GROMOV-WITTEN THEORY WITH DERIVED ALGEBRAIC GEOMETRY

Etienne Mann, Marco Robalo

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ETIENNE MANN AND MARCO ROBALO

Abstract. In this survey we add two new results that are not in our paper [MR15]. Using the idea of brane actions discovered by Toën, we construct a lax associative action of the operad of stable curves of genus zero on a smooth variety X seen as an object in correspondences in derived stacks. This action encodes the Gromov-Witten theory of X in purely geometrical terms.

Contents

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1. Introduction

This paper is a survey[1] of [MR15]. We explain without technical details the ideas of [MR15] where we use derived algebraic geometry to redefine Gromov-Witten invariants and highlight the hidden operad picture.

Gromov-Witten invariants were introduced by Kontsevich and Manin in algebraic geometry in [KM94, Kon95]. The foundations were then completed by Behrend, Fantechi and Manin in [BM96a], [BF97] and [Beh97]. In symplectic geometry, the definition is due to Y. Ruan and G. Tian in [RT94], [Rua96] and [RT97]. Mathematicians developed several techniques to compute them: via a localization formula proved by Graber and Pandharipande in [GP99], via a degeneration formula proved by J. Li in [Li02] and another one called quantum Lefschetz proved by Coates-Givental [CG07] and Tseng [Tse10].

These invariants can be encoded using different mathematical structures: quantum products, cohomological field theories (Kontsevich-Manin in [KM94]), Frobenius manifolds (Dubrovin in [Dub96]), Lagrangian cones and Quantum D-modules (Givental [Giv04]), variations of non-commutative Hodge structures (Iritani [Iri09] and Kontsevich, Katzarkov and Pantev in [KKP08]) and so on, and used to express different aspects of mirror symmetry. Another important aspect of the theory concerns the study of the functoriality of Gromov-Witten invariants via crepant resolutions or flop transitions in terms of these structures (see [Rua06], [Per07], [CIT09], [CCIT09], [BG09], [Iri10], [BCR13], [BC14], [CIJ14], etc).

We first recall the classical construction of these invariants. Let X be a smooth projective variety (or orbifold). The basic ingredient to define GW-invariants is the moduli stack of stable maps to X, denoted by $\overline{\mathcal{M}}_{g,n}(X,\beta)$, with a fixed degree $\beta \in H_2(X,\mathbb{Z})$ [2]. The evaluation at the marked points gives maps of stacks $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ and forgetting the morphism and stabilising the curve gives a map $p : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ (See Remark 2.1.3).

To construct the invariants, we integrate over "the fundamental class" of the moduli stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$. For this integration to be possible, we need this moduli stack to be proper, which was proved by Behrend-Manin [BM96a] and some form of smoothness. In general, the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not smooth and has many components with different

[1] We add two new results Theorem 5.3.11 and 5.4.2

[2] The (co)homology in this paper are the singular ones.

dimensions. Nevertheless and thanks to a theorem of Kontsevich [Kon95], it is quasi-smooth - in the sense that locally it looks like the intersection of two smooth subschemes inside an ambient smooth scheme. In genus zero however this stack is known to be smooth under some assumptions on the geometry of X, for instance, when X is the projective space or a Grassmaniann, or more generally when X is convex, *i.e.*, if for any map $f : \mathbb{P}^1 \to X$, the group $\mathrm{H}^1(\mathbb{P}^1, f^*(\mathrm{T}_X))$ vanishes. See [FP97].

This quasi-smoothness has been used by Behrend-Fantechi to define in [BF97] a "virtual fundamental class", denoted by $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$, which is a cycle in the Chow ring of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ that plays the role of the usual fundamental class.

One of the most important result of Gromov-Witten invariants is that they form a cohomological field theory, that is, there exist a family of morphisms

$$\begin{aligned} I^X_{g,n,\beta} : H^*(X)^{\otimes n} &\to H^*(\overline{\mathcal{M}}_{g,n}) \\ (\alpha_1 \otimes \ldots \otimes \alpha_n) &\mapsto \mathrm{Stb}_* \left(\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}} \cup (\cup_i ev_i^*(\alpha_i)) \right) \end{aligned} \tag{1.0.1}$$

that satisfy some properties. Another formulation of this result is that we have a morphism of operads between $\left(H_*(\overline{\mathcal{M}}_{g,n})\right)_{n\in\mathbb{N}}$ and the endomorphism operad $\mathrm{End}(H^*(X))$ (see Corollary 2.2.5). Yet a more concise way to explain this, is to say that $H^*(X)$ owns a structure of algebra over the operads $H_*(\overline{\mathcal{M}}_{g,n})$.

