcal{F}_\sigma\right] \leq \mathbb{E}\left[\underline{K}^2_T|\mathcal{F}_\sigma\right] \tag{4.10}$$

Now since, for any $p \geq 1$, $\sup_{0\leq t\leq T} \bar{K}_t$ and $\sup_{0\leq t\leq T} \underline{K}_t$ belong to L^p, it follow from Garsia and Neveu Lemma ([4] Lemma 3.3) that

$$\mathbb{E}\left[[\bar{K}]_T]^p\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[[\underline{K}]_T]^p\right] < +\infty, \quad \forall p \geq 1 \tag{4.11}$$

Since $\sup_{0\leq t\leq T} \frac{1}{\bar{K}_t}$ and $\sup_{0\leq t\leq T} \frac{1}{\underline{K}_t}$ belong to L^p for any $p \geq 1$ and using 4.11, from 4.9 we conclude by Cauchy Schwartz inequalities that for any $p \geq 1$:

$$\mathbb{E}\left[[\bar{M}]^p_T\right] \leq +\infty \quad \text{and } \mathbb{E}\left[[\underline{M}]^p_T\right] \leq +\infty$$

Thus, using the BDG inequalities, we conclude that $\bar{M}$ and $\underline{M}$ belong to $\mathcal{M}^p_0$.
Moreover if there exists a non negative constant c such that

$$\mathbb{E}\left[\exp(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s)|\mathcal{F}_\sigma\right] \leq c,$$

then from 4.7 the processes $\bar{K}$ and $\underline{K}$ are bounded, and using 4.9 and 4.10, we get that the martingales $\bar{M}$ and $\underline{M}$ are BMO-martingales.

□

4.2 Stability results of quadratic exponential semimartingale

Here, we present stability results for quadratic exponential semimartingales which we shall use for the construction of the maximal solution of a class of quadratic BSDE's with jumps. For this propose, we recall a general stability theorem of Barlow and Protter [3] for a sequence of càdlàg special semimartingales converging uniformly in L^1. We denote by $X^* := \sup_{0\leq t\leq T} |X_t|$.

Theorem 4.5. *Let X^n be a sequence of special semimartingales which belongs to $\mathcal{H}^1$ with canonical decomposition $X^n = X_0^n + M^n - V^n$, and satisfies:*

$$\mathbb{E}\Big[\int_0^T |dV_s^n|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[\left(M^n\right)^*\right] \leq C \tag{4.12}$$

for some positive constant C. Assume that:

$$\mathbb{E}\left[\left(X^n - X\right)^*\right] \longrightarrow 0, \quad \text{as} \quad n \to \infty,$$

where X is an adapted process, then X is a semimartingale in $\mathcal{H}^1$ with canonical decomposition $X = X_0 + M - V$ satisfying:

$$\mathbb{E}\Big[\int_0^T |dV_s|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[\left(M\right)^*\right] \leq C \tag{4.13}$$

and we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(V^n - V\right)^*\right] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|M^n - M\|_{\mathcal{H}^1} = 0. \tag{4.14}$$

Lemma 4.1. *Let X^n a sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales which canonical decomposition $X^n = X_0^n - V^n + M^n$ which converge in $\mathcal{H}^1$ to some process X. Therefore the process X which canonical decomposition $X = X_0 - V + M$ is an adapted càdlàg process which belongs to $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ such that:*

$$\lim_{n\to\infty} \mathbb{E}\left[\left(V^n - V\right)^*\right] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|M^n - M\|_{\mathcal{H}^1} = 0.$$

Proof. Let consider $X^n = X_0^n - V^n + (M^c)^n + U^n.\widetilde{\mu}$ a sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales. Firstly let prove that if the sequence X^n converge to the process X then this limit belongs to the space $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$.
Since, the sequence $X^n \in \mathcal{S}_Q(|\eta_T|, \Lambda, C)$, we have for each $n \in \mathbb{N}$ and for any stopping times $\sigma \leq T$:

$$-\bar{\rho}_\sigma(-\eta_T + \Lambda_{\sigma,T} + |X^n| * C_{\sigma,T}) \leq X_\sigma^n \leq \bar{\rho}_\sigma(\eta_T + \Lambda_{\sigma,T} + |X^n| * C_{\sigma,T}), \quad a.s \tag{4.15}$$

and

$$|X_\sigma^n| \leq \bar{\rho}_\sigma\left[e^{C_{\sigma,T}}|\eta_T| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s\right], \quad a.s \tag{4.16}$$

Therefore, using the dominated convergence and taking the limit of this subsequence in 4.15 and 4.16, we conclude that the limit X belongs to the space $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$. Thank's to the Dini's Theorem, the convergence of the sequence X^n in $\mathcal{H}^1$ is uniform.
Hence, since the limit X is a $\mathcal{Q}(\Lambda, C)$-semimartingale, from Theorem (3.3) the process X is càdlàg.
In the other hand, let us define the function $u \longrightarrow g(u) = e^u - u - 1$, from Remark 3.1, $2(g(u) + g(-u)) \leq (|e^u - 1|^2 + |e^{-u} - 1|^2)$. Therefore from Proposition 4.5, there exist a constant $C_1 > 0$ and $C_2 > 0$ such that:

$$\mathbb{E}\left[\int_0^T |dV_s^n|\right] \leq \mathbb{E}\left[\int_0^T \frac{1}{2} d\langle (M^c)^n\rangle_s + \int_E [g(U^n(s,x)) + g(-U^n(s,x))]\nu(ds,dx)\right] \leq C_1.$$

$$\mathbb{E}\left[\langle M^n\rangle_T^{\frac{1}{2}}\right] \leq C_2.$$

Therefore, from the stability Theorem of Barlow-Protter, the limit process X have the following canonical decomposition $X := X_0 - V + M$ satisfy for $c := \max(C_1, C_2)$

$$\mathbb{E}\Big[\int_0^T |dV_s|\Big] \leq C, \quad \text{and} \quad \mathbb{E}\left[\big(M\big)^*\right] \leq C \tag{4.17}$$

we get the expected result. □

5 Application of quadratic exponential semimartingales: quadratic BSDEs with jumps

In this section, we propose an existence results for a quadratic BSDE with unbounded terminal condition. In, [7],[**?**], [46] the autors studied Quadratic JBSDE with bounded terminal condition and a specific generator related to a stochastic optimization problem. Furthermore, Tzevadze provide a totally different approach to solve a Lipschitz-quadratic BSDE with bounded terminal condition based on a fixed point argument. In [12], also Briand and hu provided an existence result for the quadratic BSDE when the terminal condition is unbounded. However all this works exept the paper of tzevadze, are based on the seminal paper of Kobylanski. The major difficulty in the so-called Kobylanski method is the strong convergence of the martingale part which is not easy to prove.
Furthermore, Tzevadze provide a totally different approach to solve a Lipschitz-quadratic BSDE with bounded terminal condition based on a fixed point argument.

Our idea is completely different and based on Forward approach. In fact using characterization and integrability properties of the exponential quadratic semimartingale obtained in the previous section. Then by a general stability result, we prove the existence of solution of Quadratic BSDE.

5.1 Quadratic Exponential BSDE

5.2 Notations and setting

On the stochastic basis defined above $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with finite time horizon $T < +\infty$, we define $W = (W_t)_{t\geq 0}$ a d-dimensional standard Brownian motion and μ the random measure defined above such that ν is equivalent to a product measure $\lambda \otimes dt$ with density ξ satisfying

$$\nu(\omega, dt, dx) = \xi(\omega, t, x)\lambda(dx)dt,$$

where λ is a finite measure on $(E, \mathcal{E})$ satisfying $\int_E 1 \wedge |x|^2 \lambda(dx) < +\infty$ and where the density ξ is a measure, bounded nonnegative function such that for some constant C_ν:

$$0 \leq \xi(\omega, t, x) \leq C_\nu < +\infty, \quad \mathbb{P} \otimes \lambda \otimes dt - \text{ a.e .}$$

That implies in particular $\nu\left([0, T] \times E\right) \leq C_\nu T \lambda(E)$.

