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# Undecidable problems for modal definability

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## Abstract

The core of our article is the computability of the problem of deciding the modal definability of first-order sentences with respect to classes of frames. It gives a new proof of Chagrova's Theorem telling that, with respect to the class of all frames, the problem of deciding the modal definability of first-order sentences is undecidable. It also gives the proofs of new variants of Chagrova's Theorem.

*Keywords:* Modal logic, first-order logic, modal definability, Chagrova's Theorem, computability.

## 1 Introduction

Modal formulas can be used to define classes of frames. In many cases, they define classes of frames that cannot be defined by first-order sentences. As well, there exists sentences, such as the ones describing the frame properties of irreflexivity and anti-symmetry, that no modal formula can define. Hence, questions about the correspondences between modal formulas and sentences arise: which sentences are modally definable, which modal formulas are first-order definable. The study of these correspondences was begun in the 1970s. Now, it has a comprehensive literature. See [4, 15, 25] or [5, Chapter 3] for an introductory discussion about similar correspondence results. The characterization of Goldblatt and Thomason [15] tells us that a sentence is modally definable iff it is preserved under taking disjoint union, generated subframes and bounded morphic images and it reflects ultrafilter extensions. Although it can be used to give algorithmic and graph-theoretic criteria of modal definability for restricted families of sentences [19], Goldblatt–Thomason Theorem cannot be used to decide the modal definability of sentences: as shown by Chagrova [10], the problem of deciding the modal definability of sentences is undecidable. See [6, 7, 11] or [9, Chapter 17] for an introductory discussion about similar undecidability results. Concerning modal definability, the above lines seem to imply that the problem of deciding the modal definability of sentences is the only problem of interest. In our opinion, when we are using a modal language, it is also interesting to consider the problem of deciding the modal definability of sentences with respect to some specific classes of frames (the class of all reflexive frames, the class of all strict partial orders, etc.). The core of our article is the modal definability of sentences with respect to classes of frames.

Surely, the most interesting contribution of Chagrova [10] is the undecidability of the first-order definability of intuitionistic formulas. Nevertheless, in order to simplify the presentation of our article, her result about the undecidability of the modal definability of sentences will also be called 'Chagrova's Theorem'. The proof of Chagrova's Theorem—the undecidability of the modal definability of sentences—is based on the undecidability of a variant of the halting problem

concerning Minsky machines. It cannot be easily repeated for demonstrating that, with respect to different classes of frames, the problem of deciding the modal definability of sentences is undecidable too. In our article, by means of simple frame constructions, we give a new proof of Chagrova's Theorem (Corollary 1) and we repeat our proof for demonstrating that, with respect to different classes of frames, the problem of deciding the modal definability of sentences is undecidable too (Corollaries 2–14). In some sense, our method might be called direct. Using the fact that  $\perp$  is not a modal definition of a sentence  $D$  with respect to a class  $\mathcal{C}$  of frames iff there exists a frame  $\mathcal{F}$  in  $\mathcal{C}$  such that  $\mathcal{F} \models D$ , we show how to reduce the problem of deciding the validity of sentences in  $\mathcal{C}$  to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  (Theorem 1). This reduction constitutes the key result of our method. We assume that the reader has enough experience in modal logic and first-order logic to decide on the notions that we do not define in our article. From now on, a frame will be a structure of the form  $\mathcal{F} = (W, R_1, \dots, R_n)$  where  $W$  is a non-empty set of worlds and  $R_1, \dots, R_n$  are binary relations on  $W$ . In Sections 2–5 and 6.1,  $n$  will be equal to 1 whereas in Section 6.2,  $n$  will be equal to 2. We shall say that a subset  $U$  of  $W$  is  $R_i$ -closed if for all states  $s, t$  in  $\mathcal{F}$ , if  $s$  is in  $U$  and  $sR_it$  then  $t$  is in  $U$ . If  $R_i$  is an equivalence relation, an equivalence class modulo  $R_i$  is said to be degenerate if it is a singleton. A list  $s_1, \dots, s_m$  of worlds in  $\mathcal{F}$  will sometimes be written  $\bar{s}$ . We leave it to the context to determine the length of such list.

## 2 Modal language and truth

### 2.1 Modal language

Let us consider a countable set of propositional variables (with typical members denoted  $p, q, \dots$ ). The set of all modal formulas (denoted  $\phi, \psi, \dots$ ) is inductively defined as follows:

- $\phi ::= p \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$ .

We define the other Boolean constructs as usual. The modal formula  $\Diamond\phi$  is obtained as the well-known abbreviation:  $\Diamond\phi ::= \neg\Box\neg\phi$ . We adopt the standard rules for omission of the parentheses.

### 2.2 Truth

A model based on a frame  $\mathcal{F} = (W, R)$  is a triple  $\mathcal{M} = (W, R, V)$  where  $V$  is a function assigning to each propositional variable  $p$  a subset  $V(p)$  of  $W$ . Given a model  $\mathcal{M} = (W, R, V)$ , the satisfiability of a modal formula  $\phi$  at a world  $s$  in  $\mathcal{M}$ , in symbols  $\mathcal{M}, s \models \phi$ , is inductively defined as follows:

- $\mathcal{M}, s \models p$  iff  $s \in V(p)$ ,
- $\mathcal{M}, s \not\models \perp$ ,
- $\mathcal{M}, s \models \neg\phi$  iff  $\mathcal{M}, s \not\models \phi$ ,
- $\mathcal{M}, s \models \phi \vee \psi$  iff either  $\mathcal{M}, s \models \phi$ , or  $\mathcal{M}, s \models \psi$ ,
- $\mathcal{M}, s \models \Box\phi$  iff for all worlds  $t$  in  $\mathcal{M}$ , if  $sRt$  then  $\mathcal{M}, t \models \phi$ .

Obviously,  $\mathcal{M}, s \models \Diamond\phi$  iff there exists a world  $t$  in  $\mathcal{M}$  such that  $sRt$  and  $\mathcal{M}, t \models \phi$ . We shall say that a modal formula  $\phi$  is true in a model  $\mathcal{M}$ , in symbols  $\mathcal{M} \models \phi$ , if  $\phi$  is satisfied at all worlds in  $\mathcal{M}$ . A modal formula  $\phi$  is said to be valid in a frame  $\mathcal{F}$ , in symbols  $\mathcal{F} \models \phi$ , if  $\phi$  is true in all models based on  $\mathcal{F}$ . We shall say that a modal formula  $\phi$  is valid in a class  $\mathcal{C}$  of frames, in symbols  $\mathcal{C} \models \phi$ , if  $\phi$  is valid in all frames in  $\mathcal{C}$ . A frame  $\mathcal{F}$  is said to be weaker than a frame  $\mathcal{F}'$ , in symbols  $\mathcal{F} \leq \mathcal{F}'$ , if for all modal formulas  $\phi$ , if  $\mathcal{F} \models \phi$  then  $\mathcal{F}' \models \phi$ .

### 2.3 Generated subframes and bounded morphisms

Let  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  be frames. We shall say that  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$  if the following conditions are satisfied:

- $\mathcal{F}'$  is a subframe of  $\mathcal{F}$ ,
- for all worlds  $s'$  in  $\mathcal{F}'$  and for all worlds  $t$  in  $\mathcal{F}$ , if  $s'Rt$  then  $t$  is in  $\mathcal{F}'$ .

Generated subframes give rise to the following

LEMMA 1 (Generated Subframe Lemma)

Let  $\mathcal{F}, \mathcal{F}'$  be frames. If  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$  then  $\mathcal{F} \preceq \mathcal{F}'$ .

PROOF. See [5, Theorem 3.14 (ii)]. ■

Let  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  be frames. A function  $f$  assigning to each world  $s$  in  $\mathcal{F}$  a world  $f(s)$  in  $\mathcal{F}'$  is called a bounded morphism from  $\mathcal{F}$  to  $\mathcal{F}'$  if the following conditions are satisfied:

- for all worlds  $s, t$  in  $\mathcal{F}$ , if  $sRt$  then  $f(s)R'f(t)$ ; and
- for all worlds  $s$  in  $\mathcal{F}$  and for all worlds  $t'$  in  $\mathcal{F}'$ , if  $f(s)R't'$  then there exists a world  $t$  in  $\mathcal{F}$  such that  $sRt$  and  $f(t) = t'$ .

$\mathcal{F}'$  is said to be a bounded morphic image of  $\mathcal{F}$  if there exists a surjective bounded morphism from  $\mathcal{F}$  to  $\mathcal{F}'$ . Bounded morphic images give rise to the following

LEMMA 2 (Bounded Morphism Lemma)

Let  $\mathcal{F}, \mathcal{F}'$  be frames. If  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$  then  $\mathcal{F} \preceq \mathcal{F}'$ .