The main result of [MR15] is that it is possible to remove (co)homology from the previous statement. The main result of [MR15] is the following

Theorem 1.0.2 (See Theorem 3.1.2)**.** *Let X be a smooth projective variety.*
The diagrams

$$\begin{array}{ccc} & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X,\beta) & \\ {\scriptstyle p,e_1,\ldots,e_n}\swarrow & & \searrow{\scriptstyle p,e_{n+1}} \\ \overline{\mathcal{M}}_{0,n+1}\times X^n & & \overline{\mathcal{M}}_{0,n+1}\times X \end{array}$$

give a family of morphisms

$$\varphi_n : \overline{\mathcal{M}}_{0,n+1} \to \underline{\mathrm{End}}^{\mathrm{cor}}(X)[n] := \underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)$$

that forms a lax morphism of ∞-operads in the category of derived stacks.

We restrict our work to genus 0 because we lack fundamental aspects for ∞-modular operads.

In this survey we omit the technical details and we insist on the ideas behind the theorem. Nevertheless, we add some new statements with respect to [MR15] as Theorem 5.3.11 and Theorem 5.4.2 with the proofs given in the appendices.

Acknowledgements: We want to thank Bertrand Toën for the organisation of the État de la Recherche and also for some ideas to prove Theorem 5.3.11. The first author thanks Daniel Naie who explains how to make these figures.

2. Moduli space of stable maps, cohomological field theory and operads

In this section, we recall some notions and ideas related to Gromov-Witten theory. Most of them are in the book of Cox-Katz [CK99]. The mathematical story started with the paper of Kontsevich [Kon95] (see also Kontsevich-Manin [KM96]) and was followed by many more and interesting questions that we will skip here.

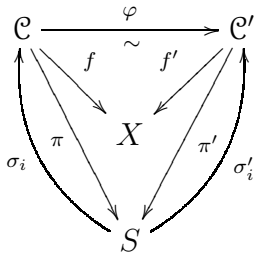
2.1. **Moduli space of stable maps.** Let X be a smooth projective variety over $\mathbb{C}$. Let $\beta \in H_2(X,\mathbb{Z})$. Let $g,n \in \mathbb{N}$. Denote by $(\mathrm{Aff}-\mathrm{sch})$ the category of affine scheme and by (Grps) the category of groupoids. We define the moduli space of stable maps by the following functor:

$$\overline{\mathcal{M}}_{g,n}(X,\beta) : (\mathrm{Aff}-\mathrm{sch})^{op} \longrightarrow (\mathrm{Grps})$$

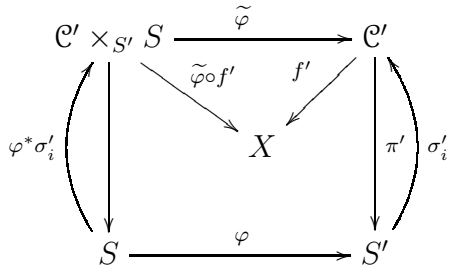
where $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ is the following groupoids. Objects are flat proper morphisms $\pi : \mathcal{C} \to S$ together with n-sections $\sigma_i : S \to \mathcal{C}$ and a morphism $f : \mathcal{C} \to X$ such that for any geometric point $s \in S$, we have

(1) the fiber $\mathcal{C}_s$ is a connected nodal curve of genus g with n distinct marked points which live on the smooth locus of $\mathcal{C}_s$.
(2) $f_s : \mathcal{C}_s \to X$ is of degree β, meaning $f_*[\mathcal{C}_s] = \beta$.
(3) the automorphism group of $\mathrm{Aut}(\mathcal{C},\underline{\sigma},f)$ is finite where we denote $\underline{\sigma} = (\sigma_1,\dots,\sigma_n)$. This condition is called *stability* condition.

For any affine scheme S, the morphism in the groupoid $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ ar the isomorphisms $\varphi : \mathcal{C} \to \mathcal{C}'$ such that the following diagram is commutative:



Let $\varphi : S \to S'$ be a morphism of affine schemes. Let $(\mathcal{C} \to S, \underline{\sigma}, f)$ be an object in $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$, then the pullback family defined by the diagram below satisfies the three conditions above that is it is in $\overline{\mathcal{M}}_{g,n}(X,\beta)(S')$



Notice that the condition (1), (2) a,d (3) are stable by pull-back.