We introduce the following spaces

- $L^{\exp}$ the space of all $\mathcal{F}_T$-measurable random variables X such that $\forall \gamma > 0$

$$\mathbb{E}\left[\exp(\gamma|x|)\right] < +\infty$$

- $\mathcal{D}_0^{\exp}$ is the space of progressively measurable processes $(X = (X)_t)_{t\leq T}$ with

$$\mathbb{E}\left[\exp\left(\gamma \sup_{0\leq t\leq T} |X|_t\right)\right] < +\infty$$

- $\mathcal{D}_1^{\exp}$ is the space of progressively measurable processes $(X = (X)_t)_{t\leq T}$ with

$$\mathbb{E}\left[\exp\left(\gamma \int_0^T |X|_t\right)\right] < +\infty$$

- $\mathcal{S}^2$ is the space of all $\mathbb{R}$-valued adapted process X such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] < +\infty$$

- $\mathcal{S}^\infty$ is the space of all $\mathbb{R}$-valued adapted process X such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|\right] < +\infty$$

- $\mathcal{H}_\lambda^{2p}$ the set of all predictable processes U such that

$$\left(\mathbb{E}\left[\int_0^T |U|_{s,\lambda}^2 ds\right]^{\frac{p}{2}}\right)^{\frac{1}{p}} < +\infty$$

where

$$|U|_{s,\lambda}^2 := \int_E |U(s,x)|^2 \xi(s,x)\lambda(dx).$$

- $\mathcal{H}^{2p}$ the set of all predictable processes Z such that

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 dt\right)^{\frac{p}{2}}\right] < \infty.$$

We will assume the following representation Theorem: for any square integrable martingale M we have

$$M_t = M_0 + \int_0^t Zs.dW_s + \int_0^t \int_E U_s(x)\tilde{\mu}(de, ds).$$

where Z and U are predictable processes in $\mathcal{H}^2$, $\mathcal{H}_\lambda^2$. Therefore, we consider a Backward stochastic Differential equation with jumps of the form

$$-dY_t = f_t(Y_t, Z_t, U_t) - Z_t dW_t - \int_E U_t(x)\widetilde{\mu}(dt, dx), \quad Y_T = \eta_T, \tag{5.18}$$

A solution of such a BSDE associated to (f, η_T) is a triple of adapted processes $(Y_t, Z_t, U_t)_{0\leq t\leq T}$. To get an existence result it is necessary to put some assumptions on The coefficient and the terminal condition of the BSDE.

5.2.1 Assumptions

- **(HC)** Continuous condition: for all $t \in [0,T], (y,z,u) \to f_t(y,z,u)$ is continuous.
- **(HL-a)** Lipschitz condition: there exists a non negative constant C such that

$$|f_t(y,z,u) - f_t(\bar{y},\bar{z},\bar{u}) \leq C\left(|y-\bar{y}| + |z-\bar{z}| + |u-\bar{u}|\right), \quad \forall t \in [0,T]$$

- **(HL-b)** Local Lipschitz condition: there exists a non negative constant C such that

$$|f_t(y,z,u) - f_t(y,\bar{z},u)| \leq C\left(1 + |z| + |z'|)(|z-\bar{z}|\right), \quad \forall t \in [0,T]$$

- **(SC)** Structure Condition: there exists a positive adapted processes $l \in \mathcal{D}_1^{\exp}$, $c \in \mathcal{S}^\infty$ and a non negative bounded constant δ such that for all $(y,z,u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2$, $t \in [0,T]$: $\mathbb{P}$-a.s,

$$\underline{q}(t,y,z,u) = \frac{1}{\delta} j_t(-\delta u) - \frac{\delta}{2}|z|^2 - l_t - c_t|y| \leq f_t(y,z,u) \leq \frac{1}{\delta} j_t(\delta u) + \frac{\delta}{2}|z|^2 + l_t + c_t|y| = \bar{q}(t,y,z,u). \tag{5.19}$$

- **(IC-a)** Integrability condition:

$$\forall \gamma > 0, \quad \mathbb{E}\left[\exp(\gamma(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s))\right] < +\infty.$$

- **(IC-b)** Integrability condition: $f(0,0,0) \in \mathcal{H}^2$
- **(DC)** Decompotision condition: there exist a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})$ measurable function $\hat{f}$ and a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+2})$ measurable function h such that:

$$f_t(y,z,u) = \hat{f}_t(y,z) + \int_E h_t(y,z,u(x))\zeta(t,x)\lambda(dx) \tag{5.20}$$

  where the function h is differentiable with respect to u and $\frac{\partial h}{\partial u} > -1$. When f is Lipschitz with respect to y, we said that f satisfies the simple (**DC**) form.

- $(\mathcal{A}_\gamma)$ condition : there exist a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2)$ measurable function γ such that

$$f_t(y,z,u) - f_t(y,z,\bar{u}) \leq \int_E \gamma_t(y,z,u(x),\bar{u}(x))\zeta(t,x)\lambda(dx) \tag{5.21}$$

  where $\gamma > -1$ and for any processes $Y \in \mathcal{S}^2$, $Z \in \mathcal{H}^2$ and $U, \bar{U} \in \mathcal{H}^2_\lambda$, the martingale $\gamma(Y,Z,U,\bar{U}).\widetilde{\mu} \in \mathcal{U}_{\exp}$.

Remark 5.2. *Under **(HL-a)**, **(HI-b)** , **(IC-b)**, Becherer [7] proved the existence of a solution of the BSDE (5.18) with the terminal value $\eta_T \in L^2(\Omega, \mathcal{F}_T)$. The author used a classical fixed point approach and a priori estimates.*
*Later, in [?] Becherer ,Buttner and Kentia, extended the result of [7] assuming (**DC**) and considering the BSDE coefficient which is not Lipschitz with respect the jump part. In fact, they work with a terminal condition bounded and since they prove the solution (Y,Z,U) of the BSDE (5.18) is such that Y and U is bounded, they give sufficient condition on the differential of the function h to get $\frac{\partial h}{\partial u}(Y,Z,U).\widetilde{\mu} \in$ BMO. Therefore, they get uniqueness of the solution using comparison result since the condition A_γ is satisfied.*

Definition 5.6. *We call Quadratic Exponential BSDE with parameters* (l, c, δ) *(*$q_{\exp}(l, c, \delta)$ *in short terms) associated to* (f, η_T)*, the BSDE* (5.17) *such that* f *satisifies the conditions* ***(HC)***, ***(SC)***, ***(IC-a)***, ***(IC-b)*** *and* ***(DC)***.

The more natural quadratic exponential BSDE is the one related to the dynamic entropy risk measure defined in first section. Let racall that, for a terminal value $\eta_T \in L^{\exp}$, the dynamic entropy risk measure $\rho(\eta_T)$ is defined by:

$$\rho_t(\eta_T) = \frac{1}{\delta} \ln \left[\mathbb{E}(\exp(\delta \eta_T) | \mathcal{F})_t \right], \quad 0 \leq t \leq T;$$

and follows the dynamics:

$$-d\rho_t(\eta_T) = f_t(\rho_t(\eta_T), Z_t, U_t) - Z_t dW_t - \int_E U(t,x) \widetilde{\mu}(dt, dx), \quad \rho_T(\eta_T) = \eta_T \tag{5.22}$$

where the BSDE coefficient f is given by:

$$f_t(y, z, u) = \frac{\delta}{2}|z|^2 + j_t(\delta u)$$

and satisfies the conditions **(HC)**, **(HL-b)**, **(SC)**, **(IC-a)**, **(IC-b)** and **(DC)** (the simple form). In fact, for any $t \in [0, T]$, $z, \bar{z} \in \mathbb{R}^d$, $y \in \mathbb{R}$ and u, $\bar{u} \in L^2$:

$$f_t(y, z, u) - f_t(y, \bar{z}, u) = \frac{\delta}{2}(|z|^2 - |\bar{z}|^2) \leq \frac{\delta}{2}|\bar{z}||z - \bar{z}|1_{\{\delta \leq 0\}} + \frac{\delta}{2}|z||z - \bar{z}|1_{\{\delta > 0\}}$$

and

$$f_t(y, z, u) - f_t(y, z, \bar{u}) = j_t(\delta u) - j_t(\delta \bar{u}) = \int_E \left([e^{\delta \bar{u}(x)} - 1]1_{\{\delta \leq 0\}} + [e^{\delta u(x)} - 1]1_{\{\delta > 0\}} \right) \xi(t,x) \lambda(dx)$$

Therefore in this particular case:

$$\gamma_t(u(x), \bar{u}(x)) = [e^{\delta \bar{u}(x)} - 1]1_{\{\delta \leq 0\}} + [e^{\delta u(x)} - 1]1_{\{\delta > 0\}} \tag{5.23}$$

Moreover the solution $(\rho(\eta_T), Z, U) \in \mathcal{S}_Q(|\eta_T|) \times \mathcal{H}^{2p} \times \mathcal{H}^{2p}_\lambda$. In fact in a forward point of view, the entropy risk measure defined by the BSDE (5.22) is a quadratic exponential seminartingale and the space of solution is deduced from Proposition (4.5.) Moreover from Theorem 3.2 the martingale $\delta Z.W + (e^{\delta U} - 1) \in \mathcal{U}_{\exp}$ hence from 5.23, the coefficient f satisfies the $(\mathcal{A}_\gamma)$ confition and the solution of the BSDE (5.22) is unique.

5.2.2 Existence of solution for quadratic BSDE with jumps

Under the above assumptions we are now able to establish the main contribution of this section. In the sample case of quadratic exponential BSDE for dynamic entropy risk measure, we characterize the space of solution using a priori estimates of quadratic exponential semimartngales given in Proposition 4.5. In the general case, we adopt the same forward point of view since if the a solution (Y, Z, U) of the quadratic exponential BSDE exists then the process Y is a quadratic exponential semimartingale and its martingale satisfies a priori-estimates. Therefore, we construct a monotone sequence of quadratic exponential semimartingale which are solution of Lipschitz BSDEs and using a priori estimates and stability we prove that the limit exist and it is a solution of a quadratic exponential BSDE.

Theorem 5.6. *There exists a triple* $(Y,Z,U) \in \mathcal{S}_Q(|\eta_T|,\Lambda,C) \times \mathcal{H}^{2p} \times \mathcal{H}^{2p}_\lambda$ *solution of the quadratic exponential BSDE associated to* (f,η_T) *with parameters* (δ,c,l) *where* ($\Lambda_t = \int_0^t l_s$, $C_t = \int_0^t c_s ds$, $t \leq T$).

