Backward doubly SDEs and semilinear stochastic PDEs in a convex domain

Anis Matoussi, Wissal Sabbagh, Tusheng Zhang

Backward Doubly SDEs and Semilinear Stochastic PDEs in a convex domain

Anis Matoussi*

University of Maine
Risk and Insurance Institute of Le Mans
Laboratoire Manceau de Mathématiques
Avenue Olivier Messiaen
e-mail: anis.matoussi@univ-lemans.fr

Wissal Sabbagh

University of Maine
Risk and Insurance Institute of Le Mans
Laboratoire Manceau de Mathématiques
Avenue Olivier Messiaen
e-mail: wissal.sabbagh@univ-lemans.fr

Tusheng Zhang

School of Mathematics, University of Manchester,
Oxford Road, Manchester M13 9PL, England, UK
e-mail: tusheng.zhang@manchester.ac.uk

Abstract: This paper presents existence and uniqueness results for reflected backward doubly stochastic differential equations (in short RBDSDEs) in a convex domain D without any regularity conditions on the boundary. Moreover, using a stochastic flow approach a probabilistic interpretation for a system of reflected SPDEs in a domain is given via such RBDSDEs. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a Sobolev space and ν is a random regular measure. The bounded variation process K, the component of the solution of the reflected BDSDE, controls the set when u reaches the boundary of D. This bounded variation process determines the measure ν from a particular relation by using the inverse of the flow associated to the the diffusion operator.

Keywords and phrases: Stochastic Partial Differential Equation, Reflected Backward Doubly Stochastic Differential Equation, Skorohod Problem, Convex Domains, Stochastic Flow, Flow of Diffeomorphisms, Regular Measure.
AMS 2000 subject classifications: Primary 60H15, 60G46; secondary 35H60.

*The first author was partially supported by chaire Risques Financiers de la fondation du risque, CMAP-École Polytechniques, Palaiseau-France

1. Introduction

Our main interest is the following system of semilinear stochastic PDE with value in $\mathbb{R}^k$,

$$du_t(x) + [\mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t\sigma(x))]dt + h_t(x, u_t(x), \nabla u_t\sigma(x)) \cdot d\overleftarrow{W}_t = 0, \tag{1.1}$$

over the time interval $[0, T]$. The final condition is given by $u_T = \Phi$, f, h are non-linear random functions and $\mathcal{L}$ is the second order differential operator which is defined component by component by

$$\mathcal{L}\varphi(x) \quad = \quad \sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial x_i}\varphi(x) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x). \tag{1.2}$$

The integral term with $d\overleftarrow{W}_t$ refers to the backward stochastic integral with respect to a d-dimensional Brownian motion on $\big(\Omega, \mathcal{F}, \mathbb{P}, (W_t)_{t\geq 0}\big)$. We use the backward notation because in the proof we will employ the backwards doubly stochastic framework introduced by Pardoux and Peng [34].
Such SPDEs appear in various applications like pathwise stochastic control problems, the Zakai equations in filtering and stochastic control with partial observations. It is well known now that BSDEs give a probabilistic interpretation for the solution of a class of semi-linear PDEs. By introducing in standard BSDEs a second nonlinear term driven by an external noise, we obtain Backward Doubly SDEs (BDSDEs in short) [34] (see also [3], [29]), which can be seen as Feynman-Kac representation of SPDEs and provide a powerful tool for probabilistic numerical schemes [1] for such SPDEs. Several generalizations to investigate more general nonlinear SPDEs have been developed following different approaches of the notion of weak solutions, namely, Sobolev's solutions [12, 16, 20, 37, 40], and stochastic viscosity solutions [25, 26, 27, 7, 8].

Given a convex domain D in $\mathbb{R}^k$, our paper is concerned with the study of weak solutions to the reflection problem for multidimensional SPDEs (1.1) in D by introducing the associated BDSDE. Inspired by the variational formulation of the obstacle problem for SPDEs and Menaldi's work [32] on reflected diffusion, we consider the solution of the reflection problem for the SPDEs (1.1) as a pair (u, ν), where ν is a random regular measure and $u \in \mathbf{L}^2\big(\Omega \times [0, T]; H^1(\mathbb{R}^d)\big)$ satisfies the following relations :

$$\begin{aligned}
&(i)\ \ u_t(x) \in \bar{D}, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.},\\
&(ii)\ \ du_t(x) + \big[\, \mathcal{L}u_t(x) + f_t(x, u_t(x), \nabla u_t\sigma(x))\,\big]\, dt + h_t(x, u_t(x), \nabla u_t\sigma(x)) \cdot d\overleftarrow{W}_t = -\nu(dt, dx), \quad a.s.,\\
&(iii)\ \ \nu(u \notin \partial D) = 0, a.s.,\\
&(iv)\ \ u_T = \Phi, \quad dx - \text{a.e.}.
\end{aligned} \tag{1.3}$$

ν is a random measure which acts only when the process u reaches the boundary of the domain D. The rigorous sense of the relation (iii) will be based on the probabilistic representation of the measure ν in terms of the bounded variation processes K, a component of the associated solution of the reflected BDSDE in the domain D. This problem is well known as a Skorohod problem for SPDEs.
In the case of diffusion processes in a domain, the reflection problem has been investigated by

severals authors (see [38], [41], [9], [28]). In the case of a convex domain this reflection problem was treated by Tanaka [39] and Menaldi [32] by using the variational inequality and the convexity properties of the domain.
In the one dimensional case, the reflection problem for nonlinear PDEs (or SPDEs) has been studied by using different approaches. The work of El Karoui et al [14] treats the obstacle problem for viscosity solution of deterministic semilinear PDEs within the framework of backward stochastic differential equations (BSDEs in short). This increasing process determines in fact the measure from the relation (ii). Bally et al [2] (see also Matoussi and Xu [31]) point out that the continuity of this process allows the classical notion of strong variational solution to be extended (see Theorem 2.2 of [5] p.238) and express the solution to the obstacle as a pair (u, ν) where ν is supported by the set $\{u = g\}$.
Matoussi and Stoica [30] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimensional Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation. They have also proved that the solution u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying the minimal Skohorod condition. In particular, they gave for the regular measure ν a probabilistic interpretation in terms of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

On the other hand, M.Pierre [35, 36] has studied parabolic PDEs with obstacles using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous with respect to the analytical capacity. Moreover he gave a representation of the reflected measure ν in terms of the associated regular potential and the approach used is based on analytical quasi-sure analysis. More recently, Denis, Matoussi and Zhang [11] have extended this approach for the obstacle problem of quasilinear SPDEs when the obstacle is regular in some sense and controlled by the solution of a SPDE.

Nualart and Pardoux [33] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, driven by an additive space-time white noise. They proved the existence and uniqueness of the solution and their method relied heavily on the results for a deterministic variational inequality. Donati- Martin and Pardoux [13] generalized the model of Nualart and Pardoux. The nonlinearity appears both in the drift and in the diffusion coefficients. They proved the existence of the solution by penalization method but they did not obtain the uniqueness result. And then in 2009, Xu and Zhang solved the problem of the uniqueness, see [42]. We note also that Zhang established in [43] the existence and uniqueness of solutions of system (1.3) in the forward case when x belongs to $[0, 1]$. He approximated the system of SPDEs by a penalized system and used a number of a priori estimates to prove the convergence of the solutions. However, in all their models, they do not consider the case where the coefficients depend on ∇u.

Our contributions in this paper are as following: first of all, reflected BDSDEs in the convex domain D are introduced and results of existence and uniqueness of such RBDSDEs are established. Next, the existence and uniqueness results of the solution (u, ν) of the reflection problem

for (1.1) are given in Theorem 3.1. Indeed, a probabilistic method based on reflected BDSDEs and stochastic flow technics are investigated in our context (see e.g. [3], [29], [24, 23] for these flow technics). The key element in [3] is to use the inversion of stochastic flow which transforms the variational formulation of the SPDEs to the associated BDSDEs. Thus it plays the same role as Itô's formula in the case of the classical solution of SPDEs.
We also mention the works [10], [17] and [18] where they have studied a Reflected BSDEs with oblique reflection in multi-dimensional case and their relations to switching problems.

This paper is organized as following: in Section 2, first the basic assumptions and the definitions of the solutions for Reflected BDSDE in a convex domain are presented. Then, existence and uniqueness of solution of RBDSDE (Theorem 2.1) is given under only convexity assumption for the domain without any regularity on the boundary. This result is proved by using penalization approximation. Thanks to the convexity properties we prove several technical lemmas, in particular the fundamental Lemma 2.2. In Section 3, we study semilinear SPDE's in a convex domain. We first provide useful results on stochastic flow associated with the forward SDEs, then in this setting as in Bally and Matoussi [3], an equivalence norm result associated to the diffusion process is given. The main result of this section Theorem 3.1 is the existence and uniqueness results of the solution of reflected SPDEs in a convex domain. The proof of this result is based on the probabilistic interpretation via the Reflected Forward-BDSDEs. The uniqueness is a consequence of the variational formulation of the SPDEs written with random test functions and the uniqueness of the solution of the Reflected FBSDE. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. In the Appendix, technical lemmas for the existence of the solution of the Reflected BDSDEs and SPDEs in a convex domain are given.

2. Backward Doubly Stochastic Differential Equations in a domain

2.1. Hypotheses and preliminaries

The euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $k \times k$ matrix A, we define $\|A\| = \sqrt{TrAA^*}$. In what folllows let us fix a positive number $T > 0$.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability product space, and let $\{W_s, 0 \leq s \leq T\}$ and $\{B_s, 0 \leq s \leq T\}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^W \vee \mathcal{N}$$

where $\mathcal{F}_t^B = \sigma\{B_r, 0 \leq r \leq t\}$, $\mathcal{F}_{t,T}^W = \sigma\{W_r - W_t, t \leq r \leq T\}$ and $\mathcal{N}$ the class of $\mathbb{P}$ null sets of $\mathcal{F}$. Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration.

2.1.1. Convexity results

Besides, we need to recall properties related to the convexity of a nonempty domain D in $\mathbb{R}^k$. Let ∂D denote the boundary of D and $\pi(x)$ the orthogonal projection of $x \in \mathbb{R}^k$ on the closure $\bar{D}$. We

have the following properties:

$$(x'-x)^*(x-\pi(x)) \leq 0, \quad \forall x \in \mathbb{R}^k, \ \forall x' \in \bar{D} \tag{2.1}$$

$$(x'-x)^*(x-\pi(x)) \leq (x'-\pi(x'))^*(x-\pi(x)), \quad \forall x, x' \in \mathbb{R}^k \tag{2.2}$$

$$\exists a \in D, \gamma > 0, \text{ such that } (x-a)^*(x-\pi(x)) \geq \gamma |x-\pi(x)|, \quad \forall x \in \mathbb{R}^k. \tag{2.3}$$

For $x \in \partial D$, we denote by $n(x)$ the set of outward normal unit vectors at the point x.

To avoid technical complications, we will focus our study on domain D which satisfy the following assumption:

Assumption 2.1. *(i) D is a regular domain (i.e. a convex domain with class C^2 boundary).*

(ii) $0 \in D$.

Remark 2.1. *If not we can approximate our convex domain D by regular convex domains Indeed, we define a sequence of regular convex domains which approximate uniformly D. Indeed, the function $h(x) = d(x, D) := \inf_{y \in D} |x-y|$ is convex and uniformly continuous in $\mathbb{R}^k$. If we denote $(g_\delta)_{0 \leq \delta \leq \delta_0}$ the approximation identity with compact supports, then $h_\delta = g_\delta * h$ is a sequence of regular convex functions which tends uniformly to h as $\delta \to 0$. For a fixed $\eta > 0$, $\{x, h_\delta(x) < \eta\}$ are regular convex domains that converge uniformly in the Hausdroff metric to $\{x, d(x, D) < \eta\}$ when δ tends to 0. Letting $\eta \to 0$, we conclude that for all $\varepsilon > 0$ there exists a regular convex domain D_ε such that*

$$\sup_{x \in D} d(x, D_\varepsilon) < \varepsilon \quad \text{and} \quad \sup_{x \in D_\varepsilon} d(x, D) < \varepsilon \tag{2.4}$$

One can find all these results in Menaldi [32], page 737.

2.1.2. Functional spaces and assumptions

Hereafter, let us define the spaces and the norms which will be needed for the formulation of the BDSDE in a domain.

- $\mathbf{L}_k^p(\mathcal{F}_T)$ the space of k-dimensional $\mathcal{F}_T$-measurable random variables ξ such that

$$\|\xi\|_{L^p}^p := \mathbb{E}(|\xi|^p) < +\infty;$$

- $\mathcal{H}_{k\times d}^2([0,T])$ the set of (classes of $d\mathbb{P}\otimes dt$ a.e. equal) $k\times d$-dimensional jointly measurable processes such that Z_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$ and

$$\|Z\|_{\mathcal{H}^2}^2 := \mathbb{E}[\int_0^T |Z_t|^2 dt] < +\infty;$$

- $\mathcal{S}_k^2([0,T])$ the space of $\mathbb{R}^k$-valued processes $Y=(Y_t)_{t\leq T}$, with continuous paths such that Y_t is $\mathcal{F}_t$-measurable and
$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{t\leq T}|Y_t|^2] < +\infty;$$

- $\mathcal{A}_k^2([0,T])$ the space of $\mathbb{R}^k$-valued processes $K=(K_t)_{t\leq T}$, with continuous and bounded variation paths such that K_t is $\mathcal{F}_t$-measurable, $K_0=0$ and
$$\|K\|^2_{\mathcal{A}^2} := \mathbb{E}[\sup_{t\leq T}|K_t|^2] < +\infty.$$

We next state our main assumptions on the terminal condition ξ and the functions f and h:

Assumption 2.2. *$\xi \in \mathbf{L}_k^2(\mathcal{F}_T)$ and $\xi \in \bar{D}$ a.s.*

Assumption 2.3. *$f:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$, $h:\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^{k\times l}$ are two random functions verifying:*

(i) *For all $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $f_t(\omega,y,z)$ and $h_t(\omega,y,z)$ are $\mathcal{F}_t$- measurable.*

(ii) $\mathbb{E}\big[\int_0^T |f_t(0,0)|^2dt\big] < +\infty \quad , \quad \mathbb{E}\big[\int_0^T \|h_t(0,0)\|^2dt\big] < +\infty.$

(iii) *There exist constants $c>0$ and $0<\alpha<1$ such that for any $(\omega,t)\in\Omega\times[0,T]$; $(y_1,z_1),(y_2,z_2)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$*
$$\begin{aligned} |f_t(y_1,z_1)-f_t(y_2,z_2)|^2 &\leq c\big(|y_1-y_2|^2+\|z_1-z_2\|^2\big)\\ \|h_t(y_1,z_1)-h_t(y_2,z_2)\|^2 &\leq c|y_1-y_2|^2+\alpha\|z_1-z_2\|^2. \end{aligned}$$

We denote by $f_t^0 := f_t(\omega,0,0)$ and $h_t^0 := h_t(\omega,0,0)$.

