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Fourier transform of Weyl fractional derivatives

P. Vigué

1 Definitions

Unlike more usual fractional operators with names now comonly agreed, like Caputo derivative or Riemann-Liouville derivative, there is no unanimous designated name for the operators presented here, that have unbounded integration intervals. Drawing inspiration from [Hil08], [Ort11] (table 2.1) et [Mai10] (section 1.4), they are named after Weyl in the rest of the document.

Let α be in $]0, 1[$, let x be a differentiable function. The Weyl fractional derivative of x (of order α) is defined as:

$$D^\alpha x(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t (t-\tau)^{-\alpha} \dot{x}(\tau) \, d\tau, \quad (1)$$

where Γ is the Euler Gamma function :

$$\Gamma(\alpha) := \int_0^\infty e^{-s} s^{\alpha-1} \, ds. \quad (2)$$

The Weyl fractional integral is similarly defined: for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$D^{-\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-\tau)^{\alpha-1} x(\tau) \, d\tau \quad (3)$$

The integration interval chosen above, $] -\infty, t[$, is not bounded. Expressions defined above exist when the function x is periodic ([Ort11], section 2.11.1), and, in the case of the fractional integral, if it is also of zero mean.

There is a link between fractional derivative and fractional integral. For $\alpha \in]0, 1[$,

$$D^\alpha x(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t (t-\tau)^{-\alpha} \dot{x}(\tau) \, d\tau \quad (4)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t (t-\tau)^{(1-\alpha)-1} \dot{x}(\tau) \, d\tau; \quad (5)$$

and given that $1-\alpha > 0$, the integral of \dot{x} of order $1-\alpha$ is

$$D^{\alpha-1} \dot{x}(t) = D^{-(1-\alpha)} \dot{x}(t) := \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t (t-\tau)^{(1-\alpha)-1} \dot{x}(\tau) \, d\tau \quad (6)$$

therefore

$$D^\alpha x = D^{\alpha-1} \dot{x} \quad (7)$$

Moreover, thanks to this link between fractional derivative and fractional integral, one can define the Weyl derivative of order $\beta \in]1, 2[$ as:

$$D^\beta x := D^{\beta-1} \dot{x} \quad (8)$$

and similarly to the standard derivative, this can be done recursively for larger orders β ; or even, skipping steps, let us define the Weyl fractional derivative of x of order $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$D^\beta x(t) := \frac{1}{\Gamma(\lceil \beta \rceil - \beta)} \int_{-\infty}^t (t-\tau)^{\lceil \beta \rceil - \beta - 1} x^{(\lceil \beta \rceil)}(\tau) \, d\tau, \quad (9)$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Setting $n := \lceil \beta \rceil - 1$ and $\alpha := \beta - n \in]0, 1[$, one gets

$$D^\beta x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t (t-\tau)^{-\alpha} x^{(n+1)}(\tau) d\tau = D^\alpha x^{(n)}, \quad (10)$$

which is why the rest of this document focuses on the operator D^α for $\alpha \in]0, 1[$.

2 Fourier transform of fractional operators

The motivation for the term “fractional” is to generalize the derivation and integration operations to non-integer orders. For a function x , the Fourier transform \mathcal{F} verifies:

$$\mathcal{F}(\dot{x})(\xi) = (i\xi)\mathcal{F}(x)(\xi). \quad (11)$$

Definitions (1), (3), generalize this property to an order α in $]0, 1[$:

$$\mathcal{F}(D^{-\alpha}x)(\xi) = (i\xi)^{-\alpha}\mathcal{F}(x)(\xi), \quad \mathcal{F}(D^\alpha x)(\xi) = (i\xi)^\alpha\mathcal{F}(x)(\xi) \quad (12)$$

Coupled with the remark (10), one can obtain that the property (12) holds for any non-integer α .

A similar property, in terms of Laplace transform, is more commonly given, but for causal functions, and for a Laplace variable s with a (strictly) positive real part. To obtain the property (12) from there, one can take the limit $s \rightarrow -i\xi$, which can be justified by theorem. This is how [Pod98], section 2.9.2, presents it. A direct proof of the property (12) for $\alpha = \frac{1}{2}$ is given in the lecture [DGP10]. The rest of this document is dedicated to give a direct¹ proof for $\alpha \in]0, 1[$.