PROOF. See [5, Theorem 3.14 (iii)]. ■

## 3 First-order language and truth

### 3.1 First-order language

Let us consider a countable set of individual variables (with typical members denoted  $x, y, \dots$ ). A list  $x_1, \dots, x_m$  of individual variables will sometimes be written  $\bar{x}$ . We leave it to the context to determine the length of such list. The set of all first-order formulas (denoted  $A, B, \dots$ ) is inductively defined as follows:

- $A ::= R_{\square}(x, y) \mid x = y \mid \neg A \mid (A \vee B) \mid \forall x A$ .

We define the other Boolean constructs as usual. The first-order formula  $\exists x A$  is obtained as the well-known abbreviation:  $\exists x A ::= \neg \forall x \neg A$ . We adopt the standard rules for omission of the parentheses. For all first-order formulas  $A$ , let  $\text{fiv}(A)$  be the set of all free individual variables occurring in  $A$ . When  $\bar{x}$  is a list of pairwise distinct individual variables, we write  $A(\bar{x})$  to denote a first-order formula  $A$  whose free individual variables belongs to  $\bar{x}$ . A first-order formula  $A$  is called a sentence if  $\text{fiv}(A) = \emptyset$ . The relativization of a first-order formula  $C$  with respect to a first-order formula  $A$  and an individual variable  $x$ , in symbols  $(C)_x^A$ , is inductively defined as follows:

- $(R_{\square}(y, z))_x^A$  is  $R_{\square}(y, z)$ ,
- $(y = z)_x^A$  is  $y = z$ ,
- $(\neg C)_x^A$  is  $\neg(C)_x^A$ ,
- $(C \vee D)_x^A$  is  $(C)_x^A \vee (D)_x^A$ ,

- $(\forall y C)_x^A$  is  $\forall y (A[x/y] \rightarrow (C)_x^A)$ .

In the above definition,  $A[x/y]$  denotes the first-order formula obtained from the first-order formula  $A$  by replacing every free occurrence of the individual variable  $x$  in  $A$  by the individual variable  $y$ . From now on, when we write  $(C)_x^A$ , we will always assume that the sets of individual variables occurring in  $A$  and  $C$  are disjoint. The reader may easily verify by induction on the first-order formula  $C$  that  $\text{fiv}((C)_x^A) \subseteq (\text{fiv}(A) \setminus \{x\}) \cup \text{fiv}(C)$ . Hence, if  $C$  is a sentence then  $\text{fiv}((C)_x^A) \subseteq \text{fiv}(A) \setminus \{x\}$ . Let  $\Gamma, \Delta$  be disjoint sets of sentences. We shall say that  $\Gamma$  and  $\Delta$  are recursively inseparable if there exists no recursive set  $\Lambda$  of sentences such that  $\Gamma \subseteq \Lambda$  and  $\Delta \cap \Lambda = \emptyset$ . In this case, remark that neither  $\Gamma$ , nor  $\Delta$  is recursive. Moreover, if  $\Gamma$  (respectively,  $\Delta$ ) is r.e. then  $\Gamma$ 's complement (respectively,  $\Delta$ 's complement) is co-r.e.-hard. See [24, Chapter 7] for an introductory discussion about disjoint pairs of sets.

### 3.2 Truth

Given a frame  $\mathcal{F} = (W, R)$ , the satisfiability of a first-order formula  $A(\bar{x})$  in  $\mathcal{F}$  with respect to a list  $\bar{s}$  of worlds in  $\mathcal{F}$ , in symbols  $\mathcal{F} \models A(\bar{x}) [\bar{s}]$ , is inductively defined as follows:

- $\mathcal{F} \models R_{\square}(x_i, x_j) [\bar{s}]$  iff  $s_i R s_j$ ,
- $\mathcal{F} \models x_i = x_j [\bar{s}]$  iff  $s_i = s_j$ ,
- $\mathcal{F} \models \neg A [\bar{s}]$  iff  $\mathcal{F} \not\models A [\bar{s}]$ ,
- $\mathcal{F} \models A \vee B [\bar{s}]$  iff either  $\mathcal{F} \models A [\bar{s}]$ , or  $\mathcal{F} \models B [\bar{s}]$ ,
- $\mathcal{F} \models \forall x A(\bar{x}, x) [\bar{s}]$  iff for all worlds  $s$  in  $\mathcal{F}$ ,  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ .

Obviously,  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$  iff there exists a world  $s$  in  $\mathcal{F}$  such that  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ . We shall say that a first-order formula  $A(\bar{x})$  is valid in a frame  $\mathcal{F}$ , in symbols  $\mathcal{F} \models A(\bar{x})$ , if  $A(\bar{x})$  is satisfied in  $\mathcal{F}$  with respect to all lists  $\bar{s}$  of worlds in  $\mathcal{F}$ . A first-order formula  $A$  is said to be valid in a class  $\mathcal{C}$  of frames, in symbols  $\mathcal{C} \models A$ , if  $A$  is valid in all frames in  $\mathcal{C}$ .

### 3.3 Relativizations

Let  $\mathcal{F}, \mathcal{F}'$  be frames. We shall say that  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  if there exists a first-order formula  $A(\bar{x}, x)$  and there exists a list  $\bar{s}$  of worlds in  $\mathcal{F}$  such that  $\mathcal{F}'$  is the restriction of  $\mathcal{F}$  to the set of all worlds  $s$  in  $\mathcal{F}$  such that  $\mathcal{F} \models A(\bar{x}, x) [\bar{s}, s]$ . In this case,  $\mathcal{F}'$  is called the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ . Obviously,  $\mathcal{F}$  possesses a relativized reduct with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  iff  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$ . Relativized reducts give rise to the following.

LEMMA 3 (Relativization Theorem)

Let  $\mathcal{F}, \mathcal{F}'$  be frames,  $A(\bar{x}, x)$  be a first-order formula and  $\bar{s}$  be a list of worlds in  $\mathcal{F}$ . If  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  then for all first-order formulas  $C(\bar{y})$  and for all lists  $\bar{t}$  of worlds in  $\mathcal{F}'$ ,  $\mathcal{F} \models (C(\bar{y}))_x^{A(\bar{x}, x)} [\bar{s}, \bar{t}]$  iff  $\mathcal{F}' \models C(\bar{y}) [\bar{t}]$ .

PROOF. See [17, Theorem 5.1.1]. ■

## 4 Modal definability

Let  $\mathcal{C}$  be a class of frames. A sentence  $A$  is said to be modally definable with respect to  $\mathcal{C}$  if there exists a modal formula  $\phi$  such that for all frames  $\mathcal{F}$  in  $\mathcal{C}$ ,  $\mathcal{F} \models A$  iff  $\mathcal{F} \models \phi$ . In this case, we shall say

that  $\phi$  is a modal definition of  $A$  with respect to  $\mathcal{C}$ . Now, we relate the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  to the problem of deciding the validity of sentences in  $\mathcal{C}$ . In this respect, a special role is played by the concept of a stable class of frames.  $\mathcal{C}$  is said to be stable if there exists a first-order formula  $A(\bar{x}, x)$  and there exists a sentence  $B$  such that

- (a) for all frames  $\mathcal{F}$  in  $\mathcal{C}$ , for all lists  $\bar{s}$  of worlds in  $\mathcal{F}$  and for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  then  $\mathcal{F}'$  is in  $\mathcal{C}$ ;
- (b) for all frames  $\mathcal{F}_0$  in  $\mathcal{C}$ , there exists frames  $\mathcal{F}, \mathcal{F}'$  in  $\mathcal{C}$  and there exists a list  $\bar{s}$  of worlds in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F} \preceq \mathcal{F}'$ .

In this case,  $(A(\bar{x}, x), B)$  is called a witness of the stability of  $\mathcal{C}$ . The following theorem states that if  $\mathcal{C}$  is stable, then the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  is at least as difficult as the problem of deciding the validity of sentences in  $\mathcal{C}$ .

#### THEOREM 1

If  $\mathcal{C}$  is stable then the problem of deciding the validity of sentences in  $\mathcal{C}$  is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$ .

PROOF. Suppose  $\mathcal{C}$  is stable. Let  $(A(\bar{x}, x), B)$  be a witness of the stability of  $\mathcal{C}$ . Let  $C$  be a sentence. Let  $D$  be the sentence  $\exists \bar{x} (\exists x A(\bar{x}, x) \wedge \neg (C)_x^{A(\bar{x}, x)}) \wedge B$ , we demonstrate  $\mathcal{C} \models C$  iff  $D$  is modally definable with respect to  $\mathcal{C}$ .