We add the following further assumption:

Assumption 2.4. (i) $\xi\in\mathbf{L}_k^4(\mathcal{F}_T)$.

(ii) *There exist $c>0$ and $0\leq\beta<1$ such that for all $(t,y,z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$*
$$h_t\,h_t^*(y,z) \leq c(Id_{\mathbb{R}^k}+yy^*)+\beta\,zz^*.$$

(iii) *f and h are uniformly bounded in (y,z).*

Remark 2.2. 1. *The Assumption 2.4* (i) and (ii) are needed to prove the uniform L^4-estimate for (Y^n,Z^n) solution of BDSDE (2.10) (see estimate (C.1) in the Appendix C). This is crucial for our proof of the fundamental Lemma 2.2.

2. The Assumption 2.4 (iii) is only added for simplicity and it can be removed by standard technics of BSDEs. The natural condition instead of (iii) is f^0 and h^0 in $\mathbf{L}^4(\Omega,\mathcal{F},\mathbb{P})$.

Now we introduce the definition of the solution of BDSDEs in a domain.

Definition 2.1. *The triplet of processes $(Y_t,Z_t,K_t)_{\{0\leq t\leq T\}}$ is a solution of the backward doubly stochastic differential equation in a convex domain D, with terminal condition ξ and coefficients f and h, if the following hold:*

(i) $Y \in \mathcal{S}_k^2([0,T])$, $Z \in \mathcal{H}_{k\times d}^2([0,T])$ *and* $K \in \mathcal{A}_k^2([0,T])$,

(ii)

$$Y_t = \xi + \int_t^T f_s(Y_s,Z_s)ds + \int_t^T h_s(Y_s,Z_s)d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t \ , \ 0 \le t \le T \ a.s. \quad (2.5)$$

(iii) $Y_t \in \bar{D}$, $0 \le t \le T$, *a.s.*

(iv) *for any continuous progressively measurable process* $(z_t)_{0\le t\le T}$ *valued in* $\bar{D}$,

$$\int_0^T (Y_t - z_t)^* dK_t \le 0, \ a.s. \quad (2.6)$$

The triplet $(Y_t, Z_t, K_t)_{\{0\le t\le T\}}$ *is called a solution of RBDSDE with data* (ξ, f, h).

Remark 2.3. *From Lemma 2.1 in [15], the condition* (2.6) *implies that the bounded variation process* K *acts only when* Y *reaches the boundary of the convex domain* D *and the so-called Skorohod condition is satisfied:*

$$\int_0^T \mathbf{1}_{\{Y_t\in D\}} dK_t = 0. \quad (2.7)$$

Moreover there exits an $\mathcal{F}_t$*-measurable process* $(\alpha_t)_{0\le t\le T}$ *valued in* $\mathbb{R}^k$ *such that*

$$K_t = \int_0^t \alpha_s d\|K_s\|_{VT} \quad \textit{and} \ -\alpha_s \in n(Y_s).$$

In the following, C will denote a positive constant which doesn't depend on n and can vary from line to line.

2.2. Existence and uniqueness of the solution

In this section we establish existence and uniqueness results for RBDSDE (2.5).

Theorem 2.1. *Let the Asumptions 2.1, 2.2, 2.3 and 2.4 hold. Then, the RBDSDE* (2.5) *has a unique solution* $(Y,Z,K) \in \mathcal{S}_k^2([0,T]) \times \mathcal{H}_{k\times d}^2([0,T]) \times \mathcal{A}_k^2([0,T])$.

Proof.

a) Uniqueness: Let (Y^1,Z^1,K^1) and (Y^2,Z^2,K^2) be two solutions of the RBDSDE (2.5). Applying generalized Itô's formula (Lemma 1.3 in [34] p.213) yields

$$\begin{aligned} |Y_t^1 - Y_t^2|^2 \ + \ & \int_t^T \|Z_s^1 - Z_s^2\|^2 ds = 2\int_t^T (Y_s^1 - Y_s^2)^*(f_s(Y_s^1,Z_s^1) - f_s(Y_s^2,Z_s^2))ds \\ + \ & 2\int_t^T (Y_s^1 - Y_s^2)^*(h_s(Y_s^1,Z_s^1) - h_s(Y_s^2,Z_s^2))d\overleftarrow{W}_s - 2\int_t^T (Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)dB_s \\ + \ & 2\int_t^T (Y_s^1 - Y_s^2)^*(dK_s^1 - dK_s^2) + \int_t^T \|h_s(Y_s^1,Z_s^1) - h_s(Y_s^2,Z_s^2)\|^2 ds. \end{aligned} \quad (2.8)$$

Moreover, under the minimality condition (iv) we have

$$\int_t^T (Y_s^1 - Y_s^2)^*(dK_s^1 - dK_s^2) \le 0, \quad \text{for all} \ \ t \in [0,T]. \quad (2.9)$$

Then, plugging (2.9) in (2.8) and taking expectation we obtain

$$\mathbb{E}[|Y_t^1 - Y_t^2|^2] + \mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds] \leq 2\mathbb{E}[\int_t^T (Y_s^1 - Y_s^2)^*(f_s(Y_s^1, Z_s^1) - f_s(Y_s^2, Z_s^2))ds]$$
$$+ \mathbb{E}[\int_t^T \|h_s(Y_s^1, Z_s^1) - h_s(Y_s^2, Z_s^2)\|^2 ds].$$

Hence from the Lipschitz Assumption on h and the inequality $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$, for all $\epsilon > 0$, it follows that

$$\mathbb{E}[|Y_t^1 - Y_t^2|^2] + \mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds] \leq (c+\epsilon)\mathbb{E}[\int_t^T |Y_s^1 - Y_s^2|^2 ds]$$
$$+ \epsilon^{-1}\mathbb{E}[\int_t^T |f_s(Y_s^1, Z_s^1) - f_s(Y_s^2, Z_s^2)|^2 ds] + \alpha\mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds],$$

where $0 < \alpha < 1$. Choosing $\epsilon = \dfrac{2c}{1-\alpha}$ and using the Lipschitz Assumption on f, we get

$$\mathbb{E}[|Y_t^1 - Y_t^2|^2] + \mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds] \leq (c + \frac{2c}{1-\alpha} + \frac{1-\alpha}{2})\mathbb{E}[\int_t^T |Y_s^1 - Y_s^2|^2 ds]$$
$$+ \frac{1-\alpha}{2}\mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds] + \alpha\mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds].$$

Consequently

$$\mathbb{E}[|Y_t^1 - Y_t^2|^2] + (\frac{1-\alpha}{2})\mathbb{E}[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds] \leq (c + \frac{2c}{1-\alpha} + \frac{1-\alpha}{2})\mathbb{E}[\int_t^T |Y_s^1 - Y_s^2|^2 ds].$$

From Gronwall's lemma, $\mathbb{E}[|Y_t^1 - Y_t^2|^2] = 0,\ 0 \leq t \leq T$, and $\mathbb{E}[\int_0^T \|Z_s^1 - Z_s^2\|^2 ds] = 0$.

b) Existence: The existence of a solution will be proved by penalisation method. For $n \in \mathbb{N}$, we consider for all $t \in [0, T]$,

$$Y_t^n = \xi + \int_t^T f_s(Y_s^n, Z_s^n)ds + \int_t^T h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s - n\int_t^T (Y_s^n - \pi(Y_s^n))ds - \int_t^T Z_s^n dB_s. \quad (2.10)$$

From Pardoux and Peng [34] (Theorem 1.1), we know that the above equation admits a unique solution with coefficient f^n and h, where $f^n(t,x,y) = f(t,x,y,z) - n(y - \pi(y))$.

Denote by $K_t^n := -n\int_0^t (Y_s^n - \pi(Y_s^n))ds$. In order to prove the convergence of the sequence (Y^n, Z^n, K^n) to the solution of our RBDSDE (2.1), we need several lemmas.

We start with the following lemma:

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\forall n \in \mathbb{N} \qquad \mathbb{E}[\int_0^T d^2(Y_s^n, D)ds] \leq \frac{C}{n}. \quad (2.11)$$

Proof. For the sake of simplicity, we treat only the case where D is a convex set with class C^2 boundary implying that the function $\rho(x) = d^2(x, D) = |x - \pi(x)|^2$ belongs to C^2 (see Subsection

2.1.1 and Appendix A). Thus, we apply generalized Itô's formula (Lemma 1.3 in [34] p.213) to $\rho(Y_t^n)$ to obtain

$$\begin{aligned}\rho(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\rho(Y_s^n)]ds &= \rho(\xi) + \int_t^T (\nabla\rho(Y_s^n))^* f_s(Y_s^n, Z_s^n)ds \\ &- \int_t^T (\nabla\rho(Y_s^n))^* Z_s^n dB_s + \int_t^T (\nabla\rho(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ &+ \frac{1}{2}\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\rho(Y_s^n)]ds - 2n\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^n - \pi(Y_s^n))ds.\end{aligned} \tag{2.12}$$

Since $\xi \in \bar{D}$ *a.s.*, we have that $\rho(\xi) = 0$. We get from the fact that $|\nabla\rho(x)|^2 = 4\rho(x)$ and the boundedness of h and the Hessian of ρ

$$\begin{aligned}\rho(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\rho(Y_s^n)]ds + 2n\int_t^T d^2(Y_s^n, D)ds \\ \leq 2\int_t^T (\rho(Y_s^n))^{1/2}|f_s(Y_s^n, Z_s^n)|ds - 2\int_t^T (Y_s^n - \pi(Y_s^n))^* Z_s^n dB_s \\ + 2\int_t^T (Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s + C.\end{aligned} \tag{2.13}$$

Now the inequality $2ab \leq a^2 + b^2$ with $a = \sqrt{\frac{n}{2}\rho(Y_s^n)}$ yields

$$(\rho(Y_s^n))^{1/2}|f_s(Y_s^n, Z_s^n)| \quad \leq \quad \frac{n}{4}\rho(Y_s^n) + \frac{1}{n}|f_s(Y_s^n, Z_s^n)|^2.$$

Then it follows that,

$$\begin{aligned}\rho(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\rho(Y_s^n)]ds + \frac{3n}{2}\int_t^T d^2(Y_s^n, D)ds \\ \leq 2\int_t^T \frac{1}{n}|f_s(Y_s^n, Z_s^n)|^2 ds - 2\int_t^T (Y_s^n - \pi(Y_s^n))^* Z_s^n dB_s \\ + 2\int_t^T (Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s + C.\end{aligned} \tag{2.14}$$

By taking expectation and using the boundedness of f, we have

$$\mathbb{E}[\rho(Y_t^n)] + \frac{1}{2}\mathbb{E}[\int_t^T trace[Z_s^n Z_s^{n*} Hess\rho(Y_s^n)]ds] + \frac{3n}{2}\mathbb{E}[\int_t^T d^2(Y_s^n, D)ds] \leq C\big(1 + \frac{1}{n}\big). \tag{2.15}$$

Hence, the required result is obtained. □

The next lemma plays a crucial role to prove the strong convergence of (Y^n, Z^n, K^n).

Lemma 2.2.

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}(d(Y_t^n, D))^4\Big] \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.16}$$

Proof. We denote by $\rho(x) = d^2(x, D)$ and $\varphi(x) = \rho^2(x)$. By applying Itô's formula to $\varphi(Y_t^n) =$

$d^4(Y_t^n, D)$, we obtain that

$$\begin{aligned}\varphi(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds = \varphi(\xi) + \int_t^T (\nabla\varphi(Y_s^n))^* f_s(Y_s^n, Z_s^n)ds \\ - \int_t^T (\nabla\varphi(Y_s^n))^* Z_s^n dB_s + \int_t^T (\nabla\varphi(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ + \frac{1}{2}\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\varphi_(Y_s^n)]ds - n\int_t^T (\nabla\varphi(Y_s^n))^*(Y_s^n - \pi(Y_s^n))ds.\end{aligned} \tag{2.17}$$

Since $\xi \in \bar{D}$ *a.s.*, we have that $\varphi(\xi) = 0$ and the chain rule of differentiation gives that

$$\nabla\varphi(x) = 2\rho(x)\nabla\rho(x) = 4\rho(x)(x - \pi(x)) \tag{2.18}$$

$$Hess\varphi(x) = 2\nabla\rho(x)(\nabla\rho(x))^* + 2\rho(x)Hess\rho(x). \tag{2.19}$$

Then it follows that

$$\begin{aligned}\varphi(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds = 4\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* f_s(Y_s^n, Z_s^n)ds \\ - 4\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* Z_s^n dB_s + 4\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ + \frac{1}{2}\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\varphi(Y_s^n)]ds - 4n\int_t^T \rho^2(Y_s^n)ds.\end{aligned} \tag{2.20}$$

By taking expectation we have

$$\begin{aligned}\mathbb{E}[\varphi(Y_t^n)] + \frac{1}{2}\mathbb{E}\Big[\int_t^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds\Big] + 4n\mathbb{E}\Big[\int_t^T \varphi(Y_s^n)ds\Big] \\ = 4\mathbb{E}\Big[\int_t^T (\rho(Y_s^n)(Y_s^n - \pi(Y_s^n))^* f_s(Y_s^n, Z_s^n)ds] \\ + \frac{1}{2}\mathbb{E}\Big[\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\varphi(Y_s^n)]ds].\end{aligned} \tag{2.21}$$

For the last term, we get from the fact that $|\nabla\rho(x)|^2 = 4\rho(x)$ and the boundedness of h and $Hess\rho$

$$\begin{aligned}\mathbb{E}\Big[\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\varphi(Y_s^n)]ds] &\le 2\mathbb{E}\Big[\int_t^T \langle h_s(Y_s^n, Z_s^n), \nabla\rho(Y_s^n)\rangle^2 ds\Big] \\ &+ \mathbb{E}\Big[\int_t^T 2\rho(Y_s^n)trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\rho(Y_s^n)]ds\Big] \\ &\le C\mathbb{E}\Big[\int_t^T |\nabla\rho(Y_s^n)|^2 ds\Big] + C\mathbb{E}\Big[\int_t^T \rho(Y_s^n)ds\Big] \\ &\le C\mathbb{E}\Big[\int_0^T (d(Y_s^n, D))^2 ds\Big].\end{aligned} \tag{2.22}$$

Now the inequality $2ab \le a^2 + b^2$ with $a = (d(Y_s^n, D))^2$ and the boundedness of f yield

$$\begin{aligned}4(d(Y_s^n, D))^3|f_s(Y_s^n, Z_s^n)| &\le 2(d(Y_s^n, D))^4 + 2(d(Y_s^n, D))^2|f_s(Y_s^n, Z_s^n)|^2 \\ &\le 2\varphi(Y_s^n) + 2C(d(Y_s^n, D))^2.\end{aligned} \tag{2.23}$$