Remark 2.1.1. Let explain the stability condition (3) in more concrete terms (See [CK99, §7.1.1 p. 169]). Denote by $\mathcal{C}_{s,i}$ the irreducible components of $\mathcal{C}_s$ and by $f_{s,i} : \mathcal{C}_{s,i} \to X$ the restrictions of the morphism. Denote by $\beta_i = (f_{s,i})_*[\mathcal{C}_{s,i}] \in H_2(X, \mathbb{Z})$ the degree of f_s on each irreducible component $\mathcal{C}_{s,i}$. On the irreducible component $\mathcal{C}_{s,i}$, a point is called *special* if it is a nodal point or a marked point. The stability condition (3) is equivalent to the following condition on each irreducible component : if $\beta_i = 0$ and the genus of $\mathcal{C}_{s,i}$ is 0 (resp. 1) then $\mathcal{C}_{s,i}$ should have at least 3 (resp. 1) special points. So for example if $\beta_i \neq 0$ or the genus is greater than 2 there is no condition on $\mathcal{C}_{s,i}$.

In this text, we will never use the coarse moduli space of $\overline{\mathcal{M}}_{g,n}(X, \beta)$, so all the morphisms that we will use are morphisms of stacks.

Example 2.1.2. Let us give an example in genus 0 (see Figure 1). Consider the following stable map in $\overline{\mathcal{M}}_{0,5}(X, \beta)$. All the C_i are isomorphic to $\mathbb{P}^1$. The stability condition on this stable map imposes only that $\beta_2 \neq 0$ because C_2 has only 2 special points.

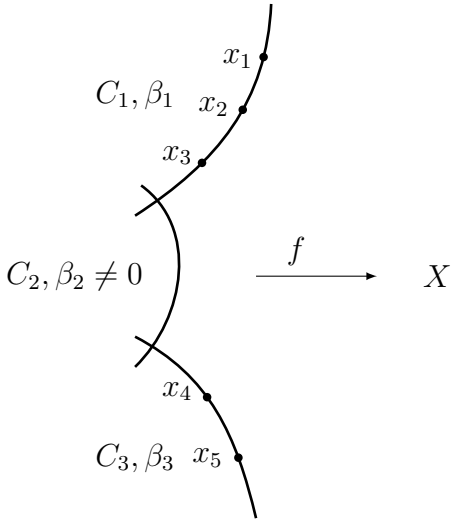


FIGURE 1. Example of a stable map

In particular, the moduli space of stable curve, denoted by $\overline{\mathcal{M}}_{g,n}$ is $\overline{\mathcal{M}}_{g,n}(\mathrm{pt}, \beta = 0)$. Notice that for $(g, n) \in \{(0, 0), (0, 1), (0, 2), (1, 0)\}$ the moduli space $\overline{\mathcal{M}}_{g,n}$ is empty.

Remark 2.1.3. There are two kinds of natural morphisms of stacks from the moduli space of stable maps.

(1) For any $i \in \{1, \ldots, n\}$, the evaluation morphism $e_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ is the evaluation at the i-th marked point i.e., it sends the geometric point $(C, x_1, \ldots, x_n, f)$ to $f(x_i)$.
(2) When $\overline{\mathcal{M}}_{g,n}$ is not empty, we define the morphism of stacks $p : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ that forgets the map and stabilises the curve that is it sends $(C, x_1, \ldots, x_n, f)$

to $(C^{\mathrm{Stab}}, x_1, \ldots, x_n)$ where C^{Stab} is obtained from C by contracting all the unstable components (see [Knu83] for the techniques). On the stable map of the example 2.1.2, forgetting the map f, the irreducible component C_2 become unstable (because it has only 2 special points). So the image by p is the following stable curve (see Figure 2).

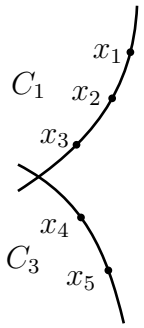


FIGURE 2. The stabilisation of the stable maps of Figure 1

Theorem 2.1.4 (Deligne-Mumford [DM69], Kontsevich-Manin [KM96], Behrend-Fantechi [BF97])**.**

(1) The moduli space $\overline{\mathcal{M}}_{g,n}$ is a proper smooth Deligne-Mumford stack of dimension $3g-3+n$.