Moreover if the terminal condition η_T *is bounded and* f *satisfies the Lipschitz local condition* ***(HL-b)*** *and the simple* ***(DC)*** *condition form then the solution* $(Y,Z,U) \in \mathcal{S}^\infty \times$ BMO $\times\, \mathcal{S}^\infty$ *is unique.*

Proof. :

We follow two steps to prove the existence. Firstly, we construct a sequence of quadratic exponential semimartingales which converges and secondly we find the convergence of the finite variation and martingale part using stability result. Finaly we prove the unicity in a particular case.

First step: Construction of the sequence of $\mathcal{S}_Q(|\eta_T|,\Lambda,C)$ semimartingales using Inf and sup convolution

Without losing any generality, let assume $\delta = 1$ and consider the $q_{\exp}(l,c,1)$-BSDE associated to (f,η_T). Since the BSDE coefficient f satisfies the decomposition condition **(DC)**, there exist a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+1})$ measurable function $\hat{f}$ and a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+2})$ measurable function g such that:

$$f_t(y,z,u) = \hat{f}_t(y,z) + \int_E h_t(y,z,u(x))\zeta(t,x)\lambda(dx)$$

For any function k, we set $k^+ = k1_{k>0}$ and $k^- = k1_{k\leq 0}$ and we define for each $n,m \in \mathbb{N}$ the sequences of coefficients:

$$f_t^{+,n}(y,z,u) = \inf_{w,r}\left\{\hat{f}_t^+(w,r) + n|w-y| + n|r-z|\right\}$$
$$+ \int_E \inf_{w,r,v(x)} \left\{h_t^+(w,r,v(x)) + n|w-y| + n|r-z| + n|v(x)-u(x)|\right\}\zeta(t,x)\lambda(dx)$$

and

$$f_t^{-,m}(y,z,u) = \sup_{w,r}\left\{\hat{f}_t^-(w,r) - m|w-y| - m|r-z|\right\}$$
$$+ \int_E \sup_{w,r,v(x)} \left\{h_t^-(w,r,v(x)) - m|w-y| - m|r-z| - m|v(x)-u(x)|\right\}\zeta(t,x)\lambda(dx)$$

For all $t \in [0,T]$, let us define the function $(y,z,u) \to q_t^+(y,z,u) = \frac{1}{2}|z|^2 + \int_E g(u(x))\zeta(t,x)\lambda(dx)$. For each $n,m \in \mathbb{N}$, let us define the sequences of coefficients:

$$q_t^{+,n}(y,z,u) = \inf_{r,w,v}\left\{+|l_t| + c|r| + \frac{1}{2}|w|^2 + +n|y-r| + n|r-z|\right\}$$
$$+ \int_E \inf_{v(x)} \left\{g(v(x)) + n|v(x)-u(x)|\right\}\zeta(t,x)\lambda(dx)$$

and

$$q_t^{-,m}(y,z,u) = \sup_{r,w,v}\left\{-|l_t| - c|r| - \frac{1}{2}|w|^2 - m|y-r| - m|w-z|\right\}$$
$$+ \int_E \sup_{v(x)} \left\{-g(-v(x)) - m|v(x)-u(x)|\right\}\zeta(t,x)\lambda(dx)$$

1. The sequence $f^{+,n}$ and $q^{+,n}$ resp(the sequence $f^{-,m}$ and $q^{-,m}$) are increasing and converge to f^+, q^+ resp (are decreasing and converge to f^-, q^-). Moreover the sequence $f^{-,m}, q^{-,m}$, $f^{+,n}$ and $q^{+,n}$ satisfy the **(HL-a)**, **(HI-b)** ,**(IC-b)** **(DC)** and $\mathcal{A}_\gamma$-conditions.

2. The sequences $(f^{+,n})_{n\in\mathbb{N}}$ resp(the sequence $(f^{-,m})_{m\in\mathbb{N}}$) satisfies $0 \leq f^{+,n} \leq q^{+,n} \leq q^+$ (resp $q^- \leq q^{-,m} \leq f^{-,m} \leq 0$), for each $n, m \in \mathbb{N}$. Moreover

$$q^- \leq f^{+,n} + f^{-,m} \leq q^+.$$

The point 1), the the proof is given in the Appendix (Properties of the BSDE's coefficient sequences). For the point 2), let consider the coefficient f associated to the $q_{exp}(l, c, 1)$ BSDE, for each $n, m \in \mathbb{N}$, $0 \leq f^{+,n} \leq q^{+,n} \leq q^+$. By similar arguments we find and $0 \geq f^{-,m} \geq q^{-,m} \geq q^-$, hence we conclude $0 \leq f^{+,n} \leq q^{+,n} \leq q^+$ and $q^- \leq q^{-,m} \leq f^{-,m} \leq 0$. Moreover, using the last inequalities we deduce $q^- \leq f^{+,n} + q^{-,m} \leq q^+$, for each $n, m \in \mathbb{N}$.

Let now consider the sequence of coefficients $f^{n,m} = f^{+,n} + f^{-,m}$ which converges to f when n, m go to infinity. We consider the BSDE associated to $(f^{n,m}, \eta_T)$:

$$-dY_t^{n,m} = f^{n;m}(t, Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m})dt - Z_t^{n,m}dW_t - \int_E U^{n,m}(t,x).\widetilde{\mu}(dt,dx), \quad Y_T^{n,m} = \eta_T.$$

Since for each $n, m \in \mathbb{N}$ the coefficient $f^{n,m}$ satisfies the **(IC-b)**, **(IC-a)** **(HL-a)**,**(HI-b)** conditions, there exists a solution $(Y^{n,m}, Z^{n,m}, U^{n,m})$ of the BSDE associated to $(f^{n,m}, \eta_T)$. Moreover $q^- \leq f^{n,m} \leq q^+$, hence $(Y^{n,m})_{n,m\in\mathbb{N}}$ is a sequence of $\mathcal{Q}(\Lambda, C)$ semimartingales. Since $q^{-,m} \leq f^{n,m} \leq q^{+,n}$ and the triple $(Y^{+,n}, Z^{+,n}, U^{+,n})$ and $(Y^{-,m}, Z^{-,m}, U^{-,m})$ solutions of the $q_{exp}(l, c, 1)$ BSDE associated to $(q^{+,n}, |\eta_T|)$ and $(q^{-,m}, -|\eta_T|)$ exist and satisfy for all stopping times $\sigma \leq T$:

$$|Y_\sigma^{+,n}| \vee |Y_\sigma^{-,m}| \leq \bar{\rho}_\sigma \left[e^{C_{\sigma,T}}|\eta_T| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s \right], \quad a.s$$

The solution exists since $q^{+,n}$ and $q^{-,m}$ satisfy the continuity, the integrability and the Lipschitz conditions for each $n, m \in \mathbb{N}$. Moreover since they also satisfy the $(\mathcal{A}_\gamma)$-condition, we find by comparison result in that Lipschitz BSDE's case $Y^{-,m} \leq Y^{n,m} \leq Y^{+,n}$ and we conclude $(Y^{n,m})_{n,m\in\mathbb{N}}$ is a sequence of $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales since for each $n, m \in \mathbb{N}$, $Y^{n,m}$ is a $\mathcal{Q}(\Lambda, C)$ semimartingale satisfying:

$$|Y_\sigma^{n,m}| \leq \bar{\rho}_\sigma \left[e^{C_{\sigma,T}}|\eta_T| + \int_\sigma^T e^{C_{\sigma,s}} d\Lambda_s \right], \quad a.s \tag{5.24}$$

This equality is given in Appendix (Properties of the BSDE's coefficient sequences). Since the coefficient $f^{n,m}$ satisfies the $(\mathcal{A}_\gamma)$ condition, we can apply comparison result in the Lipschitz BSDE case; we deduce for each $n, m \in \mathbb{N}$:

$$Y^{n+1,m} \geq Y^{n,m} \geq Y^{n,m+1}.$$

Second step: (Convergence of the semimartingale, the finite variation and the martingale part). For each $m \in \mathbb{N}$, $(Y^{n,m})_{n\geq 0}$ is an increasing sequence of bounded càdlàg $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales, with canonical decomposition $Y^{n,m} = Y_0^{n,m} - V^{n,m} - M^{n,m}$. Hence, this sequence converges, let denote Y^m its limit for each $m \in \mathbb{N}$. From stability result, Lemma 4.1, $(Y^m)_m$ is a sequence of càdlàg $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales with canonical decomposition $Y^m = Y_0^m - V^m + M^m$ where

$$\lim_{n\to\infty} \mathbb{E}\left[\left(V^{n,m} - V^m\right)^*\right] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|M^{n,m} - M^m\|_{\mathcal{H}^1} = 0.$$

For each $n, m \geq c^*$, since $Y^{n,m} \geq Y^{n,m+1}$ then $(Y^m)_m$ is a decreasing sequence of bounded càdlàg $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingales. Let Y its limit, from stability result Lemma 4.1, Y is a càdlàg $\mathcal{S}_Q(|\eta_T|, \Lambda, C)$ semimartingale with canonical decomposition $Y = Y_0 - V + M$ where

$$\lim_{m\to\infty} \mathbb{E}\left[\left(V^m - V\right)^*\right] = 0 \quad \text{and} \quad \lim_{m\to\infty} \|M^m - M\|_{\mathcal{H}^1} = 0.$$

