Expected number and distribution of critical points of real Lefschetz pencils

Michele Ancona

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Abstract

We give an asymptotic probabilistic real Riemann-Hurwitz formula computing the expected real ramification index of a random covering over the Riemann sphere. More generally, we study the asymptotic expected number and distribution of critical points of a random real Lefschetz pencil over a smooth real algebraic variety.

1 Introduction

The Riemann-Hurwitz formula says that the total ramification index R of a branched covering $f : \Sigma \to \Sigma'$ of degree d between two compact Riemann surfaces is $R = d\chi(\Sigma') - \chi(\Sigma)$. In particular, if $\Sigma' = \mathbb{C}P^1$, $R = 2d + 2g - 2$, where g is the genus of Σ.
More generally, if $p : X \dashrightarrow \mathbb{C}P^1$ is a Lefschetz pencil on a complex manifold X of dimension n, then

$$(-1)^n \#crit(p) = \chi(X) - 2\chi(F) + \chi(Y)$$

where F is a smooth fiber of p and Y is the base locus of p.

The questions that motivate this paper are the following: *how do these critical points distribute on the variety? When p is defined over $\mathbb{R}$, what about the number of real critical points of a real Lefschetz pencil?*
We answer these questions by computing the asymptotic expected number

*Institut Camille Jordan, Umr Cnrs 5208, Université Claude Bernard Lyon 1. ancona@math.univ-lyon1.fr

of real critical points of real Lefschetz pencils and also the asymptotic distribution of such points.

The chosen random setting has already been considered by Schiffman and Zelditch in [14] to study the integration current over the common zero locus of k independent random global sections of a line bundle over a complex projective manifold.

In the real case Kac [7], Kostlan [8] and Shub and Smale [15] computed the expected number of real roots of a random real polynomial.
In higher dimensions, Podkorytov [13] and Bürgisser [1] computed the expected Euler characteristic of random real algebraic submanifolds, Gayet and Welschinger estimated the Betti numbers [3], [5], [4]. (See also [10], [11] and [9]).

In [12] Nicolaescu computed the expected number of critical of a random smooth function on a Riemannian manifold have and how they distruibute.

Statements of the results

Let X be a smooth real projective manifold of dimension n, that is a complex projective manifold equipped with an anti-holomorphic involution c_X.
We denote by $\mathbb{R}X = Fix(c_X)$ its real locus. Let (L, h) be a positive real hermitian line bundle. Then, for large d, for almost every $(\alpha, \beta) \in H^0(X; L^d)^2$ (resp. $\mathbb{R}H^0(X; L^d)^2$) the map $u_{\alpha\beta} : X \dashrightarrow \mathbb{C}P^1$ defined by $x \mapsto [\alpha(x) : \beta(x)]$ is a Lefschetz pencil (resp. real Lefschetz pencil) (see Prop. 2.11).

Definition 1.1. We denote the set of critical points of $u_{\alpha\beta}$ by $\mathrm{crit}(u_{\alpha\beta})$ and by $\mathbb{R}\mathrm{crit}(u_{\alpha\beta}) = \mathrm{crit}(u_{\alpha\beta}) \cap \mathbb{R}X$ the set of real critical points.

The number of real critical points of a Lefschetz pencil depends on the pair (α, β). The main theorem of this article is the computation of the expected value of this number.

Theorem 1.2. *Let X be a smooth real projective manifold of dimension n, let (L, h) be a positive real hermitian line bundle over X. Then*

$$\lim_{d\to+\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}[\#\mathbb{R}crit(u_{\alpha,\beta})] = \begin{cases} \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \frac{\pi}{2} Vol_h(\mathbb{R}X) & \text{if } n \text{ is odd} \\ \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) Vol_h(\mathbb{R}X) & \text{if } n \text{ is even.} \end{cases}$$

In this theorem $\mathrm{Vol}_h(\mathbb{R}X)$ is the volume of $\mathbb{R}X$ with respect to the Riemannian volume $dvol_h$ induced by the metric h. The probability we consider is a natural Gaussian probability on $\mathbb{R}H^0(X; L^d)$ (see Section 2.1) and $e_{\mathbb{R}}(n)$

is the expected value of (the absolute value) of the determinant of real symmetric matrices (for the explicit values of $e_{\mathbb{R}}(n)$, see [5, Section 2]).
We recall that $e_{\mathbb{R}}(1) = \sqrt{\frac{2}{\pi}}$, then we have:

Corollary 1.3. *Let (Σ, c_Σ) be a real Riemann surface and (L, h) a real holomorphic line bundle of degree 1. Then, for every pair $(\alpha, \beta) \in \mathbb{R}H^0(X; L^d)^2$ without common zeros, $u_{\alpha\beta}$ is a degree d branched covering between Σ and $\mathbb{C}P^1$. Then the expected real total ramification index is equivalent to*

$$\sqrt{\frac{\pi}{2}} Vol_h(\mathbb{R}\Sigma)\sqrt{d}$$

as d tends to $+\infty$.

Theorem 1.2 will be a consequence of a more precise equidistribution result.
We define $\nu_{\alpha\beta} = \sum_{x \in \mathrm{crit}(u_{\alpha\beta})} \delta_x$ (resp. $\mathbb{R}\nu_{\alpha\beta} = \sum_{x \in \mathbb{R}\mathrm{crit}(u_{\alpha\beta})} \delta_x$) the Dirac measure on the (real) critical set.

Theorem 1.4. *Let X be a smooth real projective manifold of dimension n and (L, h) be a positive real hermitian line bundle over X. Then*

$$\lim_{d\to+\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}[\mathbb{R}\nu_{\alpha\beta}] = \begin{cases} \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \frac{\pi}{2} dvol_h & \text{if } n \text{ is odd} \\ \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) dvol_h & \text{if } n \text{ is even.} \end{cases}$$

weakly in the sense of distributions.

Theorem 1.4 says that for all continuous function $\varphi \in C^0(\mathbb{R}X)$

$$\lim_{d\to+\infty} \frac{1}{\sqrt{d}^n} \mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \begin{cases} \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \frac{\pi}{2} \int_{\mathbb{R}X} \varphi dvol_h & \text{if n is odd} \\ \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \int_{\mathbb{R}X} \varphi dvol_h & \text{if n is even.} \end{cases}$$

where $\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}H^0(X;L^d)^2} \sum_{x \in \mathbb{R}crit(u_{\alpha\beta})} \varphi(x) d\mu(\alpha, \beta)$.

The following equidistribution result in the complex case follows along the same lines.

Theorem 1.5. *Let X be a smooth complex projective manifold of dimension n, let (L, h) a positive hermitian line bundle over X. Then*

$$\lim_{d\to+\infty} \frac{1}{d^n} \mathbb{E}[\nu_{\alpha\beta}] = (n+1)! dvol_h$$

weakly in the sens of distribution.

As before, Theorem 1.5 says that for all continuous function φ on X, we have

$$\lim_{d\to+\infty} \frac{1}{d^n}\mathbb{E}[\nu_{\alpha\beta}](\varphi) = (n+1)n!\int_X \varphi dvol_h$$

where $dvol_h$ is the Kählerian volume associated to h.

Organisation of the paper

In Section 2.1 we introduce the Gaussian measure on $H^0(X; L^d)$ associated to a Hermitian line bundle (L, h) over a complex manifold X (see also [14], [3], [5]).
In Section 2.2 we present some classical results about Lefschetz pencils on complex manifolds.
In the rest of the Chapter 2 we introduce our main tools, namely the Hörmander peak sections (see also [5], [16]) and the incidence variety (see [15]).

Chapter 3 is completely devoted to the proofs of the Theorems 1.2, 1.4 and 1.5. In Sections 3.1 and 3.2 we prove the equidistribution of critical points of a (real) Lefschetz pencil over a (real) algebraic variety X. This will be done using coarea formula and peak sections. These ideas are taken from [5].
In Section 3.3 we will compute the universal constant by direct computation.

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2 Definitions and main tools

2.1 Notations

Let X be a smooth complex projective manifold of dimension n. Let $L \to X$ be a holomorphic line bundle equipped with a Hermitian metric h of positive curvature $\omega \in \Omega^{(1,1)}(X, \mathbb{R})$. The curvature form induces a Kähler metric on X and a volume form $dvol_h = \frac{\omega^n}{n!}$. Let $dx = \frac{\omega^n}{\int_X \omega^n}$ be the normalized volume form.

The Hermitian metric h induces a Hermitian metric h^d on L^d for every integer $d > 0$ and also a L^2-Hermitian product on the space $H^0(X; L^d)$ of global holomorphic sections of L^d denoted by $\langle , \rangle$ and defined by

$$\langle \alpha, \beta \rangle = \int_X h^d(\alpha, \beta) dx$$

for all α, β in $H^0(X; L^d)$.
This Hermitian product induces a Gaussian measure on $H^0(X; L^d)$ defined by

$$\mu(A) = \frac{1}{\pi^{N_d}} \int_A e^{-\|s\|} ds$$

for all open subset $A \subset H^0(X; L^d)$ where ds is the Lebesgue measure associated to $\langle , \rangle$ and $N_d = dim_{\mathbb{C}} H^0(X; L^d)$.
A Lefschetz pencil on X is a rational map $p : X \dashrightarrow \mathbb{C}P^1$ having only non degenerated critical points and defined by two sections of a holomorphic line bundle with smooth and transverse vanishing loci.