By plugging the estimate (2.23) and (2.22) in (2.21), we obtain thanks to lemma 2.1

$$\begin{aligned}\mathbb{E}[\varphi(Y_t^n)] + \frac{1}{2}\mathbb{E}\Big[\int_t^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds\Big] + (4n-2)\mathbb{E}\Big[\int_t^T \varphi(Y_s^n)ds\Big] \\ \leq C\mathbb{E}\Big[\int_0^T (d(Y_s^n, D))^2 ds\Big] \leq C\big(\frac{1}{n} + \frac{1}{n^2}\big).\end{aligned} \tag{2.24}$$

Notice also that Hessian of $\varphi(Y_s^n)$ is a positive semidefinite matrix since φ is a convex function, so we get that $\mathbb{E}\Big[\int_t^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds\Big] \geq 0$ and consequently,

$$\sup_{0\leq t\leq T} \mathbb{E}[\varphi(Y_t^n)] \leq C\big(\frac{1}{n} + \frac{1}{n^2}\big). \tag{2.25}$$

Moreover, we can deduce from (2.24) that, for every $t \in [0,T]$

$$\mathbb{E}\Big[\int_0^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds\Big] \longrightarrow 0, \text{ as } n \to \infty. \tag{2.26}$$

On the other hand, taking the supremum over t in the equation (2.20), by Burkholder-Davis-Gundy's inequlity and the previous calculations it follows that

$$\begin{aligned}\mathbb{E}[\sup_{0\leq t\leq T} \varphi(Y_t^n)] &\leq C\mathbb{E}[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}\Big[\int_0^T (d(Y_s^n, D))^2 ds\Big] \\ &+ C\mathbb{E}\Big[\sup_{0\leq t\leq T}\int_t^T (\rho(Y_s^n)\nabla\rho(Y_s^n))^* Z_s^n dB_s\Big] \\ &+ C\mathbb{E}\Big[\sup_{0\leq t\leq T}\int_t^T (\rho(Y_s^n)\nabla\rho(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s\Big] \\ &\leq C\mathbb{E}[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}\Big[\int_0^T (d(Y_s^n, D))^2 ds\Big] \\ &+ C\mathbb{E}\Big[\Big(\int_0^T (\rho(Y_s^n))^2\langle\nabla\rho(Y_s^n), Z_s^n\rangle^2 ds\Big)^{1/2}\Big] \\ &+ C\mathbb{E}\Big[\Big(\int_0^T (\rho(Y_s^n))^2\langle\nabla\rho(Y_s^n), h_s(Y_s^n, Z_s^n)\rangle^2 ds\Big)^{1/2}\Big].\end{aligned} \tag{2.27}$$

From the boundedness of h and the fact that $|\nabla\rho(x)|^2 = 4\rho(x)$, we have

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T (\rho(Y_s^n))^2\langle\nabla\rho(Y_s^n)), h_s(Y_s^n, Z_s^n)\rangle^2 ds\Big)^{1/2}\Big] &\leq C\mathbb{E}\Big[\Big(\int_0^T (\rho(Y_s^n))^2\rho(Y_s^n)ds\Big)^{1/2}\Big] \\ &\leq C\mathbb{E}\Big[\sup_{0\leq s\leq T}\big(\varphi(Y_s^n)\big)^{1/2}\Big(\int_0^T \rho(Y_s^n)ds\Big)^{1/2}\Big] \\ &\leq \frac{1}{4}\mathbb{E}\Big[\sup_{0\leq s\leq T}\varphi(Y_s^n)\Big] + C^2\mathbb{E}\Big[\int_0^T (d(Y_s^n, D))^2 ds\Big].\end{aligned} \tag{2.28}$$

By the Holder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T (\rho(Y_s^n))^2\langle\nabla\rho(Y_s^n)), Z_s^n\rangle^2 ds\Big)^{1/2}\Big] &\leq C\mathbb{E}\Big[\sup_{0\leq s\leq T}\big(\varphi(Y_s^n)\big)^{1/2}\Big(\int_0^T \langle\nabla\rho(Y_s^n)), Z_s^n\rangle^2 ds\Big)^{1/2}\Big] \\ &\leq \frac{1}{4}\mathbb{E}\Big[\sup_{0\leq s\leq T}\varphi(Y_s^n)\Big] + C^2\mathbb{E}\Big[\int_0^T \langle\nabla\rho(Y_s^n), Z_s^n\rangle^2 ds\Big].\end{aligned} \tag{2.29}$$

Substituting (2.28) and (2.29) in (2.27) leads to

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}\varphi(Y_t^n)] \le C\mathbb{E}[\int_0^T \varphi(Y_s^n)ds] + C\mathbb{E}\Big[\int_0^T (d(Y_s^n,D))^2 ds\Big] \\ + C^2\mathbb{E}\Big[\int_0^T \langle\nabla\rho(Y_s^n), Z_s^n\rangle^2 ds\Big].\end{aligned} \tag{2.30}$$

In the other hand, Hessian of $\rho(Y_s^n)$ is a positive semidefinite matrix since ρ is a convex function, so we get that $\mathbb{E}\big[\int_t^T trace[Z_s^n Z_s^{n*}\rho(Y_s^n)Hess\rho(Y_s^n)]ds\big] \ge 0$. By the equation (2.19), we can deduce that

$$2\mathbb{E}\Big[\int_0^T \langle\nabla\rho(Y_s^n), Z_s^n\rangle^2 ds\Big] \le \mathbb{E}\Big[\int_0^T trace[Z_s^n Z_s^{n*} Hess\varphi(Y_s^n)]ds\Big], \tag{2.31}$$

and from (2.26), we get

$$\mathbb{E}\Big[\int_0^T \langle\nabla\rho(Y_s^n), Z_s^n\rangle^2 ds\Big] \longrightarrow 0 \,\text{as}\, n\to\infty.$$

Finally, by using (2.25), (2.30) and Lemma 2.1, we get the desired result. □

Remark 2.4. *Contrary to Gegout-Petit and Pardoux [15], we have to prove the fundamental lemma with power 4 instead of 2. In fact, if we apply the genelized Itô's formula to $\rho(Y_t^n) = d^2(Y_t^n, D)$, we obtain that*

$$\begin{aligned}\rho(Y_t^n) + \frac{1}{2}\int_t^T trace[Z_s^n Z_s^{n*} Hess\rho(Y_s^n)]ds = \rho(\xi) + \int_t^T (\nabla\rho(Y_s^n))^* f_s(Y_s^n, Z_s^n)ds \\ - \int_t^T (\nabla\rho(Y_s^n))^* Z_s^n dB_s + \int_t^T (\nabla\rho(Y_s^n))^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ + \frac{1}{2}\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\rho(Y_s^n)]ds - n\int_t^T (\nabla\rho(Y_s^n))^*(Y_s^n - \pi(Y_s^n))ds.\end{aligned} \tag{2.32}$$

To prove the fundamental lemma, we need to estimate all the terms in the right hand side of the above equation in terms of quantities which depend on n. But, since $Hess\rho$ is bounded we get

$$\mathbb{E}\Big[\int_t^T trace[(h_s h_s^*)(Y_s^n, Z_s^n)Hess\rho(Y_s^n)]ds\Big] \le C\mathbb{E}\Big[\int_t^T \|h_s(Y_s^n, Z_s^n)\|^2 ds\Big].$$

Then, there is no hope to obtain the required convergence to 0.

Lemma 2.3. *The sequence (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}_k^2([0,T]) \times \mathcal{H}_{k\times d}^2([0,T])$, i.e.*

$$\mathbb{E}[\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2 + \int_0^T \|Z_t^n - Z_t^m\|^2 dt] \longrightarrow 0 \text{ as } n, m \to +\infty.$$

Proof. For all $n, m \ge 0$, we apply Itô formula to $|Y_t^n - Y_t^m|^2$

$$\begin{aligned}|Y_t^n - Y_t^m|^2 + \int_t^T \|Z_s^n - Z_s^m\|^2 ds = 2\int_t^T (Y_s^n - Y_s^m)^*(f_s(Y_s^n, Z_s^n) - f_s(Y_s^m, Z_s^m))ds \\ + 2\int_t^T (Y_s^n - Y_s^m)^*(h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m))d\overleftarrow{W}_s - 2\int_t^T (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)dB_s \\ + \int_t^T \|h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m)\|^2 ds - 2n\int_t^T (Y_s^n - Y_s^m)^*(Y_s^n - \pi(Y_s^n))ds \\ + 2m\int_t^T (Y_s^n - Y_s^m)^*(Y_s^m - \pi(Y_s^m))ds.\end{aligned} \tag{2.33}$$

By the property (2.2), we have

$$-2n\int_t^T (Y_s^n - Y_s^m)^*(Y_s^n - \pi(Y_s^n))ds + 2m\int_t^T (Y_s^n - Y_s^m)^*(Y_s^m - \pi(Y_s^m))ds \\ \leq 2(n+m)\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m))ds. \tag{2.34}$$

Hence, from the Lipschitz continuity on f and h, and taking expectation yields

$$\begin{aligned}\mathbb{E}[|Y_t^n - Y_t^m|^2] + \mathbb{E}[\int_t^T \|Z_s^n - Z_s^m\|^2 ds] &\leq 2\mathbb{E}[\int_t^T C(|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m|\|Z_s^n - Z_s^m\|)ds] \\ &+ \mathbb{E}[\int_t^T C(|Y_s^n - Y_s^m|^2 + \alpha\|Z_s^n - Z_s^m\|^2)ds] \\ &+ 2(n+m)\mathbb{E}[\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m))ds].\end{aligned} \tag{2.35}$$

For the last term, we need the following lemma whose proof is postponed in the Appendix.

Lemma 2.4. *There exists a constant $C > 0$ such that, for each $n \geq 0$,*

$$\mathbb{E}\big[\Big(n\int_0^T d(Y_s^n, D)ds\Big)^2\big] \leq C \tag{2.36}$$

Now we can deduce from the Hölder inequality and Lemma 2.4 that

$$\begin{aligned} n\mathbb{E}[\int_t^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m))ds] &\leq n\mathbb{E}[\int_t^T d(Y_s^n, D)d(Y_s^m, D))ds] \\ &\leq n\mathbb{E}[\sup_{0\leq s\leq T} d(Y_s^m, D)\int_t^T d(Y_s^n, D)ds)] \\ &\leq \Big(\mathbb{E}\big[\Big(n\int_0^T d(Y_s^n, D)ds\Big)^2\big]\Big)^{1/2}\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^m, D)]\Big)^{1/2} \\ &\leq C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^m, D)]\Big)^{1/2}. \end{aligned} \tag{2.37}$$

Substituting (2.37) in the previous inequality (2.35), we have

$$\begin{aligned}\mathbb{E}[|Y_t^n - Y_t^m|^2] + (1-\alpha-C\gamma)\mathbb{E}[\int_t^T \|Z_s^n - Z_s^m\|^2 ds] &\leq C(1+\frac{1}{\gamma})\mathbb{E}[\int_t^T |Y_s^n - Y_s^m|^2 ds] \\ &+ C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^n, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^m, D)]\Big)^{1/2}.\end{aligned}$$

Choosing $1-\alpha-C\gamma > 0$, by Gronwall's lemma, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}[|Y_t^n - Y_t^m|^2] \leq C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^m, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^n, D)]\Big)^{1/2}. \tag{2.38}$$

We deduce similarly

$$\mathbb{E}[\int_0^T \|Z_s^n - Z_s^m\|^2 ds] \leq C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^m, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\leq s\leq T} d^2(Y_s^n, D)]\Big)^{1/2}. \tag{2.39}$$

Next, by (2.33), the Burkholder-Davis-Gundy and the Cauchy-Schwarz inequalities we get

$$\mathbb{E}[\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2] \le C\mathbb{E}[\int_0^T |Y_s^n - Y_s^m||f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^m)|ds]$$
$$+ C\mathbb{E}\Big(\int_0^T |Y_s^n - Y_s^m|^2\|h_s(Y_s^n,Z_s^n) - h_s(Y_s^m,Z_s^m)\|^2 ds\Big)^{1/2} + C\mathbb{E}\Big(\int_0^T |Y_s^n - Y_s^m|^2\|Z_s^n - Z_s^m\|^2 ds\Big)^{1/2}$$
$$+ \mathbb{E}[\int_0^T C(|Y_s^n - Y_s^m|^2 + \alpha\|Z_s^n - Z_s^m\|^2)ds] + 2(n+m)\mathbb{E}[\int_0^T (Y_s^n - \pi(Y_s^n))^*(Y_s^m - \pi(Y_s^m))ds].$$

Then, it follows by the Lipschitz Assumption 2.3 on f and h and (2.37) that for any $n, m \ge 0$

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2] &\le C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^m, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^n, D)]\Big)^{1/2}\\ &\quad + C\mathbb{E}(\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2\int_0^T \|Z_s^n - Z_s^m\|^2 ds)^{1/2}\\ &\le C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^m, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^n, D)]\Big)^{1/2}\\ &\quad + C\varepsilon\mathbb{E}(\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2) + C\varepsilon^{-1}\mathbb{E}(\int_0^T \|Z_s^n - Z_s^m\|^2 ds).\end{aligned}$$

Choosing $1 - C\varepsilon > 0$ and from the inequality (2.39) we conclude that

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T}|Y_t^n - Y_t^m|^2] &\le C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^m, D)]\Big)^{1/2} + C\Big(\mathbb{E}[\sup_{0\le s\le T} d^2(Y_s^n, D)]\Big)^{1/2}\\ &\le C\Big(\mathbb{E}[\sup_{0\le s\le T} d^4(Y_s^m, D)]\Big)^{1/4} + C\Big(\mathbb{E}[\sup_{0\le s\le T} d^4(Y_s^n, D)]\Big)^{1/4} \longrightarrow 0,\end{aligned}$$

as $n, m \to \infty$, where Lemma 2.2 has been used. □

Finally, we conclude that (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}_k^2([0,T]) \times \mathcal{H}_{k\times d}^2([0,T])$ and therefore there exists a unique pair (Y_t, Z_t) of $\mathcal{F}_t$- measurable processes which valued in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$, satisfying

$$\mathbb{E}(\sup_{0\le t\le T}|Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.40}$$

Consequently, since for any $n \ge 0$ and $0 \le t \le T$,

$$\begin{aligned}K_t^n - K_t^m = Y_0^n - Y_0^m - Y_t^n - Y_t^m - \int_0^t (f_s(Y_s^n, Z_s^n) - f_s(Y_s^m, Z_s^m))ds\\ - \int_0^t (h_s(Y_s^n, Z_s^n) - h_s(Y_s^m, Z_s^m))d\overleftarrow{W}_s + \int_0^t (Z_s^n - Z_s^m)dB_s.\end{aligned} \tag{2.41}$$

we obtain from (2.40) and Burkholder-Davis-Gundy inequality,

$$\mathbb{E}(\sup_{0\le t\le T}|K_t^n - K_t^m|^2) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{2.42}$$