Let recall $dV_t^{n,m} = f^{n,m}(Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m})dt$ and consider the sequence of stopping times $(T_K)_{K\geq 0}$ defined by:

$$T_K = \inf\left\{t \geq 0, \mathbb{E}\left[\exp(e^{C_T}|\eta_T| + \int_0^T e^{C_s} d\Lambda_s)|\mathcal{F}_t\right] > K\right\}$$

The sequence $(T_K)_{K\geq 0}$ converges to infinity when K goes to infinity, moreover for $K > K_\epsilon$ large enough, $\mathbb{P}(T_K < T) \leq \frac{\epsilon}{K}$. From 5.24, we find $Y^{n,m}_{.\wedge T_K}$ lives in a compact set and its convergence to the càdlàg process Y is uniform. The same property holds for $M^{n,m}_{.\wedge T_K}$ and $V^{n,m}_{.\wedge T_K}$. Let $Z_t^{n,m,K} = Z^{n,m} 1_{t<T_K}$ and $U_t^{n,m,K} = U^{n,m} 1_{t<T_K}$ in such that $(Z^{n,m}.W)_{.\wedge T_K} = Z^{n,m,K}.W$ and $(U^{n,m}.\widetilde{\mu})_{.\wedge T_K} = U^{n,m,K}.\widetilde{\mu}$. Since the sequence $M^{n,m}_{.\wedge T_K} = (Z^{n,m}.W)_{.\wedge T_K} + (U^{n,m}.\widetilde{\mu})_{.\wedge T_K}$ strongly converges, the sequence of orthogonal martingales $(Z^{n,m,K}.W)$ and $(U^{n,m,K}.\widetilde{\mu})$ also converge in their appropriate space. Therefore, we can extract a subsequence $Z^{n,m,K}$ and $U^{n,m,K}$ converging a.s to some processes Z and U.

For $t \leq T_K$, the sequence $f^{n,m}(t, Y_t^{n,m}, Z_t^{n,m,K}, U_t^{n,m,K})$ converges to $f(t, Y_t, Z_t, U_t)dt \otimes d\mathbb{P}$ a.s . it remains to prove that $\mathbb{E}\left[\int_0^{T_K} |f^{n,m}(t, Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m}) - f(t, Y_t, Z_t, U_t)|dt\right]$ goes to zero when n, m go to infinity. Firstly we have

$$\mathbb{E}\left[\int_0^{T_K} |f^{n,m}(t, Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m}) - f(t, Y_t, Z_t, U_t)| 1_{\{[Z_t^{n,m}|+|U_t^{n,m}|\leq C\}} dt\right]$$

goes to zero when n, m go to infinity, by dominated convergence since $Y^{n,m}$ is bounded and $|f^{n,m}(t, Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m}) - f(t, Y_t, Z_t, U_t)|$ is uniformly bounded in L^1 by Lemma 4.1. Moreover for $s \leq T_K$, $\mathbb{P}\left(|Z_s^{n,m}| + |U_s^{n,m}| > C\right) \leq \frac{2}{C^2}\mathbb{E}(|Z_s^{n,m}|^2 + |U_s^{n,m}|^2)$, from Lemma 4.1, there exists a constants C_2 such that $\mathbb{E}(\langle M^{n,m}\rangle_s) \leq C_2$, therefore

$$\mathbb{E}\left[\int_0^{T_K} |f^{n,m}(t, Y_t^{n,m}, Z_t^{n,m}, U_t^{n,m}) - f(t, Y_t, Z_t, U_t)| 1_{\{[Z_t^{n,m}|+|U_t^{n,m}|> C\}} dt\right]$$

goes to zero when C goes to infinity, uniformly in n, m. As a consequence, the process V in the decomposition of the quadratic exponential semimartingale Y is given by $dV_t =$

$f(t, Y_t, z_t, U_t)dt$ on $[0, T_K]$ for any K. We conclude the triple (Y, Z, U) is a solution of the $q_{\exp}(l, c, 1)$ BSDE associated to (f, η_T). Moreover since Y belongs to the space $\mathcal{S}_Q(\eta_T, \Lambda, C)$, then from Proposition 4.5 the martingales $Z.W+(e^U-1).\widetilde{\mu}$ and $-Z.W+(e^{-U}-1).\widetilde{\mu}$ belongs to the space $\mathcal{M}_0^p$.

Third step: Unicity in a particular case: Let $(Y^1, Z^1, , U^1)$ and $(Y^2, Z^2, , U^2)$ two solutions of the quadratic exponential BSDE with bounded terminal condition. Therefore from [7], the processes Y^1, Y^2 and U^1, U^2 are bounded. Moreover from Proposition (4.5), $Z^1.W$ and $Z^2.W$ are BMO.

Let define the processes $\Delta Y = Y^1 - Y^2$, $\Delta Z = Z^1 - Z^2$, $\Delta U = U^1 - U^2$, $\bar{Z}^{1,i} = (Z^{2,1}, \ldots, Z^{2,i-1}, Z^{1,i}, \ldots, Z^{1,d})$ for $i = 2, \ldots, d$ and $\bar{Z}^{1,1} = Z^1$ and $\bar{Z}^{2,i} = (Z^{2,1}, \ldots, Z^{2,i}, Z^{1,i+1}, \ldots, Z^{1,d})$ for $i = 1, \ldots, d-1$ and $\bar{Z}^{2,d} = Z^2$. Then, we consider the following processes: for all $i = 1, \ldots, d, \quad t \leq T$

$$\beta_t^i(Z_t^1, Z_t^2) = \begin{cases} \dfrac{\hat{f}_t(Y_t^1, \bar{Z}_t^{1,i}) - \hat{f}_t(Y_t^1, \bar{Z}_t^{2,i})}{Z_t^{1,i} - Z_t^{2,i}}, & \text{if} \quad Z_t^{1,i} \neq Z_t^{2,i}, \\ 0, & \text{if} \quad Z_t^{1,i} = Z_t^{2,i}. \end{cases}$$

Then we have:

$$\hat{f}_t(Y_t^1, Z_t^1) - \hat{f}_t(Y_t^1, Z_t^2) = \sum_{i=1}^{d} \beta_t^i(Z_t^1, Z_t^2)(Z_t^{1,i} - Z_t^{2,i})$$

with $|\beta_t(Z_t^1, Z_t^2)| \leq C\left(1 + |Z_t^1| + |Z_t^2|\right)$ for all $t \in [0, T]$.

Since the coefficient f is Lipschitz with respect to y then we consider the following bounded process: for all $\quad t \leq T$

$$a_t(Y_t^1, Y_t^2) := \begin{cases} \dfrac{f_t(Y_t^1, Z_t^2, U_t^1) - f_t(Y_t^2, Z_t^2, U_t^1)}{Y_t^1 - Y_t^2}, & \text{if} \quad Y_t^1 \neq Y_t^2, \\ 0, & \text{if} \quad Y_t^1 = Y_t^2. \end{cases}$$

Moreover, since the coefficient g satisfies $\forall t \in [0, T]$ and $x \in E$:

$$\gamma_t(U_t^1(x), U_t^2(x)) := g_t(Y_t^2, U_t^1(x)) - g_t(Y_t^2, U_t^2(x)) = \int_0^1 \frac{\partial g_t}{\partial u}(Y_s^1, \nu U_t^1(x) + (1-\nu)U_t^2(x))d\nu.$$

Since Y^1, U^1, U^2 are bounded and $\frac{\partial g_t}{\partial u}$ is continuous and $\frac{\partial g_t}{\partial u} > -1$, there exists a constant $C > 0$ such that:

$$-1 < \gamma_t(U_t^1(x), U_t^2(x)) \leq C.$$

We get:

$$\begin{aligned} d\Delta Y_t = -[(f_t(Y_t^1, Z_t^1, U_t^1) - f_t(Y^1, Z_t^2, U_t^1)) + (f_t(Y_t^1, Z_t^2, U_t^1) - f_t(Y_t^2, Z_t^2, U_t^1) \\ + (f_t(Y_t^2, Z_t^2, U_t^1) - f_t(Y_t^2, Z_t^2, U_t^2))]dt + \Delta Z_t.dW_t + \int_E \Delta U_t.\widetilde{\mu}(dt, dx) \end{aligned}$$

Therefore we get:

$$-d\Delta Y_t = \beta_t(Z_t^1, Z_t^2).\Delta Z_t dt + a_t(Y_t^1, Y_t^2)\Delta Y_t dt + \int_E \gamma_t(U_t^1(x), U_t^2(x))\Delta U_t(x)\xi(t,x)\lambda(dx)$$
$$- \Delta Z_t.dW_t - \int_E \Delta U_t.\widetilde{\mu}(dt,dx)$$

hence let define the adjoint process $\frac{d\Gamma_t}{\Gamma_t} = a_t(Y_t^1, Y_t^2)dt, \quad \Gamma_0 = 1$

$$\Gamma_t\Delta Y_t \leq \mathbb{E}\Big[\sum_{i=1}^{d}\int_t^T \Gamma_s\Delta Z_s^i(dW_s^i - \beta_s^i(Z_s^1, Z_s^2)ds)$$
$$+ \int_E\int_t^T \Gamma_s\Delta U_s(x)(\widetilde{\mu}(ds,dx) - \gamma_s(U_s^1(x), U_s^2(x))\xi(s,x)\lambda(dx))\Big|\mathcal{F}_t\Big].$$