Suppose  $\mathcal{C} \models C$ . For the sake of the contradiction, suppose  $D$  is not modally definable with respect to  $\mathcal{C}$ . Let  $\mathcal{F}$  be a frame in  $\mathcal{C}$  such that  $\mathcal{F} \models D$ . Such a frame exists, otherwise  $\perp$  would be a modal definition of  $D$  with respect to  $\mathcal{C}$ . Let  $\bar{s}$  be worlds in  $\mathcal{F}$  such that  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$  and  $\mathcal{F} \not\models (C)_x^{A(\bar{x}, x)} [\bar{s}]$ . Let  $\mathcal{F}'$  be the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ . Such a frame exists, otherwise  $\mathcal{F} \not\models \exists x A(\bar{x}, x) [\bar{s}]$ . Since  $\mathcal{F}$  is in  $\mathcal{C}$ , by (a),  $\mathcal{F}'$  is in  $\mathcal{C}$ . Since  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ , by Lemma 3,  $\mathcal{F} \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$  iff  $\mathcal{F}' \models C$ . Since  $\mathcal{F} \not\models (C)_x^{A(\bar{x}, x)} [\bar{s}]$ ,  $\mathcal{F}' \not\models C$ . Since  $\mathcal{F}'$  is in  $\mathcal{C}$ ,  $\mathcal{C} \not\models C$ : a contradiction.

Suppose  $D$  is modally definable with respect to  $\mathcal{C}$ . Let  $\phi$  be a modal definition of  $D$  with respect to  $\mathcal{C}$ . For the sake of the contradiction, suppose  $\mathcal{C} \not\models C$ . Let  $\mathcal{F}_0$  be a frame in  $\mathcal{C}$  such that  $\mathcal{F}_0 \not\models C$ . Let  $\mathcal{F}, \mathcal{F}'$  be frames in  $\mathcal{C}$  and let  $\bar{s}$  be a list of worlds in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F} \preceq \mathcal{F}'$ . Since  $\mathcal{F}_0$  is in  $\mathcal{C}$ , by (b), such frames and such a list of worlds exist. Since  $\mathcal{F}' \not\models B$ ,  $\mathcal{F}' \not\models D$ . Since  $\phi$  is a modal definition of  $D$  with respect to  $\mathcal{C}$ ,  $\mathcal{F}, \mathcal{F}'$  are in  $\mathcal{C}$  and  $\mathcal{F} \preceq \mathcal{F}'$ ,  $\mathcal{F} \not\models D$ . Hence, either  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x}, x) \wedge \neg (C)_x^{A(\bar{x}, x)})$  or  $\mathcal{F} \not\models B$ . Since  $\mathcal{F} \models B$ ,  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x}, x) \wedge \neg (C)_x^{A(\bar{x}, x)})$ . Since  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ , by Lemma 3,  $\mathcal{F} \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$  iff  $\mathcal{F}_0 \models C$ . Moreover,  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$ . Since  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x}, x) \wedge \neg (C)_x^{A(\bar{x}, x)})$ , either  $\mathcal{F} \not\models \exists x A(\bar{x}, x) [\bar{s}]$ , or  $\mathcal{F} \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$ . Since  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$ ,  $\mathcal{F} \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$ . Since  $\mathcal{F} \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$  iff  $\mathcal{F}_0 \models C$ ,  $\mathcal{F}_0 \models C$ : a contradiction. ■

This tight relationship between the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  and the problem of deciding the validity of sentences in  $\mathcal{C}$  constitutes the key result of our method. The only modal-related constraint in condition (b) being that  $\mathcal{F} \preceq \mathcal{F}'$ , the modal language and its semantics are inessential in the proof of Theorem 1. A single property is really needed: the modal language should contain or define a formula, such as  $\perp$ , that is valid in no  $\mathcal{C}$ -frame. In other respect, in order to achieve the constraint that  $\mathcal{F} \preceq \mathcal{F}'$ , let us note that there is no obligation to use the frame constructions of generated subframes and bounded morphic images considered in Section 2.3: any frame construction preserving frame-validity could be used as well. Now, we will use Theorem 1 in

order to investigate the computability of the problem of deciding the modal definability of sentences with respect to the class  $\mathcal{C}_{\text{all}}$  of all frames.

**THEOREM 2**  
 $\mathcal{C}_{\text{all}}$  is stable.

**PROOF.** Let  $A(x_1, x)$  be the first-order formula  $R_{\Box}(x_1, x)$ . Let  $B$  be the sentence  $\forall y \exists z R_{\Box}(z, y)$ . Obviously,  $\mathcal{C}_{\text{all}}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{all}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{all}}$  defined as follows:

- $W = W_0 \cup \{s_1, t_1\}$ ,
- $R$  is the least relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$ ,  $(t_1, s_1)$  and  $(t_1, t_1)$ ,
- $W' = W_0 \cup \{s_1\}$ ,
- $R'$  is the least relation on  $W'$  containing  $R_0$  and  $\{s_1\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ . ■

In the above proof of Theorem 2, we assume that  $s_1$  and  $t_1$  are distinct new elements, i.e.  $s_1 \neq t_1$ ,  $s_1 \notin W_0$  and  $t_1 \notin W_0$ . Analogous assumptions will be made in the proofs of analogous theorems below. Putting Theorems 1 and 2 together leads us to a new proof of Chagrova's Theorem.

**COROLLARY 1 (Chagrova's Theorem)**

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{\text{all}}$  is undecidable.

**PROOF.** By [18], the problem of deciding the validity of sentences in a first-order language with at least one non-logical symbol, which is a predicate of arity 2 or more, is reducible to the problem of deciding the validity of sentences in a first-order language with exactly one non-logical symbol, which is a predicate of arity 2. The conclusion follows by Theorems 1 and 2 and the well-known undecidability of the problem of deciding the validity of sentences in a first-order language. ■

In the proof of Theorem 2, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}$ ,  $\mathcal{F}'$  are finite too. This immediately gives us the following.

**THEOREM 3**  
The class  $\mathcal{C}_{\text{all}}^{\text{fin}}$  of all finite frames is stable.

Putting Theorems 1 and 3 together leads us to a proof of the following new result.

**COROLLARY 2**

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{\text{all}}^{\text{fin}}$  is co-r.e.-hard.

**PROOF.** By [29], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{all}} \models A\}$  and  $\{A: \mathcal{C}_{\text{all}}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{\text{all}}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{all}}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorems 1 and 3. ■

## 5 Stable classes of frames

The frame manipulation method we have used to prove Theorem 2 is flexible. It has many interesting variations. The following theorem explores this theme.

THEOREM 4

The following classes of frames are stable: (1) the class  $\mathcal{C}_{\text{ref}}$  of all reflexive frames; (2) the class  $\mathcal{C}_{\text{sym}}$  of all symmetric frames; (3) the class  $\mathcal{C}_{\text{tra}}$  of all transitive frames; (4) the class  $\mathcal{C}_{\text{ref,sym}}$  of all reflexive symmetric frames; (5) the class  $\mathcal{C}_{\text{ref,tra}}$  of all reflexive transitive frames.

PROOF. (1) Let  $A(x_1, x)$  be the first-order formula  $R_{\square}(x_1, x) \wedge x_1 \neq x$ . Let  $B$  be the sentence  $\forall y \exists z (R_{\square}(z, y) \wedge z \neq y)$ . Obviously,  $\mathcal{C}_{\text{ref}}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{ref}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{ref}}$  defined as follows:

- $W = W_0 \cup \{s_1, t_{11}, t_{12}\}$ ,
- $R$  is the least reflexive relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$ ,  $(t_{11}, s_1)$ ,  $(t_{12}, s_1)$ ,  $(t_{12}, t_{11})$  and  $(t_{11}, t_{12})$ ,
- $W' = W_0 \cup \{s_1\}$ ,
- $R'$  is the least reflexive relation on  $W'$  containing  $R_0$  and  $\{s_1\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ .

(2) Let  $A(x_1, x_2, x)$  be the first-order formula  $R_{\square}(x_1, x) \wedge R_{\square}(x_2, x)$ . Let  $B$  be the sentence  $\forall y \exists z_1 \exists z_2 (R_{\square}(z_1, y) \wedge z_1 \neq y \wedge R_{\square}(z_2, y) \wedge z_2 \neq y \wedge z_1 \neq z_2)$ . Obviously,  $\mathcal{C}_{\text{sym}}$  and  $A(x_1, x_2, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{sym}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{sym}}$  defined as follows:

- $W = W_0 \cup \{s_1, s_2, t_{11}, t_{12}, t_{21}, t_{22}\}$ ,
- $R$  is the least symmetric relation on  $W$  containing  $R_0$ ,  $\{s_1, s_2\} \times W_0$ ,  $(t_{11}, s_1)$ ,  $(t_{12}, s_1)$ ,  $(t_{21}, s_2)$ ,  $(t_{22}, s_2)$ ,  $(t_{12}, t_{11})$  and  $(t_{22}, t_{21})$ ,
- $W' = W_0 \cup \{s'_1, s'_2, t'_1, t'_2\}$ ,
- $R'$  is the least symmetric relation on  $W'$  containing  $R_0$ ,  $\{s'_1, s'_2\} \times W_0$ ,  $(t'_1, s'_1)$ ,  $(t'_2, s'_2)$ ,  $(t'_1, t'_1)$  and  $(t'_2, t'_2)$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x_2, x)$  and  $s_1, s_2$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ .

