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SHOIQ with transitive closure of roles is decidable

Chan Le Duc¹, Myriam Lamolle¹, and Olivier Curé²

¹ LIASD Université Paris 8 - IUT de Montreuil, France
{chan.leduc, myriam.lamolle}@iut.univ-paris8.fr

² LIGM Université Paris-Est, France
ocure@univ-mlv.fr

Abstract. The Semantic Web makes an extensive use of the OWL DL ontology language, underlied by the *SHOIQ* description logic, to formalize its resources. In this paper, we propose a decision procedure for this logic extended with the transitive closure of roles in concept axioms, a feature needed in several application domains. To address the problem of consistency in this logic, we introduce a new structure for characterizing models which may have an infinite non-tree-like part.

1 Introduction

The ontology language OWL-DL [1] is widely used to formalize data resources on the Semantic Web. This language is mainly based on the description logic *SHOIN* which is known to be decidable [2]. Although *SHOIN* provides *transitive roles* to model transitivity of relations, we can find several applications in which the *transitive closure of roles*, that is more expressive than transitive roles, is needed. For instance, if we denote by R^- and R^+ the inverse and transitive closure of a role R respectively then it is obvious that the concept $\exists R^+.\forall R^-. \perp$ is unsatisfiable w.r.t. an empty TBox. If we now substitute R^+ for a transitive role R_t such that $R \sqsubseteq R_t$ (i.e. we substitute each occurrence of R^+ in axioms for R_t) then the concept $\exists R_t.\forall R^-. \perp$ is satisfiable. The point is that an instance of R^+ represents a sequence of instances of R but an instance of R_t corresponds to a sequence of instances of *itself*.

In this paper, we consider an extension of *SHOIQ* by enabling transitive closure of roles in concept axioms. In the general case, transitive closure is not expressible in the first order logic [3], the logic from which DL is a sublanguage, while the second order logic is sufficiently expressive to do so.

In the DL literature ([4]; [5]), there have been works dealing with transitive closure of roles. Recently, Ortiz [5] has proposed an algorithm for deciding consistency in the logic $ALCQIb_{reg}^+$ which allows for transitive closure of roles. However, nominals are disallowed in this logic. It is known that reasoning with a DL including number restrictions, inverse roles, nominals and transitive closure of roles is hard. The reason for this is that there exists an ontology in that DL whose models have an *infinite* non-tree-like part. Calvanese *et al.* [6] have presented an automata-based technique for dealing with the logic *ZOIQ* that includes transitive closure of roles, and showed that the sublogics *ZIQ*, *ZOQ* and *ZOI* are decidable. To obtain this result, the authors have introduced the *quasi-forest model property* to characterize models of ontologies in these sublogics.

Although they are very expressive, none of these sublogics includes \mathcal{SHOIQ} with transitive closure of roles, namely $\mathcal{SHOIQ}_{(+)}$. The following example³, noted \mathcal{K}_1 , shows that there is an ontology in $\mathcal{SHOIQ}_{(+)}$ which does not enjoy the quasi-forest model property. We consider the following axioms:

- (1) $\{o\} \sqsubseteq A$; $A \sqcap B \sqsubseteq \perp$; $A \sqsubseteq \exists R.A \sqcap \exists R'.B$; $B \sqsubseteq \exists S^+.\{o\}$
- (2) $\{o\} \sqsubseteq \forall X^-. \perp$; $\top \sqsubseteq \leq 1 X.\top$; $\top \sqsubseteq \leq 1 X^-. \top$ where $X \in \{R, R', S\}$

Figure 1 shows an infinite non-tree-like model of \mathcal{K}_1 . In fact, each individual x that satisfies $\exists S^+.\{o\}$ must have two distinct paths from x to the individual satisfying nominal o . Intuitively, we can see that (i) such a x must satisfy $\exists S^+.\{o\}$ and B , (ii) an individual satisfying B must connect to another individual satisfying A which must have a R -path to nominal o , and (iii) two concepts A and B are disjoint.

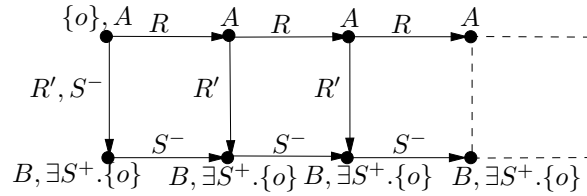


Fig. 1. An infinite non tree-like model of \mathcal{K}_1

This example shows that methods ([7], [8], [6]) based on the hypothesis which says that if an ontology is consistent it has a *quasi-forest model*, could fail to address the problem of consistency in a DL including simultaneously \mathcal{O} (nominals), \mathcal{I} (inverse roles), \mathcal{Q} (number restrictions) and transitive closure of roles.

In this paper, we propose a decision procedure for the problem of consistency in \mathcal{SHOIQ} with transitive closure of roles in concept axioms. The underlying idea of our algorithm is founded on the *star-type* and *frame* notions introduced by Pratt-Hartmann [9]. This technique uses star-types to represent individuals and “tiles” them together to form a frame for representing a model. For each star-type σ , we maintain a function $\delta(\sigma)$ which stores the number of individuals satisfying this star-type. To obtain termination condition, we introduce two additional structures into a frame : (i) the first one, namely *cycles*, describes duplicate parts of a model resulting from interactions of logic constructors in \mathcal{SHOIQ} , (ii) the second one, namely *blocking-blocked cycles*, describes parts of a model bordered by cycles which allow a frame to satisfy transitive closure of roles occurring in concepts of the form $\exists R^+.C$.

2 The Description Logic $\mathcal{SHOIQ}_{(+)}$

In this section, we present the syntax, the semantics and main inference problems of $\mathcal{SHOIQ}_{(+)}$. In addition, we introduce a tableau structure for $\mathcal{SHOIQ}_{(+)}$, which allows us to represent a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

³ This example is initially proposed by Sebastian Rudolph from an informal discussion