(2) The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper (not smooth in general) Deligne-Mumford stack. It has an expected dimension (see remark below for the meaning) which is

$$\int_\beta c_1(TX) + (1-g)\dim X + 3g - 3 + n$$

(3) There exists a class, denoted by $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$, in the Chow ring $A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ of degree equal to the expected dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which satisfies some functorial properties.

Remark 2.1.5. (1) To use standard tools of intersection on the moduli space of stable maps we need this moduli space to be proper and smooth. The smoothness would give us the existence of a well-defined fundamental class. Nevertheless, the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, which is not smooth in general, could have different irreducible components of different dimensions with some very bad singularities. So the problem is to define an ersatz of a fundamental class. This was done by Behrend-Fantechi in [BF97] where they defined the *virtual fundamental class* (see § 5.6).

(2) In some very specific case the moduli space of maps is smooth : for example only in genus 0 for homogeneous variety like $\mathbb{P}^n$, grassmannian or flag varieties. In these cases, the virtual dimension is the actual dimension and the virtual fundamental class is the fundamental class.

(3) The computation of the expected dimension comes from deformation theory. Namely, a deformation of a stable maps turns to be a deformation of the underlying curve plus a deformation of the map. As $\overline{\mathcal{M}}_{g,n}$ is smooth, the deformation functor of the curve has no obstruction and the tangent space has the dimension of $\overline{\mathcal{M}}_{g,n}$ which is $3g-3+n$. For the maps, the deformation functor has a non zero obstruction. More precisely, at a point $(C, \underline{x}, f) \in \overline{\mathcal{M}}_{g,n}(X, \beta)$, the tangent space is $H^0(C, f^*TX)$ and an obstruction is $H^1(C, f^*TX)$. Making this in family, one gets two quasi coherent sheaves that are not vector bundles. Nevertheless the Euler characteristic can be computed via the Hirzebruch-Riemann-Roch theorem:

$$\chi(C, f^*TX) = \dim H^0(C, f^*TX) - \dim H^1(C, f^*TX) = \int_C \mathrm{Td}(TC)\,\mathrm{ch}(f^*TX)$$

is constant and equals to $\int_\beta c_1(TX) + (1-g)\dim X$.

We will now introduce another moduli space which was introduce by Costello [Cos06] and which will play a crucial role latter. Let $\mathrm{NE}(X)$ be the subset of $H_2(X, \mathbb{Z})$ of classes given by the image of a curve i.e. the subset of all $f_*[C]$ for any morphism $f : C \to X$. Let define $\mathfrak{M}_{g,n\beta}$ as the moduli space of nodal curve of genus g with n marked smooth points where each irreducible component C_i has a labelled β_i (notice that this β_i is not the degree of a map because there is no map from $C \to X$, it is just a labbeled. At the end of the day, it will be related to the degree of a map but not here) such that

- $\sum_i \beta_i = \beta$
- if $\beta_i = 0$ then C_i is stable i.e., if C_i is of genus 0 then it has at least 3 special points and if the genus is 1 then it has at least 1 special point.

We have a natural morphism of stacks $p : \mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ which forgets the $(n+1)-th$ marked point and contracts the irreducible components that are not stable.

Theorem 2.1.6 ([Cos06])**.** *(1) The stack $\mathfrak{M}_{g,n,\beta}$ is a smooth Artin stack.*
(2) The morphism $p : \mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ is the universal curve.

Remark 2.1.7. (1) Notice that forgetting the last marked point and contracting the unstable component gives a morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which is also the universal curve (See [Knu83]).

(2) The Artin stack of prestable[3] curves, denoted by $\mathfrak{M}^{pre}_{g,n}$ also have a universal curve which is not $\mathfrak{M}^{\mathrm{pre}}_{g,n+1}$. As there is no stability condition on the moduli space of prestable curves, forgetting a marked point never contract a rational curve. So forgetting a marked point $\mathfrak{M}^{\mathrm{pre}}_{g,n+1} \to \mathfrak{M}^{\mathrm{pre}}_{g,n}$ is not the universal curve.