(3) Let  $A(x_1, x)$  be the first-order formula  $R_{\square}(x_1, x)$ . Let  $B$  be the sentence  $\forall y \exists z R_{\square}(z, y)$ . Obviously,  $\mathcal{C}_{\text{tra}}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{tra}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{tra}}$  defined as follows:

- $W = W_0 \cup \{s_1, t_1\}$ ,
- $R$  is the least transitive relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$ ,  $(t_1, s_1)$  and  $(t_1, t_1)$ ,
- $W' = W_0 \cup \{s_1\}$ ,
- $R'$  is the least transitive relation on  $W'$  containing  $R_0$  and  $\{s_1\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ .

(4) Similar to the proof of item (2).

(5) Let  $A(x_1, x)$  be the first-order formula  $R_{\square}(x_1, x) \wedge x_1 \neq x$ . Let  $B$  be the sentence  $\forall y \exists z (R_{\square}(z, y) \wedge z \neq y)$ . Obviously,  $\mathcal{C}_{\text{ref,tra}}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{ref,tra}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{ref,tra}}$  defined as follows:

- $W = W_0 \cup \{s_1, t_{11}, t_{12}\}$ ,
- $R$  is the least reflexive transitive relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$ ,  $(t_{11}, s_1)$ ,  $(t_{12}, s_1)$ ,  $(t_{12}, t_{11})$  and  $(t_{11}, t_{12})$ ,

- $W' = W_0 \cup \{s_1\}$ ,
- $R'$  is the least reflexive transitive relation on  $W'$  containing  $R_0$  and  $\{s_1\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ . ■

Putting Theorems 1 and 4 together leads us to a proof of

#### COROLLARY 3

The problem of deciding the modal definability of sentences with respect to the following classes of frames is undecidable: (1)  $\mathcal{C}_{\text{ref}}$ ; (2)  $\mathcal{C}_{\text{sym}}$ ; (3)  $\mathcal{C}_{\text{tra}}$ ; (4)  $\mathcal{C}_{\text{ref, sym}}$ ; (5)  $\mathcal{C}_{\text{ref, tra}}$ .

PROOF. (1) By [27], the problem of deciding the validity of sentences in lattices is undecidable. Since the first-order theory of lattices is a finite extension of the first-order theory of  $\mathcal{C}_{\text{ref}}$ , by [28, Theorem 5, Page 17], the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{ref}}$  is undecidable. The conclusion follows by Theorem 1 and item (1) of Theorem 4.

(2) By [12], the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{sym}}$  is undecidable. The conclusion follows by Theorem 1 and item (2) of Theorem 4.

(3) Similar to the proof of item (1).

(4) By [23], the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{ref, sym}}$  is undecidable. The conclusion follows by Theorem 1 and item (4) of Theorem 4.

(5) Similar to the proof of item (1). ■

$\mathcal{C}_{\text{par}}$  denoting the class of all partitions, let us remark, as proved in [1, 2], that the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{\text{par}}$  is *PSPACE*-complete. In each item of the proof of Theorem 4, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}$ ,  $\mathcal{F}'$  are finite too. This immediately gives us the following.

#### THEOREM 5

The following classes of frames are stable: (1) the class  $\mathcal{C}_{\text{ref}}^{\text{fin}}$  of all finite reflexive frames; (2) the class  $\mathcal{C}_{\text{sym}}^{\text{fin}}$  of all finite symmetric frames; (3) the class  $\mathcal{C}_{\text{tra}}^{\text{fin}}$  of all finite transitive frames; (4) the class  $\mathcal{C}_{\text{ref, sym}}^{\text{fin}}$  of all finite reflexive symmetric frames; (5) the class  $\mathcal{C}_{\text{ref, tra}}^{\text{fin}}$  of all finite reflexive transitive frames.

Putting Theorems 1 and 5 together leads us to a proof of the following new results.

#### COROLLARY 4

The problem of deciding the modal definability of sentences with respect to the following classes of frames is co-r.e.-hard: (1)  $\mathcal{C}_{\text{ref}}^{\text{fin}}$ ; (2)  $\mathcal{C}_{\text{sym}}^{\text{fin}}$ ; (3)  $\mathcal{C}_{\text{tra}}^{\text{fin}}$ ; (4)  $\mathcal{C}_{\text{ref, sym}}^{\text{fin}}$ ; (5)  $\mathcal{C}_{\text{ref, tra}}^{\text{fin}}$ .

PROOF. (1) By [26],  $\mathcal{C}_{\text{po}}$  denoting the class of all partial orders and  $\mathcal{C}_{\text{po}}^{\text{fin}}$  denoting the class of all finite partial orders, the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{po}} \models A\}$  and  $\{A: \mathcal{C}_{\text{po}}^{\text{fin}} \not\models A\}$ . Since the first-order theory of  $\mathcal{C}_{\text{po}}$  is a finite extension of the first-order theory of  $\mathcal{C}_{\text{ref}}$ , by a simple variant of [28, Theorem 5, p. 17], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{ref}} \models A\}$  and  $\{A: \mathcal{C}_{\text{ref}}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{\text{ref}}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{ref}}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorem 1 and item (1) of Theorem 5.

(2) By [20], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{sym}} \models A\}$  and  $\{A: \mathcal{C}_{\text{sym}}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{\text{sym}}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{sym}}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorem 1 and item (2) of Theorem 5.

(3) Similar to the proof of item (1).

(4) By [20], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{ref},\text{sym}} \models A\}$  and  $\{A: \mathcal{C}_{\text{ref},\text{sym}}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{\text{ref},\text{sym}}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{ref},\text{sym}}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorem 1 and item (4) of Theorem 5. ■

(5) Similar to the proof of item (1). ■

The following theorem illustrates again the flexibility of the frame manipulation method we have used to prove Theorem 2.

#### THEOREM 6

The following classes of frames are stable: (1) the class  $\mathcal{C}_{\text{spo}}$  of all strict partial orders; (2)  $\mathcal{C}_{\text{po}}$ .

PROOF. (1) Let  $A(x_1, x_2, x)$  be the first-order formula  $R_{\square}(x_1, x) \wedge R_{\square}(x_2, x)$ . Let  $B$  be the sentence  $\exists y \forall z (y = z \vee R_{\square}(y, z))$ . Obviously,  $\mathcal{C}_{\text{spo}}$  and  $A(x_1, x_2, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{\text{spo}}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{\text{spo}}$  defined as follows:

- $W = W_0 \cup \{s_1, s_2, t\}$ ,
- $R$  is the least transitive relation on  $W$  containing  $R_0$ ,  $\{s_1, s_2\} \times W_0$ ,  $(t, s_1)$  and  $(t, s_2)$ ,
- $W' = W_0 \cup \{s_1, s_2\}$ ,
- $R'$  is the least transitive relation on  $W'$  containing  $R_0$  and  $\{s_1, s_2\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x_2, x)$  and  $s_1, s_2$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ .

(2) Similar to the proof of item (1). ■

Putting Theorems 1 and 6 together leads us to a proof of the following new results.

#### COROLLARY 5

The problem of deciding the modal definability of sentences with respect to the following classes of frames is undecidable: (1)  $\mathcal{C}_{\text{spo}}$ ; (2)  $\mathcal{C}_{\text{po}}$ .

PROOF. (1) By [27], the problem of deciding the validity of sentences in lattices is undecidable. Since the first-order theory of lattices is a finite extension of the first-order theory of  $\mathcal{C}_{\text{spo}}$ , by [28, Theorem 5, p. 17], the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{spo}}$  is undecidable. The conclusion follows by Theorem 1 and item (1) of Theorem 6.

(2) Similar to the proof of item (1). ■

In each item of the proof of Theorem 6, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}$ ,  $\mathcal{F}'$  are finite too. This immediately gives us the following

#### THEOREM 7

The following classes of frames are stable: (1) the class  $\mathcal{C}_{\text{spo}}^{\text{fin}}$  of all finite strict partial orders; (2)  $\mathcal{C}_{\text{po}}^{\text{fin}}$ .

Putting Theorems 1 and 7 together leads us to a proof of the following new results.