Define the probability measure $\mathbb{Q}$ with the Radon-Nikodym density $Z^{\mathbb{Q}}$ with respect to $\mathbb{P}$ given by:

$$dZ_t^{\mathbb{Q}} = Z_{t^-}^{\mathbb{Q}}\left(\beta_t(Z_t^1, Z_t^2)\cdot dW_t + \int_E \gamma_t(U_t^1(x), U_t^2(x))\widetilde{\mu}(dt,dx)\right).$$

Since $-1 \leq \gamma_t^i(U_t^1, U_t^2) \leq C$ and $|\beta_t(Z_t^1, Z_t^2)| \leq C\left(1 + [Z_t^1| + |Z_t^2|\right)$ for all $t \in [0,T]$, we have that $\beta \star W + \gamma \star \widetilde{\mu}$ is BMO-martingale, hence $Z^{\mathbb{Q}}$ is uniformly integrable. Therefore, we have that $\Delta Y_t \leq 0$, and then $Y_t^1 \leq Y_t^2$ a.s. We use the same arguments to prove $Y_t^2 \leq Y_t^1$ a.s. permutting the role of Y^1 and Y^2. □

□

6 Appendix

6.1 Tehnical lemma

Lemma 6.2. *For any $k \geq 1$ and any local martingale M:*

$$j_t(k\Delta M_t) \geq k j_t(\Delta M_t), \quad 0 \leq t \leq T$$

Proof. Let recall that for any local martingale $M = M^c + M^d$ from representation theorem, there exists $U \in \mathcal{G}_{loc}(\mu)$ such that $M^d = U.\widetilde{\mu}$, then $j(\Delta M^d) = (e^U - U - 1).\nu$. Therefore from representation theorem it is sufficient to prove the following function f_k: $x \longrightarrow (e^{kx} - kx - 1) - k(e^x - x - 1)$ is positive to find the result. For any $x \in \mathbb{R}$, since $f_k'(x) = ke^x(e^{(k-1)x} - 1)$, then we conclude the function f_k is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$. Therefore, for any $x \in \mathbb{R}$, $f_k(x) \geq f_k(0) = 0$. □

6.2 Properties of the BSDE's coefficients sequence

6.2.1 Properties for the bound sequences $q^{+,n}$ and $q^{-,m}$

Lemma 6.3. *Let us define the following quadractic exponential coefficients by $q_t^+(y,z,u) = c|y| + |l| + \frac{1}{2}|z|^2 + j_t(u)$ and $q_t^-(y,z,u) = -c|y| - |l| - \frac{1}{2}|z|^2 - j_t(-u)$ and define the sequence $q^{+,n}$ and $q^{-,m}$ by the inf-convolution and sup-convolution for $n, m \geq c^* = \sup_{t\in[0,T]} c_t$:*

$$q^{+,n}(y,z,u) = \inf_{r,w,v} \{q^+(r,w,v) + n|y-r| + n|z-w| + n|u-v|\}, \quad |v| := \int_E |v(x)|\xi(t,x)\lambda(dx)$$
$$q^{-,m}(y,z,u) = \sup_{r,w,v} \{q^-(r,w,v) - m|y-r| - m|z-w| - m|u-v|\}$$

then:

i) The sequences $q^{+,n}$ and $q^{-,m}$ satisfy the structure condition $\mathcal{Q}(\Lambda, C)$.

ii) There exists a unique solution $(Y^{+,n}, Z^{+,n}, U^{+,n})$ (resp. $((Y^{-,m}, Z^{-,m}, U^{-,m}))$ of the BSDE's associated to $(q^{+,n}, |\xi_T|)$ (resp. to $(q^{-,m}, -|\xi_T|)$.

iii) The processes $Y^{+,n}$ and $Y^{-,m}$ are values processes of the following robust optimization problem,, for any $\sigma \leq T$:

$$Y_\sigma^{+,n} = \sup_{\{\mathbb{Q}\ll\mathbb{P}, |\beta|\leq n; -1\leq\kappa\leq n\}} \mathbb{E}_\sigma^{\mathbb{Q}} \left[S_{\sigma,T}^c |\xi_T| + \int_\sigma^T S_{\sigma,t}^c |l_t| dt + \int_\sigma^T c_t S_{\sigma,t}^c \ln\left(\frac{Z_t^{\mathbb{Q}}}{Z_\sigma^{\mathbb{Q}}}\right) dt + S_{\sigma,T}^c \ln\left(\frac{Z_T^{\mathbb{Q}}}{Z_\sigma^{\mathbb{Q}}}\right) \right]$$
$$Y_\sigma^{-,m} = \inf_{\{\mathbb{Q}\ll\mathbb{P}, |\beta|\leq m; -1\leq\kappa\leq m\}} \mathbb{E}_\sigma^{\mathbb{Q}} \left[-S_{\sigma,T}^c |\xi_T| - \int_\sigma^T S_{\sigma,t}^c |l_t| dt - \int_\sigma^T c_t S_{\sigma,t}^c \ln\left(\frac{Z_t^{\mathbb{Q}}}{Z_\sigma^{\mathbb{Q}}}\right) dt + S_{\sigma,T}^c \ln\left(\frac{Z_T^{\mathbb{Q}}}{Z_\sigma^{\mathbb{Q}}}\right) \right]$$

where $S_{\sigma,t}^c = \exp(\int_\sigma^t c_s ds)$ and the Radon Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}_T$ is $Z_T^{\mathbb{Q}}$, the process $Z_t^{\mathbb{Q}} := \mathbb{E}\left[Z_T^{\mathbb{Q}} | \mathcal{G}_t\right] = \mathcal{E}(\beta.W + \kappa.\widetilde{\mu})_t$.

Moreover, we have the following estimates:

$$Y_\sigma^{-,m} \le Y_\sigma^{n,m} \le Y_\sigma^{+,n}, \quad \sigma \le T.$$

and

$$|Y_\sigma^{-,m}| \vee |Y_\sigma^{+,n}| \le \rho_\sigma \left[S_{\sigma,T}^c |\xi_T| + \int_\sigma^T S_{\sigma,s}^c |l_s| ds \right], \quad \sigma \le T.$$

Proof: i) We compute explicitly the sequence of functions $\bar{q}^n$ and $\underline{q}^m$ which satisfy the structure condition $\mathcal{Q}_{\exp}(\Lambda, C)$ for every $n, m \ge c^*$. By definition we have

$$\begin{aligned} q_t^{+,n}(y,z,u) &= \inf_{r,w,v} \{ q_t^+(y,w,v) + n|y-r| + n|z-w| + n|u-v| \} \\ &= c|y| + |l| + \inf_w \{ \frac{1}{2}|w|^2 + n|z-w| \} + \inf_v \{ j_t(v) + n|u-v| \} \end{aligned}$$

Obviously one can find the explicit form of $q^{+,n}$ which is given by

$$\begin{aligned} &q^{+,n}(y,z,u) = c|y| + |l| + \frac{1}{2}|z|^2 \mathbf{1}_{\{|z| \le n\}} + n(|z| - \frac{n}{2}) \mathbf{1}_{\{|z|>n\}} \\ &+ \int_E \Big[g(u(e)) \mathbf{1}_{\{e^{u(e)}-1 \le n\}} + \big(-(n+1)\ln(n+1) + n(u(e)+1) \big) \mathbf{1}_{\{e^{u(e)}-1>n\}} \Big] \zeta_t(e) \rho(de) \end{aligned}$$

where we recall $g(x) = e^x - x - 1$, using the similar arguments we find the explicitely form of $\underline{q}_m$:

$$\begin{aligned} &q^{-,m}(y,z,u) = -c|y| - |l| - \frac{1}{2}|z|^2 \mathbf{1}_{\{|z| \le m\}} - m(|z| - \frac{m}{2}) \mathbf{1}_{\{|z|>m\}} \\ &+ \int_E \Big[g(-u(e)) \mathbf{1}_{\{e^{-u(e)}-1 \le m\}} + \big((m+1)\ln(m+1) + m(u(e)-1) \big) \mathbf{1}_{\{e^{-u(e)}-1>m\}} \Big] \zeta_t(e) \rho(de) \end{aligned}$$

then we conclude for each $n, m \ge c^*$, $q^{+,n}$ and $q^{-,m}$ satisfy the structure condition $\mathcal{Q}_{exp}(\Lambda, C)$.