Definition 1. Let \mathbf{R} be a non-empty set of role names and $\mathbf{R}_+ \subseteq \mathbf{R}$ be a set of transitive role names. We use $\mathbf{R}_1 = \{P^- \mid P \in \mathbf{R}\}$ to denote a set of inverse roles, and $\mathbf{R}_\oplus = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_1\}$ to denote a set of transitive closure of roles. Each element of $\mathbf{R} \cup \mathbf{R}_1 \cup \mathbf{R}_\oplus$ is called a $\mathcal{SHOI}Q_{(+)}$ -role. A role inclusion axiom is of the form $R \sqsubseteq S$ for two $\mathcal{SHOI}Q_{(+)}$ -roles R and S such that $R \notin \mathbf{R}_\oplus$ and $S \notin \mathbf{R}_\oplus$. A role hierarchy \mathcal{R} is a finite set of role inclusion axioms. An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$ (domain) and a function $\cdot^{\mathcal{I}}$ which maps each role name to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ such that

$$\begin{aligned} R^{-\mathcal{I}} &= \{\langle x, y \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \langle y, x \rangle \in R^{\mathcal{I}}\} \text{ for all } R \in \mathbf{R}, \\ \langle x, z \rangle \in S^{\mathcal{I}}, \langle z, y \rangle \in S^{\mathcal{I}} &\text{ implies } \langle x, y \rangle \in S^{\mathcal{I}} \text{ for each } S \in \mathbf{R}_+, \text{ and} \\ (Q^+)^{\mathcal{I}} &= \bigcup_{n>0} (Q^n)^{\mathcal{I}} \text{ with } (Q^1)^{\mathcal{I}} = Q^{\mathcal{I}}, \end{aligned}$$

$$(Q^n)^{\mathcal{I}} = \{\langle x, y \rangle \in (\Delta^{\mathcal{I}})^2 \mid \exists z \in \Delta^{\mathcal{I}}, \langle x, z \rangle \in (Q^{n-1})^{\mathcal{I}}, \langle z, y \rangle \in Q^{\mathcal{I}}\} \text{ for } Q^+ \in \mathbf{R}_\oplus$$

* An interpretation \mathcal{I} satisfies a role hierarchy \mathcal{R} if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$ for each $R \sqsubseteq S \in \mathcal{R}$. Such an interpretation is called a model of \mathcal{R} , denoted by $\mathcal{I} \models \mathcal{R}$. To simplify notations for nested inverse roles and transitive closures of roles, we define two functions \cdot^\ominus and \cdot^\oplus as follows:

$$R^\ominus = \begin{cases} R^- & \text{if } R \in \mathbf{R}; \\ S & \text{if } R = S^- \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = S^+, S \in \mathbf{R}, \\ S^+ & \text{if } R = (S^-)^+, S \in \mathbf{R} \end{cases} \quad R^\oplus = \begin{cases} R^+ & \text{if } R \in \mathbf{R}; \\ S^+ & \text{if } R = (S^+)^+ \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = S^- \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = (S^+)^- \text{ and } S \in \mathbf{R} \end{cases}$$

* A relation \boxsubseteq is defined as the transitive-reflexive closure \mathcal{R}^+ of \sqsubseteq on $\mathbf{R} \cup \{R^\ominus \sqsubseteq S^\ominus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^\oplus \sqsubseteq S^\oplus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^\oplus \mid Q \in \mathbf{R} \cup \mathbf{R}_1\}$. We define a function $\text{Trans}(R)$ which returns true iff there is some $Q \in \mathbf{R}_+ \cup \{P^\ominus \mid P \in \mathbf{R}_+\} \cup \{P^\oplus \mid P \in \mathbf{R} \cup \mathbf{R}_1\}$ such that $Q \boxsubseteq R \in \mathcal{R}^+$. A role R is called simple w.r.t. \mathcal{R} if $\text{Trans}(R) = \text{false}$.

The reason for the introduction of two functions \cdot^\ominus and \cdot^\oplus in Definition 1 is that they avoid using R^{-} and R^{++} . Moreover, it remains a unique nested case $(R^-)^+$. According to Definition 1, axiom $R \sqsubseteq Q^\oplus$ is not allowed in a role hierarchy \mathcal{R} since this may lead to undecidability [10]. Notice that the closure \mathcal{R}^+ may contain $R \sqsubseteq Q^\oplus$ if $R \sqsubseteq Q$ belongs to \mathcal{R}^+ .

Definition 2 (terminology). Let \mathbf{C} be a non-empty set of concept names with a non-empty subset $\mathbf{C}_o \subseteq \mathbf{C}$ of nominals. The set of $\mathcal{SHOI}Q_{(+)}$ -concepts is inductively defined as the smallest set containing all C in \mathbf{C} , \top , $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n S.C)$ and $(\geq n S.C)$ where n is a positive integer, C and D are $\mathcal{SHOI}Q_{(+)}$ -concepts, R is an $\mathcal{SHOI}Q_{(+)}$ -role and S is a simple role w.r.t. a role hierarchy. We denote \perp for $\neg \top$. The interpretation function $\cdot^{\mathcal{I}}$ of an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ maps each concept name to a subset of $\Delta^{\mathcal{I}}$ such that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $|\{o^{\mathcal{I}}\}| = 1$ for all $o \in \mathbf{C}_o$, $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$, $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$, $(\geq n S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y \rangle \in S^{\mathcal{I}}\}| \geq n\}$, $(\leq n S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y \rangle \in S^{\mathcal{I}}\}| \leq n\}$ where $|S|$ is denoted for the cardinality of a set S . An axiom $C \sqsubseteq D$ is called a general concept inclusion (GCI)

where C, D are $\mathcal{SHOIQ}_{(+)}$ -concepts (possibly complex), and a finite set of GCIs is called a terminology \mathcal{T} . An interpretation \mathcal{I} satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and \mathcal{I} satisfies a terminology \mathcal{T} if \mathcal{I} satisfies each GCI in \mathcal{T} . Such an interpretation is called a model of \mathcal{T} , denoted by $\mathcal{I} \models \mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called a $\mathcal{SHOIQ}_{(+)}$ knowledge base where \mathcal{R} is a $\mathcal{SHOIQ}_{(+)}$ role hierarchy and \mathcal{T} is a $\mathcal{SHOIQ}_{(+)}$ terminology. A knowledge base $(\mathcal{T}, \mathcal{R})$ is said to be consistent if there is a model \mathcal{I} of both \mathcal{T} and \mathcal{R} , i.e., $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{R}$. A concept C is called satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$ iff there is some interpretation \mathcal{I} such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a model of C w.r.t. $(\mathcal{T}, \mathcal{R})$. A concept D subsumes a concept C w.r.t. $(\mathcal{T}, \mathcal{R})$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model \mathcal{I} of $(\mathcal{T}, \mathcal{R})$. \triangleleft