To this end, three lemmas are first established. The first is adapted from Jordan’s lemma. The second one is proved very similarly to the first one. These two first lemmas are used to prove the thid one, a Fourier transform used for the final proof, which establishes the property (12).

Lemma (Jordan). *In the complex plane, let C_R be the quarter circle of center 0, radius R , endpoints R and iR , positively oriented. Let $\alpha \in]0, 1[$, let $\xi < 0$. Then the integral*

$$I_R := \int_{C_R} z^{\alpha-1} e^{-i\xi z} dz \quad (13)$$

tends to 0 when R tends to infinity.

Proof. After choosing the polar parametrization of C_R : $z = Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$

$$\left| \int_{C_R} z^{\alpha-1} e^{-i\xi z} dz \right| = \left| \int_0^{\pi/2} R^{\alpha-1} e^{i\theta(\alpha-1)} e^{-i\xi(R \cos \theta + iR \sin \theta)} iR e^{i\theta} d\theta \right| \quad (14)$$

$$\leq \int_0^{\pi/2} R^{\alpha-1} \left| e^{i\theta(\alpha-1)} \right| \left| e^{-i\xi R \cos \theta} \right| e^{\xi R \sin \theta} |i| R |e^{i\theta}| d\theta \quad (15)$$

$$\leq R \int_0^{\pi/2} R^{\alpha-1} e^{\xi R \sin \theta} d\theta \quad (16)$$

Let $\varepsilon > 0$. Since $\alpha - 1 < 0$, there exists R_0 such that for $R > R_0$, $R^{\alpha-1} < \varepsilon$. On the other hand, the function \sin is over $\left[0, \frac{\pi}{2}\right]$, so its graph is above the line segment from $(0; 0)$ to $\left(\frac{\pi}{2}; 1\right)$, yielding

$$\forall \theta \in \left[0, \frac{\pi}{2}\right], \quad \frac{2}{\pi}\theta \leq \sin \theta \quad (17)$$

Since $\xi < 0$,

$$\forall \theta \in \left[0, \frac{\pi}{2}\right], \quad \xi \frac{2}{\pi}\theta \geq \xi \sin \theta \quad (18)$$

¹In a sense of: as self-contained as possible, using explicit proofs rather than externalizing the burden to lesser known theorems!

Therefore, for $R > R_0$,

$$|I_R| \leq R\varepsilon \int_0^{\pi/2} e^{\xi R\theta/2/\pi} d\theta \quad (19)$$

$$\leq R\varepsilon \frac{e^{\xi R} - 1}{\xi R 2/\pi} = \varepsilon \frac{\pi}{2} \frac{1 - e^{\xi R}}{|\xi|} \xrightarrow{R \rightarrow +\infty} 0 \quad (20)$$

□

Lemma (contour around 0). *In the complex plane, let C_0 be the quarter circle of center 0, radius $\frac{1}{R}$, endpoints $\frac{1}{R}$ and $\frac{i}{R}$, negatively oriented. Let $\alpha \in]0, 1[$, $\xi < 0$. Then the integral*

$$I_0 := \int_{C_0} z^{\alpha-1} e^{-i\xi z} dz \quad (21)$$

tends to 0 when R tends to infinity.

Proof. Again, a polar parametrization $z = \frac{1}{R} e^{i\theta}$ yields:

$$\left| \int_{\pi/2}^0 \left(\frac{1}{R} e^{i\theta} \right)^{\alpha-1} e^{-i\xi \frac{1}{R} (\cos\theta + i \sin\theta)} i \frac{1}{R} e^{i\theta} d\theta \right| \leq \int_0^{\pi/2} R^{1-\alpha} \exp\left(\frac{\xi \sin\theta}{R}\right) \frac{1}{R} d\theta \quad (22)$$

$$\leq R^{-\alpha} \int_0^{\pi/2} \exp\left(\frac{\xi}{R} \frac{2}{\pi} \theta\right) d\theta \quad (23)$$

(from Eq. (18))

$$\leq R^{-\alpha} \frac{R\pi}{2\xi} \left(\exp\left(\frac{\xi}{R}\right) - 1 \right) \quad (24)$$

$$\leq R^{1-\alpha} \frac{\pi}{2} \frac{1 - e^{\xi/R}}{|\xi|} \xrightarrow{R \rightarrow +\infty} 0 \quad (25)$$

since when $R \rightarrow +\infty$,

$$R^{1-\alpha} (1 - e^{\xi/R}) \sim -R^{1-\alpha} \frac{\xi}{R} = -R^{-\alpha} \xi \xrightarrow{R \rightarrow +\infty} 0 \quad (26)$$