Let us explain the meaning of being an universal curve of $\mathfrak{M}_{g,n,\beta}$. Let C be a curve of genus g with 4 marked points with a label β. This is equivalent by definition to

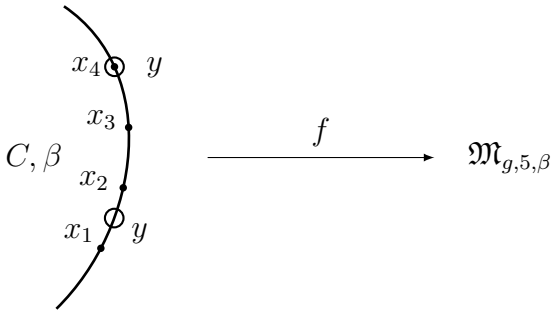
[3]where we do not ask any stability condition on irreducible components see [CK99, p.179].

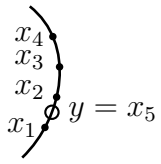
a morphism pt $\to \mathfrak{M}_{g,4,\beta}$. Being a universal curve means that we have the $C = \mathfrak{M}_{g,5,\beta} \times_{\mathfrak{M}_{g,4,\beta}} \text{pt}$ that is the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & \mathfrak{M}_{g,5,\beta} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathfrak{M}_{g,4,\beta} \end{array}$$

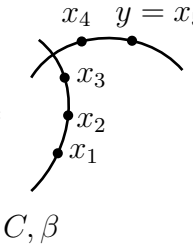
is cartesian. Let explain the morphism φ. To a smooth point $y \in C \setminus \{x_1, \ldots, x_4\}$, $f(y)$ is the curve C where y is now x_5. If $y = x_i$, then $\varphi(y)$ is the curve C where we attach a $\mathbb{P}^1$ at x_i (let's say at 0 of this $\mathbb{P}^1$) with $\beta = 0$ and you marked x_i and x_5 at 1 and ∞. If y is a node which is the intersection with C_i and C_j, then we replace the node by a $\mathbb{P}^1$ with degree 0 which meet C_i at 0, C_j at ∞ and we marked the point 1 by x_5 on this $\mathbb{P}^1$.

Here is a picture that we hope makes this clearer (see Figure 3). Forgetting the last point makes the component $(\mathbb{P}^1, \beta = 0)$ unstable so one should contract it and we get back C.



$f(y) =$ 

if y is not a marked point

$f(y) =$ 

if y is the marked point x_4

FIGURE 3. Universal curve

2.2. **Gromov-Witten classes and cohomological field theory.** We first define the Gromov-Witten classes. Let $\alpha_1, \dots, \alpha_n \in H^*(X)$. Let $\beta \in H_2(X, \mathbb{Z})$. We define the following morphism

$$\begin{aligned} \varphi_{g,n,\beta} : H^*(X) \times \cdots \times H^*(X) &\longrightarrow H^*(\overline{\mathcal{M}}_{g,n}) \\ (\alpha_1, \dots, \alpha_n) &\longmapsto p_* \left(\prod_{i=1}^{n} e_i^* \alpha_i \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \right) \end{aligned}$$

Theorem 2.2.1 (Kontsevich-Manin [KM96])**.** *All these maps* $\{\varphi_{g,n,\beta}\}_{g,n\in\mathbb{N},\beta\in H_2(X,\mathbb{Z})}$ *together form a cohomological field theory.*

Remark 2.2.2. (1) We refer to [KM96] for a complete definition of a cohomological field theory.

(2) Unwindy the definition, is the so-called splitting property. Let $g_1, g_2, n_1, n_2 \in \mathbb{N}$. Denote by $g = g_1 + g_2$ and $n = n_1 + n_2$. Consider the gluing morphism of stacks

$$\begin{aligned} g : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} &\to \overline{\mathcal{M}}_{g,n} \\ (C_1, C_2) &\longmapsto C_1 \circ C_2 \end{aligned} \tag{2.2.3}$$

that identifies the $n_2 + 1$-th marked point of C_2 with the first marked point of C_1. Notice that the gluing morphism above is given by the pushout. More precisely, let $(C_1 \to S, \underline{\sigma})$ in $\overline{\mathcal{M}}_{g_1,n_1+1}(S)$ and $(C_2 \to S, \underline{\sigma})$ in $\overline{\mathcal{M}}_{g_2,n_2+1}(S)$ then $C_1 \circ C_2$ is the pushout[4] $C_1 \coprod_S C_2$ given by the two closed immersion given by the marking $\sigma_1 : S \to C_1$ and $\sigma_{n_2+1} : S \to C_2$.