Since unsatisfiability, subsumption and consistency w.r.t. a $\mathcal{SHOIQ}_{(+)}$ knowledge base can be reduced to each other, it suffices to study knowledge base consistency. For the ease of construction, we assume all concepts to be in *negation normal form* (NNF), i.e., negation occurs only in front of concept names. Any $\mathcal{SHOIQ}_{(+)}$ -concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. According to [12], $\text{nnf}(C)$ can be computed in polynomial time in the size of C . For a concept C , we denote the nnf of C by $\text{nnf}(C)$ and the nnf of $\neg C$ by \dot{C} . Let D be a $\mathcal{SHOIQ}_{(+)}$ -concept in NNF. We define $\text{cl}(D)$ to be the smallest set that contains all sub-concepts of D including D . For a knowledge base $(\mathcal{T}, \mathcal{R})$, we reuse $\text{cl}(\mathcal{T}, \mathcal{R})$, which was introduced by Horrocks *et al.* [7], to denote all sub-concepts occurring in the axioms of $(\mathcal{T}, \mathcal{R})$. We have $\text{cl}(\mathcal{T}, \mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ [7]. To translate *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for \mathcal{C}^2 into those for \mathcal{SHOIQ} , we need to add new sets of concepts, denoted $\text{cl}_1(\mathcal{T}, \mathcal{R})$ and $\text{cl}_2(\mathcal{T}, \mathcal{R})$, to the signature of a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$.

$$\begin{aligned} \text{cl}_1(\mathcal{T}, \mathcal{R}) = & \{ \leq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \text{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n \} \cup \\ & \{ \geq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \text{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n \} \end{aligned}$$

For a generating concept $(\geq nS.C)$ and a set $I \subseteq \{0, \dots, \log n + 1\}$, we denote $\mathcal{C}_{(\geq nS.C)}^I = \prod_{i \in I} C_{(\geq nS.C)}^i \sqcap \prod_{j \notin I} \neg C_{(\geq nS.C)}^j$ where $C_{(\geq nS.C)}^i$ are new concept names

for $0 \leq i \leq \log n + 1$. We define $\text{cl}_2(\mathcal{T}, \mathcal{R})$ as follows:

$$\begin{aligned} \text{cl}_2(\mathcal{T}, \mathcal{R}) = & \{ C_{(\geq nS.C)}^i \mid (\geq nS.C) \in \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}), 0 \leq i \leq \log n + 1 \} \cup \\ & \{ \mathcal{C}_{(\geq nS.C)}^I \mid (\geq nS.C) \in \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}), I \subseteq \{0, \dots, \log n + 1\} \} \end{aligned}$$

Remark 1. If numbers are encoded in binary then the number of new concept names $C_{(\geq nS.C)}^i$ for $0 \leq i \leq \log n + 1$, is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ since n is bounded by $\mathcal{O}(2^{|\mathcal{T}, \mathcal{R}|})$. This implies that $|\text{cl}_2(\mathcal{T}, \mathcal{R})|$ is bounded by $\mathcal{O}(2^{|\mathcal{T}, \mathcal{R}|})$. Note that two concepts $\mathcal{C}_{(\geq nS.C)}^I$ and $\mathcal{C}_{(\geq nS.C)}^J$ are disjoint for all $I, J \subseteq \{0, \dots, \log n + 1\}$, $I \neq J$. The concepts $\mathcal{C}_{(\exists S.C)}$ and $\mathcal{C}_{(\geq nS.C)}^I$ will be used for building chromatic star-types. This notion will be clarified after introducing the frame structure (Definition 5).

Finally, we denote $\mathbf{CL}(\mathcal{T}, \mathcal{R}) = \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}) \cup \text{cl}_2(\mathcal{T}, \mathcal{R})$, and use $\mathbf{R}(\mathcal{T}, \mathcal{R})$ to denote the set of all role names occurring in \mathcal{T}, \mathcal{R} with their inverse. The definition of $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ is inspired from the Fischer-Ladner closure that was introduced in [13]. The closure $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ contains not only sub-concepts syntactically obtained from \mathcal{T}

but also sub-concepts that are semantically derived from \mathcal{T} w.r.t. \mathcal{R} . For instance, if $\forall S.C$ is a sub-concept from \mathcal{T} and $R \sqsubseteq S \in \mathcal{R}$ then $\forall R.C \in \mathbf{CL}(\mathcal{T}, \mathcal{R})$.

To describe a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base in a more intuitive way, we use a tableau structure that expresses semantic constraints resulting directly from the logic constructors in $\mathcal{SHOIQ}_{(+)}$. A tableau definition for $\mathcal{SHOIQ}_{(+)}$ can be found in [14].

3 A Decision Procedure For $\mathcal{SHOIQ}_{(+)}$

This section starts by translating *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for \mathcal{C}^2 into those for $\mathcal{SHOIQ}_{(+)}$.

Definition 3 (star-type). Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A star-type is a pair $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$, where $\lambda(\sigma) \in 2^{\mathbf{CL}(\mathcal{T}, \mathcal{R})}$ is called core label, $\xi(\sigma) = \langle \langle r_1, l_1 \rangle, \dots, \langle r_d, l_d \rangle \rangle$ is a d -tuple over $2^{\mathbf{R}(\mathcal{T}, \mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T}, \mathcal{R})}$. A pair $\langle r, l \rangle$ is a ray of σ if $\langle r, l \rangle = \langle r_i, l_i \rangle$ for some $1 \leq i \leq d$. We use $\langle r(\rho), l(\rho) \rangle$ to denote a ray $\rho = \langle r, l \rangle$ where $r(\rho) = r$ and $l(\rho) = l$.

- A star-type σ is *nominal* if $o \in \lambda(\sigma)$ for some $o \in \mathbf{C}_o$.
- A star-type σ is *chromatic* if $\rho \neq \rho'$ implies $l(\rho) \neq l(\rho')$ for two rays ρ, ρ' of σ . When a star-type σ is chromatic, $\xi(\sigma)$ can be considered as a set of rays.
- Two star-types σ, σ' are *equivalent* if $\lambda(\sigma) = \lambda(\sigma')$, and there is a bijection π between $\xi(\sigma)$ and $\xi(\sigma')$ such that $\pi(\rho) = \rho'$ implies $r(\rho') = r(\rho)$ and $l(\rho') = l(\rho)$.

We denote Σ for the set of all star-types for $(\mathcal{T}, \mathcal{R})$. ◁

Note that for a chromatic star-type σ , $\xi(\sigma)$ can be considered as a set of rays since rays are distinct and not ordered. We can think of a star-type σ as the set of individuals x satisfying all concepts in $\lambda(\sigma)$, and each ray ρ of σ corresponds to a “neighbor” individual x_i of x such that $r(\rho)$ is the label of the link between x and x_i ; and x_i satisfies all concepts in $l(\rho)$. In this case, we say that x *satisfies* σ .