#### COROLLARY 6

The problem of deciding the modal definability of sentences with respect to the following classes of frames is co-r.e.-hard: (1)  $\mathcal{C}_{\text{spo}}^{\text{fin}}$ ; (2)  $\mathcal{C}_{\text{po}}^{\text{fin}}$ .

PROOF. (1) By [26], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{\text{spo}} \models A\}$  and  $\{A: \mathcal{C}_{\text{spo}}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{\text{spo}} \models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{\text{spo}}^{\text{fin}}$  is

co-r.e.-hard. The conclusion follows by Theorem 1 and item (1) of Theorem 7.

(2) Similar to the proof of item (1). ■

## 6 Lattices, partitions and linear orders

Up to now, the classes of frames that we have considered were always definable by a universal sentence. Moreover, we have always considered a modal language with only one modal connective and a first-order language with only one relation symbol. In Section 6.1, we study the computability of the problem of deciding the modal definability of sentences with respect to a class of frames definable by a universal-existential sentence. In this respect, we will consider the class of all lattices, i.e. partial orders in which each pair of worlds has a greatest lower bound and a least upper bound. In Section 6.2, we study the computability of the problem of deciding the modal definability of sentences when the modal language has two modal connectives and the first-order language has two relation symbols. In this respect, we will consider the class of all bi-partitions, i.e. frames with two equivalence relations, the class of all linear partitions, i.e. frames with a linear order and an equivalence relation, and the class of all bilinear orders, i.e. frames with two linear orders.

### 6.1 Lattices

Remark that universal sentences are preserved under subframes. For this reason, obviously, each of the classes of frames considered in Theorems 2, 4 and 6 satisfies the condition (a) for any first-order formula  $A(\bar{x}, x)$ . Since the class  $\mathcal{C}_{lat}$  of all lattices is definable by a universal-existential sentence, the formula  $A(x_1, x)$  used in the proof of the following theorem has to be more specific.

THEOREM 8

$\mathcal{C}_{lat}$  is stable.

PROOF. Let  $A(x_1, x)$  be the first-order formula  $A' \wedge (A'')^A_x$  where  $A'$  denotes the first-order formula  $R_{\square}(x_1, x) \wedge x_1 \neq x$  and  $A''$  denotes the conjunction of the following sentences:

- $\forall y R_{\square}(y, y)$ ,
- $\forall y \forall z (R_{\square}(y, z) \wedge R_{\square}(z, y) \rightarrow y = z)$ ,
- $\forall y \forall z \forall t (R_{\square}(y, z) \wedge R_{\square}(z, t) \rightarrow R_{\square}(y, t))$ ,
- $\forall y \forall z \exists t (R_{\square}(t, y) \wedge R_{\square}(t, z) \wedge \forall u (R_{\square}(u, y) \wedge R_{\square}(u, z) \rightarrow R_{\square}(u, t)))$ ,
- $\forall y \forall z \exists t (R_{\square}(y, t) \wedge R_{\square}(z, t) \wedge \forall u (R_{\square}(y, u) \wedge R_{\square}(z, u) \rightarrow R_{\square}(t, u)))$ .

Let  $B$  be the sentence  $\neg \exists y \exists z (y \neq z \wedge \forall t (R_{\square}(y, t) \wedge (y \neq t \rightarrow R_{\square}(z, t))))$ . Remark that for all frames  $\mathcal{F}$  in  $\mathcal{C}_{lat}$ , for all worlds  $s_1$  in  $\mathcal{F}$  and for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$  then  $\mathcal{F}'$  is in  $\mathcal{C}_{lat}$ . Hence,  $\mathcal{C}_{lat}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{lat}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{lat}$  defined as follows:

- $W = W_0 \cup \{s_1, t_1, u_1, u_2, v\}$ ,
- $R$  is the least reflexive transitive relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$ ,  $(t_1, s_1)$ ,  $(u_1, t_1)$ ,  $(u_2, t_1)$  and  $(v, u_1)$ ,
- $W' = W_0 \cup \{s_1, t_1\}$ ,
- $R'$  is the least reflexive transitive relation on  $W'$  containing  $R_0$ ,  $\{s_1\} \times W_0$  and  $(t_1, s_1)$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ . ■

Putting Theorems 1 and 8 together leads us to a proof of the following new result.

COROLLARY 7

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{lat}$  is undecidable.

PROOF. By [27], the problem of deciding the validity of sentences in lattices is undecidable. The conclusion follows by Theorems 1 and 8. ■

In the proof of Theorem 8, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}$ ,  $\mathcal{F}'$  are finite too. This immediately gives us the following.

THEOREM 9

The class  $\mathcal{C}_{lat}^{\text{fin}}$  of all finite lattices is stable.

Putting Theorems 1 and 9 together leads us to a proof of the following new result.

COROLLARY 8

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{lat}^{\text{fin}}$  is co-r.e.-hard.

PROOF. By [26], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{lat} \models A\}$  and  $\{A: \mathcal{C}_{lat}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{lat}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{lat}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorem 1 and 9. ■

## 6.2 Partitions and linear orders

Let us consider the variant of our modal language with two modal connectives  $\Box_1$  and  $\Box_2$ . Given a model  $\mathcal{M} = (W, R_1, R_2, V)$ , the satisfiability of the modal formulas  $\Box_1\phi$  and  $\Box_2\phi$  at a world  $s$  in  $\mathcal{M}$  is defined by means of the binary relations  $R_1$  and  $R_2$  as expected. The concepts of a generated subframe and a bounded morphic image being accordingly defined, invariance results such as the ones described in Lemmas 1 and 2 can be easily obtained. Let us consider the variant of our first-order language with two relation symbols  $R_{\Box_1}$  and  $R_{\Box_2}$ , the concept of a relativization being accordingly defined. Given a frame  $\mathcal{F} = (W, R_1, R_2)$ , the satisfiability of the first-order formulas  $R_{\Box_1}(x_i, x_j)$  and  $R_{\Box_2}(x_i, x_j)$  with respect to a list  $\bar{s}$  of worlds in  $\mathcal{F}$  is defined by means of the binary relations  $R_1$  and  $R_2$  as expected. The concept of a relativized reduct being accordingly defined, an invariance result such as the one described in Lemma 3 can be easily obtained. A frame  $\mathcal{F} = (W, R_1, R_2)$  is said to be a bi-partition if  $R_1$  and  $R_2$  are equivalence relations. We shall say that a frame  $\mathcal{F} = (W, R_1, R_2)$  is a linear partition if  $R_1$  is a linear order and  $R_2$  is an equivalence relation. A frame  $\mathcal{F} = (W, R_1, R_2)$  is said to be a bilinear order if  $R_1$  and  $R_2$  are linear orders. Now, defining the concept of modal definability as in Section 4, let us see if results such as the ones described in Corollaries 1 and 2 can be obtained too for bi-partitions and bilinear orders. We do not know whether the class  $\mathcal{C}_{bip}$  of all bi-partitions is stable. Nevertheless, it is still possible to prove the following.

THEOREM 10

The problem of deciding the validity of sentences in  $\mathcal{C}_{bip}$  is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{bip}$ .

PROOF. Let  $A(x_1, x)$  be the first-order formula  $\neg R_{\square_1}(x_1, x)$ . Let  $C$  be a sentence. Let  $B$  be the sentence  $\exists y_1 \dots \exists y_{d+1} B'$  where  $d$  denotes the quantifier depth of  $C$  and  $B'$  denotes the conjunction of the following first-order formulas:

- $\bigwedge \{R_{\square_1}(y_i, y_j): 1 \leq i < j \leq d+1\}$ ,
- $\bigwedge \{\neg R_{\square_2}(y_i, y_j): 1 \leq i < j \leq d+1\}$ ,
- $\forall z \forall z' (R_{\square_1}(y_1, z) \wedge R_{\square_2}(z, z') \rightarrow R_{\square_1}(y_1, z'))$ ,
- $\forall z (R_{\square_1}(y_1, z) \rightarrow \exists z' (R_{\square_2}(z, z') \wedge z \neq z'))$ .

Remark that for all frames  $\mathcal{F} = (W, R_1, R_2)$  in  $\mathcal{C}_{bip}$ ,  $\mathcal{F} \models B$  iff there exists an  $R_2$ -closed equivalence class modulo  $R_1$  containing  $d+1$  pairwise disjoint equivalence classes modulo  $R_2$  and containing no degenerate equivalence class modulo  $R_2$ . Let  $D$  be the sentence  $\exists x_1 (\exists x A(x_1, x) \wedge \neg (C)_x^{A(x_1, x)}) \wedge B$ , we demonstrate  $\mathcal{C}_{bip} \models C$  iff  $D$  is modally definable with respect to  $\mathcal{C}_{bip}$ . Contrary to what was the situation in the proof of Theorem 1, remark that the  $B$ -part of  $D$  now depends on the given  $C$ .