Hence, there exists a $\mathcal{F}_t$- adapted continuous process $(K_t)_{0\le t\le T}$ (with $K_0 = 0$) such that

$$\mathbb{E}(\sup_{0\le t\le T}|K_t - K_t^n|^2) \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, (2.36) shows that the total variation of K^n is uniformly bounded. Thus, K is also of uniformly bounded variation. Passing to the limit in (2.10), the processes $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ satisfy

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds + \int_t^T h_s(Y_s, Z_s)d\overleftarrow{W}_s - \int_t^T Z_s dB_s + K_T - K_t \ , \ 0 \leq t \leq T.$$

Since we have from Lemma 2.2 that Y_t is in $\bar{D}$, it remains to check the minimality property for (K_t), namely i.e., for any continuous progressively measurable process (z_t) valued in $\bar{D}$,

$$\int_0^T (Y_t - z_t)^* dK_t \leq 0.$$

We note that (2.1) gives us

$$\int_0^T (Y_t^n - z_t)^* dK_t^n = -n \int_0^T (Y_t^n - z_t)^* (Y_t^n - \pi(Y_t^n))dt \leq 0.$$

Therefore, we will show that we can extract a subsequence such that $\int_0^T (Y_t^n - z_t)^* dK_t^n$ converge a.s. to $\int_0^T (Y_t - z_t)^* dK_t$. Following the proof of Lemma B.1 in Appendix, we have

$$\begin{aligned} 2\gamma\|K^n\|_{VT} &\leq |\xi - a|^2 + 2\int_0^T (Y_s^n - a)f_s(Y_s^n, Z_s^n)ds + 2\int_0^T (Y_s^n - a)h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ &+ \int_0^T \|h_s(Y_s^n, Z_s^n)\|^2 ds - 2\int_0^T (Y_s^n - a)Z_s^n dB_s. \end{aligned} \tag{2.43}$$

Notice that the right hand side tends in probability as n goes to infinity to

$$|\xi - a|^2 + 2\int_0^T (Y_s - a)f_s(Y_s, Z_s)ds + 2\int_0^T (Y_s - a)h_s(Y_s, Z_s)d\overleftarrow{W}_s + \int_0^T \|h_s(Y_s, Z_s)\|^2 ds - 2\int_0^T (Y_s - a)Z_s dB_s.$$

Thus, there exists a subsequence $(\phi(n))_{n\geq 0}$ such that the convergence is almost surely and $\|K^{\phi(n)}\|_{VT}$ is bounded. Moreover, due to the convergence in $\mathbb{L}^2$ of $\sup_{0\leq t\leq T} |Y_t^n - Y_t|^2$ to 0, we can extract a subsequence from $(\phi(n))_{n\geq 0}$ such that $Y^{\phi(\psi(n))}$ converges uniformly to Y. Hence, we apply Lemma 5.8 in [15] and we obtain

$$\int_0^T (Y_t^{\phi(\psi(n))} - z_t)^* dK_t^{\phi(\psi(n)))} \longrightarrow \int_0^T (Y_t - z_t)^* dK_t \quad a.s. \text{ as } n \to \infty$$

which is the required result. □

3. Weak solution of semilinear SPDE in a convex domain

The aim of this section is to give a Feynman-Kac's formula for the weak solution of a semilinear reflected SPDEs (1.3) in a given convex domain D via Markovian class of RBDSDEs studied in the last section. As explained in the introduction, the solution of such SPDEs is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a Sobolev space and ν is a signed Radon regular measure. The bounded variation processes K component of the solution of the reflected BDSDE controls the set when u reaches the boundary of D. In fact, this bounded variation process determines the measure ν from a particular relation by using the inverse of the flow associated to the diffusion operator.

3.1. Notations and Hypothesis

Let us first introduce some notations:
- $C^n_{l,b}(\mathbb{R}^p, \mathbb{R}^q)$ the set of C^n-functions which grow at most linearly at infinity and whose partial derivatives of order less than or equal to n are bounded.
- $\mathbf{L}^2_w\left(\mathbb{R}^d\right)$ will be a Hilbert with the inner product,

$$(u,v)_w = \int_{\mathbb{R}^d} u\,(x)\, v\,(x)\, w(x)dx,\ \|u\|_2 = \left(\int_{\mathbb{R}^d} u^2\,(x)\, w(x)dx\right)^{\frac{1}{2}}.$$

Assumption 3.1. *We assume that w is the weight function that satisfy the following conditions:*

- *w is a continuous positive function.*
- *w is integrable and $\dfrac{1}{w}$ is locally integrable.*

In general, we shall use for the usual L^2-scalar product

$$(u,v) = \int_{\mathbb{R}^d} u(x)v(x)\,dx,$$

where u, v are measurable functions defined in $\mathbb{R}^d$ and $uv \in \mathbf{L}^1(\mathbb{R}^d)$.
Our evolution problem will be considered over a fixed time interval $[0,T]$ and the norm for an element of $\mathbf{L}^2_w\left([0,T]\times\mathbb{R}^d\right)$ will be denoted by

$$\|u\|_{2,2} = \left(\int_0^T \int_{\mathbb{R}^d} |u(t,x)|^2 w(x)dxdt\right)^{\frac{1}{2}}.$$

We assume the following hypotheses :

Assumption 3.2. $\Phi : \mathbb{R}^d \to \mathbb{R}^k$ *belongs to* $\mathbf{L}^2_w(\mathbb{R}^d)$ *and* $\Phi(x) \in \bar{D}$ *a.e.* $\forall x \in \mathbb{R}^d$*;*

Assumption 3.3. (i) $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \to \mathbb{R}^k$ *and* $h : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$ *are measurable in* (t,x,y,z) *and satisfy* $f^0, h^0 \in \mathbf{L}^2_w\left([0,T]\times\mathbb{R}^d\right)$ *where* $f^0_t(x) := f(t,x,0,0)$, $h^0_t := h(t,,x,0,0)$.

(ii) *There exist constants* $c > 0$ *and* $0 < \alpha < 1$ *such that for any* $(t,x) \in [0,T]\times\mathbb{R}^d$; $(y_1,z_1),(y_2,z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$\begin{aligned} |f_t(x,y_1,z_1) - f_t(x,y_2,z_2)|^2 &\leq c\big(|y_1-y_2|^2 + \|z_1-z_2\|^2\big) \\ \|h_t(x,y_1,z_1) - h_t(x,y_2,z_2)\|^2 &\leq c|y_1-y_2|^2 + \alpha\|z_1-z_2\|^2. \end{aligned}$$

Assumption 3.4. *The coefficients* b *and* σ *of the second order differential operator* $\mathcal{L}$ *(1.2) satisfy* $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^{d\times d})$.

Assumption 3.5. (i) $\Phi \in \mathbf{L}^4_w(\mathbb{R}^d)$.

(ii) *There exist* $c > 0$ *and* $0 \leq \beta < 1$ *such that for all* $(t,x,y,z) \in [0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$

$$h_t\, h_t^*(x,y,z) \leq c(Id_{\mathbb{R}^k} + yy^*) + \beta\, zz^*.$$

(iii) f *and* h *are uniformly bounded in* (x,y,z).

3.2. Weak formulation for a solution of Stochastic PDEs

The space of test functions which we employ in the definition of weak solutions of the evolution equations (1.1) is $\mathcal{D}_T = \mathcal{C}^\infty([0,T]) \otimes \mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$, where

- $\mathcal{C}^\infty([0,T])$ denotes the space of real functions which can be extended as infinitely differentiable functions in the neighborhood of $[0,T]$,
- $\mathcal{C}_c^\infty\left(\mathbb{R}^d\right)$ is the space of infinite differentiable functions with compact supports in $\mathbb{R}^d$.

We denote by $\mathcal{H}_T$ the space of $\mathcal{F}_{t,T}^W$-progressively measurable processes (u_t) with values in the weighted Sobolev space $H_w^1(\mathbb{R}^d)$ where

$$H_w^1(\mathbb{R}^d) := \{v \in \mathbf{L}_w^2(\mathbb{R}^d) \mid \nabla v\sigma \in \mathbf{L}_w^2(\mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T}^2 = \mathbb{E}\left[\sup_{0\le t\le T}\|u_s\|_2^2 + \int_{\mathbb{R}^d}\int_0^T |\nabla u_s(x)\sigma(x)|^2 ds w(x)dx\right],$$

where we denote the gradient by $\nabla u(t,x) = \left(\partial_1 u(t,x), \cdots, \partial_d u(t,x)\right)$. Here, the derivative is defined in the weak sense (Sobolev sense).

Definition 3.1 (**Weak solution of regular SPDE**). *We say that $u \in \mathcal{H}_T$ is a Sobolev solution of SPDE* (1.1) *if the following relation holds, for each $\varphi \in \mathcal{D}_T$,*

$$\begin{aligned}&\int_t^T (u(s,x), \partial_s\varphi(s,x))ds + (u(t,x),\varphi(t,x)) - (\Phi(x),\varphi(T,x)) - \int_t^T (u(s,x), \mathcal{L}^*\varphi(s,x))ds\\ &= \int_t^T (f_s(x,u(s,x),\nabla u(s,x)\sigma(x)),\varphi(s,x))ds + \int_t^T (h_s(x,u(s,x),\nabla u(s,x)\sigma(x)),\varphi(s,x))d\overleftarrow{W}_s.\end{aligned} \tag{3.1}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. We denote by $u := \mathcal{U}(\Phi,f,h)$ the solution of SPDEs with data (Φ,f,h).

The existence and uniqueness of weak solution for SPDEs (3.1) is ensured by Theorem 3.1 in Bally and Matoussi [3] or Denis and Stoica [12].

3.3. Stochastic flow of diffeomorphisms and random test functions

We are concerned in this part with solving our problem by developing a stochastic flow method which was first introduced in Kunita [21], [22] and Bally, Matoussi [3]. We recall that $\{X_{t,s}(x), t \le s \le T\}$ is the diffusion process starting from x at time t and is the strong solution of the equation:

$$X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x))dr + \int_t^s \sigma(X_{t,r}(x))dB_r. \tag{3.2}$$

The existence and uniqueness of this solution was proved in Kunita [21]. Moreover, we have the following properties:

Proposition 3.1. *Under Assumption 3.4 and for each $t > 0$, there exists a version of $\{X_{t,s}(x);\ x \in \mathbb{R}^d,\ s \geq t\}$ such that $X_{t,s}(\cdot)$ is a $C^2(\mathbb{R}^d)$-valued continuous process which satisfies the flow property: $X_{t,r}(x) = X_{s,r} \circ X_{t,s}(x)$, $0 \leq t < s < r$. Furthermore, for all $p \geq 2$, there exists M_p such that for all $0 \leq t < s$, $x, x' \in \mathbb{R}^d$, $h, h' \in \mathbb{R}\backslash\{0\}$,*

$$\begin{aligned}
&\mathbb{E}(\sup_{t\leq r\leq s}|X_{t,r}(x)-x|^p) \leq M_p(s-t)(1+|x|^p),\\
&\mathbb{E}(\sup_{t\leq r\leq s}|X_{t,r}(x)-X_{t,r}(x')-(x-x')|^p) \leq M_p(s-t)(|x-x'|^p),\\
&\mathbb{E}(\sup_{t\leq r\leq s}|\Delta_h^i[X_{t,r}(x)-x]|^p) \leq M_p(s-t),\\
&\mathbb{E}(\sup_{t\leq r\leq s}|\Delta_h^i X_{t,r}(x)-\Delta_{h'}^i X_{t,r}(x')|^p) \leq M_p(s-t)(|x-x'|^p+|h-h'|^p),
\end{aligned}$$

where $\Delta_h^i g(x) = \frac{1}{h}(g(x+he_i)-g(x))$, and $(e_1, \cdots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$.

For all $v \in \mathbf{L}^2(\mathbb{R}^d)$, the process $(v \circ X_{t,s}(x))_{t\leq s\leq T}$ defined as the composition of v with the stochastic flow $(X_{t,s}(x))_{t\leq s\leq T}$ is well defined thanks to the following equivalence norms results (see [4] and [3] for the proofs).

Proposition 3.2. *There exists two constants $c > 0$ and $C > 0$ such that for every $t \leq s \leq T$ and $\varphi \in L^1(\mathbb{R}^d, dx)$,*

$$c\int_{\mathbb{R}^d}|\varphi(x)|w(x)dx \leq \int_{\mathbb{R}^d}\mathbb{E}(|\varphi(X_{t,s}(x))|)w(x)dx \leq C\int_{\mathbb{R}^d}|\varphi(x)|w(x)dx. \tag{3.3}$$

Moreover, for every $\Psi \in L^1([0,T]\times\mathbb{R}^d, dt\otimes dx)$,

$$c\int_{\mathbb{R}^d}\int_t^T|\Psi(s,x)|dsw(x)dx \leq \int_{\mathbb{R}^d}\int_t^T\mathbb{E}(|\Psi(s,X_{t,s}(x))|)dsw(x)dx \leq C\int_{\mathbb{R}^d}\int_t^T|\Psi(s,x)|dsw(x)dx. \tag{3.4}$$

This equivalence of norms result plays also an important role in the proof of the existence of the solution for SPDE as a connection between the functional norms for such solutions and random norms from BDSDE solutions.

Under regular conditions (Assumption 3.4) on the diffusion, it is known that the stochastic flow solution of a continuous SDE satisfies the homeomorphic property (see Bismut [6], Kunita [21], [22]). We have the following result whose proof can be found in [21].

Proposition 3.3. *Let Assumption 3.4 holds. Then $\{X_{t,s}(x); x \in \mathbb{R}^d\}$ is a C^2-diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow satisfies the following backward SDE*

$$X_{t,s}^{-1}(y) = y - \int_t^s \widehat{b}(X_{r,s}^{-1}(y))dr - \int_t^s \sigma(X_{r,s}^{-1}(y))d\overleftarrow{B}_r. \tag{3.5}$$

for any $t < s$, where

$$\widehat{b}(x) = b(x) - \sum_{i,j}\frac{\partial\sigma^j(x)}{\partial x_i}\sigma^{ij}(x). \tag{3.6}$$

We denote by $J(X_{t,s}^{-1}(x))$ the determinant of the Jacobian matrix of $X_{t,s}^{-1}(x)$, which is positive and $J(X_{t,t}^{-1}(x)) = 1$. For $\varphi \in C_c^\infty(\mathbb{R}^d)$, we define a process $\varphi_t : \Omega\times[t,T]\times\mathbb{R}^d \to \mathbb{R}$ by

$$\varphi_t(s,x) := \varphi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x)). \tag{3.7}$$

By a change of variable formula, we have for all $v \in \mathbf{L}^2(\mathbb{R}^d)$,

$$(v \circ X_{t,s}(\cdot), \varphi) = \int_{\mathbb{R}^d} v(X_{t,s}(x))\varphi(x)dx = \int_{\mathbb{R}^d} v(y)\varphi(X_{t,s}^{-1}(y))J(X_{t,s}^{-1}(y))dy = (v, \varphi_t(s,\cdot)),\ \mathbb{P}-a.s.$$

Since $(\varphi_t(s,x))_{t\leq s}$ is a process, we may not use it directly as a test function because $\int_t^T (u(s,\cdot), \partial_s\varphi_t(s,\cdot))ds$ has no sense. However $\varphi_t(s,x)$ is a semimartingale and we have the following decomposition of $\varphi_t(s,x)$ where the proof can be found in [3] (proof of Lemma 2.1. p.135), see also Kunita [24], [23] for the use of such random test functions.