□

Lemma (Fourier transform). *For $\alpha \in]0, 1[$ and $\xi \in \mathbb{R}$*

$$\int_0^{+\infty} e^{-i\xi s} s^{\alpha-1} ds = (i\xi)^{-\alpha} \Gamma(\alpha) \quad (27)$$

Proof. Starting with the case $\xi < 0$, let C be the closed curve shown in Fig. 1. The holomorphic function

$$z \mapsto e^{-i\xi z} z^{\alpha-1}$$

has only one pole, $z = 0$, and it is outside of the contour C . Cauchy Integral Theorem² gives:

$$\int_C e^{-i\xi z} z^{\alpha-1} dz = 0 \quad (28)$$

This integral is now split into four:

$$\int_C = \int_{1/R}^R + I_R + \int_{iR}^{i/R} + I_0 \quad (29)$$

²Voir par ex. Weisstein, E. W. : *Cauchy Integral Theorem*. From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/CauchyIntegralTheorem.html>.

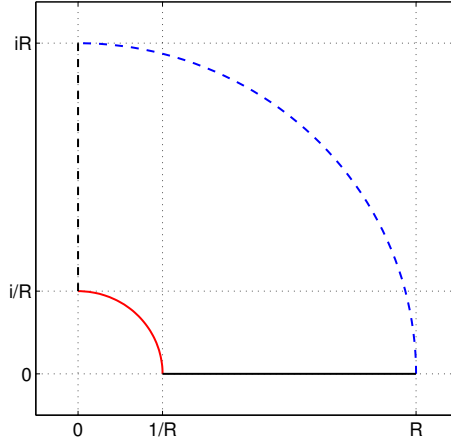


Figure 1: Contour C , concatenation of four simple curves. Blue, dash: quarter circle C_R (first lemma). Solid, black: when $R \rightarrow +\infty$, positive part of the real axis. Black, dash: when $R \rightarrow +\infty$, positive part of the imaginary axis. Red: quarter circle C_0 (second lemma).

The previous lemmas established that the second and fourth integrals, namely I_R and I_0 , tend to 0 when R tend to infinity. Therefore, the limit $R \rightarrow +\infty$ of Eq. (29) yields:

$$\int_0^{+\infty} e^{-i\xi z} z^{\alpha-1} dz + \int_{+i\infty}^0 e^{-i\xi z} z^{\alpha-1} dz = 0 \quad (30)$$

The integrals are parametrized as follows : $z \in \mathbb{R}_+$ for the first one, $z = it$ and $t \in \mathbb{R}_+$ for the second one, one obtains:

$$\int_0^{+\infty} e^{-i\xi s} s^{\alpha-1} ds = \int_0^{+\infty} e^{-i\xi(it)} (it)^{\alpha-1} i dt \quad (31)$$

$$= \int_0^{+\infty} e^{\xi t} i^\alpha t^{\alpha-1} dt \quad (32)$$

$$= i^\alpha \int_0^{+\infty} e^{-y} \left(\frac{y}{-\xi} \right)^{\alpha-1} \frac{1}{-\xi} dy \quad (33)$$

substitution $y = -\xi t$ ($\xi < 0$)

$$= i^\alpha (-\xi)^{-\alpha} \int_0^{+\infty} e^{-y} y^{\alpha-1} dy \quad (34)$$

$$= (i\xi)^{-\alpha} \Gamma(\alpha) \quad (35)$$

since $i^\alpha (-1)^{-\alpha} = e^{i\alpha\pi/2} e^{-i\alpha\pi} = e^{i\pi/2(-\alpha)} = i^{-\alpha}$. By denoting

$$p^\alpha : s \mapsto 0 \quad \text{if } s < 0, \quad s^{\alpha-1} \quad \text{otherwise} \quad (36)$$

one obtains that for $\xi < 0$,

$$\mathcal{F}(p^\alpha)(\xi) = \int_0^{+\infty} e^{-i\xi s} s^{\alpha-1} ds = (i\xi)^{-\alpha} \Gamma(\alpha) \quad (37)$$