Suppose  $\mathcal{C}_{bip} \models C$ . For the sake of the contradiction, suppose  $D$  is not modally definable with respect to  $\mathcal{C}_{bip}$ . Following an argument similar to the one considered in the first part of the proof of Theorem 1, the reader may easily obtain a contradiction.

Suppose  $D$  is modally definable with respect to  $\mathcal{C}_{bip}$ . Let  $\phi$  be a modal definition of  $D$  with respect to  $\mathcal{C}_{bip}$ . For the sake of the contradiction, suppose  $\mathcal{C}_{bip} \not\models C$ . Let  $\mathcal{F}_0 = (W_0, R_{01}, R_{02})$  be a frame in  $\mathcal{C}_{bip}$  such that  $\mathcal{F}_0 \not\models C$ . We shall say that a  $R_{02}$ -closed equivalence class  $U$  modulo  $R_{01}$  is  $d$ -reparable if  $U$  contains  $d+1$  pairwise disjoint equivalence classes modulo  $R_{02}$  and  $U$  contains no degenerate equivalence class modulo  $R_{02}$ . Let  $\tilde{\cdot}$  be a function assigning to each  $d$ -reparable  $R_{02}$ -closed equivalence class  $U$  modulo  $R_{01}$  a subset  $\tilde{U}$  of  $U$  consisting of exactly  $d$  equivalence classes modulo  $R_{02}$ . Let  $\mathcal{F}_1 = (W_1, R_{11}, R_{12})$  be the subframe of  $\mathcal{F}_0$  obtained by replacing each  $d$ -reparable  $R_{02}$ -closed equivalence class  $U$  modulo  $R_{01}$  by  $\tilde{U}$ . Obviously,  $\mathcal{F}_1 \not\models B$ . Moreover, player  $\exists$  has a winning strategy in the Ehrenfeucht–Fraïssé game of length  $d$  on  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . As is well-known, the  $d$ -round Ehrenfeucht–Fraïssé game captures elementary equivalence up to quantifier depth  $d$ . See [13, Chapter 1] for an introductory discussion about similar characterization results. Since  $C$  is a sentence of quantifier depth  $d$  and  $\mathcal{F}_0 \not\models C$ ,  $\mathcal{F}_1 \not\models C$ . Consider the frames  $\mathcal{F} = (W, R_1, R_2)$ ,  $\mathcal{F}' = (W', R'_1, R'_2)$  in  $\mathcal{C}_{bip}$  defined as follows:

- $W = W_1 \cup \{s_1, \dots, s_{d+1}, s_{d+2}, \dots, s_{2d+2}\}$ ,
- $R_1$  is the least equivalence relation on  $W$  containing  $R_{11}$  and  $\{(s_i, s_j): 1 \leq i < j \leq 2d+2\}$ ,
- $R_2$  is the least equivalence relation on  $W$  containing  $R_{12}$  and  $\{(s_i, s_{i+d+1}): 1 \leq i \leq d+1\}$ ,
- $W' = W_1 \cup \{s_1, \dots, s_{d+1}\}$ ,
- $R'_1$  is the least equivalence relation on  $W'$  containing  $R_{11}$  and  $\{(s_i, s_j): 1 \leq i < j \leq d+1\}$ ,
- $R'_2$  is the least equivalence relation on  $W'$  containing  $R_{12}$ .

Obviously,  $\mathcal{F}_1$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ . Following an argument similar to the one considered in the end of the second part of the proof of Theorem 1, the reader may easily obtain a contradiction. ■

Theorem 10 alone leads us to a proof of the following new result.

COROLLARY 9

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{bip}$  is undecidable.

PROOF. By [23], the problem of deciding the validity of sentences in  $\mathcal{C}_{bip}$  is undecidable. The conclusion follows by Theorem 10. ■

Let  $\mathcal{C}_{bip}^{\text{fin}}$  be the class of all finite bi-partitions. In the proof of Theorem 10, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}, \mathcal{F}'$  are finite too. This immediately gives us the following

**THEOREM 11**

The problem of deciding the validity of sentences in  $\mathcal{C}_{bip}^{\text{fin}}$  is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{bip}^{\text{fin}}$ .

Theorem 11 alone leads us to a proof of the following new result.

**COROLLARY 10**

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{bip}^{\text{fin}}$  is co-r.e.-hard.

**PROOF.** By [20], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{bip} \models A\}$  and  $\{A: \mathcal{C}_{bip}^{\text{fin}} \not\models A\}$ . Since  $\{A: \mathcal{C}_{bip}^{\text{fin}} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{bip}^{\text{fin}}$  is co-r.e.-hard. The conclusion follows by Theorem 11. ■

As for the class  $\mathcal{C}_{lip}$  of all linear partitions, we have the following.

**THEOREM 12**

$\mathcal{C}_{lip}$  is stable.

**PROOF.** Let  $A(x_1, x)$  be the first-order formula  $R_{\square_1}(x_1, x) \wedge x_1 \neq x$ . Let  $B$  be the sentence  $\exists y (\forall z R_{\square_1}(y, z) \wedge \exists z (R_{\square_2}(y, z) \wedge y \neq z))$ . Obviously,  $\mathcal{C}_{lip}$  and  $A(x_1, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_{01}, R_{02})$  be a frame in  $\mathcal{C}_{lip}$ . Consider the frames  $\mathcal{F} = (W, R_1, R_2)$ ,  $\mathcal{F}' = (W', R'_1, R'_2)$  in  $\mathcal{C}_{lip}$  defined as follows:

- $W = W_0 \cup \{s_1, t\}$ ,
- $R_1$  is the least reflexive transitive relation on  $W$  containing  $R_{01}$ ,  $\{s_1\} \times W_0$  and  $(t, s_1)$ ,
- $R_2$  is the least equivalence relation on  $W$  containing  $R_{02}$  and  $(t, s_1)$ ,
- $W' = W_0 \cup \{s'\}$ ,
- $R'_1$  is the least reflexive transitive relation on  $W'$  containing  $R_{01}$  and  $\{s'\} \times W_0$ ,
- $R'_2$  is the least equivalence relation on  $W'$  containing  $R_{02}$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x)$  and  $s_1$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ . ■

Putting Theorems 1 and 12 together leads us to a proof of the following new results.

**COROLLARY 11**

The problem of deciding the modal definability of sentences with respect  $\mathcal{C}_{lip}$  is undecidable.

**PROOF.** By [22, 23], the problem of deciding the validity of sentences in  $\mathcal{C}_{lip}$  is undecidable. The conclusion follows by Theorems 1 and 12. ■

In the proof of Theorem 12, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}, \mathcal{F}'$  are finite too. This immediately gives us the following.

**THEOREM 13**

The class  $\mathcal{C}_{lip}^{\text{fin}}$  of all finite linear partitions is stable.

Putting Theorems 1 and 13 together leads us to a proof of the following new results.

COROLLARY 12

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{lip}^{fin}$  is co-r.e.-hard.

PROOF. By [20], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{lip} \models A\}$  and  $\{A: \mathcal{C}_{lip}^{fin} \not\models A\}$ . Since  $\{A: \mathcal{C}_{lip}^{fin} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{lip}^{fin}$  is co-r.e.-hard. The conclusion follows by Theorems 1 and 13. ■

Concerning the class  $\mathcal{C}_{bil}$  of all bilinear orders, we have the following.

THEOREM 14

$\mathcal{C}_{bil}$  is stable.

PROOF. Let  $A(x_1, x_2, x)$  be the first-order formula  $R_{\square_1}(x_1, x) \wedge R_{\square_2}(x_2, x)$ . Let  $B$  be the sentence  $\neg \exists y \forall z (R_{\square_1}(y, z) \wedge R_{\square_2}(y, z))$ . Obviously,  $\mathcal{C}_{bil}$  and  $A(x_1, x_2, x)$  satisfy the condition (a). As for the condition (b), let  $\mathcal{F}_0 = (W_0, R_0)$  be a frame in  $\mathcal{C}_{bil}$ . Consider the frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F}' = (W', R')$  in  $\mathcal{C}_{bil}$  defined as follows:

- $W = W_0 \cup \{s_1, s_2\}$ ,
- $R_1$  is the least reflexive transitive relation on  $W$  containing  $R_0$ ,  $\{s_1\} \times W_0$  and  $(s_2, s_1)$ ,
- $R_2$  is the least reflexive transitive relation on  $W$  containing  $R_0$ ,  $\{s_2\} \times W_0$  and  $(s_1, s_2)$ ,
- $W' = W_0 \cup \{s'\}$ ,
- $R'_1$  is the least reflexive transitive relation on  $W'$  containing  $R_0$  and  $\{s'\} \times W_0$ ,
- $R'_2$  is the least reflexive transitive relation on  $W'$  containing  $R_0$  and  $\{s'\} \times W_0$ .