Definition 4 (valid star-type). Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. Let σ be a star-type for $(\mathcal{T}, \mathcal{R})$ where $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$. The star-type σ is *valid* if σ is chromatic and the following conditions are satisfied:

1. If $C \sqsubseteq D \in \mathcal{T}$ then $\text{nnf}(\neg C \sqcup D) \in \lambda(\sigma)$;
2. $\{A, \neg A\} \not\subseteq \lambda$ for every concept name A where $\lambda = \lambda(\sigma)$ or $\lambda = l(\rho)$ for each $\rho \in \xi(\sigma)$;
3. If $C_1 \sqcap C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \subseteq \lambda(\sigma)$;
4. If $C_1 \sqcup C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \cap \lambda(\sigma) \neq \emptyset$;
5. If $\exists R.C \in \lambda(\sigma)$ then there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $R \in r(\rho)$;
6. If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $S \in r(\rho)$ then $C \in l(\rho)$ or $\dot{\neg}C \in l(\rho)$;
7. If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $S \in r(\rho)$ then there is some $1 \leq m \leq n$ such that $\{(\leq mS.C), (\geq mS.C)\} \subseteq \lambda(\sigma)$;
8. For each ray $\rho \in \xi(\sigma)$, if $R \in r(\rho)$ and $R \sqsubseteq S$ then $S \in r(\rho)$;

9. If $\forall R.C \in \lambda(\sigma)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $C \in l(\rho)$;
10. If $\forall R.D \in \lambda(\sigma)$, $S \sqsubseteq R$, $\text{Trans}(S)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $\forall S.D \in l(\rho)$;
11. If $\forall Q^\oplus.C \in \lambda(\sigma)$, $R \sqsubseteq Q$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $\forall Q^\oplus.C \in l(\rho)$;
12. If $\exists Q^\oplus.C \in \lambda(\sigma)$ then $(\exists Q.C \sqcup \exists Q^\oplus.C) \in \lambda(\sigma)$;
13. If $(\geq nS.C) \in \lambda(\sigma)$ then there are n distinct rays $\rho_1, \dots, \rho_n \in \xi(\sigma)$ such that $\{C, \mathcal{C}_{(\geq nS.C)}^{I_i}\} \subseteq l(\rho_i)$, $S \in r(\rho_i)$ for all $1 \leq i \leq n$; and $I_j, I_k \subseteq \{0, \dots, \log n + 1\}$, $I_j \neq I_k$ for all $1 \leq j < k \leq n$.
14. If $(\leq nS.C) \in \lambda(\sigma)$ and there do not exist $n + 1$ rays $\rho_0, \dots, \rho_n \in \xi(\sigma)$ such that $C \in l(\rho_i)$ and $S \in r(\rho_i)$ for all $0 \leq i \leq n$. \triangleleft

Roughly speaking, a star-type σ is valid if each individual x satisfies *semantically* all concepts in $\lambda(\sigma)$. In fact, each condition in Definition 4 represents the semantics of a constructor in $\mathcal{SHOIQ}_{(+)}$ except for transitive closure of roles. From valid star-types, we can “tile” a model instead of using expansion rules for generating nodes as described in tableau algorithms. Before presenting how to “tile” a model from star-types, we need some notation that will be used in the remainder of the paper.

Notation 1 We call $\mathcal{P} = \langle (\sigma_1, \rho_1, d_1), \dots, (\sigma_k, \rho_k, d_k) \rangle$ a sequence where $\sigma_i \in \Sigma$, $\rho_i \in \xi(\sigma_i)$ and $d_i \in \mathbb{N}$ for $1 \leq i \leq k$.

- $\text{tail}(\mathcal{P}) = (\sigma_k, \rho_k, d_k)$, $\text{tail}_\sigma(\mathcal{P}) = \sigma_k$, $\text{tail}_\rho(\mathcal{P}) = \rho_k$, $\text{tail}_\delta(\mathcal{P}) = d_k$ and $|\mathcal{P}| = k$. We denote $\mathcal{L}(\mathcal{P}) = \lambda(\text{tail}_\sigma(\mathcal{P}))$.
- $\mathbf{p}^i(\mathcal{P}) = (\sigma_i, \rho_i, d_i)$, $\mathbf{p}_\sigma^i(\mathcal{P}) = \sigma_i$, $\mathbf{p}_\rho^i(\mathcal{P}) = \rho_i$ and $\mathbf{p}_\delta^i(\mathcal{P}) = d_i$ for each $1 \leq i \leq k$.
- an operation $\text{add}(\mathcal{P}, (\sigma, \rho, d))$ extends \mathcal{P} to a new sequence with $\text{add}(\mathcal{P}, (\sigma, \rho, d)) = \langle \mathcal{P}, (\sigma, \rho, d) \rangle$.

Definition 5 (frame). Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A frame for $(\mathcal{T}, \mathcal{R})$ is a tuple $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$, where

1. \mathcal{N} is a set of valid star-types such that σ is not equivalent to σ' for all $\sigma, \sigma' \in \mathcal{N}$;
2. $\mathcal{N}_o \subseteq \mathcal{N}$ is a set of nominal star-types;
3. Ω is a function that maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ to a sequence $\Omega(\sigma, \rho) = \langle (\sigma_1, \rho_1, d_1), \dots, (\sigma_m, \rho_m, d_m) \rangle$ with $\sigma_i \in \mathcal{N}$, $\rho_i \in \xi(\sigma_i)$, $d_i \in \mathbb{N}$ for $1 \leq i \leq m$ such that for each σ_i with $1 \leq i \leq m$, it holds that $l(\rho) = \lambda(\sigma_i)$, $l(\rho_i) = \lambda(\sigma)$ and $r(\rho_i) = r^-(\rho)$ where $r^-(\rho) = \{R^\ominus \mid R \in r(\rho)\}$.
4. δ is a function $\delta : \mathcal{N} \rightarrow \mathbb{N}$. By abuse of notation, we also use δ to denote a function which maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ into a number in \mathbb{N} , i.e., $\delta(\sigma, \rho) \in \mathbb{N}$. \triangleleft

The frame structure, as introduced in Definition 5, allows us to compress individuals of a model into star-types. For each star-type σ and each ray $\rho \in \xi(\sigma)$, a list $\Omega(\sigma, \rho)$ of triples (σ_i, ρ_i, d_i) with $\rho_i \in \xi(\sigma_i)$ is maintained where σ_i is a “neighbor” star-type of σ via $\rho \in \xi(\sigma)$, and d_i indicates the d_i -th “layer” of rays of σ_i . We can think a layer of rays of σ_i as an individual that connects to its neighbor individuals via the rays of σ_i . The following definition presents how to connect such layers to form paths in a frame.