To conclude whatever the sign of ξ , let $\xi < 0$, by property of the Fourier transform, since p^α is a real function,

$$\mathcal{F}(p^\alpha)(-\xi) = \overline{\mathcal{F}(p^\alpha)(\xi)} \quad (38)$$

$$= \overline{i^{-\alpha} \xi^{-\alpha} \Gamma(\alpha)} \quad (39)$$

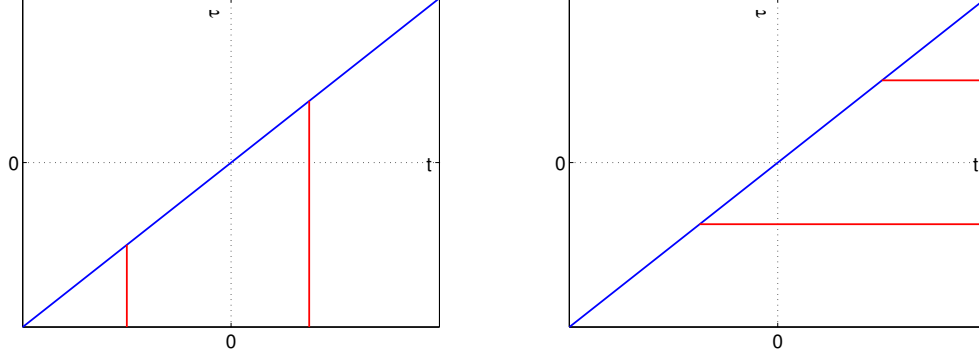


Figure 2: Integration domain under the line $t = \tau$ (blue), parametrized on the left side by the red half-lines $(t, \tau) \in \mathbb{R} \times]-\infty, t]$, on the right side by the half-lines $(\tau, t) \in \mathbb{R} \times [\tau, +\infty[$.

and

$$\overline{i^{-\alpha}} = \overline{e^{-i\alpha\pi/2}} = e^{i(-\alpha)(-\pi/2)} = (-i)^{-\alpha} \quad (40)$$

so

$$\mathcal{F}(p^\alpha)(-\xi) = (-i)^{-\alpha} \xi^{-\alpha} \Gamma(\alpha) = (i(-\xi))^{-\alpha} \Gamma(\alpha) \quad (41)$$

□

Proof of the property (12).

$$\mathcal{F}(D^{-\alpha}x)(\xi) = \int_{-\infty}^{+\infty} D^{-\alpha}x(t)e^{-i\xi t} dt \quad (42)$$

$$= \int_{t=-\infty}^{+\infty} \frac{1}{\Gamma(\alpha)} \int_{\tau=-\infty}^t (t-\tau)^{\alpha-1} x(\tau) d\tau e^{-i\xi t} dt \quad (43)$$

$$= \frac{1}{\Gamma(\alpha)} \int_{\tau=-\infty}^{+\infty} \int_{t=\tau}^{+\infty} (t-\tau)^{\alpha-1} x(\tau) e^{-i\xi t} dt d\tau \quad (44)$$

by using the new parametrization of the half-plane under the line $t = \tau$ (see Fig. 2)

$$= \frac{1}{\Gamma(\alpha)} \int_{\tau=-\infty}^{+\infty} \int_{s=0}^{+\infty} s^{\alpha-1} x(\tau) e^{-i\xi(s+\tau)} ds d\tau \quad (45)$$

$$\text{(substitution } s = t - \tau) \quad (46)$$

$$= \frac{1}{\Gamma(\alpha)} \int_{s=0}^{+\infty} s^{\alpha-1} e^{-i\xi s} ds \int_{\tau=-\infty}^{+\infty} x(\tau) e^{-i\xi \tau} d\tau \quad (47)$$

$$= (i\xi)^{-\alpha} \mathcal{F}(x)(\xi) \quad (48)$$

using the third lemma for the last equality. On the other hand,

$$\mathcal{F}(D^\alpha x)(\xi) = \mathcal{F}(D^{\alpha-1}\dot{x})(\xi) \quad \text{by property (7)} \quad (49)$$

$$= (i\xi)^{\alpha-1} \mathcal{F}(\dot{x})(\xi) \quad (\alpha - 1 < 0, \text{ established result for the integral } D^{\alpha-1}) \quad (50)$$

$$= (i\xi)^\alpha \mathcal{F}(x)(\xi) \quad (51)$$

□

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