All these definitions have a real counterpart. Let X a smooth real projective manifold of dimension n, that is a smooth complex projective manifold equipped with an anti-holomorphic involution c_X. We denote by
$\mathbb{R}X = Fix(c_X)$ its real locus.
A real holomorphic line bundle $p : L \to X$ is a line bundle equipped with an anti-holomorphic involution c_L such that $p \circ c_L = c_X \circ p$ and c_L is complex-antilinear in the fibers. We denote by $\mathbb{R}H^0(X; L)$ the real vector space of real global section of L, i.e. sections $s \in H^0(X; L)$ such that $s \circ c_X = c_L \circ s$.
A real Hermitian metric on L is a Hermitian metric h_L such that $c_L^* h_L = \bar{h}_L$.
If (L, h_L) is a line bundle over X with positive curvature ω, then $\omega(., i.)$ is a Hermitian metric over X which restricts to a Riemannian metric over $\mathbb{R}X$. With a slight abuse of notation, we denote also the Riemannian volume form by $dvol_h$. The L^2-Hermitian product $\langle , \rangle$ on $H^0(X; L^d)$ restricts to a L^2-scalar product on $\mathbb{R}H^0(X; L^d)$, denoted also by $\langle , \rangle$. Then, as in the complex case, also in the real case we have a natural Gaussian measure on $\mathbb{R}H^0(X; L^d)$ defined by

$$\mu(A) = \frac{1}{\sqrt{\pi}^{N_d}} \int_A e^{-\|s\|} ds$$

for all open subset $A \subset \mathbb{R}H^0(X; L^d)$ where ds is the Lebesgue measure associated to $\langle , \rangle$ and $N_d = \dim_{\mathbb{C}} H^0(X; L^d) = \dim_{\mathbb{R}} \mathbb{R}H^0(X; L^d)$.
If (X, c_X) is a smooth real projective manifold then a real Lefschetz pencil

is a Lefschetz pencil $p : X \dashrightarrow \mathbb{C}P^1$ such that $p \circ c_X = conj \circ p$.
We conclude this section by introducing some notation on symmetric matrices.

Definition 2.1. For every $n \in \mathbb{N}^*$, denote by $Sym(n, \mathbb{R})$ the real vector space of real symmetric matrices of size $n \times n$. The vector of dimension of these vector spaces is $\frac{n(n+1)}{2}$ and we equip them with the basis $\mathcal{B}$ given by $\tilde{E}_{jj}$ and $\tilde{E}_{ij} = E_{ij} + E_{ji}$ for $1 \leqslant i < j \leqslant n$, where for every k, l with $1 \leqslant k, l \leqslant n$, E_{kl} is the elementary matrix whose entry at the i-th row and j-th column equals 1 if $(i, j) = (k, l)$ and 0 otherwise.

We equip $Sym(n, \mathbb{R})$ with the scalar product turning $\mathcal{B}$ into an orthonormal basis. Let $\mu_{\mathbb{R}}$ the associated Gaussian probability measure. We then set

$$e_{\mathbb{R}}(n) = \int_{A \in Sym(n,\mathbb{R})} \mid \det A \mid d\mu_{\mathbb{R}}(A).$$

2.2 Lefschetz pencils

In this section, we compute the asymptotic value of the number of critical points of a Lefschetz pencil (see also [3, Section 1]).
Recall that a Lefschetz fibration is a map $X \to \mathbb{C}P^1$ with only non degenerate critical points. The following proposition is a kind of Riemann-Hurwitz formula for Lefschetz pencils, for a proof see for example Proposition 1 of [3].

Proposition 2.2. *Let X be a smooth complex projective manifold of positive dimension n equipped with a Lefschetz fibration $p : X \to \mathbb{C}P^1$ and let F be a regular fiber of p. Then we have the following equality:*

$$\chi(X) = 2\chi(F) + (-1)^n \# crit(p).$$

Remark that if $p : X \dashrightarrow \mathbb{C}P^1$ a Lefschetz pencil and we blow-up the base locus $Base(p) \doteqdot Y$, then we obtain a Lefschetz fibration $\tilde{p} : \tilde{X} \doteqdot Bl_Y X \to \mathbb{C}P^1$. By additivity of the Euler characteristic, we have that $\chi(\tilde{X}) = \chi(X) + \chi(Y)$, then by Proposition 2.2 we have

$$\chi(X) = 2\chi(F) - \chi(Y) + (-1)^n \# \mathrm{crit}(p). \tag{1}$$

Proposition 2.3. *Let L be an ample line bundle over a complex manifold X of dimension n. For almost all global sections $\alpha, \beta \in H^0(X; L^d)$, the map $u_{\alpha\beta}$ defined as $x \mapsto [\alpha(x) : \beta(x)]$ is a Lefschetz pencil (see Prop. 2.11).
Then, as d goes to infinity, we have*

$$\# crit(u_{\alpha\beta}) = (n+1) \left(\int_X c_1(L)^n \right) d^n + O(d^{n-1}). \tag{2}$$

Proof. We will follow the lines of Lemma 2, Lemma 3 and Proposition 4 of [3].
We have $\chi(F) = \int_F c_{n-1}(F)$ and $\chi(Y) = \int_Y c_{n-2}(Y)$.
We remark that the base locus is the intersection of the zero locus of α and β, that is $Y = Z_\alpha \cap Z_\beta$.
A regular fiber F over $[a, b] \in \mathbb{C}P^1$ is the zero locus of the section $b\alpha - a\beta \in H^0(X; L^d)$, thus the normal bundle $N_{X/F}$ is $L^d_{|F}$ To compute $\chi(F)$ we will use the adjunction formula. We have

$$0 \to TF \to TX_{|F} \to N_{X/F} \to 0$$

then we have $c(X)_{|F} = c(F) \wedge c(L^d)_{|F}$, i.e. $(1 + c_1(X) + ... + c_n(X))_{|F} = (1 + c_1(F) + ... + c_{n-1}(F)) \wedge (1 + dc_1(L))$.
If we develop this we have $c_1(X) = c_1(F) + dc_1(L)$ and, for $j \in \{2, ..., n-1\}$, we have $c_j(X)_{|F} = c_j(F) + dc_1(L)_{|F} \wedge c_{j-1}(F)$.
Then, summing up the term,

$$c_j(F) = \sum_{k=0}^{j} (-1)^k d^k c_1(L)^k_{|F} \wedge c_{j-k}(X)_{|F}.$$

In particular, for $j = n - 1$ we have

$$c_{n-1}(F) = \sum_{k=0}^{n-1} (-1)^k d^k c_1(L)^k_{|F} \wedge c_{n-k-1}(X)_{|F}.$$

Then $\chi(F)$ is equal to $\int_F \sum_{k=0}^{n-1} (-1)^k d^k c_1(L)^k)_{|F} \wedge c_{n-k-1}(X)_{|F}$
But, for $\alpha \in H^{2n-2}_{dR}(X)$, we have that

$$\int_F \alpha_{|F} = \int_X \alpha \wedge c_1(L^d)$$

so,

$$\chi(F) = \sum_{k=0}^{n-1} \int_X (-1)^k d^{k+1} c_1(L)^{k+1} \wedge c_{n-k-1}(X)$$

and asymptotically

$$\chi(F) \sim (-1)^{n-1} (\int_X c_1(L)^n) d^n.$$

For $Y = Z_\alpha \cap Z_\beta$, the same argument gives us

$$c_j(Y) = \sum_{k=0}^{j} (-1)^k d^k c_1(L)^k_{|Y} \wedge c_{j-k}(Z_\alpha)_{|Y}.$$

But, as before,

$$c_{j-k}(Z_\alpha) = \sum_{h=0}^{j-k} (-1)^h d^h c_1(L)^h \wedge c_{j-k-h}(X).$$

and so, replacing in the above equation

$$c_j(Y) = \sum_{k=0}^{j} (-1)^k d^k c_1(L)^k_{|Y} \wedge (\sum_{h=0}^{j-k} (-1)^h d^h c_1(L)^h_{|Y} \wedge c_{j-k-h}(X)_{|Y})$$

For $j = n-2$ we have

$$c_{n-2}(Y) = \sum_{k=0}^{n-2} (-1)^k d^k c_1(L)^k_{|Y} \wedge (\sum_{h=0}^{n-2-k} (-1)^h d^h c_1(L)^h_Y \wedge c_{n-2-k-h}(X)_{|Y})$$

and this is equivalent to

$$\sum_{k=0}^{n-2} (-1)^{n-2} d^{n-2} c_1(L)^{n-2}_{|Y} = (-1)^{n-2}(n-1) d^{n-2} c_1(L)^{n-2}_{|Y}$$

as $d \to \infty$.
So we have,as $d \to \infty$,

$$\chi(Y) \sim (-1)^{n-2}(n-1) d^{n-2} \int_Y c_1(L)^{n-2}_{|Y} =$$

$$= (-1)^{n-2}(n-1) d^{n-1} \int_{Z_\alpha} c_1(L)^{n-2} \wedge c_1(L) =$$

$$= (-1)^{n-2}(n-1)(\int_X c_1(L)^n) d^n.$$

Combining this with $\chi(X) = 2\chi(F) - \chi(Y) + (-1)^n \#\text{crit}(u_{\alpha\beta})$ we obtain the result. □

2.3 Hörmander's peak sections

Here we introduce the Hörmander's peak sections, an essential tool for our proofs of Theorems 1.4 and 1.5 (see also [6], [16], [5]). Let L be a holomorphic line bundle over a smooth complex projective manifold equipped with a Hermitian metric h of positive curvature ω and let $dx = \frac{\omega^n}{\int_X \omega^n}$ be the normalized volume form. Let x be a point of X. There exists, in the neighborhood of x, a holomorphic trivialization e of L such that the associated potential $\phi = -\log h(e,e)$ reaches a local minimum at x with Hessian of type $(1,1)$. The following result was proved in [16] (see also [5]) .