ii) Since the coefficients $q^{+,n}$ and $q^{-,m}$ are Lipschitz we deduce there exists a solution $(Y^{+,n}, Z^{+,n}, U^{+,n})$ resp $(Y^{-,m}, Z^{-,m}, U^{-,m})$ associated to $(q^{+,n}, |\xi_T|)$ resp $(q^{-,m}, -|\xi_T|)$. Let now prove these coefficients satisfy the $(\mathcal{A}_\gamma)$ condition. To prove this result let first remark that for all $x \in \mathbb{R}$:

$$-(n+1)\ln(n+1) + n(x+1) = g[\ln(n+1)] + n(x - \ln(n+1))$$

Let $u, \bar{u}$, we set $E = \bigcup_{i=1}^4 A_i$ where

$$\begin{aligned} A_1 &= \{x \in E, e^{u(x)} - 1 \le n, e^{\bar{u}(x)} - 1 \le n\}, \quad A_2 = \{x \in E, e^{u(x)} - 1 \le n, e^{\bar{u}(x)} - 1 > n\} \\ A_3 &= \{x \in E, e^{u(x)} - 1 > n, e^{\bar{u}(x)} - 1 \le n\}, \quad A_4 = \{x \in E, e^{u(x)} - 1 > n, e^{\bar{u}(x)} - 1 > n\} \end{aligned}$$

Therefore we find for all y, z:

$$\begin{aligned} &q^{+,n}(y,z,u) - q^{+,n}(y,z,\bar{u}) \\ &= \int_{A_1} [g(u(x)) - g(\bar{u}(x))] \zeta_t(x) \rho(dx) + \int_{A_2} [g(u(x)) - g(\ln(n+1)) - n(\bar{u}(x) - \ln(n+1))] \zeta_t(x) \rho(dx) \\ &+ \int_{A_3} [n(u(x) - \ln(n+1)) + g(\ln(n+1)) - g(\bar{u}(x))] \zeta_t(x) \rho(dx) + \int_{A_4} n(u(x) - \bar{u}(x)) \zeta_t(x) \rho(dx) \end{aligned}$$

We now find differents inequalities on every given subset of E:

On the set A_1: we use the convex property of the coefffient g and we deduce:

$$\int_{A_1}[g(u(x))-g(\bar{u}(x))]\zeta_t(x)\rho(dx)\leq\int_{A_1}(e^{\bar{u}(x)}-1)(u(x)-\bar{u}(x))\zeta_t(x)\rho(dx) \tag{6.25}$$

On the set A_2: Since on A_2, the function g is increasing we find $\forall x\in A_2$, $g(u(x))\leq g(\ln(n+1))$, moreover we have $\bar{u}(x)-\ln(n+1)\geq 0$ for $x\in A_2$ then we conclude:

$$\int_{A_2}[g(u(x))-g(\ln(n+1))-n(\bar{u}(x)-\ln(n+1))]\zeta_t(x)\rho(dx)\leq 0. \tag{6.26}$$

On the set A_3: we use the convex property of the coefficient g and we find $g(\ln(n+1))-g(\bar{u}(x))\leq -n(\bar{u}(x)-\ln(n+1)),\quad \forall x\in A_3$. Therefore, we find:

$$\int_{A_3}[n(u(x)-\ln(n+1)+g(\ln(n+1))-g(\bar{u}(x))]\zeta_t(x)\rho(dx)\leq\int_{A_3}n(u(x)-\bar{u}(x))\zeta_t(x)\rho(dx) \tag{6.27}$$

Hence using (6.25), (6.26) and (6.27), we find:

$$q^{+,n}(y,z,u)-q^{+,n}(y,z,\bar{u})=\int_E\gamma^n(u(x),\bar{u}(x))(u(x)-\bar{u}(x))\zeta_t(x)\rho(dx)$$

where $\gamma^n(u(x),\bar{u}(x))=n\mathbf{1}_\{e^{u(x)}-1>n\}+(e^{\bar{u}(x)}-1)1_{A_1}$, then $-1\leq\gamma^n\leq n$ hence $\gamma^n\in\mathcal{U}^{\exp}$. Therefore the sequence $q^{+,n}$ satisfies the $\mathcal{A}_\gamma$ condition. We use similar arguments to prove the sequence $q^{-,m}$ satisfies also the $\mathcal{A}_\gamma$ condition. We conclude from Comparison Theorem, the uniqueness of the triple $(Y^{+,n},Z^{+,n},U^{+,n})$ resp($(Y^{-,m},Z^{-,m},U^{-,m})$ solution of the BSDE associated to $(q^{+,n},|\xi_T|)$ resp$((q^{-,m},-|\xi_T|))$.

iii) Let define the cost functional of the robust optimization problems defined in Lemma 6.3-iii), $J^{\mathbb{Q},+,n}$ and $J^{\mathbb{Q},-,m}$, and define the value processes $V^{+,n}$, $V^{-,m}$. Assume $|\xi_T|\in\mathcal{L}^{\exp}$, $|l|\in\mathcal{D}^1_{\exp}$ and c bounded then the value processes of the robust optimization exist see Bordigoni, Matoussi and Schweizer [10] for more details. Moreover we deduce for any $\mathbb{Q}\ll\mathbb{P}$:

$$\begin{aligned}J^{\mathbb{Q},+,n}_t&=S^c_tV^{+,n}_t+\int_0^tS^c_s|l_s|ds+\int_0^tc_sS^c_s\ln(Z^{\mathbb{Q}}_s)ds-S^c_t\ln(Z^{\mathbb{Q}}_t)\\J^{\mathbb{Q},-,m}_t&=S^c_tV^{-,m}_t-\int_0^tS^c_s|l_s|ds-\int_0^tc_sS^c_s\ln(Z^{\mathbb{Q}}_s)ds+S^c_t\ln(Z^{\mathbb{Q}}_t)\end{aligned} \tag{6.28}$$

moereover the value processes $V^{+,n}$ and $V^{-,m}$ are special semimartingales the following the representation theorem there exist a predictable process $Z^{+,n}$, $U^{+,n}$ and an predicatble process $A^{\bar{V}^{+,n}}$ resp ($Z^{-,m}$,$U^{-,m}$ and an predictable process $A^{V^{+,n}}$) such that $d\bar{V}^{+,n}_t=dA^{V^{+,n}}_t+Z^{+,n}_t.dW_t+\int_E\bar{U}^{+,n}_t(e).\widetilde{\mu}(dt,de)$ resp($dV^{-,m}_t=dA^{\mathrm{V}^{-,m}}_t+Z^{-,m}_t.dW_t+\int_E U^{-,m}_t(e).\widetilde{\mu}(dt,de)$). we define the dynamics of $Z^{\mathbb{Q}}$,

$$dZ^{\mathbb{Q}}_t=Z^{\mathbb{Q}}_{t^-}\left(\beta_t.dW_t+\int_E\kappa_t.\widetilde{\mu}(dx,dt)\right)$$

For every $n, m \in \mathbb{N}^*$, for every $\mathbb{Q} \ll \mathbb{P}$, $J^{\mathbb{Q},+,n}$ resp($J^{\mathbb{Q},-,m}$) are submartingales and martingales for the optimal resp(surmartingales and martingale for the optimal) then we get:

$$A_t^{V^{+,n}} = -\left\{\int_0^t \max_{\{|\beta|\leq n\}} (|l_s| + c_s V_s^{+,n}) + \int_0^t \left(\langle Z^{+,n} - \beta, \beta\rangle_s + \frac{1}{2}|\beta_s|^2\right) ds\right\}$$
$$- \left\{\int_0^t \int_E \max_{\{-1\leq\kappa\leq n\}} (\kappa_s(x)(v_s(x)+1) - (1+\kappa_s(x))\ln(1+\kappa_s(x)))\,\zeta_s(x)\rho(dx)ds\right\}$$

resp

$$A_t^{V^{-,m}} = -\left\{\int_0^t \min_{\{|\beta|\leq m\}} (-|l_s| + c_s V_s^{-,m}) + \int_0^t \left(\langle Z^{-,m} + \beta, \beta\rangle_s - \frac{1}{2}|\beta_s|^2\right) ds\right\}$$
$$- \left\{\int_0^t \int_E \min_{\{-1\leq\kappa\leq m\}} (\kappa_s(x)(v_s(x)-1) + (1+\kappa_s(x))\ln(1+\kappa_s(x)))\,\zeta_s(x)\rho(dx)ds\right\}$$

Using first order condition, we find for the first optimization problem $\kappa^* = \left(e^{U^{+,n}} - 1\right)\mathbf{1}_{\{e^{U^{+,n}}-1\leq n\}} + n\mathbf{1}_{\{e^{U^{+,n}}-1>n\}}$, then we deduce $(V^{+,n}, Z^{+,n}, U^{+,n})$ is the solution associated to the BSDE $(h^{+,n}, |\xi_T|)$:

$$-dV_t^{+,n} = h^{+,n}(V^{+,n}, Z_t^{+,n}, U_t^{+,n})dt - Z_t^{+,n}.dW_t - U_t^{+,n}(x).\widetilde{\mu}(dt,dx), \quad V_T^{+,n} = |\xi_T|.$$

where

$$h^{+,n}(y,z,u) = cy + |l| + \frac{1}{2}|z|^2\mathbf{1}_{\{|z|\leq n\}} + n(|z| - \frac{n}{2})\mathbf{1}_{\{|z|>n\}}$$
$$+ \int_E \left[(e^{u(e)} - u(e) - 1)\mathbf{1}_{\{e^{u(e)}-1\leq n\}} + \big(-(n+1)\ln(n+1) + n(u+1)\big)\mathbf{1}_{\{e^{u(e}-1>n\}}\right]\zeta_t(e)\rho(de)$$