Lemma 3.1. *For every function $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$\varphi_t(s,x) \quad = \varphi(x) + \int_t^s \mathcal{L}^*\varphi_t(r,x)dr - \sum_{j=1}^d \int_t^s \left(\sum_{i=1}^d \frac{\partial}{\partial x_i}(\sigma^{ij}(x)\varphi_t(r,x))\right) dB_r^j, \tag{3.8}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Thanks to the above lemma, we can replace $\partial_s\varphi\, ds$ by the Itô stochastic integral with respect to $d\varphi_t(s,x)$. This allows us to give the following

Definition 3.2. *For every $s \in [t,T]$, $u \in \mathcal{H}_T$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$, we define*

$$\int_s^T (u_r, d\varphi_t(r,.)) \quad = \int_s^T (u_r, \mathcal{L}^*\varphi_t(r,.))\, dr - \sum_{j=1}^d \int_s^T \left(\sum_{i=1}^d \left(u_r, \frac{\partial}{\partial x_i}(\sigma^{ij}(x)\varphi_t(r,.))\right)\right) dB_r^j, \tag{3.9}$$

3.4. Existence and uniqueness of solutions for the reflected SPDE

In order to provide a probabilistic representation to the solution of the RSPDEs (1.3), we introduce the following Markovian RBDSDE:

$$\begin{cases} (i)\ Y_s^{t,x} = \Phi(X_{t,T}(x)) + \displaystyle\int_s^T f_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x})dr + \int_s^T h_r(X_{t,r}(x), Y_r^{t,x}, Z_r^{t,x})d\overleftarrow{W}_r + K_T^{t,x} - K_s^{t,x} \\ \qquad\qquad - \displaystyle\int_s^T Z_r^{t,x}dB_r,\ \mathbb{P}\text{-}a.s.,\ \forall\, s \in [t,T] \\ (ii)\ Y_s^{t,x} \in \bar{D} \quad \mathbb{P}\text{-}a.s. \\ (iii) \displaystyle\int_0^T (Y_s^{t,x} - v_s(X_{t,s}(x)))^* dK_s^{t,x} \leq 0.,\ \mathbb{P}\text{-}a.s., \\ \text{for any continuous } \mathcal{F}_t - \text{random function } v\ :\ [0,T]\times\Omega\times\mathbb{R}^d \longrightarrow \bar{D}. \end{cases} \tag{3.10}$$

Moreover, using Assumptions 3.2 and 3.3 and the equivalence of norm results (3.3) and (3.4), we get

$$\Phi(X_{t,T}(x)) \in \mathbf{L}^2(\mathcal{F}_T) \text{ and } \Phi(X_{t,T}(x)) \in \bar{D},$$
$$f_s^0(X_{t,s}(x)) \in \mathcal{H}_k^2(t,T) \text{ and } h_s^0(X_{t,s}(x)) \in \mathcal{H}_{k\times d}^2(t,T).$$

Therefore under Assumption 3.2-3.5 and according to Theorem 2.1, there exists a unique triplet $(Y^{t,x}, Z^{t,x}, K^{t,x})$ solution of the RBDSDE (3.4) associated to (Φ, f, h).

We now consider the following definition of weak solutions for the reflected SPDE (1.3):

Definition 3.3 (**Weak solution of RSPDE**). *We say that* $(u,\nu) := (u^i,\nu^i)_{1\le i\le k}$ *is the weak solution of the reflected SPDE (1.3) associated to* (Φ,f,h), *if for each* $1\le i\le k$

(i) $u\in\mathcal{H}_T$, $u_t(x)\in\bar{D}, dx\otimes dt\otimes d\mathbb{P}$ *a.e., and* $u(T,x)=\Phi(x)$.

(ii) ν^i *is a signed Random measure on* $[0,T]\times\mathbb{R}^d$ *concentrated on* $\{u\in\partial D\}$ *such that:*

a) ν^i *is adapted in the sense that for any measurable and* $|\nu^i|$*-integrable function* $\psi:[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ *and for each* $s\in[t,T]$, $\int_s^T\int_{\mathbb{R}^d}\psi(r,x)\nu^i(dr,dx)$ *is* $\mathcal{F}^W_{s,T}$*-measurable.*

b) $\mathbb{E}\big[\int_0^T\int_{\mathbb{R}^d} w(x)|\nu^i|(dt,dx)\big]<\infty$.

(iii) For every $\varphi\in\mathcal{D}_T$,

$$\begin{aligned}&\int_t^T\int_{\mathbb{R}^d}u^i(s,x)\partial_s\varphi(s,x)dxds+\int_{\mathbb{R}^d}(u^i(t,x)\varphi(t,x)-\Phi^i(x)\varphi(T,x))dx-\int_t^T\int_{\mathbb{R}^d}u^i(s,x)\mathcal{L}^*\varphi(s,x)dxds\\&=\int_t^T\int_{\mathbb{R}^d}f_s(x,u(s,x),\nabla u(s,x)\sigma(x))\varphi(s,x)dxds+\int_t^T\int_{\mathbb{R}^d}h_s(x,u(s,x),\nabla u(s,x)\sigma(x))\varphi(s,x)dxd\overleftarrow{W}_s\\&+\int_t^T\int_{\mathbb{R}^d}\varphi(s,x)1_{\{u\in\partial D\}}(s,x)\nu^i(ds,dx),\ \ \mathbb{P}-a.s.\end{aligned}\tag{3.11}$$

For the sake of simplicity we will omit in the sequel the subscript i.

The main result of this section is the following:

Theorem 3.1. *Let Assumptions 2.1 and 3.2-3.5 hold and* $w(x)=(1+|x|)^{-p}$ *with* $p>d+1$. *Then there exists a weak solution* (u,ν) *of the reflected SPDE (1.3) associated to* (Φ,f,h) *such that,* $u(t,x):=Y_t^{t,x}$, $dt\otimes d\mathbb{P}\otimes w(x)dx-a.e.$, *and*

$$Y_s^{t,x}=u(s,X_{t,s}(x)),\qquad Z_s^{t,x}=(\nabla u\sigma)(s,X_{t,s}(x)),\quad ds\otimes d\mathbb{P}\otimes w(x)dx-a.e..\tag{3.12}$$

Moreover, ν^i *is a regular measure in the following sense: for every measurable bounded and positive functions* φ *and* ψ,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T\varphi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)1_{\{u\in\partial D\}}(s,x)\nu^i(ds,dx)\\&=\int_{\mathbb{R}^d}\int_t^T\varphi(s,x)\psi(s,X_{t,s}(x))dK_s^{t,x,i}dx,\ a.s.\end{aligned}\tag{3.13}$$

where $(Y_s^{t,x},Z_s^{t,x},K_s^{t,x})_{t\le s\le T}$ *is the solution of RBDSDE (3.4) and satisfying the probabilistic interpretation* (3.13).
If $(\overline{u},\overline{\nu})$ *is another solution of the reflected SPDE (1.3) such that* $\overline{\nu}$ *satisfies (3.13) with some* $\overline{K}$ *instead of* K, *where* $(\overline{K}_s^{t,x})_{t\le s\le T}$ *is a continuous finite variation process for all* (t,x), *then* $\overline{u}=u$ *and* $\overline{\nu}=\nu$.
In other words, there is a unique Randon regular measure with support $\{u\in\partial D\}$ *which satisfies (3.13).*

Remark 3.1. *The expression (3.13) gives us the probabilistic interpretation (Feymamn-Kac's formula) for the measure* ν *via the nondecreasing process* $K^{t,x}$ *of the RBDSDE. This formula was first introduced in Bally et al. [2] (see also [31]) in the context of obstacle problem for PDEs. Here we adapt this notion to the case of SPDEs in a convex domain.*

We give now the following result which allows us to link in a natural way the solution of RSPDE with the associated RBDSDE. Roughly speaking, for each test function $\varphi \in C_c^\infty(\mathbb{R}^d)$, the variational formulation (3.11) written with the random test functions $\varphi_t(\cdot,\cdot)$ gives the RBDSDE integrated against the test function φ , this dictionary can be understood as the dual formulation of the variational equation for SPDE. Pardoux and Peng [34] have proved the probabilistic representation of classical solution u for semilinear SPDE's via BDSDEs by using the classical Itô's formula for $u(s, X_{t,s}(x))$. However, since we consider Sobolev solutions for RSPDEs, the following proposition plays the rule of Itô's formula applied to the random test function $\varphi_t(s,x)$.

Proposition 3.4. *Let Assumptions 3.2-3.5 hold and (u,ν) be a weak solution of the reflected SPDE (1.3) associated to (Φ, f, h), then for $s \in [t,T]$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^T u(r,x)d\varphi_t(r,x)dx + (u(s,\cdot),\varphi_t(s,\cdot)) - (\Phi(\cdot),\varphi_t(T,\cdot)) - \int_{\mathbb{R}^d}\int_s^T u(r,x)\mathcal{L}^*\varphi_t(r,x))drdx\\ &= \int_{\mathbb{R}^d}\int_s^T f_r(x,u(r,x),\nabla u(r,x)\sigma(x))\varphi_t(r,x)drdx + \int_{\mathbb{R}^d}\int_s^T h_r(x,u(r,x),\nabla u(r,x)\sigma(x))\varphi_t(r,x)d\overleftarrow{W}_r dx\\ &+ \int_{\mathbb{R}^d}\int_s^T \varphi_t(r,x)1_{\{u\in\partial D\}}(r,x)\nu(dr,dx) \quad a.s.\end{aligned} \tag{3.14}$$

where $\int_{\mathbb{R}^d}\int_s^T u(r,x)d\varphi_t(r,x)dx$ is well defined in the semimartingale decomposition result (Lemma 3.1).

The proof of the proposition is the same as the proof of Proposition 2.3, p. 137 in Bally and Matoussi [3]. This latter is based on Lemma 4.1 p.147 and Lemma 4.2. p.148 which involve the Wong-Zakai approximation of the Itô stochastic integral appearing in the semimartingale decomposition of the random test functions given by (3.7) (see [19], chap. 6, section 7 , p.480-517). The main idea is to use $(\varphi_t(s,x))$ as a test function in the (3.11). The problem is that $(\varphi_t(s,x))$ is not differentiable in the time variable s, so that $\int\int_t^T u_s\partial_s\varphi_t(s,x)dsdx$ has no sense. However $(\varphi_t(s,x))$ a semimartingale and one can use Wong-Zakai approximation (see [3], Lemma 4.2 p.148) to handel with this point . Of course, the WZ approximation converges to Stratonovitch stochastic integral, but thanks to regularity assumption on the diffusion coefficient $\sigma(x)$, one can write the result explicitly as Itô's stochastic integral with drift term which disappear with a part of the drift term in the semimartingale decomposition of $(X_{t,s}^{-1}(x))$. We note that Kunita ([24], [23]) has developed a theory of distribution valued semimartingales to study a class of linear SPDEs using these random test functions.

Proof of Theorem 3.1.

a) Existence: The existence of a solution will be proved in two steps. For the first step, we suppose that h doesn't depend on y, z, then we are able to apply the classical penalization method. In the second step, we study the case when h depends on y, z with the result obtained in the first step.

Step 1 : We will use the penalization method. For $n \in \mathbb{N}$, we consider for all $s \in [t,T]$,

$$\begin{aligned}Y_s^{n,t,x} = \Phi(X_{t,T}(x)) &+ \int_s^T f_r(X_{t,r}(x), Y_r^{n,t,x}, Z_r^{n,t,x})dr + \int_s^T h_r(X_{t,r}(x))d\overleftarrow{W}_r\\ &- n\int_s^T (Y_r^{n,t,x} - \pi(Y_r^{n,t,x}))dr - \int_s^T Z_r^{n,t,x}dB_r.\end{aligned}$$

From Theorem 3.1 in Bally and Matoussi [3], we know that $u_n(t,x) := Y_t^{n,t,x}$, is a solution of the SPDE (Φ, f_n, h) (1.1), with $f_n(t,x,y) = f(t,x,y,z) - n(y - \pi(y))$, i.e. for every $\varphi \in \mathcal{D}_T$

$$\begin{aligned}\int_t^T (u^n(s,\cdot), \partial_s\varphi(s,\cdot))ds + (u^n(t,\cdot),\varphi(t,\cdot)) - (\Phi(\cdot),\varphi(T,\cdot)) - \int_t^T (u^n(s,\cdot), \mathcal{L}^*\varphi(s,\cdot))ds \\ = \int_t^T (f_s(\cdot, u^n(s,\cdot), \sigma^*\nabla u_n(s,\cdot)), \varphi(s,\cdot))ds + \int_t^T (h_s(\cdot),\varphi(s,\cdot))d\overleftarrow{W}_s \\ - n\int_t^T ((u^n - \pi(u^n))(s,\cdot),\varphi(s,\cdot))ds. \qquad (3.15)\end{aligned}$$

Moreover from Theorem 3.1 in Bally and Matoussi [3], we also have

$$Y_s^{n,t,x} = u_n(s, X_{t,s}(x)) \ , \ Z_s^{n,t,x} = (\nabla u_n \sigma)(s, X_{t,s}(x)), \ ds \otimes d\mathbb{P} \otimes w(x)\,dx - a.e. \qquad (3.16)$$

Set $K_s^{n,t,x} = -n\int_t^s (Y_r^{n,t,x} - \pi(Y_r^{n,t,x}))dr$. Then by (3.16), we have that

$$K_s^{n,t,x} = -n\int_t^s (u_n - \pi(u_n))(r, X_{t,r}(x))dr. \qquad (3.17)$$

Following the estimates and convergence results for $(Y^{n,t,x}, Z^{n,t,x}, K^{n,t,x})$ in Section 2 and estimate (B.1), we get :

$$\sup_n \mathbb{E}\left[\sup_{t\le s\le T} |Y_s^{n,t,x}|^2 + \int_t^T \|Z_s^{n,t,x}\|^2 ds + \|K^{n,t,x}\|_{VT}\right] \le C(T,x), \qquad (3.18)$$

where

$$C(T,x) := C\,\mathbb{E}\Big[|\Phi(X_{t,T}(x))|^2 + \int_t^T \left(|f_s^0(X_{t,s}(x))|^2 + |h_s^0(X_{t,s}(x))|^2\right) ds\Big],$$

and

$$\begin{aligned}&\mathbb{E}[\sup_{t\le s\le T} |Y_s^{n,t,x} - Y_s^{m,t,x}|^2] + \mathbb{E}[\int_t^T \|Z_s^{n,t,x} - Z_s^{m,t,x}\|^2 ds] \\ &+ \mathbb{E}[\sup_{t\le s\le T} |K_s^{n,t,x} - K_s^{m,t,x}|^2] \quad \longrightarrow 0, \text{ as } n,m \longrightarrow +\infty.\end{aligned}$$