Lemma 2.4. *Let (L,h) be a holomorphic Hermitian line bundle of positive curvature ω over a smooth complex projective manifold X. Let $x \in X$, $(p_1,...,p_n) \in \mathbb{N}^n$ and $p' > p_1 + ... + p_n$. There exists $d_0 \in \mathbb{N}$ such that for every $d > d_0$, the bundle L^d has a global holomorphic section σ satisfying $\int_X h^d(\sigma,\sigma)dx = 1$ and*

$$\int_{X\setminus B(x,\frac{\log d}{\sqrt{d}})} h^d(\sigma,\sigma)dx = O(\frac{1}{d^{2p'}})$$

Moreover, if $(x_1,...,x_n)$ are local holomorphic coordinates in the neighborhood of x, we can assume that in a neighborhood of x,

$$\sigma(x_1,...,x_n) = \lambda(x_1^{p_1}\cdots x_n^{p_n} + O(\mid x \mid^{2p'}))e^d(1+O(\frac{1}{d^{2p'}}))$$

where

$$\lambda^{-2} = \int_{B(x,\frac{\log d}{\sqrt{d}})} \mid x_1^{p_1}\cdots x_n^{p_n} \mid^2 h^d(e^d,e^d)dx$$

and e is a holomorphic trivialization of L in the neighborhood of x whose potential $\phi = -\log h(e,e)$ reaches a local minimum at x with Hessian equal to $\pi\omega(.,i.)$.

Here is an analogue of Lemma 2.4 in the real case:

Lemma 2.5. *Let (L,h) be a real holomorphic Hermitian line bundle of positive curvature ω over a smooth rel projective manifold X. Let $x \in \mathbb{R}X$, $(p_1,...,p_n) \in \mathbb{N}^n$ and $p' > p_1 + ... + p_n$. There exists $d_0 \in \mathbb{N}$ such that for every $d > d_0$, the bundle L^d has a global holomorphic section σ satisfying $\int_X h^d(\sigma,\sigma)dx = 1$ and*

$$\int_{X\setminus B(x,\frac{\log d}{\sqrt{d}})} h^d(\sigma,\sigma)dV_h = O(\frac{1}{d^{2p'}})$$

Moreover, if $(x_1,...,x_n)$ are local real holomorphic coordinates in the neighborhood of x, we can assume that in a neighborhood of x,

$$\sigma(x_1,...,x_n) = \lambda(x_1^{p_1}\cdots x_n^{p_n} + O(\mid x \mid^{2p'}))e^d(1+O(\frac{1}{d^{2p'}}))$$

where

$$\lambda^{-2} = \int_{B(x,\frac{\log d}{\sqrt{d}})} \mid x_1^{p_1}\cdots x_n^{p_n} \mid^2 h^d(e^d,e^d)dx$$

and e is a real trivialization of L in the neighborhood of x whose potential $\phi = -\log h(e,e)$ reaches a local minimum at x with Hessian $\pi\omega(.,i.)$.

This real counterpart follows from Lemma 2.4 by averaging the peak sections with the real structure.
Let σ_0 be the section given by the Lemma 2.5 with $p' = 3$ and $p_i = 0$ for all i, σ_i the section given by Lemma 2.5 with $p' = 3$ and $p_j = \delta_{ij}$, σ_{ij} the section given by Lemma 2.5 with $p_i = p_j = 1$ and $p_k = 0$ otherwise and σ_{kk} the section given by the Lemma 2.5 with $p_k = 2$ and $p_l = 0$ for $l \neq k$.
These sections are called *peak sections*. Their Taylor expansions are:

$$\sigma_0(y) = (\lambda_0 + O(\| y \|^6))e^d(1 + O(\frac{1}{d^6}));$$

$$\sigma_i(y) = (\lambda_i y_i + O(\| y \|^6))e^d(1 + O(\frac{1}{d^6})) \quad \forall i;$$

$$\sigma_{ij}(y) = (\lambda_{ij} y_i y_j + O(\| y \|^6))e^d(1 + O(\frac{1}{d^6})) \quad \forall i \neq j;$$

$$\sigma_{kk}(y) = (\lambda_{kk} y_k^2 + O(\| y \|^6))e^d(1 + O(\frac{1}{d^6})) \quad \forall k.$$

The following lemma provides the asymptotic of the constants λ_0, λ_i, λ_{ij} et λ_{kk}.

Lemma 2.6. *[5, Lemma 2.5] Under the hypothesis of Lemma 2.4 or 2.5, we have*

$$\lim_{d\longrightarrow\infty} \frac{1}{\sqrt{d}^n}\lambda_0 = \sqrt{\delta_L}$$

$$\lim_{d\longrightarrow\infty} \frac{1}{\sqrt{d}^{n+1}}\lambda_i = \sqrt{\pi}\sqrt{\delta_L}$$

$$\lim_{d\longrightarrow\infty} \frac{1}{\sqrt{d}^{n+2}}\lambda_{ij} = \pi\sqrt{\delta_L}$$

$$\lim_{d\longrightarrow\infty} \frac{1}{\sqrt{d}^{n+2}}\lambda_{kk} = \frac{\pi}{\sqrt{2}}\sqrt{\delta_L}$$

for the L^2-product induced by $dx = \frac{\omega^n}{\int_X \omega^n}$ where $\delta_L = \int_X c_1(L)^n$ is the degree of the line bundle L.

Let

$$H_{2x} = \{s \in H^0(X; L^d) \mid s(x) = 0, \nabla s(x) = 0, \nabla^2 s(x) = 0\}$$

$$\left(\text{resp.} \quad \mathbb{R}H_{2x} = \{s \in \mathbb{R}H^0(X; L^d) \mid s(x) = 0, \nabla s(x) = 0, \nabla^2 s(x) = 0\}\right).$$

This space is formed by sections whose 2-jet vanishes at x. The sections $(\sigma_i)_{0\leq i\leq n}$ $(\sigma_{ij})_{1\leq i\leq j\leq n}$ provide a basis of a complement of H_{2x}. This basis is not orthonormal and its spanned subspace is not orthogonal to H_{2x}. However, this basis is aymptotically an orthonormal basis and its spanned subspace is asymptotically orthonormal to H_{2x}, in the following sense:

Proposition 2.7. *[16, Lemma 3.1] The section $(\sigma_i)_{0\leq i\leq n}$ and $(\sigma_{ij})_{1\leq i\leq j\leq n}$ have L^2-norm equal to 1 and their pairwise scalar product are $O(\frac{1}{d})$. Likewise, their scalar products with every unitary element of H_{2x} are $O(\frac{1}{d^{3/2}})$.*

2.4 Incidence varieties

Following [15] we define an incidence variety associated to the complex (resp. real) manifold X and to the (real) positive line bundle L. We will use this incidence variety to prove that, for global sections $\alpha, \beta \in H^0(X; L^d)$ (resp. $\alpha, \beta \in \mathbb{R}H^0(X; L^d)$), the map $u_{\alpha\beta}$ defined as $x \mapsto [\alpha(x) : \beta(x)]$ is almost surely a Lefschetz pencil (Proposition 2.3).

Let (L, h) be a real Hermitian line bundle with positive curvature ω over a real manifold X of dimension n.

Definition 2.8. Let $\alpha, \beta \in H^0(X; L^d)$ (resp. $\mathbb{R}H^0(X; L^d)$) be (resp. real) global sections such that the map $x \mapsto [\alpha(x) : \beta(x)]$ is a Lefschetz pencil. We define

1. the *base locus* of a Lefschetz pencil as the points x such that $\alpha(x) = \beta(x) = 0$;

2. the *critical points* as the points $x \in X \setminus Base(u_{\alpha\beta})$ such that $(\alpha\nabla\beta - \beta\nabla\alpha)(x) = 0$ (this expression doesn't depend on the choice of a connexion ∇ on L).
   We denote by $\mathrm{crit}(u_{\alpha\beta})$ the set of critical points of $(u_{\alpha\beta})$ and by $\mathbb{R}\mathrm{crit}(u_{\alpha\beta}) = \mathrm{crit}(u_{\alpha\beta}) \cap \mathbb{R}X$ the set of real critical points)

We denote by Δ (resp.$\mathbb{R}\Delta$) the set of $(\alpha, \beta, x) \in H^0(X; L^d)^2 \times X$ (resp. $(\alpha, \beta, x) \in \mathbb{R}H^0(X; L^d)^2 \times \mathbb{R}X$) such that $\alpha(x) = \beta(x) = 0$. Set

$$\mathcal{I} = \{(\alpha, \beta, x) \in (H^0(X; L^d)^2 \times X)\backslash\Delta \mid x \in crit(u_{\alpha\beta})\}$$

$$\text{(resp.} \quad \mathbb{R}\mathcal{I} = \{(\alpha, \beta, x) \in (\mathbb{R}H^0(X; L^d)^2 \times \mathbb{R}X)\backslash\mathbb{R}\Delta \mid x \in \mathrm{crit}(u_{\alpha\beta})\})$$

Proposition 2.9. *Let L a (real) holomorphic line bundle over a smooth complex (resp. real) projective manifold X. If L^d is 1-ample, that is if the 1-jet map*

$$H^0(X; L^d) \times X \to J^1(L^d)$$
$$(s, x) \mapsto j^1_x(s) = (s(x), \nabla s(x))$$

is surjective, then $\mathcal{I}$ (resp. $\mathbb{R}\mathcal{I}$) is a manifold of complex (resp. real) dimension $2N_d$, where $N_d = \dim H^0(X; L^d)$.