Definition 6 (path). Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. A path is inductively defined as follows:

1. A sequence $\langle \emptyset, (\sigma, \rho, 1) \rangle$ is a path if $\sigma \in \mathcal{N}_o$ and $\rho \in \xi(\sigma)$;
2. A sequence $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ with $\mathcal{P} \neq \emptyset$ and $\text{tail}(\mathcal{P}) = (\sigma_0, \rho_0, d_0)$, is a path if $(\sigma, \rho, d) = \mathbf{p}^{d_0}(\Omega(\sigma_0, \rho'))$ for each $\rho' \neq \rho_0$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is the ρ' -neighbor of \mathcal{P} , and two paths $\mathcal{P}, \langle \mathcal{P}, (\sigma, \rho, d) \rangle$ are neighbors. Additionally, if $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a ρ' -neighbor of \mathcal{P} and $Q \in r(\rho')$ then $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a Q -neighbor of \mathcal{P} . In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a Q -neighbor of \mathcal{P} , or \mathcal{P} is a Q^\ominus -neighbor of $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$.

We define $\mathcal{P} \sim \mathcal{P}'$ if $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$ and $\text{tail}_\delta(\mathcal{P}) = \text{tail}_\delta(\mathcal{P}')$. Since \sim is an equivalence relation over the set of all paths, we use \mathcal{P} to denote the set of all equivalence classes $[\mathcal{P}]$ of paths in \mathcal{F} . For $[\mathcal{P}], [\mathcal{Q}] \in \mathcal{P}$, we define:

1. $[\mathcal{P}]$ is a neighbor (ρ' -neighbor) of $[\mathcal{Q}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that \mathcal{Q}' is a neighbor (ρ' -neighbor) of \mathcal{P}' ;
2. $[\mathcal{Q}]$ is a reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \dots, [\mathcal{P}_n] \in \mathcal{P}$ such that $[\mathcal{P}_{i+1}]$ is a neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$.
3. $[\mathcal{Q}]$ is a Q -neighbor of $[\mathcal{P}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that \mathcal{Q}' is a Q -neighbor of \mathcal{P}' , or \mathcal{P}' is a Q^\ominus -neighbor of \mathcal{Q}' ;
4. $[\mathcal{Q}]$ is a Q -reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \dots, [\mathcal{P}_n] \in \mathcal{P}$ such that $[\mathcal{P}_{i+1}]$ is a Q -neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$ where $[\mathcal{P}_1] = [\mathcal{P}]$ and $[\mathcal{Q}] = [\mathcal{P}_n]$. \triangleleft

Note that for two paths $\mathcal{P}, \mathcal{P}'$ with $\text{tail}_\rho(\mathcal{P}) \neq \text{tail}_\rho(\mathcal{P}')$, we have $\mathcal{P} \sim \mathcal{P}'$ if $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$ and $\text{tail}_\delta(\mathcal{P}) = \text{tail}_\delta(\mathcal{P}')$. This does not allow for extending $\text{tail}_\rho(\mathcal{P})$ to $\text{tail}_\rho([\mathcal{P}])$. As a consequence, there may be several “predecessors” of an equivalence class $[\mathcal{P}]$. However, we can define $\text{tail}_\sigma([\mathcal{P}]) = \text{tail}_\sigma(\mathcal{P})$, $\text{tail}_\delta([\mathcal{P}]) = \text{tail}_\delta(\mathcal{P})$ and $\mathcal{L}([\mathcal{P}]) = \mathcal{L}(\mathcal{P})$. In the sequel, we use \mathcal{P} instead of $[\mathcal{P}]$ whenever it is clear from the context.

Definition 7 (cycle). Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set \mathcal{P} of paths in \mathcal{F} .

1. A cycle is a set Θ of triples $(\mathcal{P}, \rho, \mathcal{Q})$ with $\mathcal{P}, \mathcal{Q} \in \mathcal{P}$ and $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ such that for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ the following conditions are satisfied:
 - (a) $\text{tail}_\delta(\mathcal{P}) > 1$ and $\text{tail}_\delta(\mathcal{Q}) > 1$;
 - (b) If \mathcal{P}' is the ρ -neighbor of \mathcal{P} then $\text{tail}_\sigma(\mathcal{P}') = \text{tail}_\sigma(\mathcal{Q})$;
 - (c) for each sequence $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathcal{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, \mathcal{P}_2 is not the ρ -neighbor of \mathcal{P} , and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta$ such that
 - i. either $\mathcal{Q}'' = \mathcal{P}_j$, $\text{tail}_\sigma(\mathcal{P}_{j+1}) = \text{tail}_\sigma(\mathcal{P}'')$, $\text{tail}_\delta(\mathcal{P}_{j+1}) \geq \text{tail}_\delta(\mathcal{P}'')$ and \mathcal{P}_j is the ρ -neighbor of \mathcal{P}_{j+1} for some $1 < j < n$,
 - ii. or there are $\mathcal{P}_{n+1}, \dots, \mathcal{P}_{n+m} \in \mathcal{P}$ with $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\text{tail}_\sigma(\mathcal{P}_{n+m}) = \text{tail}_\sigma(\mathcal{P}'')$, $\text{tail}_\delta(\mathcal{P}_{n+m}) \geq \text{tail}_\delta(\mathcal{P}'')$ and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $n \leq i < n+m$.

In this case, we say that \mathcal{Q} is cycled by \mathcal{P} via ρ .

2. A cycle Θ' is a reachable cycle of Θ if for each $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and for each sequence $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathcal{P}$ such that $\mathcal{P}_1 = \mathcal{P}$, \mathcal{P}_2 is not a ρ -neighbor of \mathcal{P} , and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $1 \leq i < n$, there is some $(\mathcal{P}'', \rho'', \mathcal{Q}'') \in \Theta'$ such that

- (a) either $\mathcal{Q}'' = \mathcal{P}_j$, $\text{tail}_\sigma(\mathcal{P}_{j+1}) = \text{tail}_\sigma(\mathcal{P}'')$, $\text{tail}_\delta(\mathcal{P}_{j+1}) \geq \text{tail}_\delta(\mathcal{P}'')$ and \mathcal{P}_j is a ρ -neighbor of \mathcal{P}_{j+1} for some $1 < j < n$,
- (b) or there are $\mathcal{P}_{n+1}, \dots, \mathcal{P}_{n+m} \in \mathcal{P}$ with $\mathcal{Q}'' = \mathcal{P}_{n+m-1}$, $\text{tail}_\sigma(\mathcal{P}_{n+m}) = \text{tail}_\sigma(\mathcal{P}'')$, $\text{tail}_\delta(\mathcal{P}_{n+m}) \geq \text{tail}_\delta(\mathcal{P}'')$ and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $n \leq i < n+m$. \triangleleft