Obviously,  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(x_1, x_2, x)$  and  $s_1, s_2$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ . ■

Putting Theorems 1 and 14 together leads us to a proof of the following new results.

COROLLARY 13

The problem of deciding the modal definability of sentences with respect  $\mathcal{C}_{bil}$  is undecidable.

PROOF. By [22, 23], the problem of deciding the validity of sentences in  $\mathcal{C}_{bil}$  is undecidable. The conclusion follows by Theorem 1 and item Theorem 14. ■

In the proof of Theorem 14, if the frame  $\mathcal{F}_0$  is finite then the frames  $\mathcal{F}$ ,  $\mathcal{F}'$  are finite too. This immediately gives us the following.

THEOREM 15

The class  $\mathcal{C}_{bil}^{fin}$  of all finite bilinear orders is stable.

Putting Theorems 1 and 15 together leads us to a proof of the following new results.

COROLLARY 14

The problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{bil}^{fin}$  is co-r.e.-hard.

PROOF. By [20], the following sets of sentences are recursively inseparable:  $\{A: \mathcal{C}_{bil} \models A\}$  and  $\{A: \mathcal{C}_{bil}^{fin} \not\models A\}$ . Since  $\{A: \mathcal{C}_{bil}^{fin} \not\models A\}$  is r.e., the problem of deciding the validity of sentences in  $\mathcal{C}_{bil}^{fin}$  is co-r.e.-hard. The conclusion follows by Theorems 1 and 15. ■

## 7 Final remarks

We shall say that a first-order formula

- $A(\bar{x})$  is preserved under taking generated subframes if for all frames  $\mathcal{F}$ , if  $\mathcal{F} \models A(\bar{x})$  then for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$  then  $\mathcal{F}' \models A(\bar{x})$ ,
- $A(\bar{x})$  is preserved under taking bounded morphisms if for all frames  $\mathcal{F}$ , if  $\mathcal{F} \models A(\bar{x})$  then for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$  then  $\mathcal{F}' \models A(\bar{x})$ .

Let  $\mathcal{C}$  be a class of frames. Theorem 1 can be changed in such a way that it states that if  $\mathcal{C}$  is stable then the problem of deciding the preservability under taking generated subframes of sentences with respect to  $\mathcal{C}$  and the problem of deciding the preservability under taking bounded morphisms of sentences with respect to  $\mathcal{C}$  are at least as difficult as the problem of deciding the validity of sentences in  $\mathcal{C}$ . In this respect, we need to change our definition of stability.  $\mathcal{C}$  is said to be gs-stable (for ‘generated subframe stable’) if there exists a first-order formula  $A(\bar{x}, x)$  and there exists a sentence  $B$  such that

- for all frames  $\mathcal{F}$  in  $\mathcal{C}$ , for all lists  $\bar{s}$  of worlds in  $\mathcal{F}$  and for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  then  $\mathcal{F}'$  is in  $\mathcal{C}$ ,
- for all frames  $\mathcal{F}_0$  in  $\mathcal{C}$ , there exists frames  $\mathcal{F}, \mathcal{F}'$  in  $\mathcal{C}$  and there exists a list  $\bar{s}$  of worlds in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ .

We shall say that  $\mathcal{C}$  is bm-stable (for ‘bounded morphism stable’) if there exists a first-order formula  $A(\bar{x}, x)$  and there exists a sentence  $B$  such that

- for all frames  $\mathcal{F}$  in  $\mathcal{C}$ , for all lists  $\bar{s}$  of worlds in  $\mathcal{F}$  and for all frames  $\mathcal{F}'$ , if  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$  then  $\mathcal{F}'$  is in  $\mathcal{C}$ ,
- for all frames  $\mathcal{F}_0$  in  $\mathcal{C}$ , there exists frames  $\mathcal{F}, \mathcal{F}'$  in  $\mathcal{C}$  and there exists a list  $\bar{s}$  of worlds in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a bounded morphic image of  $\mathcal{F}$ .

In both cases,  $(A(\bar{x}, x), B)$  is called a witness of the stability of  $\mathcal{C}$ . The following theorem states that if  $\mathcal{C}$  is gs-stable then the problem of deciding the preservability undertaking generated subframes of sentences with respect to  $\mathcal{C}$  is at least as difficult as the problem of deciding the validity of sentences in  $\mathcal{C}$ . As the reader will see, its proof is similar to the proof of Theorem 1.

### THEOREM 16

If  $\mathcal{C}$  is gs-stable then the problem of deciding the validity of sentences in  $\mathcal{C}$  is reducible to the problem of deciding the preservability undertaking generated subframes of sentences with respect to  $\mathcal{C}$ .

PROOF. Suppose  $\mathcal{C}$  is gs-stable. Let  $(A(\bar{x}, x), B)$  be a witness of the gs-stability of  $\mathcal{C}$ . Let  $C$  be a sentence. Let  $D$  be the sentence  $\exists \bar{x} (\exists x A(\bar{x}, x) \wedge \neg (C)_x^{A(\bar{x}, x)}) \wedge B$ , we demonstrate  $\mathcal{C} \models C$  iff  $D$  is preserved under taking generated subframes with respect to  $\mathcal{C}$ .

Suppose  $\mathcal{C} \models C$ . For the sake of the contradiction, suppose  $D$  is not preserved under taking generated subframes with respect to  $\mathcal{C}$ . Let  $\mathcal{F}$  be a frame in  $\mathcal{C}$  such that  $\mathcal{F} \models D$ . Such a frame exists, otherwise  $D$  would be preserved under taking generated subframes with respect to  $\mathcal{C}$ . Let  $\bar{s}$  be worlds in  $\mathcal{F}$  such that  $\mathcal{F} \models \exists x A(\bar{x}, x) [\bar{s}]$  and  $\mathcal{F} \not\models (C)_x^{A(\bar{x}, x)} [\bar{s}]$ . Let  $\mathcal{F}'$  be the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ . Such a frame exists, otherwise  $\mathcal{F} \not\models \exists x A(\bar{x}, x) [\bar{s}]$ . Since  $\mathcal{F}$  is in  $\mathcal{C}$ , by (a),  $\mathcal{F}'$  is in  $\mathcal{C}$ . Since  $\mathcal{F}'$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x}, x)$  and  $\bar{s}$ , by Lemma 3,  $\mathcal{F}' \models (C)_x^{A(\bar{x}, x)} [\bar{s}]$

iff  $\mathcal{F}' \models C$ . Since  $\mathcal{F} \not\models (C)_x^{A(\bar{x},x)} [\bar{s}]$ ,  $\mathcal{F}' \not\models C$ . Since  $\mathcal{F}'$  is in  $\mathcal{C}$ ,  $\mathcal{C} \not\models C$ : a contradiction.

Suppose  $D$  is preserved under taking generated subframes with respect to  $\mathcal{C}$ . For the sake of the contradiction, suppose  $\mathcal{C} \not\models C$ . Let  $\mathcal{F}_0$  be a frame in  $\mathcal{C}$  such that  $\mathcal{F}_0 \not\models C$ . Let  $\mathcal{F}, \mathcal{F}'$  be frames in  $\mathcal{C}$  and let  $\bar{s}$  be a list of worlds in  $\mathcal{F}$  such that  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x},x)$  and  $\bar{s}$ ,  $\mathcal{F} \models B$ ,  $\mathcal{F}' \not\models B$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ . Since  $\mathcal{F}_0$  is in  $\mathcal{C}$ , by (b), such frames and such a list of worlds exist. Since  $\mathcal{F}' \not\models B$ ,  $\mathcal{F}' \not\models D$ . Since  $D$  is preserved under taking generated subframes with respect to  $\mathcal{C}$ ,  $\mathcal{F}, \mathcal{F}'$  are in  $\mathcal{C}$  and  $\mathcal{F}'$  is a generated subframe of  $\mathcal{F}$ ,  $\mathcal{F} \not\models D$ . Hence, either  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x},x) \wedge \neg (C)_x^{A(\bar{x},x)})$ , or  $\mathcal{F} \not\models B$ . Since  $\mathcal{F} \models B$ ,  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x},x) \wedge \neg (C)_x^{A(\bar{x},x)})$ . Since  $\mathcal{F}_0$  is the relativized reduct of  $\mathcal{F}$  with respect to  $A(\bar{x},x)$  and  $\bar{s}$ , by Lemma 3,  $\mathcal{F} \models (C)_x^{A(\bar{x},x)} [\bar{s}]$  iff  $\mathcal{F}_0 \models C$ . Moreover,  $\mathcal{F} \models \exists x A(\bar{x},x) [\bar{s}]$ . Since  $\mathcal{F} \not\models \exists \bar{x} (\exists x A(\bar{x},x) \wedge \neg (C)_x^{A(\bar{x},x)})$ , either  $\mathcal{F} \not\models \exists x A(\bar{x},x) [\bar{s}]$ , or  $\mathcal{F} \models (C)_x^{A(\bar{x},x)} [\bar{s}]$ . Since  $\mathcal{F} \models \exists x A(\bar{x},x) [\bar{s}]$ ,  $\mathcal{F} \models (C)_x^{A(\bar{x},x)} [\bar{s}]$ . Since  $\mathcal{F} \models (C)_x^{A(\bar{x},x)} [\bar{s}]$  iff  $\mathcal{F}_0 \models C$ ,  $\mathcal{F}_0 \models C$ : a contradiction. ■

The following theorem states that if  $\mathcal{C}$  is bm-stable then the problem of deciding the preservability under taking bounded morphisms of sentences with respect to  $\mathcal{C}$  is at least as difficult as the problem of deciding the validity of sentences in  $\mathcal{C}$ .