Proof. We study the differential of the map

$$q : (H^0(X; L^d)^2 \times X)\backslash\Delta \to T^*X \otimes L^{2d}$$

defined by

$$(\alpha, \beta, x) \mapsto (\alpha\nabla\beta - \beta\nabla\alpha)(x) \in T^*_x X \otimes L^{2d}_x$$

defining $\mathcal{I}$. If we prove that 0 is a regular value, then, by Implicit Function Theorem, we have the result. Now, for $(\alpha, \beta, x) \in \mathcal{I}$ we have

$$d_{|(\alpha,\beta,x)}q\cdot(\dot\alpha, \dot\beta, \dot x) = (\dot\alpha\nabla\beta - \beta\nabla\dot\alpha + \alpha\nabla\dot\beta - \dot\beta\nabla\alpha + \alpha\nabla^2_{(\dot x,.)}\beta - \beta\nabla^2_{(\dot x,.)}\alpha + \nabla_{\dot x}\alpha\nabla\beta - \nabla_{\dot x}\beta\nabla\alpha)(x).$$

Let $\eta \in T^*_x X \otimes L^{2d}_x$ we have to prove that there exists $(\dot\alpha, \dot\beta, \dot x)$ such that $d_{|(\alpha,\beta,x)}q \cdot (\dot\alpha, \dot\beta, \dot x) = \eta$. As $(\alpha, \beta, x) \notin \Delta$, we know that at least one between $\alpha(x)$ and $\beta(x)$ is not zero. Without loss of generality, suppose that $\alpha(x) \neq 0$, then, as L^d is 1-ample, there exists $\dot\beta$ such that $\dot\beta(x) = 0$ and $\alpha(x)\nabla\dot\beta(x) = \eta$, then $d_{|(\alpha,\beta,x)}q \cdot (0, \dot\beta, 0) = \eta$. □

If L is ample, then, for large d, L^d is 1-ample. So, in our case $\mathcal{I}$ (resp. $\mathbb{R}\mathcal{I}$) is a smooth variety for large d, called the *incidence variety.* The tangent space of $\mathcal{I}$ at (α, β, x) is

$$T_{(\alpha,\beta,x)}\mathcal{I} = \{\ (\dot\alpha, \dot\beta, \dot x) \in H^0(X; L^d) \times H^0(X; L^d) \times T_x X \mid \left(\dot\alpha\nabla\beta - \beta\nabla\dot\alpha + \alpha\nabla\dot\beta - \dot\beta\nabla\alpha + \alpha\nabla^2_{(\dot x,.)}\beta - \beta\nabla^2_{(\dot x,.)}\alpha\right)(x) = 0\}\,.$$

$$\text{(resp.}\quad T_{(\alpha,\beta,x)}\mathbb{R}\mathcal{I} = \{(\dot\alpha\ , \dot\beta, \dot x) \in \mathbb{R}H^0(X; L^d) \times \mathbb{R}H^0(X; L^d) \times T_x\mathbb{R}X \mid (\dot\alpha\nabla\beta - \beta\nabla\dot\alpha + \alpha\nabla\dot\beta - \dot\beta\nabla\alpha + \alpha\nabla^2_{(\dot x,.)}\beta - \beta\nabla^2_{(\dot x,.)}\alpha\ )(x) = 0\}\ )\,.$$

Remark 2.10.

- *In the equation defining the tangent space there is also the term* $(\nabla_{\dot x}\alpha\nabla\beta - \nabla_{\dot x}\beta\nabla\alpha)(x)$ *but it equals zero both in the complex and real case because on* $\mathcal{I}$ *and* $\mathbb{R}\mathcal{I}$ *we have the condition* $(\alpha\nabla\beta - \beta\nabla\alpha)(x) = 0$ *and then* $(\nabla_{\dot x}\alpha\nabla\beta - \nabla_{\dot x}\beta\nabla\alpha)(x) = \left((\nabla_{\dot x}\alpha\frac{\beta}{\alpha} - \nabla_{\dot x}\beta)\nabla\alpha\right)(x) = 0.$
- *The incidence variety comes equipped with two natural projections*

$$\pi_H : \mathcal{I} \to H^0(X; L^d) \times H^0(X; L^d) \quad \textit{and} \quad \pi_X : \mathcal{I} \to X$$

$$\left(\textit{resp.}\quad \pi_{\mathbb{R}H} : \mathbb{R}\mathcal{I} \to \mathbb{R}H^0(X; L^d) \times \mathbb{R}H^0(X; L^d) \quad \textit{and} \quad \pi_{\mathbb{R}X} : \mathbb{R}\mathcal{I} \to \mathbb{R}X\right)$$

Proposition 2.11. *Let L an ample holomorphic line bundle (resp. real holomorphic) over a smooth complex projective manifold X (resp. real projective). For every couple $\alpha, \beta \in H^0(X; L^d)$ (resp. $\in \mathbb{R}H^0(X; L^d)$) the map*

$$u_{\alpha\beta} : X \dashrightarrow \mathbb{C}P^1$$
$$x \mapsto [\alpha(x) : \beta(x)].$$

is, for large d, almost surely a Lefschetz pencil (resp. real Lefschetz pencil).

Proof. The critical points of π_H (resp. $\pi_{\mathbb{R}H}$) are exactly the triples (α, β, x) such that $(\alpha\nabla^2\beta - \beta\nabla^2\alpha)(x)$ is degenerate. By Sard's theorem $\text{valcrit}(\pi_H)$ has zero Lebesgue and Gaussian measure. Also, for large d, the set Γ composed by the pairs $(\alpha, \beta) \in H^0(X; L^d) \times H^0(X; L^d)$ such that $\{x \in X, \alpha(x) = \beta(x) = 0\}$ is not smooth has zero Lebesgue and Gaussian measure (see for example [10, Section 2.2]).
Then $(\Gamma \cup \text{valcrit}(\pi_H))$ has zero measure and its complement is exactly the set of couple of sections defining a Lefschetz pencil. □

3 Proof of the theorems

In this chapter we prove Theorems 1.2, 1.4 and 1.5.

3.1 Coarea formula

Here we will use the incidence variety defined in Section 2.4 and the coarea formula to see the expected distribution of critical points of a (real) Lefschetz pencil as an integral over X (resp. $\mathbb{R}X$).

Definition 3.1. The *normal jacobian* $\text{Jac}_N f$ of a submersion $f : M \to N$ between Riemannian manifolds is the determinant of the differential of the map restricted to the orthogonal of its kernel. Equivalently, if df_p is the differential of f at p, then the normal jacobian is equal to $\sqrt{\det(df_p df_p^*)}$, where df_p^* is the adjoint of df_p with respect to the scalar product on T_pM and $T_{f(p)}N$.

Let X be a smooth complex (resp. real) projective manifold of dimension n and (L, h) be a (real) holomorphic line bundle of positive curvature ω.

Definition 3.2. We define a Dirac measure for (real) critical points of a (real) Lefschetz pencil $u_{\alpha\beta}$ associated to a couple $(\alpha, \beta) \in H^0(X; L^d)^2$ (resp. $\in \mathbb{R}H^0(X; L^d)^2$) by

$$\nu_{\alpha\beta} = \sum_{x \in \text{crit}(u_{\alpha\beta})} \delta_x \qquad (\text{resp.} \quad \mathbb{R}\nu_{\alpha\beta} = \sum_{x \in \mathbb{R}\text{crit}(u_{\alpha\beta})} \delta_x).$$

Let φ be a continuous function on $\mathbb{R}X$. Then, by definition,

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}H^0(X;L^d)^2} \sum_{x\in crit(u_{\alpha\beta})} \varphi(x) d\mu(\alpha,\beta).$$

Proposition 3.3. *Following the notation of Section 2.4, we have*

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}X} \varphi(x) \int_{\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} \frac{1}{\mid Jac_N(\pi_{\mathbb{R}X}) \mid} d\mu_{\mid \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} dvol_h. \quad (3)$$

where the measure $d\mu_{\mid \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))}$ is the following: first we restrict the scalar product $\langle , \rangle$ on $\mathbb{R}H^0(X;L^d)^2$ to $\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))$, that is a codimension n submanifold, then we consider the Riemannian measure associated to this metric, and finally we multiply it by the factor $\frac{1}{\pi^{N_d}} e^{-\|\alpha\|^2 - \|\beta\|^2}$, where $N_d = \dim H^0(X;L^d)$.

Proof. We pull-back the integral

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}H^0(X;L^d)^2} \sum_{x\in \mathrm{crit}(u_{\alpha\beta})} \varphi(x) d\mu(\alpha,\beta)$$

on the incidence variety $\mathbb{R}\mathcal{I}$ and we obtain

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}\mathcal{I}} (\pi_{\mathbb{R}X}^* \varphi)(\alpha,\beta,x)(\pi_{\mathbb{R}H}^* d\mu)(\alpha,\beta,x).$$

Here $\pi_{\mathbb{R}H}^* d\mu$ is well defined because $\pi_{\mathbb{R}H}$ is (almost everywhere) a local isomorphism.
Now we use the coarea formula (see [2, Lemma 3.2.3] or [15, Theorem 1]) for the map $\pi_{\mathbb{R}X}$ and we obtain

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}X} \varphi(x) \int_{\pi_{\mathbb{R}X}^{-1}(x)} \frac{1}{\mid \mathrm{Jac}_N(\pi_{\mathbb{R}X}) \mid} (\pi_{\mathbb{R}H}^* d\mu)_{\mid \pi_{\mathbb{R}X(x)}^{-1}} dvol_h$$

where the measure $(\pi_{\mathbb{R}H}^* d\mu)_{\mid \pi_X^{-1}(x)}$ is the following: first we restrict the (singular) metric $\pi_H^* \langle , \rangle$ on $\mathbb{R}\mathcal{I}$ to $\pi_{\mathbb{R}X}^{-1}(x)$, that is a codimension n submanifold, then we consider the Riemannian measure associated to this metric, and finally we multiply it by the factor $\frac{1}{\pi^{N_d}} e^{-\|\alpha\|^2 - \|\beta\|^2}$, where $N_d = \dim H^0(X;L^d)$. Then another application of coarea formula gives us the result.