Note that cycles may encapsulate loops if $\text{tail}_\delta(\mathcal{P}_{j+1}) = \text{tail}_\delta(\mathcal{P}'')$ holds in Conditions 1(c)i and 1(c)ii, Definition 7. Let Θ be a cycle in a frame. Definition 7 ensures that each reachable path of some path \mathcal{P} with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ goes through a star-type $\sigma = \text{tail}_\sigma(\mathcal{Q})$ with some $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta$. Such a cycle, which is similar to blocking-blocked nodes in completion graphs for *SHOIQ* [7], allows for “unravelling” infinitely the frame to obtain a model of a KB in *SHOIQ* (without transitive closure of roles). This means that we can extend the set \mathcal{P} of paths by adding infinitely paths which lengthen \mathcal{Q} such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ and \mathcal{P} is not a neighbor of \mathcal{Q} . However, such a cycle structure is not sufficient to represent models of a KB with transitive closure of roles since a concept such as $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$ can be satisfied by a Q -reachable path \mathcal{P}' of \mathcal{P} which is arbitrarily far from \mathcal{P} . There are the following possibilities for an algorithm which builds a frame: (i) the algorithm stops building the frame as soon as a cycle Θ is detected such that each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ for each cycling path \mathcal{P} of Θ is satisfied, i.e., \mathcal{P} has a Q -reachable path \mathcal{P}' with $\exists Q.D \in \mathcal{L}(\mathcal{P})$, (ii) despite of several detected cycles, the algorithm continues building the frame until each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycling path \mathcal{P} of Θ . If we adopt the first possibility, the completeness of such an algorithm cannot be established since there are models in which paths satisfying concepts of the form $\exists Q^\oplus.D$ can spread over several “iterative structures” such as cycles. For this reason, we adopt the second possibility by introducing into frames an additional structure, namely *blocking-blocked cycles*, which determines a sequence of cycles $\Theta_1, \dots, \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i . Reachability of cycles allows for “unravelling” the frame between cycled paths \mathcal{Q}' with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_k$ and cycling paths \mathcal{P} with $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_1$.

Definition 8 (blocking). Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a *SHOIQ*₍₊₎ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set \mathcal{P} of all paths in \mathcal{F} .

1. A cycle Θ' is blockable by a cycle Θ if Θ' is a reachable cycle of Θ , and for each $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta'$ there is some $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta$ such that $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$, $\mathcal{L}(\mathcal{Q}) = \mathcal{L}(\mathcal{Q}')$ and $r(\rho) = r(\rho')$. In this case, we say that \mathcal{Q}' is blockable by \mathcal{P} via ρ .
2. A cycle Θ' is blocked by a cycle Θ if there are $\Theta_1, \dots, \Theta_k$ with $\Theta = \Theta_1$, $\Theta' = \Theta_k$ such that Θ_{i+1} is blockable by Θ_i for $1 \leq i < k$, and for each $(\lambda, s, \lambda') \in 2^{\text{CL}(\mathcal{T}, \mathcal{R})} \times 2^{\mathbf{R}(\mathcal{T}, \mathcal{R})} \times 2^{\text{CL}(\mathcal{T}, \mathcal{R})}$ with $\exists Q^\oplus.D \in \lambda$,
 - if there is some path \mathcal{P}_k with $(\mathcal{P}_k, \rho_k, \mathcal{Q}_k) \in \Theta_k$, $\mathcal{L}(\mathcal{P}_k) = \lambda$, $\mathcal{L}(\mathcal{Q}_k) = \lambda'$, $s = r(\rho_k)$ such that \mathcal{P}_k has no Q -reachable path \mathcal{P}'_k with $\exists Q.D \in \mathcal{L}(\mathcal{P}'_k)$,
 - then there is some \mathcal{P}_1 with $(\mathcal{P}_1, \rho_1, \mathcal{Q}_1) \in \Theta_1$, $\mathcal{L}(\mathcal{P}_1) = \lambda$, $\mathcal{L}(\mathcal{Q}_1) = \lambda'$, $r(\rho_1) = s$ such that for each $\exists P^\oplus.C \in \mathcal{L}(\mathcal{P}_1)$,
 - there is a P -reachable path \mathcal{Q}'' of \mathcal{P}_1 with $\exists P.C \in \mathcal{L}(\mathcal{Q}'')$, and

- there are two triples $(\mathcal{P}_j, \rho_j, \mathcal{Q}_j) \in \Theta_j$ and $(\mathcal{P}_{j+1}, \rho_{j+1}, \mathcal{Q}_{j+1}) \in \Theta_{j+1}$ for some $1 \leq j < k$, which satisfy that \mathcal{Q}'' is a reachable path of \mathcal{Q}_j and \mathcal{Q}_{j+1} is a reachable path of \mathcal{Q}'' .

In this case, we say that \mathcal{P}_1 blocks \mathcal{P}_k via ρ_k . \triangleleft

According to Definition 8, there are a sequentially reachable cycles between a blocking cycle Θ_1 and a blocked cycle Θ_k , which allows for unravelling the frame between Θ_k and Θ_1 . Condition 2 Definition 8 says that if $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$ and $\mathcal{L}(\mathcal{P})$ contains concepts of the form $\exists Q^\oplus.D$ which are not satisfied by reachable paths of \mathcal{P} then there exists some \mathcal{P}' with $(\mathcal{P}', \rho', \mathcal{Q}') \in \Theta_1$ which allows for satisfying these concepts $\exists Q^\oplus.D$ in $\mathcal{L}(\mathcal{P})$ by unravelling. We would like to note that a path \mathcal{P} is blocked if there is some blocked cycle Θ_k such that $(\mathcal{P}, \rho, \mathcal{Q}) \in \Theta_k$.

Definition 9 (valid frame). Let $(\mathcal{T}, \mathcal{R})$ be a SHOIQ knowledge base. A frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ is valid if the following conditions are satisfied:

1. For each $o \in \mathbf{C}_o$ there is a unique $\sigma_o \in \mathcal{N}_o$ such that $o \in \lambda(\sigma_o)$ and $\delta(\sigma_o) = 1$;
2. For each $\sigma \in \mathcal{N}$, σ is valid;
3. If $\exists Q^\oplus.C \in \lambda(\text{tail}_\sigma(\mathcal{P}_0))$ for some $\mathcal{P}_0 \in \mathcal{P}$ then there are $\mathcal{P}, \mathcal{P}' \in \mathcal{P}$ such that one of the following conditions is satisfied:
 - (a) $\mathcal{P}_0 = \mathcal{P} = \mathcal{P}'$ and $\exists Q.C \in \mathcal{L}(\mathcal{P}_0)$;
 - (b) \mathcal{P}' is a Q -reachable of \mathcal{P} , and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or \mathcal{P} blocks \mathcal{P}_0 ;
 - (c) \mathcal{P} is a Q^\ominus -reachable of \mathcal{P}' , and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or \mathcal{P} blocks \mathcal{P}_0 . \triangleleft