Moreover, the equivalence of norms results (3.4) yield:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T w(x)(|u_n(s,x) - u_m(s,x)|^2 + |(\nabla u_n\sigma)(s,x) - (\nabla u_m\sigma)(s,x)|^2)dsdx \\ \le\ & \frac{1}{k_2}\int_{\mathbb{R}^d} w(x)\mathbb{E}\int_t^T (|Y_s^{n,t,x} - Y_s^{m,t,x}|^2 + \|Z_s^{n,t,x} - Z_s^{m,t,x}\|^2)dsdx \longrightarrow 0.\end{aligned}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_T$, and the limit $u = \lim_{n\to\infty} u_n$ belongs to $\mathcal{H}_T$.
Moreover, using standard computations from BSDEs and SPDEs technics we can prove that $\mathbb{E}[\sup_{s\in[0,T]} \|u_s^n - u_s^m\|_2^2] \longrightarrow 0$ as $n,m \to \infty$.
Denote $\nu_n(dt,dx) = -n(u_n - \pi(u_n))(t,x)dtdx$ and $\pi_n(dt,dx) = w(x)\nu_n(dt,dx)$, then by the equivalence norm result (3.4) we get

$$\begin{aligned}\mathbb{E}\big[|\pi_n|([t,T]\times\mathbb{R}^d)\big] &= \int_{\mathbb{R}^d}\int_t^T \mathbb{E}\big[n|(u_n - \pi(u_n))(s,x)|\big]dsw(x)dx \\ &\le \frac{1}{k_2}\int_{\mathbb{R}^d}\int_t^T \mathbb{E}\big[n|(u_n - \pi(u_n))(s,X_{t,s}(x))|\big]dsw(x)dx.\end{aligned}$$

Finally, using (3.17) and (3.18), we obtain

$$\mathbb{E}\big[|\pi_n|([0,T]\times\mathbb{R}^d)\big] \quad \leq \quad \frac{1}{k_2}\int_{\mathbb{R}^d} w(x)\mathbb{E}\left\|K^{n,0,x}\right\|_{VT} dx \leq C\int_{\mathbb{R}^d} w(x)dx < \infty.$$

It follows that

$$\sup_n \mathbb{E}\big[|\pi_n|([0,T]\times\mathbb{R}^d)\big] < \infty. \tag{3.19}$$

Moreover by Lemma D.1 (see Appendix D), the sequence of measures $(\pi_n(\omega,ds,dx))_{n\in\mathbb{N}}$ is tight $\mathbb{P}$-a.s. in $\omega\in\Omega$. Therefore, there exits a subsequence such that $(\pi_n(\omega,ds,dx))_{n\in\mathbb{N}}$ converges weakly to a measure $\pi(\omega,ds,dx)$. Define $\nu := w^{-1}\pi$, it remains now to prove that the limit $\nu(\omega,ds,dx)$ is measurable with respect to ω and satisfying Definition 3.3-(ii). We note first that from (3.19) and Fatou's Lemma, we get $\mathbb{E}\big[|\pi|([0,T]\times\mathbb{R}^d)\big] < \infty$. We have also for $\varphi\in\mathcal{D}_T$ with compact support in x,

$$\int_{\mathbb{R}^d}\int_t^T \varphi d\nu_n = \int_{\mathbb{R}^d}\int_t^T \frac{\phi}{w}d\pi_n \to \int_{\mathbb{R}^d}\int_t^T \frac{\phi}{w}d\pi = \int_{\mathbb{R}^d}\int_t^T \varphi d\nu.$$

Passing to the limit in the SPDE (Φ, f_n, h) (3.15), we get that (u,ν) satisfies the following equation, i.e. for every $\varphi\in\mathcal{D}_T$, we have

$$\begin{aligned}
&\int_t^T (u(s,\cdot),\partial_s\varphi(s,\cdot))ds + (u(t,\cdot),\varphi(t,\cdot)) - (\Phi(\cdot),\varphi(T,\cdot)) - \int_t^T (u(s,\cdot),\mathcal{L}^*\varphi(s,\cdot))ds \\
- &\int_t^T (f_s(\cdot,u(s,\cdot),\sigma^*\nabla u(s,\cdot)),\varphi(s,\cdot))ds - \int_t^T (h_s(\cdot),\varphi(s,\cdot))\overleftarrow{dW}_s = \int_t^T\int_{\mathbb{R}^d}\varphi(s,x)\nu(ds,dx),\ \mathbb{P}-a.s.
\end{aligned} \tag{3.20}$$

Thus, the term in the right hand side of the last equation is measuble with respect to ω since it is the sum of measurable terms obtained as the limit of sequence of measurable processes. Therefore, the pair (u,ν) satisfies the reflected SPDE associated to (Φ,f,h).

The last point is to prove that ν satisfies the probabilistic interpretation (3.13). Since $K^{n,t,x}$ converges to $K^{t,x}$ uniformly in t, the measure $dK^{n,t,x}$ converges to $dK^{t,x}$ weakly in probability. Fix two continuous functions φ, $\psi : [0,T]\times\mathbb{R}^d \to \mathbb{R}^+$ which have compact support in x and a continuous function with compact support $\theta : \mathbb{R}^d \to \mathbb{R}^+$, from Bally et al [2] (The proof of Theorem 4), we have (see also Matoussi and Xu [31])

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T \varphi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\theta(x)\nu(ds,dx) \\
= &\lim_{n\to\infty} -\int_{\mathbb{R}^d}\int_t^T \varphi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\theta(x)n(u_n-\pi(u_n))(s,x)dsdx \\
= &\lim_{n\to\infty} -\int_{\mathbb{R}^d}\int_t^T \varphi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))n(u_n-\pi(u_n))(t,X_{t,s}(x))dtdx \\
= &\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T \varphi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))dK_s^{n,t,x}dx \\
= &\int_{\mathbb{R}^d}\int_t^T \varphi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))dK_s^{t,x}dx.
\end{aligned}$$

We take $\theta=\theta_R$ to be the regularization of the indicator function of the ball of radius R and pass to the limit with $R\to\infty$, to get that

$$\int_{\mathbb{R}^d}\int_t^T \varphi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\nu(ds,dx) = \int_{\mathbb{R}^d}\int_t^T \varphi(s,x)\psi(s,X_{t,s}(x))dK_s^{t,x}dx. \tag{3.21}$$

From Section 2, it follows that $dK_s^{t,x} = 1_{\{u\in\partial D\}}(s, X_{t,s}(x))dK_s^{t,x}$. Again by regularization procedure we can set $\psi = 1_{\{u\in\partial D\}}$ in (3.21) to obtain

$$\int_{\mathbb{R}^d}\int_t^T \varphi(s, X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))1_{\{u\in\partial D\}}(s,x)\nu(ds,dx)$$
$$= \int_{\mathbb{R}^d}\int_t^T \varphi(s, X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\nu(ds,dx), \text{ a.s.}$$

Note that the family of functions $A(\omega) = \{(s,x) \to \varphi(s, X_{t,s}^{-1}(x)) : \varphi \in C_c^\infty\}$ is an algebra which separates the points (because $x \to X_{t,s}^{-1}(x)$ is a bijection). Given a compact set G, $A(\omega)$ is dense in $C([0,T]\times G)$. It follows that $J(X_{t,s}^{-1}(x))1_{\{u\in\partial D\}}(s,x)\nu(ds,dx) = J(X_{t,s}^{-1}(x))\nu(ds,dx)$ for almost every ω. While $J(X_{t,s}^{-1}(x)) > 0$ for almost every ω, we get $\nu(ds,dx) = 1_{\{u\in\partial D\}}(s,x)\nu(ds,dx)$, and (3.13) follows.
Then we get easily that $Y_s^{t,x} = u(s, X_{t,s}(x))$ and $Z_s^{t,x} = (\nabla u\sigma)(s, X_{t,s}(x))$, in view of the convergence results for $(Y_s^{n,t,x}, Z_s^{n,t,x})$ and the equivalence of norms. So $u(s, X_{t,s}(x)) = Y_s^{t,x} \in \bar{D}$. Specially for $s = t$, we have $u(t,x) \in \bar{D}$.

Step 2 : *The nonlinear case where h depends on y and z.*
Let $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ the unique solution of RBDSDE (2.5) associated to (Φ, f, h). In order to avoid standard fixed point arguments, we define $H(s,x) \triangleq h(x, X_{t,s}(x), Y_s^{t,x}, Z_s^{t,x})$. Due to the fact that h is Lipschitz with respect to (y,z), we have

$$|H(s,x)| \le |h_s^0(X_{t,s}(x))| + C(|Y_s^{t,x}| + |Z_s^{t,x}|).$$

Besides, by standard computations similar to estimate (3.18) we get

$$\mathbb{E}\left[|Y_s^{t,x}|^2 + \int_t^T \left\|Z_r^{t,x}\right\|^2 dr + \|K^{t,x}\|_{VT}\right] \le C\,(T,x)\,, \tag{3.22}$$

where

$$C(T,x) := C\,\mathbb{E}\Big[\,|\Phi(X_{t,T}(x))|^2 + \int_t^T \left(\,\left|f_s^0(X_{t,s}(x))\right|^2 + \left|h_s^0(X_{t,s}(x))\right|^2\,\right) ds\Big].$$

Integrating the estimate (3.22) over x with respect to the measure $w(x)dx$ and thanks to the equivalence norm results, we conclude that H belongs to $L^2([0,T]\times\Omega\times\mathbb{R}^d; dt\otimes d\mathbb{P}\otimes w(x)dx)$. Then, applying the result of Step 1 yields that there exists (u,ν) where $u(t,x) := Y_t^{t,x}$, $dt\otimes d\mathbb{P}\otimes w(x)dx - a.e.$ and satisfying the SPDE with obstacle (Φ, f, H), i.e. for every $\varphi \in \mathcal{D}_T$, we have

$$\begin{aligned}
&\int_t^T (u(s,\cdot), \partial_s\varphi(s,\cdot))ds + (u(t,\cdot),\varphi(t,\cdot)) - (\Phi(\cdot),\varphi(T,\cdot)) - \int_t^T (u(s,\cdot), \mathcal{L}^*\varphi(s,\cdot))ds \\
= &\int_t^T (f_s(\cdot, u(s,\cdot), \sigma^*\nabla u(s,\cdot)), \varphi(s,\cdot))ds + \int_t^T (H_s(\cdot),\varphi(s,\cdot))d\overleftarrow{W}_s \\
+ &\int_t^T\int_{\mathbb{R}^d} \phi(s,x)1_{\{u\in\partial D\}}(s,x)\nu(ds,dx).
\end{aligned} \tag{3.23}$$

Then by the uniqueness of the solution to the RBDSDE $(\Phi(X_{t,T}(x)),\ f,\ h)$, we get easily that $Y_s^{t,x} = u(s, X_{t,s}(x))$, $Z_s^{t,x} = (\nabla u\sigma)(s, X_{t,s}(x))$, and ν satisfies the probabilistic interpretation (3.13). So $u(s, X_{t,s}(x)) = Y_s^{t,x} \in \bar{D}$. Specially for $s = t$, we have $u(t,x) \in \bar{D}$, which is the desired result.

b) Uniqueness : Set $(\overline{u},\overline{\nu})$ to be another weak solution of the reflected SPDE (1.3) associated to (Φ,f,h); with $\overline{\nu}$ verifies (3.13) for a continuous finite variation process $(\overline{K}_s^{t,x})_{t\leq s\leq T}$. We fix $\varphi:\mathbb{R}^d\to\mathbb{R}^k$, a smooth function in $C_c^2(\mathbb{R}^d)$ with compact support and denote $\varphi_t(s,x)=\varphi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))$. From Proposition 3.4, one may use $\varphi_t(s,x)$ as a test function in the SPDE (Φ,f,h) with $\partial_s\varphi(s,x)ds$ replaced by a stochastic integral with respect to the semimartingale $\varphi_t(s,x)$. Then we get, for $t\leq s\leq T$

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^T\overline{u}(r,x)d\varphi_t(r,x)dx+\int_{\mathbb{R}^d}\overline{u}(s,x)\varphi_t(s,x)dx-\int_{\mathbb{R}^d}\Phi(x)\varphi_t(T,x)dx-\int_s^T\int_{\mathbb{R}^d}\overline{u}(r,x)\mathcal{L}^*\varphi_t(r,x)drdx\\&=\int_s^T\int_{\mathbb{R}^d}f_r(x,\overline{u}(r,x),(\nabla\overline{u}\sigma)(r,x))\varphi_t(r,x)drdx+\int_s^T\int_{\mathbb{R}^d}h_r(x,\overline{u}(r,x),(\nabla\overline{u}\sigma)(r,x))\varphi_t(r,x)dx\overleftarrow{dW}_r\\&+\int_s^T\int_{\mathbb{R}^d}\varphi_t(r,x)1_{\{\overline{u}\in\partial D\}}(r,x)\overline{\nu}(dr,dx).\end{aligned}\tag{3.24}$$

By (3.9) in Lemma 3.1, we have

$$\int_{\mathbb{R}^d}\int_s^T\overline{u}(r,x)d\varphi_t(r,x)dx=\int_s^T(\int_{\mathbb{R}^d}(\nabla\overline{u}\sigma)(r,x)\varphi_t(r,x)dx)dB_r+\int_s^T\int_{\mathbb{R}^d}\overline{u}(r,x)\mathcal{L}^*\varphi_t(r,x)drdx.$$

Substituting this equality in (3.24), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\overline{u}(s,x)\varphi_t(s,x)dx=\int_{\mathbb{R}^d}\Phi(x)\varphi_t(T,x)dx-\int_s^T(\int_{\mathbb{R}^d}(\nabla\overline{u}\sigma)(r,x)\varphi_t(r,x)dx)dB_r\\&+\int_s^T\int_{\mathbb{R}^d}f_r(x,\overline{u}(r,x),(\nabla\overline{u}\sigma)(r,x))\varphi_t(r,x)drdx+\int_s^T\int_{\mathbb{R}^d}h_r(x,\overline{u}(r,x),(\nabla\overline{u}\sigma)(r,x))\varphi_t(r,x)dx\overleftarrow{dW}_r\\&+\int_s^T\int_{\mathbb{R}^d}\varphi_t(r,x)1_{\{\overline{u}\in\partial D\}}(r,x)\overline{\nu}(dr,dx).\end{aligned}$$