We use the same arguments of first order condition to deduce the solution of the second optimization problem; We get the triple $(V^{-,m}, Z^{-,m}, U^{-,m})$ is the solution of the BSDE associated to $(h^{-,m}, -|\xi_T|)$:

$$-dV_t^{-,m} = h^{-,m}(V^{-,m}, Z_t^{-,m}, U_t^{-,m})dt - Z_t^{-,m}.dW_t - \int_E U_t^{-,m}(x).\widetilde{\mu}(dx,dt), \quad \underline{V}_T^{-,m} = -|\xi_T|.$$

where

$$h^{-,m}(y,z,u) = cy - |l| - \frac{1}{2}|z|^2\mathbf{1}_{\{|z|\leq m\}} - m(|z| - \frac{m}{2})\mathbf{1}_{\{|z|>m\}}$$
$$+ \int_E \left[g(-u(e))\mathbf{1}_{\{e^{-u(e)}-1\leq m\}} + \big((m+1)\ln(m+1) + m(u(e)-1)\big)\mathbf{1}_{\{e^{-u(e)}-1>m\}}\right]\zeta_t(x)\rho(dx)$$

To finish the proof, we find $Y^{+,n} \geq 0$ and $Y^{+,m} \leq 0$ since $|\xi_T| \geq 0$ and $-|\xi_T| \leq 0$ by comparaison theorem in Lipschitz case. Then we conclude $h^{+,n} = q^{+,n}$ and $h^{-,m} = q^{-,m}$. for each $n, m \in \mathbb{N}$. By uniqueness of the solution of the BSDE associated to $(q^{+,n}, |\xi_T|)$ and $(q^{-,m}, -|\xi_T|)$, we conclude $V^{+,n} = Y^{+,n}$ and $V^{-,m} = Y^{-,m}$ moreover since $q^{-,m} \leq f^{n,m} \leq q^{+,n}$ by Comparison Theorem, we find $Y^{-,m} \leq Y^{n,m} \leq Y^{+,n}$. However, from the dual

representation of $Y^{+,n}$ (resp $Y^{-,m}$) given by $V^{+,n}$ resp($V^{-,m}$), we conclude by Proposition 4.2 of [4] that for any $\sigma \leq T$:

$$Y_\sigma^{+,n} \leq \rho_\sigma \left[S^c_{\sigma,T}|\xi_T| + \int_\sigma^T S^c_{\sigma,s}|l_s|ds \right] \text{ and } \quad Y_\sigma^{-,m} \leq \rho_\sigma \left[S^c_{\sigma,T}|\xi_T| + \int_\sigma^T S^c_{\sigma,s}|l_s|ds \right].$$

□

6.2.2 Properties for the sequences $f^{+,n}$, $f^{-,m}$ and $f^{n,m}$

Let recall that $\forall (t,y,z,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^2$, we have:

$$f_t^{n,m}(y,z,u) = \hat{f}_t^{n,m}(y,z) + \int_E h_t^{n,m}(y,z,u(x))\zeta_t(t,x)\lambda(x) \tag{6.29}$$

where

$$\hat{f}_t^{n,m}(y,z) = \hat{f}_t^{+,n}(y,z) + \hat{f}_t^{-,m}(y,z), \quad \text{and} \quad h_t^{n,m}(y,z,u(x)) = h_t^{+,n}(y,z,u(x)) + h_t^{-,m}(y,z,u(x)).$$

Since:

$$0 \leq f_t^{+,n}(y,z,u) \leq q_t^{+,n}(y,z,u) \quad \text{and} \quad q_t^{-,m}(y,z,u) \leq f_t^{-,m}(y,z,u) \leq 0$$

we get

$$q_t^{-,m}(y,z,u) \leq f_t^{n,m}(y,z,u) \leq q_t^{+,n}(y,z,u).$$

Moreover the sequence $f^{n,m}$ is increasing with respect to n and decreasing with respect to m. To conclude, let prove for decomposition of $f^{n,m}$ given by 6.29, $\frac{\partial h^{n,m}}{\partial u} > -1$ and $\frac{\partial h^{n,m}}{\partial u}(Y^{n,m}, Z^{n,m}, U^{n,m}).\widetilde{\mu} \in \mathcal{U}_{\text{exp}}$. Let us define $x \in E$ and $u(x), \bar{u}(xi \in L^2$ and the following sets:

$$A_1 = \{x \in E, h_t(y,z,u(x)) \geq 0, h_t(y,z,\bar{u}(x)) \geq 0\}, \quad A_2 = \{x \in E, h_t(y,z,u(x)) < 0, h_t(y,z,\bar{u}(x)) < 0\}$$
$$A_3 = \{x \in E, h_t(y,z,u(x)) > 0, h_t(y,z,\bar{u}(x)) < 0\}, \quad A_4 = \{x \in E, h_t(y,z,u(x)) < 0, h_t(y,z,\bar{u}(x)) > 0\}$$

We know that:

$$0 \leq h_t^{+,n}(y,z,u(x)) \leq h_t^{+}(y,z,u(x)), \quad \text{and } h_t^{-}(y,z,u(x)) \leq h_t^{-,m}(y,z,u(x)) \leq 0 \tag{6.30}$$

Therefore if $h_t^+(y,z,u(x)) \leq 0$ then $h_t^{+,n}(y,z,u(x)) = 0$ and $h_t^{-,m}(y,z,u(x)) \leq 0$ and if $h_t^+(y,z,u(x)) > 0$, $h_t^{+,n}(y,z,u(x)) \geq 0$ and $h_t^{-,m}(y,z,u(x)) = 0$. Using theses properties, we get:

$$\begin{aligned} h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,\bar{u}(x)) &= [h_t^{+,n}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x))]1_{A_1} \\ &= [h_t^{-,m}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x))]1_{A_2} \\ &= [h_t^{+,n}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x))]1_{A_3} \\ &= [h_t^{-,m}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x))]1_{A_4} \end{aligned} \tag{6.31}$$

The functions $h^{+,n}$ and $h^{-,m}$ are Lipschitz with respect to u hence differentiable. We get:

$$h_t^{+,n}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x)) = (u(x)-\bar{u}(x))\int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu u(x)+(1-\nu)\bar{u}(x))d\nu$$

and

$$h_t^{-,m}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x)) = (u(x)-\bar{u}(x))\int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu u(x)+(1-\nu)\bar{u}(x))d\nu$$

Let define

$$\gamma_t^{1,n,m}(y,z,u(x),\bar{u}(x)) = \int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu u(x)+(1-\nu)\bar{u}(x))d\nu \tag{6.32}$$

and

$$\gamma_t^{2,n,m}(y,z,u(x),\bar{u}(x)) = \int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu u(x)+(1-\nu)\bar{u}(x))d\nu \tag{6.33}$$

On A_3 since the function h is continue with respect to u, using "the valeurs intermdiaires Theorem", we can find $\hat{u}(x)$ such that $h_t(y,z,\hat{u}(x)) = 0$. Using 6.30, $h_t^{-,m}(y,z,\hat{u}(x)) = h_t^{+,n}(y,z,\hat{u}(x)) = 0$, then on the set A_3:

$$\begin{aligned}
h_t^{+,n}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x)) &= [h_t^{+,n}(y,z,u(x)) - h_t^{+,n}(y,z,\hat{u}(x))] \\
&\quad + [h_t^{-,m}(y,z,\hat{u}(x)) - h_t^{-,m}(y,z,\bar{u}(x))] \\
&= (u(x)-\hat{u}(x))\int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu u(x)+(1-\nu)\hat{u}(x))d\nu \\
&\quad + (\hat{u}(x)-\bar{u}(x))\int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu \hat{u}(x)+(1-\nu)\bar{u}(x))d\nu
\end{aligned}$$

Let

$$\gamma_t^{+,3,n}(y,z,u(x),\hat{u}(x)) = \int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu u(x)+(1-\nu)\hat{u}(x))d\nu,$$

and

$$\gamma_t^{-,3,m}(y,z,\hat{u}(x),\bar{u}(x)) = \int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu \hat{u}(x)+(1-\nu)\bar{u}(x))d\nu.$$

Therefore, we get

$$h_t^{+,n}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x)) = \gamma_t^{3,n,m}(y,z,u(x),\hat{u}(x))(u(x)-\bar{u}(x)) \tag{6.34}$$

where

$$\gamma_t^{3,n,m}(y,z,u(x),\bar{u}(x)) = \left[\gamma_t^{+,3,n}(y,z,u(x),\hat{u}(x)) + [\gamma_t^{-,3,m} - \gamma_t^{+,3,n}](y,z,\hat{u}(x),\bar{u}(x))\frac{(\hat{u}(x)-\bar{u}(x))}{u(x)-\bar{u}(x)}\right]$$

Since $\hat{u}(x) \in]u(x) \wedge u(x), u(x) \vee \bar{u}(x)[$, then $\frac{(\hat{u}(x)-\bar{u}(x))}{u(x)-\bar{u}(x)} \in [0,1]$. Hence on the set A_3, we have:

$$\gamma_t^{-,3,m} \wedge \gamma_t^{+,3,n} \leq \gamma_t^{3,n,m} \leq \gamma_t^{-,3,m} \vee \gamma_t^{+,3,n} \tag{6.35}$$