Conditions 1-3 in Definition 9 ensure satisfaction of tableau properties [14]. In particular, Condition 3 takes into account satisfaction of transitive closure of roles. In fact, for each blocking path \mathcal{P} with $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$, this condition says that \mathcal{P} must be satisfied by a Q -reachable path \mathcal{P}' of \mathcal{P} between a blocking cycle Θ_1 and a blocked cycle Θ_k . If there is some path \mathcal{Q} between Θ_1 and Θ_k with $\exists S^\oplus.C \in \mathcal{L}(\mathcal{Q})$ that is not satisfied, due to the construction of star-types and frames, a concept $\exists S^\oplus.C$ is propagated along S -reachable paths of \mathcal{Q} . This implies that \mathcal{Q} has a S -reachable path \mathcal{Q}' such that \mathcal{Q}' is blocked and $\exists S^\oplus.C \in \mathcal{L}(\mathcal{Q}')$. By unravelling (details will be given in soundness proof), we can build an (extended) S -reachable path \mathcal{Q}'' of \mathcal{Q}' such that $\exists S.C \in \mathcal{L}(\mathcal{Q}'')$.

We now present Algorithm 1 for building a frame which is valid if the conditions in Definition 9 are satisfied. This algorithm starts by adding nominal star-types to the frame. For each non blocked path \mathcal{P} with a ray $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ such that $\delta(\text{tail}_\sigma(\mathcal{P}))$ is minimal and there is a difference between $\delta(\text{tail}_\sigma(\mathcal{P}))$ and $\delta(\text{tail}_\sigma(\mathcal{P}), \rho)$, the algorithm picks in a nondeterministic way a valid star-type ω that matches $\text{tail}_\sigma(\mathcal{P})$ via ρ , and updates $\Omega(\text{tail}_\sigma(\mathcal{P}), \rho)$, $\Omega(\omega, \rho')$, $\delta(\text{tail}_\sigma(\mathcal{P}), \rho)$, $\delta(\omega, \rho')$, eventually, $\delta(\text{tail}_\sigma(\mathcal{P}))$ and $\delta(\omega)$ by calling `updateFrame(\dots)` [14]. The algorithm terminates when a blocked cycle is detected.

Figure 2 depicts a frame when executing Algorithm 1 for \mathcal{K}_1 in the example presented in Section 1. The algorithm builds a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ where $\mathcal{N} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathcal{N}_o = \{\sigma_0\}$. The dashed arrows indicate how the function

Require: A $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$
Ensure: A frame $\langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ for $(\mathcal{T}, \mathcal{R})$

- 1: Let Σ be the set of all star-types for $(\mathcal{T}, \mathcal{R})$
- 2: **for all** $o \in \mathbf{C}_o$ **do**
- 3: **if** there is no $\sigma \in \mathcal{N}$ such that $o \in \lambda(\sigma)$ **then**
- 4: Choose a star-type $\sigma_o \in \Sigma$ such that $o \in \lambda(\sigma_o)$
- 5: Set $\delta(\sigma_o) = 1$, $\mathcal{N} = \mathcal{N} \cup \{\sigma_o\}$ and $\mathcal{N}_o = \mathcal{N}_o \cup \{\sigma_o\}$
- 6: Set $\delta(\sigma_o, \rho) = 0$, $\Omega(\sigma_o, \rho) = \emptyset$ for all $\rho \in \xi(\sigma_o)$
- 7: **end if**
- 8: **end for**
- 9: **while** there is a path \mathcal{P} that is not blocked and a ray $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ such that
 $\text{tail}_\delta(\mathcal{P}) = \delta(\text{tail}_\sigma(\mathcal{P}), \rho) + 1$ and $\delta(\text{tail}_\sigma(\mathcal{P})) \leq \delta(\omega)$ for all $\omega \in \mathcal{N}$ **do**
- 10: Choose a star-type $\sigma' \in \Sigma$ such that there is a ray $\rho' \in \xi(\sigma')$ satisfying
 $l(\rho) = \lambda(\sigma')$, $l(\rho') = \lambda(\sigma)$, $r(\rho') = r^-(\rho)$, and
 $\sigma' \in \mathcal{N}$ implies $\delta(\sigma') = \delta(\sigma', \rho') + 1$
- 11: updateFrame($\sigma, \rho, \sigma', \rho'$)
- 12: **end while**

Algorithm 1: An algorithm for building a frame

$\Omega(\sigma, \rho)$ can be built. For example, $\Omega(\sigma_0, \rho_0) = \{(\sigma_1, \nu_0, 1)\}$, $\Omega(\sigma_0, \rho_1) = \{(\sigma_2, \rho'_0, 1)\}$ where ρ_0 and ρ_1 are the respective horizontal and vertical rays of σ_0 ; ν_0 is the left ray of σ_1 ; ρ'_0 is the vertical ray of σ_2 . Moreover, the directed dashed arrow from σ_0 to σ_1 indicates that the ray ρ_0 of σ_0 can match the ray ν_0 on the left ray of σ_1 since $l(\rho_0) = \lambda(\sigma_1)$, $r(\nu_0) = \lambda(\sigma_0)$, $r(\nu_0) = r^-(\rho_0)$.

Then, the algorithm generates $\delta(\sigma_0) = 1$, $\delta(\sigma_1) = 1$, $\delta(\sigma_2) = 1$ and forms a cycle Θ consisting of the following triples : $((\sigma_3, 2), \rho_1, (\sigma_3, 3))$ (ρ_1 is the left ray of σ_3), $((\sigma_3, 2), \rho_3, (\sigma_4, 1))$ (ρ_3 is the vertical ray of σ_3), $((\sigma_4, 1), \rho_4, (\sigma_4, 2))$ (ρ_4 is the left ray of σ_4) and $((\sigma_4, 1), \rho_5, (\sigma_3, 2))$ (ρ_5 is the vertical ray of σ_4). Note that for the sake of brevity, we use just $\text{tail}_\sigma(\mathcal{P})$ and $\text{tail}_\delta(\mathcal{P})$ to denote a path in the triples. We can check that any path that is an extension of a path \mathcal{P} gets through a path \mathcal{Q} where \mathcal{P} is the first component of a triple and \mathcal{Q} is the third component of a triple.

The algorithm may add some more paths that go through σ_3 and σ_4 to form a blocked cycle. A model of the ontology can be built by starting from σ_0 and getting (i) σ_4 via σ_1 , (ii) σ_3 via σ_1 , and (iii) σ_3 via σ_2 . From σ_3 and σ_4 , the model goes through σ_3 and σ_4 infinitely. Note that from any individual x satisfying σ_3 (or σ_4), i.e. the “label” of x contains $\exists Q^+.\{o\}$, there is a path containing S which goes back the individual satisfying σ_0 . Thus, the concept $\exists Q^+.\{o\}$ is satisfied for each individual whose label contains $\exists Q^+.\{o\}$.