Then by changing of variable $y=X_{t,r}^{-1}(x)$ and applying (3.13) for $\overline{\nu}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\overline{u}(s,X_{t,s}(y))\varphi(y)dy=\int_{\mathbb{R}^d}\Phi(X_{t,T}(y))\varphi(y)dy\\&+\int_{\mathbb{R}^d}\int_s^T\varphi(y)f_r(X_{t,r}(y),\overline{u}(r,X_{t,r}(y)),(\nabla\overline{u}\sigma)(r,X_{t,r}(y)))drdy\\&+\int_{\mathbb{R}^d}\int_s^T\varphi(y)h_r(X_{t,r}(y),\overline{u}(r,X_{t,r}(y)),(\nabla\overline{u}\sigma)(r,X_{t,r}(y)))dy\overleftarrow{dW}_r\\&+\int_s^T\int_{\mathbb{R}^d}\phi(y)1_{\{\overline{u}\in\partial D\}}(r,X_{t,s}(y))d\overline{K}_r^{t,y}dy-\int_s^T\int_{\mathbb{R}^d}\varphi(y)(\nabla\overline{u}\sigma)(r,X_{t,r}(y))dydB_r\end{aligned}$$

Since φ is arbitrary, we can prove that for $w(y)dy$ almost every y, $(\overline{u}(s,X_{t,s}(y)),(\nabla\overline{u}\sigma)(s,X_{t,s}(y)),\widehat{K}_s^{t,y})$ solves the RBDSDE $(\Phi(X_{t,T}(y)),f,h)$. Here $\widehat{K}_s^{t,y}=\int_t^s1_{\{\overline{u}\in\partial D\}}(r,X_{t,r}(y))d\overline{K}_r^{t,y}$. Then by the uniqueness of the solution of the RBDSDE, we know $\overline{u}(s,X_{t,s}(y))=Y_s^{t,y}=u(s,X_{t,s}(y))$, $(\nabla\overline{u}\sigma)(s,X_{t,s}(y))=Z_s^{t,y}=(\nabla u\sigma)(s,X_{t,s}(y))$, and $\widehat{K}_s^{t,y}=K_s^{t,y}$. Taking $s=t$ we deduce that $\overline{u}(t,y)=u(t,y)$, $w(y)dy$-a.s. and by the probabilistic interpretation (3.13), we obtain

$$\int_s^T\int_{\mathbb{R}^d}\varphi_t(r,x)1_{\{\overline{u}\in\partial D\}}(r,x)\overline{\nu}(dr,dx)=\int_s^T\int_{\mathbb{R}^d}\varphi_t(r,x)1_{\{u\in\partial D\}}(r,x)\nu(dr,dx).$$

So $1_{\{\overline{u}\in\partial D\}}(r,x)\overline{\nu}(dr,dx) = 1_{\{u\in\partial D\}}(r,x)\nu(dr,dx)$.

□

Appendix A: Some properties of convexity

In this section, we will list some properties of convex domains. Denote $\rho(x) \triangleq (d(x,D))^2 = |x - \pi(x)|^2$, the square of the distance to the domain D and π the projection on the closure $\bar{D}$. If D is a convex domain, then the function ρ is convex. Moreover, if D is a regular domain, ρ is two times differentiable on the complement of D and we obtain:

$$\forall x \notin D, \quad \nabla\rho(x) = 2(x - \pi(x)).$$

From this expression of gradient $\nabla\rho$, we remark that the hessian matrix $Hess\rho(x)$ has the following form:

$$Hess\rho(x) = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (M) & \\ 0 & & & \end{pmatrix}$$

where M is a positive semidefinite matrix. We deduce also that:

$$\forall z \in \mathbb{R}^{k\times d} \quad \text{trace}[zz^* Hess\rho(x)] \geq 0 \tag{A.1}$$

Since $Hess$ is a positive matrix we have for every unit outward normal $n(x)$

$$\begin{aligned} |z(n(x))^*|^2 &\leq C\frac{1}{2}\,\text{trace}\Big[zz^* \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (M) & \\ 0 & & & \end{pmatrix}\Big] \\ &\leq C\ \text{trace}[zz^* Hess\rho(x)] \end{aligned} \tag{A.2}$$

Lemma A.1. *If D_ε is a convex domain which satisfies (2.4), then $\exists c > 0$ such that $\forall \varepsilon < 1, \forall x \in \mathbb{R}^k$*

$$|\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon^2 + \varepsilon d(x, D_\varepsilon)} \quad \textit{and} \quad |\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon^2 + \varepsilon d(x, D)}$$

where $\pi_\varepsilon(x)$ is the projection of $x \in \mathbb{R}^k$ on the closure $\bar{D}_\varepsilon$

Lemma A.2.

$$\exists c > 0 \quad \textit{such that} \quad \forall \varepsilon < 1,\ \forall x \in \mathbb{R}^k \quad |\pi(x) - \pi_\varepsilon(x)| < c\sqrt{\varepsilon}(1 + d(x, D_\varepsilon))$$

$$\exists c > 0 \quad \textit{such that} \quad \forall \varepsilon < 1,\ \forall x \in \mathbb{R}^k \quad |\pi(x) - \pi_\varepsilon(x)|\mathbf{1}_{\{d(x,D_\varepsilon)>\varepsilon\}} < c\sqrt{\varepsilon}\sqrt{d(x, D_\varepsilon)}\mathbf{1}_{\{d(x,D_\varepsilon)>\varepsilon\}}$$

Appendix B: A priori estimates

In this section, we provide a priori estimates which are uniform in n on the solutions of (2.10).

Lemma B.1. *There exists a constant $C > 0$, independent of n, such that for all n large enough*

$$\sup_n \mathbb{E}\big[\sup_{0\le t\le T}|Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds + \|K^n\|_{VT}\big] \le C\mathbb{E}\big[|\xi^2| + \int_t^T \big(|f_s^0|^2 + |h_s^0|^2\big)ds\big]. \tag{B.1}$$

Proof. For a given point $a \in D$, that satisfies condition (2.3), we apply generalized Itô's formula to get

$$\begin{aligned}|Y_t^n - a|^2 + \int_t^T \|Z_s^n\|^2 ds &= |\xi - a|^2 + 2\int_t^T (Y_s^n - a)^* f_s(Y_s^n, Z_s^n)ds + 2\int_t^T (Y_s^n - a)^* h_s(Y_s^n, Z_s^n)d\overleftarrow{W}_s \\ &\quad - 2\int_t^T (Y_s^n - a)^* Z_s^n dB_s + \int_t^T \|h_s(Y_s^n, Z_s^n)\|^2 ds - 2n\int_t^T (Y_s^n - a)^*(Y_s^n - \pi(Y_s^n))ds.\end{aligned} \tag{B.2}$$

The stocastic integrals have both zero expectations since (Y^n, Z^n) belongs to $\mathcal{S}_k^2([0,T])\times\mathcal{H}_{k\times d}^2([0,T])$. We take expectation in (B.2) and we use conditions (2.1), (2.3) and the Lipschitz Assumption 2.3 in order to obtain

$$\begin{aligned}\mathbb{E}[|Y_t^n - a|^2] + \mathbb{E}[\int_t^T \|Z_s^n\|^2 ds] &\le \mathbb{E}[|\xi - a|^2] + 2C\mathbb{E}[\int_t^T |Y_s^n - a|(|f_s(a,0)| + |Y_s^n - a| + \|Z_s^n\|)ds] \\ &\quad + \mathbb{E}[\int_t^T |h_s(a,0)|^2 ds] + C\mathbb{E}[\int_t^T |Y_s^n - a|^2 ds] + \alpha\mathbb{E}[\int_t^T \|Z_s^n\|^2 ds] \\ &\le C(1 + \mathbb{E}[\int_t^T (|f_s(a,0)|^2 + |h_s(a,0)|^2)ds] + C(1+\varepsilon^{-1})\mathbb{E}[\int_t^T |Y_s^n - a|^2 ds] \\ &\quad + (\alpha + \varepsilon)\mathbb{E}[\int_t^T \|Z_s^n\|^2 ds].\end{aligned} \tag{B.3}$$

Thus, if we choose $\varepsilon = \dfrac{1-\alpha}{2}$, we have

$$\mathbb{E}[|Y_t^n - a|^2] + (\frac{1-\alpha}{2})\mathbb{E}[\int_t^T \|Z_s^n\|^2 ds] \le C(1 + \mathbb{E}[\int_t^T |Y_s^n - a|^2 ds]).$$

Then, it follows from Gronwall's lemma that

$$\sup_{0\le t\le T}\mathbb{E}[|Y_t^n - a|^2] \le Ce^{CT}.$$

Therefore we can deduce

$$\sup_{0\le t\le T}\mathbb{E}[|Y_t^n|^2] \le C \quad \text{and} \quad \mathbb{E}[\int_0^T \|Z_s^n\|^2 ds] \le C. \tag{B.4}$$

On the other hand, the uniform estimate on Y^n is obtained by taking the supremum over t in the equation (B.2), using the previous calculations and Burkholder-Davis-Gundy inequality. Thus, we get for all $n \ge 0$

$$\mathbb{E}[\sup_{0\le t\le T}|Y_t^n - a|^2] \le C \quad \text{and} \quad \mathbb{E}[\sup_{0\le t\le T}|Y_t^n|^2] \le C.$$

Finally, the total variation of the process K^n is given by

$$\|K^n\|_{VT} = n\int_0^T |Y_s^n - \pi(Y_s^n)|ds.$$

But from the property (2.3) and the equation (B.2) we have

$$\begin{aligned}2n\int_t^T \gamma|Y_s^n-\pi(Y_s^n)|ds &\le 2n\int_t^T |(Y_s^n-a)^*(Y_s^n-\pi(Y_s^n))|ds\\ &\le |\xi-a|^2+2\int_t^T (Y_s^n-a)^* f_s(Y_s^n,Z_s^n)ds+2\int_t^T (Y_s^n-a)^* h_s(Y_s^n,Z_s^n)d\overleftarrow{W}_s\\ &\quad -2\int_t^T (Y_s^n-a)^* Z_s^n dB_s+\int_t^T \|h_s(Y_s^n,Z_s^n)\|^2 ds.\end{aligned}$$

Hence it follows from previous estimates that

$$\mathbb{E}[\|K^n\|_{VT}]\le C,$$

and the proof of Lemma B.1 is complete. □

Appendix C: Proof of Lemma 2.4

Let first recall that (Y^n,Z^n) is solution of the BDSDE (2.10) associated to (ξ,f^n,h) where $f_s^n(y,z)=f_s(y,z)-n(y-\pi(y))$, for each $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$. Note that , since we have assumed that $0\in D$ (Assumption 2.1), $f_s^n(0,0)=f_s(0,0):=f_s^0$. Therefore, thanks to L^p-estimate for BDSDE (Theorem 4.1 in [34] applied for $p=4$), we have the following estimate

$$\sup_n \mathbb{E}[\sup_{0\le t\le t}|Y_t^n|^4+\Big(\int_0^T\|Z_s^n\|^2ds\Big)^2]\le C\mathbb{E}\Big[|\xi|^4+\int_0^T(|f_s^0|^4+|h_s^0|^4)ds\Big]<\infty. \tag{C.1}$$

Now, we apply generalized Itô's formula to get

$$\begin{aligned}|Y_t^n|^2+\int_t^T\|Z_s^n\|^2ds+2n\int_t^T(Y_s^n)^*(Y_s^n-\pi(Y_s^n))ds&=|\xi|^2+2\int_t^T(Y_s^n)^*f_s(Y_s^n,Z_s^n)ds\\ &\quad+2\int_t^T(Y_s^n)^*h_s(Y_s^n,Z_s^n)d\overleftarrow{W}_s-2\int_t^T(Y_s^n)^*Z_s^ndB_s+\int_t^T\|h_s(Y_s^n,Z_s^n)\|^2ds.\end{aligned} \tag{C.2}$$

Without loss of generality we apply the property (2.3) for $a=0$ since $0\in D$ (If not, we apply the generalized Itô formula to $|Y_t^n-a|^2$) to obtain

$$\begin{aligned}2n\int_0^T \gamma|Y_s^n-\pi(Y_s^n)|ds &\le 2n\int_0^T |(Y_s^n)^*(Y_s^n-\pi(Y_s^n))|ds\\ &\le |\xi|^2+2\int_0^T (Y_s^n)^* f_s(Y_s^n,Z_s^n)ds+2\int_0^T (Y_s^n)^* h_s(Y_s^n,Z_s^n)d\overleftarrow{W}_s\\ &\quad -2\int_0^T (Y_s^n)^* Z_s^n dB_s+\int_0^T \|h_s(Y_s^n,Z_s^n)\|^2 ds.\end{aligned}$$

Then, taking the square and the expectation yields

$$\begin{aligned}\mathbb{E}\Big[\Big(n\int_0^T|Y_s^n-\pi(Y_s^n)|ds\Big)^2\Big]&\le C\mathbb{E}\Big[|\xi|^4\Big]+C\mathbb{E}\Big[\Big(\int_0^T(Y_s^n)^*f_s(Y_s^n,Z_s^n)ds\Big)^2\Big]\\ &\quad+C\mathbb{E}\Big[\Big(\int_0^T(Y_s^n)^*h_s(Y_s^n,Z_s^n)d\overleftarrow{W}_s\Big)^2\Big]+C\mathbb{E}\Big[\Big(\int_0^T(Y_s^n)^*Z_s^ndB_s\Big)^2\Big]\\ &\quad+C\mathbb{E}\Big[\Big(\int_0^T\|h_s(Y_s^n,Z_s^n)\|^2ds\Big)^2\Big].\end{aligned}$$

By using the isometry property and the boundedness of f and h, we obtain

$$\mathbb{E}\Big[\Big(n\int_0^T |Y_s^n - \pi(Y_s^n)|ds\Big)^2\Big] \leq C\mathbb{E}\Big[|\xi|^4\Big] + C\mathbb{E}\Big[\Big(\int_0^T |Y_s^n|ds\Big)^2\Big] + C\mathbb{E}\Big[\int_0^T |Y_s^n h_s(Y_s^n, Z_s^n)|^2 ds\Big] + C\mathbb{E}\Big[\int_0^T |Y_s^n Z_s^n|^2 ds\Big] + C.$$

Finally, we deduce from Holder inequality and boundedness of h that

$$\mathbb{E}\Big[\Big(n\int_0^T |Y_s^n - \pi(Y_s^n)|ds\Big)^2\Big] \leq C\mathbb{E}\Big[|\xi|^4 + \int_0^T |Y_s^n|^2 ds + \sup_{0\leq t\leq T}|Y_t^n|^4 + \Big(\int_0^T \|Z_s^n\|^2 ds\Big)^2\Big] + C.$$

Thus, from the estimate (C.1) we get the desired result. □

Appendix D: Proof of the tightness of the sequence $(\pi_n)_{n\in\mathbb{N}}$

Recall first that $\nu_n(dt,dx) = -n(u_n - \pi(u_n))(t,x)dtdx$ and $\pi_n(dt,dx) = w(x)\nu_n(dt,dx)$ where u_n is the solution of the SPDEs (3.15).