On the set A_4, $h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,u(x)) \leq 0$ and using "the valeurs intermdiaires Theorem", we can find $\hat{u}(x)$ such that $h_t(y,z,\hat{u}(x)) = 0$. Using 6.30, $h_t^{-,m}(y,z,\hat{u}(x)) = h_t^{+,n}(y,z,\hat{u}(x)) = 0$, then on the set A_4:

$$\begin{aligned}
h_t^{-,m}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x)) &= [h_t^{-,m}(y,z,u(x)) - h_t^{-,m}(y,z,\hat{u}(x))] \\
&\quad + [h_t^{+,n}(y,z,\hat{u}(x)) - h_t^{+,n}(y,z,\bar{u}(x))] \\
&= (u(x)-\hat{u}(x)) \int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu u(x) + (1-\nu)\hat{u}(x))d\nu \\
&\quad + (\hat{u}(x)-\bar{u}(x)) \int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu \hat{u}(x) + (1-\nu)\bar{u}(x))d\nu
\end{aligned}$$

Let

$$\gamma_t^{-,4,n}(y,z,u(x),\hat{u}(x)) = \int_0^1 \frac{\partial h_t^{-,m}}{\partial u}(y,z,\nu u(x) + (1-\nu)\hat{u}(x))d\nu,$$

and

$$\gamma_t^{+,4,n}(y,z,\hat{u}(x),\bar{u}(x)) = \int_0^1 \frac{\partial h_t^{+,n}}{\partial u}(y,z,\nu \hat{u}(x) + (1-\nu)\bar{u}(x))d\nu.$$

Therefore, we get

$$h_t^{-,m}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x)) = \gamma_t^{4,n,m}(y,z,u(x),\hat{u}(x))(u(x)-\bar{u}(x)) \tag{6.36}$$

where

$$\gamma_t^{4,n,m}(y,z,u(x),\bar{u}(x)) = \left[\gamma_t^{-,4,m}(y,z,u(x),\hat{u}(x)) + [\gamma_t^{+,4,n} - \gamma_t^{-,4,n}](y,z,\hat{u}(x),\bar{u}(x))\frac{(\hat{u}(x)-\bar{u}(x))}{u(x)-\bar{u}(x)}\right]$$

Since $\hat{u}(x) \in]u(x) \wedge u(x), u(x) \vee \bar{u}(x)[$, then $\frac{(\hat{u}(x)-\bar{u}(x))}{u(x)-\bar{u}(x)} \in [0,1]$. Hence on the set A_4, we have:

$$\gamma_t^{-,4,m} \wedge \gamma_t^{+,4,n} \leq \gamma_t^{4,n,m} \leq \gamma_t^{-,4,m} \vee \gamma_t^{+,4,n}$$

Using (6.31), (6.32), (6.33), (6.34) and (6.36), we get:

$$h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,\bar{u}(x)) = \sum_{i=1}^{4} 1_{A_i}\gamma_t^{i,n,m}(y,z,u(x),\bar{u}(x))[u(x)-\bar{u}(x)]$$

and

$$\left[\sum_{i=1}^{2} 1_{A_i}\gamma_t^{i,n,m}(y,z,u(x),\bar{u}(x)) + 1_{A_4}\gamma_t^{4,n,m}(y,z,u(x),\bar{u}(x))\right][u(x)-\bar{u}(x)] \\ \leq h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,\bar{u}(x)) \tag{6.37}$$

and

$$h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,\bar{u}(x)) \leq \left[\sum_{i=1}^{3} 1_{A_i}\gamma_t^{i,n,m}(y,z,u(x),\bar{u}(x))\right][u(x)-\bar{u}(x)] \tag{6.38}$$

Therefore, since the Lipschitz constant of $h^{+,n}$ (resp $h^{-,m}$) is bounded by n (resp m), we conclude $\frac{\partial h^{+,n}}{\partial u} \leq n$ (resp $\frac{\partial h^{-,m}}{\partial u} \leq m$), we get from (6.32), (6.33),(6.35 and (6.38):

$$\sum_{i=1}^{3} 1_{A_i}\gamma_t^{i,n,m}(y,z,u(x),\bar{u}(x)) \leq (n \vee m) \tag{6.39}$$

For the other hand, let us recall that for any functions ρ and ψ, we have:

$$\inf_x \psi(x) - \inf_x \rho(x) \geq \sup_x[\psi(x)-\rho(x)], \quad \text{and } \inf_x \psi(x) - \inf_x \rho(x) \geq \sup_x[\psi(x)-\rho(x)]$$

Hence, we get for any $u(x), \bar{u}(x) \in \mathbb{R}$:

$$h_t^{+,n}(y,z,u(x)) - h_t^{+,n}(y,z,\bar{u}(x)) \geq \sup_{v(x)}\{h_t^{+}(y,z,u(x)-v(x)) - h_t^{+}(y,z,\bar{u}(x)-v(x))\} \tag{6.40}$$

and

$$h_t^{-,m}(y,z,u(x)) - h_t^{-,m}(y,z,\bar{u}(x)) \geq \sup_{v(x)}\{h_t^{-}(y,z,u(x)-v(x)) - h_t^{-}(y,z,\bar{u}(x)-v(x))\} \tag{6.41}$$

We consider

$$B_1 = \{x \in E, h_t(y,z,u(x)-v(x)) \geq 0, h_t(y,z,\bar{u}(x)-v(x)) \geq 0\},$$

$$B_2 = \{x \in E, h_t(y,z,u(x)-v(x)) < 0, h_t(y,z,\bar{u}(x)-v(x)) < 0\}$$

$$B_3 = \{x \in E, h_t(y,z,u(x)-v(x)) > 0, h_t(y,z,\bar{u}(x)-v(x)) < 0\},$$

$$B_4 = \{x \in E, h_t(y,z,u(x)-v(x)) < 0, h_t(y,z,\bar{u}(x)-v(x)) > 0\}$$

On the set B_1:

$$h_t^+(y,z,u(x)-v(x))-h_t^+(y,z,\bar{u}(x)-v(x))=h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))$$

and

$$h_t^-(y,z,u(x)-v(x))-h_t^-(y,z,\bar{u}(x)-v(x))=0$$

On the set B_2:

$$h_t^+(y,z,u(x)-v(x))-h_t^+(y,z,\bar{u}(x)-v(x))=0$$

and

$$h_t^-(y,z,u(x)-v(x))-h_t^-(y,z,\bar{u}(x)-v(x))=h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))$$

On the set B_3:

$$h_t^+(y,z,u(x)-v(x))-h_t^+(y,z,\bar{u}(x)-v(x))\geq 0$$

and

$$h_t^-(y,z,u(x)-v(x))-h_t^-(y,z,\bar{u}(x)-v(x))\geq 0$$

On the B_4:

$$h_t^+(y,z,u(x)-v(x))-h_t^+(y,z,\bar{u}(x)-v(x))\geq h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))$$

and

$$h_t^-(y,z,u(x)-v(x))-h_t^-(y,z,\bar{u}(x)-v(x))\geq h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))$$

Using (6.42) and (6.42), we get:

$$h_t^{+,n}(y,z,u(x))-h_t^{+,n}(y,z,\bar{u}(x))\geq \sup_{v(x)}\{1_{B_1\cap B_4}[h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))]\} \tag{6.42}$$

and

$$h_t^{-,m}(y,z,u(x))-h_t^{-,m}(y,z,\bar{u}(x))\geq \sup_{v(x)}\{1_{B_2\cap B_4}[h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))]\} \tag{6.43}$$

Since

$$h_t(y,z,u(x)-v(x))-h_t(y,z,\bar{u}(x)-v(x))=(u(x)-\bar{u}(x))\int_0^1\frac{\partial h_t}{\partial u}[y,z,\nu(u(x)-v(x))+(1-\nu)(u(x)-v(x)))d\nu$$

The BSDE coefficient satisfies **(DC)** condition then $\frac{\partial h_t}{\partial u} > -1$ and taking $u(x) = \bar{u}(x) + h$, we condlude:

$$\frac{\partial h_t^{+,n}}{\partial u} > -1 \quad \text{and} \ \frac{\partial h_t^{-,m}}{\partial u} > -1.$$

Therefore we get from (6.38): , $\frac{\partial h_t^{n,m}}{\partial u} > -1$ and we finally conclude from (6.39) $-1 < \frac{\partial h_t^{n,m}}{\partial u} \leq n \vee m$ and

$$h_t^{n,m}(y,z,u(x)) - h_t^{n,m}(y,z,\bar{u}(x)) = (u(x) - \bar{u}(x)) \int_0^1 \frac{\partial h_t^{n,m}}{\partial u}(y,z,\nu u(x) + (1-\nu)\bar{u}(x))d\nu$$

Therefore for any triple $(Y,Z,U) \in \mathcal{S}^2 \times H^2 \times \mathcal{H}^2_\lambda$, $\frac{\partial h_t^{n,m}}{\partial u}(Y,Z,U).\widetilde{\mu} \in$ BMO then belongs to $\mathcal{U}_{\exp}$. The sequence $f^{n,m}$ satisfies the $\mathcal{A}_\gamma$ condition.

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