□

The space $\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))$ is formed by pairs $(\alpha,\beta) \in \mathbb{R}H^0(X;L^d)^2$ such that $x \in \mathbb{R}\mathrm{crit}(u_{\alpha\beta})$. In the next section we will identify this space with an

intersection of some quadrics in the vector space $\mathbb{R}H^0(X;L^d)^2$.
In the complex case, the same argument gives us, for every continous function φ on X

$$\mathbb{E}[\nu_{\alpha\beta}](\varphi) = \int_X \varphi(x) \int_{\pi_H(\pi_X^{-1}(x))} \frac{1}{\mid \mathrm{Jac}_N(\pi_X) \mid} d\mu_{\mid \pi_H(\pi_X^{-1}(x))} dvol_h. \qquad (4)$$

3.2 Computation of the normal jacobian

In this section we compute the normal jacobian that appears in (3) and (4). We follow the notations of Sections 2.1, 2.4 and 3.1. The main result of this section is the following proposition:

Proposition 3.4. *Following the notation of Sections 2.4 and 3.1, under the hypothesis of Theorem 1.4, we have:*

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi) = \int_{\mathbb{R}X} \varphi(x) R_d(x) dvol_h,$$

where

$$R_d(x) = (\sqrt{\pi d}^n)(\int_Q \frac{\mid \det(a_0 b_{ij} - b_0 a_{ij}) \mid}{\sqrt{\det\left((a_i a_j + b_i b_j) + (a_0^2 + b_0^2) Id\right)}} d\mu_Q + O(\frac{1}{\sqrt{d}}))$$

and $Q \subset \mathbb{R}^{2(n+1)+n(n+1)}$ is the product of the intersection of quadrics

$$\tilde{Q} = \{(a_0, b_0, ..., a_n, ..., b_n) \in \mathbb{R}^{2(n+1)} \mid a_0 b_i - a_i b_0 = 0 \quad \forall i = 1, ..., n\}$$

with the vector space $\mathbb{R}^{n(n+1)}$ of coordinates a_{ij} and b_{ij} for $1 \leq i \leq j \leq n$ and $d\mu_Q = \frac{e^{-\sum_i a_i^2 - \sum_i b_i^2 - \sum_{i,j} a_{ij}^2 - \sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q$.

The remaining part of this section is devoted to the proof of this proposition. Our main tool will be the peak sections defined in Section 2.3.

We fix a point $x \in X$ (resp. $x \in \mathbb{R}X$) and want to compute the integral

$$\int_{\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} \frac{1}{\mid \mathrm{Jac}_N(\pi_{\mathbb{R}X}) \mid} d\mu_{\mid \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} \qquad (5)$$

that appears in (3).

We recall that the tangent space of $\mathcal{I}$ (resp. $\mathbb{R}\mathcal{I}$) at (α, β, x) is

$$\{(\dot{\alpha}, \dot{\beta}, \dot{x}) \in H^0(X;L^d) \times H^0(X;L^d) \times T_x X \mid$$

$$(\dot\alpha\nabla\beta-\beta\nabla\dot\alpha+\alpha\nabla\dot\beta-\dot\beta\nabla\alpha+\alpha\nabla^2_{(\dot x,.)}\beta-\beta\nabla^2_{(\dot x,.)}\alpha)(x)=0\}$$

$$\begin{aligned}(\text{resp.}\quad &\{(\dot\alpha,\dot\beta,\dot x)\in\mathbb{R}H^0(X;L^d)\times\mathbb{R}H^0(X;L^d)\times T_x\mathbb{R}X\mid\\ &(\dot\alpha\nabla\beta-\beta\nabla\dot\alpha+\alpha\nabla\dot\beta-\dot\beta\nabla\alpha+\alpha\nabla^2_{(\dot x,.)}\beta-\beta\nabla^2_{(\dot x,.)}\alpha)(x)=0)(x)=0\}.\end{aligned}$$

We remark that $d\pi_{H|(\alpha,\beta,x)}$ is (almost everywhere) an isometry, because on $\mathcal{I}$ we put the (singular) metric $\pi_H^*\langle,\rangle$. We compute $\mathrm{Jac}_N(\pi_X)$ (resp. $\mathrm{Jac}_N(\pi_{\mathbb{R}X})$) at (α,β,x) using the following two linear maps:

$$A:H^0(X;L^d)\times H^0(X;L^d)\to T^*_xX\otimes L^{2d}_x \tag{6}$$

$$(\text{resp.}\quad A:\mathbb{R}H^0(X;L^d)\times\mathbb{R}H^0(X;L^d)\to\mathbb{R}(T^*X\otimes L^{2d})_x)$$

and

$$B:T_xX\to T^*_xX\otimes L^{2d}_x \tag{7}$$

$$(\text{resp.}\quad B:T_x\mathbb{R}X\to\mathbb{R}(T^*X\otimes L^{2d})_x\big)$$

defined by

$$A(\dot\alpha,\dot\beta)=(\dot\alpha\nabla\beta-\beta\nabla\dot\alpha+\alpha\nabla\dot\beta-\dot\beta\nabla\alpha)(x)$$

and

$$B(\dot x)=\left(\alpha\nabla^2_{(\dot x,.)}\beta-\beta\nabla^2_{(\dot x,.)}\alpha\right)(x)$$

On $T^*_xX\otimes L^{2d}_x$ (resp. $\mathbb{R}(T^*X\otimes L^{2d})_x$) we have the Hermitian (resp. scalar) product induced by h.

Proposition 3.5. *Following the notation of Sections 2.4 and 3.1, we have, for all $(\alpha,\beta,x)\in\mathcal{I}$ (resp. $\mathbb{R}\mathcal{I}$), $Jac_N(\pi_X)=\frac{Jac_N(A)}{Jac(B)}$.*

Proof. We have $d\pi\mid_{(\alpha,\beta,x)}=B^{-1}\circ A$ and this implies $d\pi\mid_{(\alpha,\beta,x)}\mid_{(kerA)^\perp}=B^{-1}\circ A\mid_{(kerA)^\perp}$. It follows that $\mathrm{Jac}(d\pi\mid_{(\alpha,\beta,x)}\mid_{(kerA)^\perp})=\mathrm{Jac}(B^{-1})\mathrm{Jac}(A\mid_{(kerA)^\perp})$ Now, $\mathrm{Jac}(B^{-1})=\mathrm{Jac}(B)^{-1}$ and $\mathrm{Jac}(A\mid_{(kerA)^\perp})=\mathrm{Jac}_N(A)$. □

Fix real holomorphic coordinates $(x_1,...,x_n)$ in a neighborhood of a point $x\in\mathbb{R}X$ such that $(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n})$ is an orthonormal basis of T_xX (resp. $T_x\mathbb{R}X$). We want to compute the integral

$$\int_{\pi_{\mathbb{R}H}(\pi^{-1}_{\mathbb{R}X}(x))}\frac{1}{\mid\mathrm{Jac}_N(\pi_{\mathbb{R}X})\mid}d\mu_{\mid\pi_{\mathbb{R}H}(\pi^{-1}_{\mathbb{R}X}(x))} \tag{8}$$

that appears in (3).

Let $(\alpha, \beta) \in H^0(X; L^d)^2$ (resp. $\mathbb{R}H^0(X; L^d)^2$) then

$$\alpha = \sum_{i=0}^{n} a_i \sigma_i + \sum_{1 \leqslant k \leqslant l \leqslant n} a_{kl} \sigma_{kl} + \tau$$

$$\beta = \sum_{i=0}^{n} b_i \sigma_i + \sum_{1 \leqslant k \leqslant l \leqslant n} b_{kl} \sigma_{kl} + \tau'$$

where $\tau, \tau' \in ker J_x^2$ and σ_i, σ_{kl} are the peak section of Lemma 2.5.
We remark that $(\alpha, \beta) \in \pi_H(\pi_X^{-1}(x))$ if and only if $a_0 b_i - a_i b_0 = 0 \; \forall i = 1, ..., n$, and also that the definition of $\mathrm{Jac}_N(\pi_X)$ involves only the 2-jets of sections. With this remark in mind we define the following spaces:

- $K_2 \doteqdot (Ker J_x^2 \times Ker J_x^2) \subset H^0(X; L^d)^2$ (resp. $\mathbb{R}H^0(X; L^d)^2$);
- $H_2 \doteqdot Vect\{(\sigma_i, 0), (\sigma_{kl}, 0), (0, \sigma_i), (0, \sigma_{kl})\} \subset H^0(X : L^d)^2$ (resp. $\mathbb{R}H^0(X; L^d)^2$) for $i = 0, ..., n$ and $1 \leq l \leq k \leq n$;
- $Q = H_2 \cap \pi_H(\pi_X^{-1}(x))$.

We see Q as the product of the intersection of quadrics:

$$\tilde{Q} = \{(a_0, b_0, ..., a_n, b_n) \in \mathbb{R}^{2(n+1)} \mid a_0 b_i - a_i b_0 = 0 \quad \forall i = 1, ..., n\}$$

with the vector space $\mathbb{R}^{n(n+1)}$ of coordinates a_{ij} and b_{ij} for $1 \leq i \leq j \leq n$.
Let $\pi_2 : K_2^{\perp} \to H_2$ be the orthogonal projection. A consequence of Proposition 2.7 is that, for large d, π_2 is invertible.