Lemma 1. Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

1. Algorithm 1 terminates.
2. If Algorithm 1 can build a valid frame for $(\mathcal{T}, \mathcal{R})$ then $(\mathcal{T}, \mathcal{R})$ is consistent.
3. If $(\mathcal{T}, \mathcal{R})$ is consistent then Algorithm 1 can build a valid frame \mathcal{F} for $(\mathcal{T}, \mathcal{R})$.

Proof (sketch). Since the functions $\delta(\sigma)$ and $\delta(\sigma, \langle r, l \rangle)$ is increased monotonously by Algorithm 1, termination of the algorithm can be proved if we can show that : (i) the

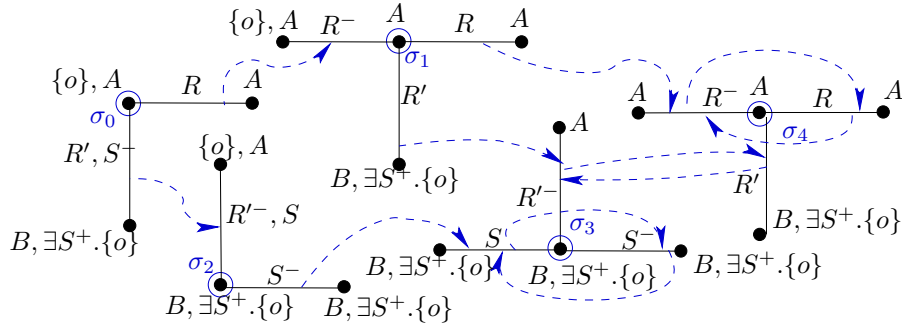


Fig. 2. A frame obtained by Algorithm 1 for \mathcal{K}_1 in the example in Section 1

number of different star-types is bounded; (ii) the detection of a blocked cycle according to Definition 8 terminates. For the soundness of Algorithm 1, we can extend the set \mathcal{P} of paths to a set $\widehat{\mathcal{P}}$ of extended paths by “unravelling” the frame between blocking-blocked cycles. A tableau [14] can be built from $\widehat{\mathcal{P}}$. The main argument is that when extending a path \mathcal{P} if $\text{tail}_\sigma(\mathcal{P}) \neq \text{tail}_\sigma(\mathcal{P}')$ for all blocked path \mathcal{P}' then this extension process can be continued up to a star-type $\sigma' = \text{tail}_\sigma(\mathcal{P}'')$ for some blocked path \mathcal{P}'' . This holds due to the definition of cycles and blockable cycles. Otherwise, i.e., $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$ for some blocked path \mathcal{P}' by Q then \mathcal{P} can be extended by getting through $\text{tail}_\sigma(Q)$.

Regarding completeness, a tableau can guide the algorithm (i) to choose valid star-types, (ii) to ensure that $\delta(\sigma) = 1$ for each nominal star-type σ , and (iii) to detect a pair (Θ_1, Θ_k) of blocking and blocked cycles as soon as each concept of the form $\exists Q^\oplus.D$ in Θ_1 is satisfied. We refer the readers to [14] for a complete proof of Lemma 1.

The following theorem is a consequence of Lemma 1.

Theorem 1. $SHOIQ_{(+)}$ is decidable.

4 Conclusion

In this paper, we have presented a decision procedure for the description logic $SHOIQ$ with transitive closure of roles in concept axioms, whose decidability was not known. The most significant feature of our contribution is to introduce a structure for characterizing models which have an infinite non-tree-like part. This structure would provide an insight into regularity of such models which would be enjoyed by a more expressive DL, such as $ZOIQ$ [6], whose decidability remains open. In future work, we aim to improve the algorithm by making it more goal-directed and aim to investigate another open question about the hardness of $SHOIQ_{(+)}$.

References

1. Patel-Schneider, P., Hayes, P., Horrocks, I.: OWL web ontology language semantics and abstract syntax. In: W3C Recommendation. (2004)
2. Tobies, S.: The complexity of reasoning with cardinality restrictions and nominals in expressive description logics. *Journal of Artificial Intelligence Research* **12** (2000) 199–217
3. Aho, A.V., Ullman, J.D.: Universality of data retrieval languages. In: Proceedings of the 6th of ACM Symposium on Principles of Programming Language. (1979)
4. Baader, F.: Augmenting concept languages by transitive closure of roles: An alternative to terminological cycles. In: Proceedings of the Twelfth International Joint Conference on Artificial Intelligence. (1991)
5. Ortiz, M.: An automata-based algorithm for description logics around *SRIQ*. In: Proceedings of the fourth Latin American Workshop on Non-Monotonic Reasoning 2008, CEUR-WS.org (2008)
6. Calvanese, D., Eiter, T., Ortiz, M.: Regular path queries in expressive description logics with nominals. In: IJCAI. (2009) 714–720
7. Horrocks, I., Sattler, U.: A tableau decision procedure for *SHOIQ*. *Journal Of Automated Reasoning* **39**(3) (2007) 249–276
8. Motik, B., Shearer, R., Horrocks, I.: Hypertableau reasoning for description logics. *J. of Artificial Intelligence Research* **36** (2009) 165–228
9. Pratt-Hartmann, I.: Complexity of the two-variable fragment with counting quantifiers. *Journal of Logic, Language and Information* **14**(3) (2005) 369–395
10. Le Duc, C., Lamolle, M.: Decidability of description logics with transitive closure of roles. In: Proceedings of the 23rd International Workshop on Description Logics (DL 2010), CEUR-WS.org (2010)
11. Horrocks, I., Sattler, U., Tobies, S.: Practical reasoning for expressive description logics. In: Proceedings of the International Conference on Logic for Programming, Artificial Intelligence and Reasoning (LPAR 1999), Springer (1999)
12. Baader, F., Nutt, W.: Basic description logics. In: *The Description Logic Handbook: Theory, Implementation and Applications* (2nd edition), Cambridge University Press (2007) 47–104
13. Fischer, M.J., Ladner, R.I.: Propositional dynamic logic of regular programs. *Journal of Computer and System Sciences* **18**(18) (1979) 174–211
14. Le Duc, C., Lamolle, M., Curé, O.: A decision procedure for *SHOIQ* with transitive closure of roles in concept axioms. In: Technical Report, <http://www.iut.univ-paris8.fr/~leduc/papers/RR-SHOIQTr.pdf> (2013)