Lemma D.1. *$\mathbb{P}$-a.s. in $\omega\in\Omega$, the sequence of random measure $(\pi_n(\omega,ds,dx))_{n\in\mathbb{N}}$ is tight.*

Proof. We shall prove that for every $\epsilon > 0$, there exists some constant K such that

$$\mathbb{E}\big[\int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x|\geq 2K\}}|\pi_n|(ds,dx)\big] \leq \epsilon,\ \forall n\in\mathbb{N}. \tag{D.1}$$

We first write

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x|\geq 2K\}}|\pi_n|(ds,dx)\\
&= \int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x|\geq 2K\}}\left(\mathbf{1}_{\{|X_{0,s}^{-1}(x)-x|\leq K\}} + \mathbf{1}_{\{|X_{0,s}^{-1}(x)-x|\geq K\}}\right)|\pi_n|(ds,dx)\\
&:= I_K^n + L_K^n, \quad \mathbb{P}-a.s.
\end{aligned}$$

Taking expectation yields

$$\mathbb{E}\big[\int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x|\geq 2K\}}\pi_n(ds,dx)\big] = \mathbb{E}\big[I_K^n\big] + \mathbb{E}\big[L_K^n\big]. \tag{D.2}$$

From Definition 3.3$(ii)a$), we have $\pi_n(\omega,ds,dx)$ is a $\mathcal{F}_{s,T}^W$-adapted measure and we know that the inverse of the flow depends only on the noise B. Then, since W and B are mutually independant we get by (3.19) and for $K \geq 2\|b\|_\infty T$, we get

$$\begin{aligned}
\mathbb{E}\big[L_K^n\big] &\leq \mathbb{P}\left(\sup_{0\leq r\leq T}\big|X_{0,r}^{-1}(x) - x\big| \geq K\right)\mathbb{E}\big[\int_0^T\int_{\mathbb{R}^d}|\pi_n|(ds,dx)\big]\\
&\leq \big(C_1\,\exp(-C_2K^2) + C_3\,\exp(-C_4K)\big)\,\mathbb{E}\big[|\pi_n|([0,T]\times\mathbb{R}^d)\big]\\
&\leq C_1'\,\exp(-C_2K^2) + C_3'\,\exp(-C_4K).
\end{aligned}$$

Finally, for K sufficiently large we obtain

$$\mathbb{E}[L_K^n] \leq \epsilon.$$

On the other hand, if $|x| \geq 2K$ and $\left|X_{0,s}^{-1}(x) - x\right| \leq K$ then $\left|X_{0,s}^{-1}(x)\right| \geq K$. Therefore

$$\begin{aligned}\mathbb{E}[I_K^n] &\leq \mathbb{E}[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)|\geq K\}} w(x)|\nu_n|(ds,dx)] \\ &= \mathbb{E}[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|X_{0,s}^{-1}(x)|\geq K\}} w(x)n|u_n - \pi(u_n)|(s,x)dsdx]\end{aligned}$$

which, by the change of variable $y = X_{0,s}^{-1}(x)$, becomes

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|y|\geq K\}} w(X_{0,s}(y))J(X_{0,s}(y))n|u_n - \pi(u_n)|(s,X_{0,s}(y))dsdy\right] \\ &\leq \mathbb{E}\left[\int_{\mathbb{R}^d} w(x)\left(w(x)^{-1}\mathbf{1}_{\{|x|\geq K\}} \sup_{0\leq r\leq T} w(X_{0,r}(x))J(X_{0,r}(x))\right)\|K^{n,0,x}\|_{VT}dx\right] \\ &\leq \left(\mathbb{E}\left[\int_{\mathbb{R}^d} \|K^{n,0,x}\|_{VT}^2 w(x)dx\right]\right)^{1/2} \\ &\quad \left(\mathbb{E}\left[\int_{\mathbb{R}^d}\left(w(x)^{-1}\mathbf{1}_{\{|x|\geq K\}} \sup_{0\leq r\leq T} w(X_{0,r}(x))J(X_{0,r}(x))\right)^2 w(x)dx\right]\right)^{1/2} \\ &\leq C\left(\mathbb{E}\left[\int_{\mathbb{R}^d}\left(w(x)^{-1}\mathbf{1}_{\{|x|\geq K\}} \sup_{0\leq r\leq T} w(X_{0,r}(x))J(X_{0,r}(x))\right)^2 w(x)dx\right]\right)^{1/2}.\end{aligned}$$

where the last inequality is a consequence of (B.1). It is now sufficient to prove that

$$\int_{\mathbb{R}^d} w(x)^{-1}\mathbb{E}\left[\left(\sup_{0\leq r\leq T} w(X_{0,r}(x))J(X_{0,r}(x))\right)^2\right]dx < \infty. \tag{D.3}$$

Note that

$$\begin{aligned}&\mathbb{E}\left[\left(\sup_{0\leq r\leq T} w(X_{0,r}(x))J(X_{0,r}(x))\right)^2\right] \\ &\leq \left[\mathbb{E}\left(\sup_{0\leq r\leq T}|w(X_{0,r}(x))|\right)^4\right]^{1/2}\left[\mathbb{E}\left(\sup_{0\leq r\leq T}|J(X_{0,r}(x))|\right)^4\right]^{1/2} \\ &\leq C\left[\mathbb{E}\left(\sup_{0\leq r\leq T}|w(X_{0,r}(x))|\right)^4\right]^{1/2}.\end{aligned}$$

Therefore it is sufficient to prove that:

$$\int_{\mathbb{R}^d}\frac{1}{w(x)}\left(\mathbb{E}\left[\sup_{t\leq r\leq T}|w(X_{t,r}(x))|^4\right]\right)^{1/2}dx < \infty.$$

Since $w(x) \leq 1$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq r\leq T}|w(X_{t,r}(x))|^4\right] &\leq \mathbb{E}\left[\sup_{t\leq r\leq T}|w(X_{t,r}(x))|^4 \mathbf{1}_{\left\{\sup_{t\leq r\leq T}|X_{t,r}(x)-x|\leq\frac{|x|}{2}\right\}}\right] \\ &\quad +\mathbb{P}\left(\sup_{t\leq r\leq T}|X_{t,r}(x)-x| \geq \frac{|x|}{2}\right) \\ &= : A(x) + B(x).\end{aligned}$$

If $\sup_{t\leq r\leq T}|X_{t,r}(x)-x| \leq \frac{|x|}{2}$ then $|X_{t,r}(x)| \geq \frac{|x|}{2}$ and so $|w(X_{t,r}(x))| \leq \left(1+\frac{|x|}{2}\right)^{-p}$. Thus we have that $A(x) \leq \left(1+\frac{|x|}{2}\right)^{-4p}$ and so $\int_{\mathbb{R}^d}(1+|x|)^p A(x)^{1/2}dx < \infty$. On the other hand, if $|x| \geq 4\|b\|_\infty T$, then (the same argument as in the existence proof step 2 of Theorem 4 in [2] for the Itô integral with respect to the Brownian motion)

$$\begin{aligned} B(x) &\leq \mathbb{P}\left(\sup_{t\leq s\leq T}\left|\int_0^s \sigma(X_{0,r}(x))dW_r\right| \geq \frac{|x|}{8}\right) \\ &\leq C_1\,\exp(-C_2|x|^2) \end{aligned}$$

and so $\int_{\mathbb{R}^d}(1+|x|)^p B(x)^{1/2}dx < \infty$.

References

[1] Bachouch, A., Lasmar, A. B., Matoussi, A., and Mnif, M. Numerical scheme for semilinear SPDEs via backward doubly SDEs. *arXiv :1302.0440*.

[2] Bally, V., Caballero, M., Karoui, N. E., and Fernandez, B. Reflected BSDE's, PDE's and variational inequalities. *Rapport de Recherche INRIA*, 4455 (2002).

[3] Bally, V., and Matoussi, A. Weak solutions for SPDEs and backward doubly stochastic differential equations. *J. Theoret. Probab. 14*, 1 (2001), 125–164.

[4] Barles, G., and Lesigne, E. SDE, BSDE and PDE. In *Backward stochastic differential equations (Paris, 1995–1996)*, vol. 364 of *Pitman Res. Notes Math. Ser.* Longman, Harlow, 1997, pp. 47–80.

[5] Bensoussan, A., and Lions, J.-L. *Applications des Inéquations variationnelles en contrôle stochastique*. Dunod, Paris, 1978.

[6] Bismut, J. Mécanique aléatoire. *Ecole d'été de Probabilité de Saint-Flour, Lect. Notes Math*, 929 (1980), 5–100.

[7] Buckdahn, R., and Ma, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I. *Stochastic Process. Appl. 93*, 2 (2001), 181–204.

[8] Buckdahn, R., and Ma, J. Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II. *Stochastic Process. Appl. 93*, 2 (2001), 205–228.

[9] Chaleyat-Maurel, M., El Karoui, N., and Marchal, B. Réflexion discontinue et systèmes stochastiques. *Ann. Probab. 8*, 6 (1980), 1049–1067.

[10] Chassagneux, J.-F., Elie, R., and Kharroubi, I. A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electron. Commun. Probab. 16* (2011), 120–128.

[11] Denis, L., Matoussi, A., and Zhang, J. The obstacle problem for quasilinear stochastic PDEs: analytical approach. *Ann. Probab. 42*, 3 (2014), 865–905.

[12] Denis, L., and Stoica, L. A general analytical result for non-linear SPDE's and applications. *Electron. J. Probab. 9* (2004), no. 23, 674–709 (electronic).

[13] Donati-Martin, C., and Pardoux, É. White noise driven SPDEs with reflection. *Probab. Theory Related Fields 95*, 1 (1993), 1–24.

[14] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., and Quenez, M. C. Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab. 25*, 2 (1997), 702–737.

[15] Gegout-Petit, A., and Pardoux, E. Equations différentielles stochastiques rétrogrades réfléchies dans un convexe. *Stochastics and Stochastics Reports 57* (1995), 11–128.

[16] Gyöngy, I., and Rovira, C. On L^p-solutions of semilinear stochastic partial differential equations. *Stochastic Process. Appl. 90*, 1 (2000), 83–108.

[17] Hamadène, S., and Zhang, J. Switching problem and related system of reflected backward SDEs. *Stochastic Process. Appl. 120*, 4 (2010), 403–426.

[18] Hu, Y., and Tang, S. Multi-dimensional BSDE with oblique reflection and optimal switching. *Probab. Theory Related Fields 147*, 1-2 (2010), 89–121.

[19] Ikeda, N., and Watanabe, S. Stochastic differential equations and diffusion processes. *North-Holland/Kodansha, second edition* (1989).

[20] Krylov, N. V. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, vol. 64 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 1999, pp. 185–242.

[21] Kunita, H. Stochastic differential equations and stochastic flows of diffeomorphisms. In *École d'été de probabilités de Saint-Flour, XII—1982*, vol. 1097 of *Lecture Notes in Math.* Springer, Berlin, 1984, pp. 143–303.

[22] Kunita, H. *Stochastic flows and stochastic differential equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[23] Kunita, H. Generalized solutions of a stochastic partial differential equation. *J. Theoret. Probab. 7*, 2 (1994), 279–308.

[24] Kunita, H. Stochastic flow acting on schwartz distributions. *Journal of Theoretical Probability 7*, 2 (1994), 279–308.

[25] Lions, P.-L., and Souganidis, P. E. Fully nonlinear viscosity stochastic partial differential equations: non-smooth equations and applications. *C.R. Acad. Sci. Paris 327*, 1 (1998), 735–741.

[26] Lions, P.-L., and Souganidis, P. E. Équations aux dérivées partielles stochastiques nonlinéaires et solutions de viscosité. Exp. No. I, 15.

[27] Lions, P.-L., and Souganidis, P. E. Viscosity solutions of fully nonlinear stochastic partial differential equations. *Sūrikaisekikenkyūsho Kōkyūroku*, 1287 (2002), 58–65.

[28] Lions, P.-L., and Sznitman, A.-S. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math. 37*, 4 (1984), 511–537.

[29] Matoussi, A., and Scheutzow, M. Stochastic PDEs driven by nonlinear noise

and backward doubly SDEs. *J. Theoret. Probab. 15*, 1 (2002), 1–39.

[30] Matoussi, A., and Stoica, L. The obstacle problem for quasilinear stochastic PDE's. *Ann. Probab. 38*, 3 (2010), 1143–1179.

[31] Matoussi, A., and Xu, M. Sobolev solution for semilinear pde with obstacle under monotocity condition. *Electronic Journal of Probability 13*, 35 (2008), 1035–1067.

[32] Menaldi, J.-L. Stochastic variational inequality for reflected diffusion. *Indiana Univ. Math. J. 32*, 5 (1983), 733–744.

[33] Nualart, D., and Pardoux, É. White noise driven quasilinear SPDEs with reflection. *Probab. Theory Related Fields 93*, 1 (1992), 77–89.

[34] Pardoux, É., and Peng, S. G. Backward doubly stochastic differential equations and systems of quasilinear SPDEs. *Probab. Theory Related Fields 98*, 2 (1994), 209–227.

[35] Pierre, M. Problèmes d'evolution avec contraintes unilaterales et potentiels parabolique. *Comm. in Partial Differential Equations 4*, 10 (1979), 1149–1197.

[36] Pierre, M. Représentant précis d'un potentiel parabolique. *Séminaire de Théorie du Potentiel Paris Lecture Notes in Math 814*, 5 (1980), 186–228.

[37] Sanz-Solé, M., and Vuillermot, P.-A. Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. Inst. H. Poincaré Probab. Statist. 39*, 4 (2003), 703–742.

[38] Skorohod, A. V. Stochastic equations for diffusion processes with a boundary. *Teor. Verojatnost. i Primenen. 6* (1961), 287–298.

[39] Tanaka, H. Stochastic differential equations with reflecting boundary condition in convex regions. *Hiroshima Math. J. 9*, 1 (1979), 163–177.

[40] Walsh, J. B. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, vol. 1180 of *Lecture Notes in Math.* Springer, Berlin, 1986, pp. 265–439.

[41] Watanabe, S. On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. *J. Math. Kyoto Univ. 11* (1971), 169–180.

[42] Xu, T., and Zhang, T. White noise driven SPDEs with reflection: existence, uniqueness and large deviation principles. *Stochastic Process. Appl. 119*, 10 (2009), 3453–3470.

[43] Zhang, T. Systems of stochastic partial differential equations with reflection: existence and uniqueness. *Stochastic Process. Appl. 121*, 6 (2011), 1356–1372.