Proposition 3.6. *Following the notation of Section 2.4 and 3.1, let A and B be the linear application defined in (6) and (7). Then, in the complex case, under the hypothesis of Theorem 1.5,*

$$(\pi_2^{-1})_* Jac_N(A) = \det((\pi \delta_L^2 d^{2n+1})((a_i \bar{a}_j + b_i \bar{b}_j) E_{ij} + (\mid a_0 \mid^2 + \mid b_0 \mid^2) Id + O(\frac{1}{\sqrt{d}}))$$

$$(\pi_2^{-1})_* Jac(B) = \mid \det(\pi \delta_L \sqrt{d}^{2n+2})((a_0 b_{ij} - b_0 a_{ij}) \tilde{E}_{ij} + O(\frac{1}{\sqrt{d}})) \mid^2$$

and, in the real case, under the hypothesis of the Theorem 1.4,

$$(\pi_2^{-1})_* Jac_N(A) = \sqrt{\det(\pi \delta_L^2 d^{2n+1})((a_i a_j + b_i b_j) E_{ij} + (a_0^2 + b_0^2) Id + O(\frac{1}{\sqrt{d}}))}$$

$$(\pi_2^{-1})_* Jac(B) = \det((\pi\delta_L\sqrt{d}^{2n+2})((a_0b_{ij} - b_0a_{ij})\tilde{E}_{ij} + O(\frac{1}{\sqrt{d}})))$$

where $\tilde{E}_{ij}$ for $1 \leqslant i \leqslant j \leqslant n$ and E_{ij} for $i, j = 1, ..., n$ are the matrices defined in Definition 2.1.

Proof. Let e be a local trivialization of L at x as in Section 2.2 and let $(\sigma_i)_{i=0,...,n}$, $(\sigma_{kl})_{1\leqslant k\leqslant l\leqslant n}$ as in Lemma 2.4 (resp. Lemma 2.5).
Let $(\alpha, \beta) \in H^0(X; L^d)^2$ (resp. $\mathbb{R}H^0(X; L^d)^2$) then

$$\alpha = \sum_{i=0}^{n} a_i\sigma_i + \sum_{1\leqslant k\leqslant l\leqslant n} a_{kl}\sigma_{kl} + \tau$$

$$\beta = \sum_{i=0}^{n} b_i\sigma_i + \sum_{1\leqslant k\leqslant l\leqslant n} b_{kl}\sigma_{kl} + \tau'$$

where $\tau, \tau' \in ker J_x^2$.
In particular we have

$$\alpha(x) = a_0\sigma_0(x), \qquad \beta(x) = b_0\sigma_0(x),$$

$$\nabla\alpha(x) = \sum_{i=0}^{n} a_i\nabla\sigma_i(x), \qquad \nabla\beta(x) = \sum_{i=0}^{n} b_i\nabla\sigma_i(x),$$

$$\nabla^2\alpha(x) = \sum_{i=0}^{n} a_i\nabla^2\sigma_i(x) + \sum_{k,l} a_{kl}\nabla^2\sigma_{kl}(x), \quad \nabla^2\beta(x) = \sum_{i=0}^{n} b_i\nabla^2\sigma_i(x) + \sum_{k,l} b_{kl}\nabla^2\sigma_{kl}(x).$$

As basis for T_xX and $T_x^*X \otimes L_x^{2d}$ (resp. $T_x\mathbb{R}X$ and $\mathbb{R}(T_x^*X \otimes L_x^{2d})$) we choose $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$ and $(dx_1 \otimes e^{2d}, ..., dx_n \otimes e^{2d})$ respectively. We choose $(\sigma_i, 0)$ and $(0, \sigma_i)$, $i = 0, ..., n$, as a basis of a complement of $ker J_x^1 \times ker J_x^1$. Thanks to Lemma 2.7, this basis is asymptotically orthonormal for the L^2-Hermitian product $\langle , \rangle$. By definition it is an orthonormal basis for the scalar product $(\pi_2^{-1})_*\langle , \rangle$ restricted to H_2 . Then we obtain, using Lemma 2.6,

$$\langle A(\sigma_0, 0), dx_j \otimes e^{2d}\rangle = b_j\sqrt{\pi}\delta_L\sqrt{d}^{2n+1} + O(\sqrt{d}^{2n});$$

$$\langle A(\sigma_i, 0), dx_j \otimes e^{2d}\rangle = -b_0\sqrt{\pi}\delta_L\sqrt{d}^{2n+1}\delta_{ij} + O(\sqrt{d}^{2n}) \quad \text{for} \quad i = 1, ...n;$$

$$\langle A(0, \sigma_0), dx_j \otimes e^{2d}\rangle = -a_j\sqrt{\pi}\delta_L\sqrt{d}^{2n+1} + O(\sqrt{d}^{2n});$$

$$\langle A(0, \sigma_i), dx_j \otimes e^{2d}\rangle = a_0\sqrt{\pi}\delta_L\sqrt{d}^{2n+1}\delta_{ij} + O(\sqrt{d}^{2n}) \quad \text{for} \quad i = 1, ...n;$$

$$\left\langle B(\frac{\partial}{\partial x_i}), dx_j \otimes e^{2d}\right\rangle = (a_0b_{ij} - b_0a_{ij})\pi\delta_L\sqrt{d}^{2n+2} + O(\sqrt{d}^{2n+1}) \quad \text{for} \quad i \neq j;$$

$$\left\langle B(\frac{\partial}{\partial x_k}), dx_k \otimes e^{2d} \right\rangle = \sqrt{2}(a_0 b_{kk} - b_0 a_{kk})\pi\delta_L \sqrt{d}^{2n+2} + O(\sqrt{d}^{2n+1}).$$

where the Hermitian (resp. scalar) product on $T^*_x X \otimes L^{2d}_x$
(resp. $\mathbb{R}(T^*_x X \otimes L^{2d}_x)$) is induced by h.
What we have just computed are the coefficients of the matrices of A and B with respect to our choice of basis and with respect to the scalar product $(\pi_2^{-1})_* \langle , \rangle$. We recall that B is a square matrix and that $\mathrm{Jac}_N(A) = \sqrt{\mathrm{Jac}(AA^*)}$.
More precisely, as $d \to \infty$, A is equivalent to the following matrix:

$$\sqrt{\pi}\delta_L \sqrt{d}^{2n+1} \begin{bmatrix} b_1 & -b_0 & 0 & \dots & 0 & -a_1 & a_0 & 0 & \dots & 0 \\ b_2 & 0 & -b_0 & \dots & 0 & -a_2 & 0 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_n & 0 & 0 & \dots & -b_0 & -a_n & 0 & 0 & \dots & a_0 \end{bmatrix}$$

and B to the following one:

$$\pi\delta_L \sqrt{d}^{2n+2} \begin{bmatrix} \sqrt{2}(a_0 b_{11} - b_0 a_{11}) & a_0 b_{12} - b_0 a_{12} & \dots & a_0 b_{1n} - b_0 a_{1n} \\ a_0 b_{21} - b_0 a_{21} & \sqrt{2}(a_0 b_{22} - b_0 a_{22}) & \dots & a_0 b_{2n} - b_0 a_{2n} \\ \dots & \dots & \dots & \dots \\ a_0 b_{n1} - b_0 a_{n1} & a_0 b_{n2} - b_0 a_{n2} & \dots & \sqrt{2}(a_0 b_{nn} - b_0 a_{nn}) \end{bmatrix}$$

A direct computation shows us that AA^* is the matrix

$$(\pi\delta_L^2 d^{2n+1})((a_i \bar{a}_j + b_i \bar{b}_j)E_{ij} + (\mid a_0 \mid^2 + \mid b_0 \mid^2)Id + O(\frac{1}{d}).$$

The results follows. □

The last proposition implies

$$(\pi_2^{-1})_* \frac{1}{\mathrm{Jac}_N(\pi_X)} = (\pi d)^n (\frac{\mathrm{Jac}_{\mathbb{R}}((a_0 b_{ij} - b_0 a_{ij})\tilde{E}_{ij})}{\det((a_i \bar{a}_j + b_i \bar{b}_j)E_{ij} + (\mid a_0 \mid^2 + \mid \ b_0 \mid^2)Id)} + O(\frac{1}{\sqrt{d}}))$$

$$(\pi_2^{-1})_* \frac{1}{\mathrm{Jac}_N(\pi_{\mathbb{R}X})} = (\sqrt{\pi d}^n) \, (\, \frac{\det((a_0 b_{ij} - b_0 a_{ij})\tilde{E}_{ij})}{\sqrt{\det\left((a_i a_j + b_i b_j)E_{ij} + (a_0^2 + b_0^2)Id\right)}} + O(\frac{1}{\sqrt{d}}))$$

We want to integrate this quantity over $\pi_H(\pi_X^{-1}(x))$. We recall that the measure $d\mu_{|\pi_{\mathbb{R}H}(\pi^{-1}_{\mathbb{R}X}(x))}$ is the following one: first we restrict the scalar product $\langle , \rangle$ on $\mathbb{R}H^0(X; L^d)^2$ to $\pi_{\mathbb{R}H}(\pi^{-1}_{\mathbb{R}X}(x))$, that is a codimension n submanifold, then

we consider the Riemannian measure associated to this metric, and finally we multiply it by the factor $\frac{1}{\pi^{N_d}} e^{-\|\alpha\|^2 - \|\beta\|^2}$, where $N_d = \dim H^0(X; L^d)$. Then (8) is equal to

$$\begin{aligned}
&\int_{\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} \frac{|\operatorname{Jac}(B)|}{|\operatorname{Jac}_N(A)|} d\mu \,|_{\pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} = \\
&\quad = \int_{K_2^{\perp} \cap \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x)) \oplus K_2} \frac{|\operatorname{Jac}(B)|}{|\operatorname{Jac}_N(A)|} d\mu \,|_{\pi_H(\pi_{\mathbb{R}X}^{-1}(x))} = \\
&\quad = \int_{K_2^{\perp} \cap \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} \frac{|Jac(B)|}{|\operatorname{Jac}_N(A)|} d\mu \,|_{K_2^{\perp} \cap \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))} = \\
&\quad = \int_Q (\pi_2^{-1})_* \frac{|\operatorname{Jac}(B)|}{|\operatorname{Jac}_N(A)|} (\pi_{2*} d\mu \,|_{K_2^{\perp} \cap \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))}). \quad (9)
\end{aligned}$$