#### THEOREM 17

If  $\mathcal{C}$  is bm-stable then the problem of deciding the validity of sentences in  $\mathcal{C}$  is reducible to the problem of deciding the preservability under taking bounded morphisms of sentences with respect to  $\mathcal{C}$ .

PROOF. Similar to the proof of Theorem 16. ■

From the proofs of Theorems 2, 4, 6, 8, 10, 12 and 14, the reader may easily conclude that:

#### THEOREM 18

The following classes of frames are gs-stable: (1)  $\mathcal{C}_{\text{all}}$ ; (2)  $\mathcal{C}_{\text{ref}}$ ; (3)  $\mathcal{C}_{\text{tra}}$ ; (4)  $\mathcal{C}_{\text{ref,tra}}$ ; (5)  $\mathcal{C}_{\text{spo}}$ ; (6)  $\mathcal{C}_{\text{po}}$ ; (7)  $\mathcal{C}_{\text{lat}}$ .

#### THEOREM 19

The following classes of frames are bm-stable: (1)  $\mathcal{C}_{\text{sym}}$ ; (2)  $\mathcal{C}_{\text{ref,sym}}$ ; (3)  $\mathcal{C}_{\text{bip}}$ ; (4)  $\mathcal{C}_{\text{lip}}$ ; (5)  $\mathcal{C}_{\text{bil}}$ .

Hence,

#### COROLLARY 15

The problem of deciding the preservability under taking generated subframes of sentences with respect to the following classes of frames is undecidable: (1)  $\mathcal{C}_{\text{all}}$ ; (2)  $\mathcal{C}_{\text{ref}}$ ; (3)  $\mathcal{C}_{\text{tra}}$ ; (4)  $\mathcal{C}_{\text{ref,tra}}$ ; (5)  $\mathcal{C}_{\text{spo}}$ ; (6)  $\mathcal{C}_{\text{po}}$ ; (7)  $\mathcal{C}_{\text{lat}}$ .

#### COROLLARY 16

The problem of deciding the preservability under taking bounded morphisms of sentences with respect to the following classes of frames is undecidable: (1)  $\mathcal{C}_{\text{sym}}$ ; (2)  $\mathcal{C}_{\text{ref,sym}}$ ; (3)  $\mathcal{C}_{\text{bip}}$ ; (4)  $\mathcal{C}_{\text{lip}}$ ; (5)  $\mathcal{C}_{\text{bil}}$ .

TABLE 1. Computability of the problem of deciding the modal definability of sentences with respect to classes of frames

Class of all ...	Computability	References
frames	undecidable	[10] and Corollary 1
fin. frames	co-r.e.-hard	Corollary 2
reflexive frames	undecidable	Item (1) of Corollary 3
symmetric frames	undecidable	Item (2) of Corollary 3
transitive frames	undecidable	Item (3) of Corollary 3
reflexive symmetric frames	undecidable	Item (4) of Corollary 3
reflexive transitive frames	undecidable	Item (5) of Corollary 3
partitions	<i>PSPACE</i> -complete	[1, 2]
fin. reflexive frames	co-r.e.-hard	Item (1) of Corollary 4
fin. symmetric frames	co-r.e.-hard	Item (2) of Corollary 4
fin. transitive frames	co-r.e.-hard	Item (3) of Corollary 4
fin. reflexive symmetric frames	co-r.e.-hard	Item (4) of Corollary 4
fin. reflexive transitive frames	co-r.e.-hard	Item (5) of Corollary 4
strict partial orders	undecidable	Item (1) of Corollary 5
partial orders	undecidable	Item (2) of Corollary 5
fin. strict partial orders	co-r.e.-hard	Item (1) of Corollary 6
fin. partial orders	co-r.e.-hard	Item (2) of Corollary 6
lattices	undecidable	Corollary 7
fin. lattice	co-r.e.-hard	Corollary 8
bi-partitions	undecidable	Corollary 9
fin. bi-partitions	co-r.e.-hard	Corollary 10
linear partitions	undecidable	Corollary 11
fin. linear partitions	co-r.e.-hard	Corollary 12
bilinear orders	undecidable	Corollary 13
fin. bilinear orders	co-r.e.-hard	Corollary 14

## 8 Conclusion and open problems

By means of the notion of a stable class of frames, we have given a new proof of Chagrova's Theorem (Corollary 1) and we have repeated our proof for demonstrating that, with respect to different classes of frames, the problem of deciding the modal definability of sentences is undecidable too (Corollaries 2–14). See Table 1 for a review of the results we know so far. The key result of our method is the fact that for all stable classes  $\mathcal{C}$  of frames, the problem of the  $\mathcal{C}$ -validity of sentences is reducible to the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  (Theorem 1). Hence, if the problem of the  $\mathcal{C}$ -validity of sentences is undecidable, then the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$  is also undecidable. For example, by demonstrating, as in Theorem 2, that  $\mathcal{C}_{\text{all}}$  is stable, Chagrova's Theorem appears as a particular case of Theorem 1. Moreover, if the problem of the  $\mathcal{C}$ -validity of sentences is decidable then any lower bound on its complexity is also a lower bound of the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}$ . For example, by demonstrating, as in [1, 2], that the problem of the  $\mathcal{C}_{\text{par}}$ -validity of sentences is *PSPACE*-hard and by demonstrating that  $\mathcal{C}_{\text{par}}$  is stable, the reader may easily conclude as a particular case of Theorem 1 that the problem of deciding the modal definability of sentences with respect to  $\mathcal{C}_{\text{par}}$  is *PSPACE*-hard. In fact, as proved in [1, 2], this problem is *PSPACE*-complete. In order to show the stability of a class of frames, note that there is no obligation to use the frame constructions of generated subframes and bounded morphic images, the only important thing in condition (b) being that  $\mathcal{F} \preceq \mathcal{F}'$ . See also the remark after the proof of Theorem 1.

Much remains to be done. Firstly, the frame constructions of generated subframes and bounded morphic images that we have used to prove our stability theorems have counterparts in modal languages with modal connectives of arity 2 and more. See [5, Definition 3.13]. We believe that results similar to the theorem of Chagrova and our undecidability results can be proved in the case of modal languages with modal connectives of arity 2 and more. Secondly, between worlds in a frame, the universal relation and the difference relation are important relations that the ordinary modal language cannot express. See [16, 21] or [5, Chapter 7]. What become the theorem of Chagrova and our undecidability results in modal languages enriched by the universal modality (interpreted by the universal relation), or the difference modality (interpreted by the difference relation)? Thirdly, the proof of the undecidability of other properties (first-order definability, Kripke completeness, etc.) has also been based on the undecidability of the halting problem concerning Minsky machines. See [6, 7, 11] or [9, Chapter 17]. By means of the notion of a stable class of frames, can we give a new proof of these undecidability results? Fourthly, in [1, 2], Balbiani and Tinchev have proved that, with respect to  $\mathcal{C}_{par}$ , the problem of deciding the modal definability of sentences is *PSPACE*-complete. What happens, with respect to the class of all symmetric transitive frames? Fifthly, in [3], Balbiani *et al.* have introduced modal logics for region-based theories of space. Within their context, it also makes sense to consider the modal definability of sentences. Can we obtain for these modal logics undecidability results similar to the theorem of Chagrova and our undecidability results? Sixthly, some of the stable classes considered in Theorems 2, 4 and 6 are definable by universal Horn formulas. See [14]. Can we demonstrate that all such Horn classes are stable? Seventhly, are there any counter-examples to the equivalence between the decidability of validity in a class of frames and the decidability of modal definability with respect to the same class of frames?

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