Thanks to the Proposition 2.7, the pushforward measure $(\pi_2)_*(\mu_{|K_2^{\perp}})$ on H_2 coincides with the Gaussian measure associated to the orthonormal basis $\{(\sigma_i, 0), (\sigma_{kl}, 0), (0, \sigma_i), (0, \sigma_{kl})\}$ up to a $O(\frac{1}{\sqrt{d}})$ term. As a consequence we have that $(\pi_{2*} d\mu \,|_{K_2^{\perp} \cap \pi_{\mathbb{R}H}(\pi_{\mathbb{R}X}^{-1}(x))})$ is equal to $\frac{e^{-\sum_i a_i^2 - \sum_i b_i^2 - \sum_{i,j} a_{ij}^2 - \sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q$ up to a $O(\frac{1}{\sqrt{d}})$ term. We then have that (9) is equal to

$$\begin{aligned}
&\int_Q (\pi_2^{-1})_* \frac{|\operatorname{Jac}(B)|}{|\operatorname{Jac}_N(A)|} d\mu_Q + O(\frac{1}{\sqrt{d}}) = \\
&= \int_{\substack{a_i, b_i, a_{ij}, b_{ij} \\ a_0 b_i - b_0 a_i = 0}} (\sqrt{\pi d}^n)(\frac{|\det((a_0 b_{ij} - b_0 a_{ij}) \tilde{E}_{ij})|}{\sqrt{\det((a_i a_j + b_i b_j) E_{ij} + (a_0^2 + b_0^2) Id)}} d\mu_Q + O(\frac{1}{\sqrt{d}}))
\end{aligned} \quad (10)$$

where $d\mu_Q = \frac{e^{-\sum_i a_i^2 - \sum_i b_i^2 - \sum_{i,j} a_{ij}^2 - \sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q$.
Putting (10) in (3), we obtain Proposition 3.4.

3.3 Computation of the universal constant

The purpose of this section is the explicit computation of the function $R_d(x)$ that appears in Proposition 3.4. We use the notation of Section 3.2.

To understand $R_d(x)$, we have to compute

$$\sqrt{\pi d}^n \int_Q \frac{|\det((a_0 b_{ij} - b_0 a_{ij}) \tilde{E}_{ij})|}{\sqrt{\det((a_i a_j + b_i b_j) E_{ij} + (a_0^2 + b_0^2) Id)}} \frac{e^{-\sum_i a_i^2 - \sum_i b_i^2 - \sum_{i,j} a_{ij}^2 - \sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q.$$

The main result of this section is the following computation:

Proposition 3.7. *Let Q be as in Proposition 3.4. Then*

$$\sqrt{\pi d}^n \int_Q \frac{\mid \det((a_0 b_{ij} - b_0 a_{ij})\tilde{E}_{ij}) \mid}{\sqrt{det((a_i a_j + b_i b_j)E_{ij} + (a_0^2 + b_0^2)Id)}} \frac{e^{-\sum_i a_i^2 - \sum_i b_i^2 - \sum_{i,j} a_{ij}^2 - \sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q$$

is equal to

$$\begin{cases} \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \frac{\pi}{2} \sqrt{d}^n & \text{if } n \text{ is odd} \\ \frac{n!!}{(n-1)!!} e_{\mathbb{R}}(n) \sqrt{d}^n & \text{if } n \text{ is even.} \end{cases} \tag{11}$$

where $\tilde{E}_{ij}$ and E_{ij} are the matrices of Definition 2.1 and $e_{\mathbb{R}}(n)$ is the expected value of the determinant of (the absolute value of) the real symmetric matrices.

We recall that $Q \subset \mathbb{R}^{2(n+1)+n(n+1)}$ is the product of the intersection of quadrics:

$$\tilde{Q} = \{(a_0, b_0, ..., a_n, ...b_n) \in \mathbb{R}^{2(n+1)} \mid a_0 b_i - a_i b_0 = 0 \quad \forall i = 1, ..., n\}$$

with $\mathbb{R}^{n(n+1)}$ of coordinates a_{ij} and b_{ij} for $1 \leq i \leq j \leq n$. We consider the parametrization $\psi : \mathbb{R}^{(n+2)} \to \tilde{Q}$ defined by

$$\psi(a, b, t_1, ..., t_n) = (a, b, at_1, bt_1, ..., at_n, bt_n).$$

Lemma 3.8. *We have $Jac(\psi) = \sqrt{1 + \sum_i t_i^2} \sqrt{(a^2 + b^2)}^n$.*

Proof. A computation gives us

$$\text{Jac}\psi\text{Jac}\psi^t = \det \begin{bmatrix} 1 + \sum_{i=1}^n t_i^2 & 0 & t_1 a & t_2 a & \dots & \dots & t_n a \\ 0 & 1 + \sum_{i=1}^n t_i^2 & t_1 b & t_2 b & \dots & \dots & t_n b \\ t_1 a & t_1 b & a^2 + b^2 & 0 & \dots & \dots & 0 \\ t_2 a & t_2 b & 0 & a^2 + b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_n a & t_n b & 0 & 0 & \dots & \dots & a^2 + b^2 \end{bmatrix}$$

We develop the last line and we obtain

$$(a^2+b^2) \det \begin{bmatrix} 1 + \sum_{i=1}^n t_i^2 & 0 & t_1 a & t_2 a & \dots & \dots & t_{n-1} a \\ 0 & 1 + \sum_{i=1}^n t_i^2 & t_1 b & t_2 b & \dots & \dots & t_{n-1} b \\ t_1 a & t_1 b & a^2 + b^2 & 0 & \dots & \dots & 0 \\ t_2 a & t_2 b & 0 & a^2 + b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n-1} a & t_{n-1} b & 0 & 0 & \dots & \dots & a^2 + b^2 \end{bmatrix}$$

$$+(-1)^n t_n b \det \begin{bmatrix} 1+\sum_{i=1}^n t_i^2 & t_1 a & t_2 a & \dots & \dots & t_n a \\ 0 & t_1 b & t_2 b & \dots & \dots & t_n b \\ t_1 a & a^2+b^2 & 0 & \dots & \dots & 0 \\ t_2 a & 0 & a^2+b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ t_{n-1} a & 0 & 0 & \dots & a^2+b^2 & 0 \end{bmatrix}$$

$$+(-1)^{n+1} t_n a \det \begin{bmatrix} 0 & t_1 a & t_2 a & \dots & \dots & t_n a \\ 1+\sum_{i=1}^n t_i^2 & t_1 b & t_2 b & \dots & \dots & t_n b \\ t_1 b & a^2+b^2 & 0 & \dots & \dots & 0 \\ t_2 b & 0 & a^2+b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ t_{n-1} b & 0 & 0 & \dots & a^2+b^2 & 0 \end{bmatrix}$$

For the second matrix we have:

$$\det \begin{bmatrix} 1+\sum_{i=1}^n t_i^2 & t_1 a & t_2 a & \dots & \dots & t_n a \\ 0 & t_1 b & t_2 b & \dots & \dots & t_n b \\ t_1 a & a^2+b^2 & 0 & \dots & \dots & 0 \\ t_2 a & 0 & a^2+b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ t_{n-1} a & 0 & 0 & \dots & a^2+b^2 & 0 \end{bmatrix}$$

$$= (1+\sum_{i=1}^n t_i^2) \det \begin{bmatrix} t_1 b & t_2 b & \dots & \dots & \dots & t_n b \\ a^2+b^2 & 0 & \dots & \dots & 0 & 0 \\ 0 & a^2+b^2 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a^2+b^2 & 0 & 0 \\ 0 & \dots & \dots & 0 & a^2+b^2 & 0 \end{bmatrix}$$

$$= (-1)^{n+1}(1+\sum_{i=1}^n t_i^2) t_n b (a^2+b^2)^{n-1}.$$

where the first equality is obtained by developping the first column and remarking that, in the development, each time we clear the i-th line, the $(i-1)$-th column and the last column are linearly equivalent.

Similarly,

$$\det\begin{bmatrix} 0 & t_1a & t_2a & \dots & \dots & t_na \\ 1+\sum_{i=1}^n t_i^2 & t_1b & t_2b & \dots & \dots & t_nb \\ t_1b & a^2+b^2 & 0 & \dots & \dots & 0 \\ t_2b & 0 & a^2+b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ t_{n-1}b & 0 & 0 & \dots & a^2+b^2 & 0 \end{bmatrix}$$

$$= (-1)^n(1+\sum_{i=1}^n t_i^2)t_na(a^2+b^2)^{n-1}.$$

Then we have

$$\mathrm{Jac}\psi\mathrm{Jac}\psi^t = (a^2+b^2)\det\begin{bmatrix} 1+\sum_{i=1}^n t_i^2 & 0 & t_1a & t_2a & \dots & \dots & t_{n-1}a \\ 0 & 1+\sum_{i=1}^n t_i^2 & t_1b & t_2b & \dots & \dots & t_{n-1}b \\ t_1a & t_1b & a^2+b^2 & 0 & \dots & \dots & 0 \\ t_2a & t_2b & 0 & a^2+b^2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ t_{n-1}a & t_{n-1}b & 0 & 0 & \dots & \dots & a^2+b^2 \end{bmatrix}$$

$$-(1+\sum_{i=1}^n t_i^2)t_n^2(a^2+b^2)^n.$$

Continuing to develop in the same way, we obtain by induction

$$\mathrm{Jac}\psi\mathrm{Jac}\psi^t = (1+\sum_{i=1}^n t_i^2)^2(a^2+b^2)^n-(1+\sum_{i=1}^n t_i^2)\sum_{i=1}^n t_i^2(a^2+b^2)^n = (1+\sum_{i=1}^n t_i^2)(a^2+b^2)^n.$$

Passing to the square root we obtain the result. □

Remark 3.9. *In the following we will not write the symbols $\tilde{E}_{ij}$ E_{ij} to simplify the notation.*

After this change of variables, we have:

$$\sqrt{\pi d}^n \int_Q \frac{|\det(a_0b_{ij}-b_0a_{ij})|}{\sqrt{det((a_ia_j+b_ib_j)+(a_0^2+b_0^2)Id)}} \frac{e^{-\sum_i a_i^2-\sum_i b_i^2-\sum_{i,j} a_{ij}^2-\sum_{i,j} b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dvol_Q =$$

$$= \sqrt{\pi d}^n \int_{a,b,a_{ij},b_{ij},t_i\in\mathbb{R}} \frac{|\det(ab_{ij}-ba_{ij})|}{\sqrt{\det((a^2+b^2)((t_it_j)+Id))}} \times$$

$$\times \frac{e^{-(1+\sum_i t_i^2)(a^2+b^2)-\sum_{i,j}(a_{ij}^2+b_{ij}^2)}}{\pi^{n+1+\frac{n(n+1)}{2}}} \sqrt{(1+\sum_i t_i^2)} \sqrt{(a^2+b^2)}^n dadbda_{ij}db_{ij}dt_i$$

Now $\det((a^2+b^2)((t_it_j)+Id)) = (1+\sum_i t_i^2)(a^2+b^2)^n$ so we obtain

$$\sqrt{\pi d}^n \int_{a,b,a_{ij},b_{ij},t_i\in\mathbb{R}} |\det(ab_{ij}-ba_{ij}))| \frac{e^{-(1+\sum_i t_i^2)(a^2+b^2)-\sum_{i,j}(a_{ij}^2+b_{ij}^2)}}{\pi^{n+1+\frac{n(n+1)}{2}}} dadbda_{ij}db_{ij}dt_i.$$

It is more practical to see (a,b) as a complex number $c\in\mathbb{C}$ and also (a_{ij},b_{ij}) as $e_{ij}\in\mathbb{C}$. With a slight abuse of notation, we denote dc and de_{ij} instead of $\frac{-1}{2i}dcd\bar{c}$ and $\frac{-1}{2i}de_{ij}d\bar{e}_{ij}$. Then we have

$$\sqrt{\pi d}^n \int_{c\in\mathbb{C},e_{ij}\in\mathbb{C},t_i\in\mathbb{R}} |\det(\mathrm{Im}\,(\bar{c}e_{ij}))| \frac{e^{-(1+\sum_i t_i^2)|c|^2-\sum_{i,j}|e_{ij}|^2}}{\pi^{n+1+\frac{n(n+1)}{2}}} dcde_{ij}dt_i.$$

Now, set $\tilde{c} = (\sqrt{1+\sum_i t_i^2})c$ and then $\tilde{c} = re^{i\vartheta}$. We obtain

$$\sqrt{\pi d}^n \int_{\tilde{c}\in\mathbb{C},e_{ij}\in\mathbb{C}} |\det(\mathrm{Im}\,(\bar{\tilde{c}}e_{ij}))| \frac{e^{-|\tilde{c}|^2-\sum_{i,j}|e_{ij}|^2}}{\pi^{1+\frac{n(n+1)}{2}}} d\tilde{c}de_{ij}$$
$$\times \int_{t_i\in\mathbb{R}} \frac{1}{\pi^n\sqrt{1+\sum_i t_i^2}^{n+1}} dt_i =$$
$$\sqrt{\pi d}^n \int_{\vartheta\in(0,2\pi],e_{ij}\in\mathbb{C}} |\det(\mathrm{Im}\,(e^{-i\vartheta}e_{ij}))| \frac{e^{-\sum_{i,j}|e_{ij}|^2}}{\pi^{\frac{n(n+1)}{2}}} de_{ij}d\vartheta$$
$$\times \int_{r=0}^{+\infty} \frac{r^{n+1}e^{-r^2}}{\pi} dr \times \int_{t_i\in\mathbb{R}} \frac{1}{\pi^n\sqrt{1+\sum_i t_i^2}^{n+1}} dt_i.$$

so we have to compute these three integrals. For the first term, we have

$$\int_{\vartheta\in(0,2\pi],e_{ij}\in\mathbb{C}} |\det(\mathrm{Im}\,(e^{-i\vartheta}c_{ij}))| \frac{e^{-\sum_{i,j}|e_{ij}|^2}}{\pi^{\frac{n(n+1)}{2}}} de_{ij}d\vartheta =$$
$$= 2\pi \int_{e_{ij}\in\mathbb{C}} |\det(\mathrm{Im}\,e_{ij})| \frac{e^{-\sum_{i,j}|e_{ij}|^2}}{\pi^{\frac{n(n+1)}{2}}} de_{ij} =$$
$$= 2\pi \int_{b_{ij}\in\mathbb{R}} |\det(b_{ij})| \frac{e^{-\sum_{i,j} b_{ij}^2}}{\sqrt{\pi}^{\frac{n(n+1)}{2}}} db_{ij} = 2\pi e_{\mathbb{R}}(n).$$

Here, $e_{\mathbb{R}}(n) = \int_{B \in Sym(n,\mathbb{R})} \mid \det B \mid d\mu_{\mathbb{R}}(B)$ (see the end of Section 2.1). For the explicit value of $e_{\mathbb{R}}(n)$, see [5, Section 2].
For the second term, we consider the change of variable $r^2 = \rho$ and we obtain

$$\int_{r=0}^{+\infty} \frac{r^{n+1}e^{-r^2}}{\pi} dr = \frac{1}{2} \int_{\rho=0}^{+\infty} \frac{\rho^{\frac{n}{2}} e^{-\rho}}{\pi} d\rho = \frac{\Gamma(\frac{n}{2}+1)}{2\pi}$$

where Γ is the Gamma function. For the third term we use spherical coordinates and we obtain

$$\int_{t_i \in \mathbb{R}} \frac{1}{\sqrt{1+\sum_i t_i^2}^{n+1}} dt_i = \mathrm{Vol}(S^{n-1}) \int_{t=0}^{+\infty} \frac{t^{n-1}}{\sqrt{(1+t^2)}^{n+1}} dt.$$

where

$$\mathrm{Vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

is the volume of the $(n-1)$-dimensional sphere. For

$$\int_{t=0}^{+\infty} \frac{t^{n-1}}{(\sqrt{1+t^2})^{n+1}} dt = \int_{t=0}^{+\infty} \sqrt{\frac{t^2}{(1+t^2)}}^{n-1} \frac{1}{1+t^2} dt$$

we change $\frac{t^2}{1+t^2} = 1 - u^2$ and we obtain

$$\int_0^1 \sqrt{1-u^2}^{n-2} du.$$

Finally, set $u = sin\theta$ and we have

$$\int_0^{\pi/2} cos^{n-1}\theta d\theta.$$

The formula

$$\int cos^{n-1}(\theta) du = \frac{sin(\theta)cos^{n-2}(\theta)}{n-1} + \frac{n-2}{n-1} \int cos^{n-3}(\theta) du$$

tells us that $\int_0^{\pi/2} cos^{n-1}\theta d\theta$ is equal to

$$\frac{(n-2)!!}{(n-1)!!}$$

if n is even and it is equal to

$$\frac{(n-2)!!}{(n-1)!!}\frac{\pi}{2}$$

if n is odd. Putting together all these values and using that

$$\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})}=\frac{n}{2},$$

we obtain Proposition 3.7.

3.4 End of the proofs

From Proposition 3.4 we have

$$\mathbb{E}[\mathbb{R}\nu_{\alpha\beta}](\varphi)=\int_{\mathbb{R}X}\varphi(x)(\int_Q\frac{\mid \mathrm{Jac}_N(A)\mid}{\mid \mathrm{Jac}(B)\mid}d\mu(a_i,b_i,a_{kl},b_{kl})+O(\frac{1}{d}))dvol_h=$$

$$=\int_{\substack{a_i,b_i,a_{ij},b_{ij}\\ a_0b_i-b_0a_i=0}}(\sqrt{\pi d}^n)\,(\,\frac{\det(a_0b_{ij}-b_0a_{ij})}{\sqrt{det\,((a_ia_j+b_ib_j)+(a_0^2+b_0^2)Id)}}+O(\frac{1}{\sqrt{d}})d\mu_Q)dvol_h$$

where $d\mu_Q=e^{-\sum_i a_i^2-\sum_i b_i^2-\sum_{i,j}a_{ij}^2-\sum_{i,j}b_{ij}^2}dvol_Q$.

In Section 3.3 we proved that (see Proposition 3.7 eq. 11)

$$\sqrt{\pi d}^n\int_Q\frac{\mid \det(a_0b_{ij}-b_0a_{ij})\mid}{\sqrt{\det((a_ia_j+b_ib_j)+(a_0^2+b_0^2)Id)}}\frac{e^{-\sum_i a_i^2-\sum_i b_i^2-\sum_{i,j}a_{ij}^2-\sum_{i,j}b_{ij}^2}}{\pi^{n+1+\frac{n(n+1)}{2}}}dvol_Q$$

is equal to

$$\begin{cases}\frac{n!!}{(n-1)!!}e_{\mathbb{R}}(n)\frac{\pi}{2}\sqrt{d}^n & \text{if n is odd}\\ \frac{n!!}{(n-1)!!}e_{\mathbb{R}}(n)\sqrt{d}^n & \text{if n is even.}\end{cases}$$

Theorem 1.4 is obtained by dividing by $\sqrt{d}^n$ and by passing to the limit.
Theorem 1.2 is Theorem 1.4 for $\varphi=1$.
The proof of Theorem 1.5 is similar to the proof of Theorem 1.4. For the computation of the universal constant we put $\varphi=1$ and use Proposition 2.3.

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