This corresponds to the following picture

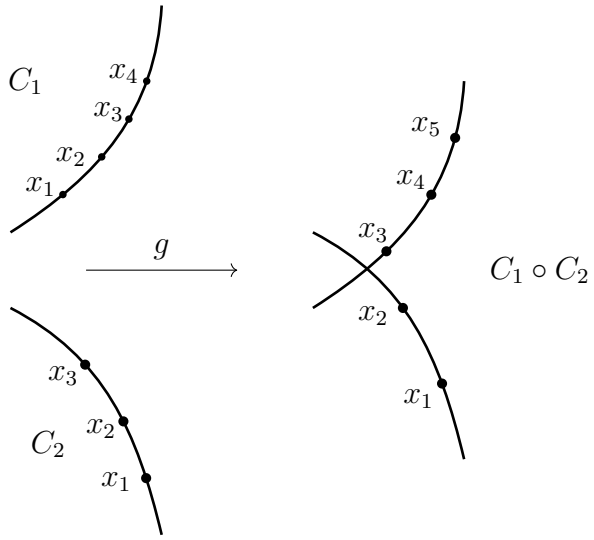


FIGURE 4. Gluing curves: the output of C_2, that is x_3, with the first input of C_1

[4]Notice that pushouts do not exist for any morphisms of schemes in the category of schemes but pushout along closed immersion does exist.

The splitting formula is the following

(2.2.4)

$$g^*\varphi_{g,n,\beta}(\alpha_1,\dots,\alpha_n)=\sum_{\substack{g_1+g_2=g\\ \beta_1+\beta_2=\beta}}\sum_{a=0}^{s}\varphi_{g_1,n_1+1,\beta_1}(\alpha_1,\dots,\alpha_{n_1},T_a)\varphi_{g_2,n_2+1,\beta_2}(T^a,\alpha_{n_1+1},\dots,\alpha_n)$$

where $(T_a)_{a\in\{0,\dots,s\}}$ is a basis of $H^*(X)$ and (T^a) is its Poincaré dual basis.

Beyond this formula, the idea is that we can control the behaviour of the virtual fondamental class when we glue curves. We will see this again later.

Restricting to genus 0, we can reformulate this equality (2.2.4) by the following statement.

Corollary 2.2.5. *We have a morphism of operads in vector spaces*

$$\psi_{n,\beta}: H_*(\overline{\mathcal{M}}_{0,n+1})\to \mathrm{End}(H^*(X))[n]:=\mathrm{Hom}(H^*(X)^{\otimes n},H^*(X))$$

given by

$$\psi_{0,n,\beta}(\gamma)(\alpha_1,\dots,\alpha_n)=(e_{n+1})_*\left(p^*\gamma\cup\prod_{i=1}^{n}e_i^*\alpha_i\cap[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}\right)$$

Another way of expressing exactly the same statement is to say that the cohomology $H^*(X)$ is an $\{H_*(\overline{\mathcal{M}}_{0,n+1})\}_{n\geq 2}$-algebra. The goal of this survey is to explain how to remove the (co)homology from this corollary and doing this at the geometrical level.

2.3. **Reviewed on operads.** We add this section for completeness as operads are not so well known to algebraic geometer[5].

An operad is the following data :

(1) A family of objects in a category (vector spaces, schemes or Deligne-Mumford stacks) $\mathcal{O}(n)$ for all $n\in\mathbb{N}$. The example that one should have in mind for this note is $\mathcal{O}(n)=\overline{\mathcal{M}}_{0,n+1}$. We should think that $\mathcal{O}(n)$ as a collection of operations, each with n inputs and one output. In the case of $\overline{\mathcal{M}}_{0,n+1}$, the marked points $x_1,\dots,x_n$ can be thought as the inputs and the last marked points, x_{n+1}, is thought as the output.

(2) A collection of operations: putting the output of $\mathcal{O}(b)$ with the i-th input of $\mathcal{O}(a)$. Let $a,b\in\mathbb{N}$, for any $i\in\{1,\dots,a\}$, we have

(2.3.1)
$$\circ_i:\mathcal{O}(a)\times\mathcal{O}(b)\to\mathcal{O}(a+b-1)$$

satisfying some relations like associativity of the compositions.

Example 2.3.2. We give three examples of operads that we will use in the next sections.

[5]The first author did not know this notion before the working seminar in Montpellier where these ideas were first discussed.

(1) The example $\mathcal{O}(n) = \overline{\mathcal{M}}_{0,n+1}$ is an operads in DM stacks where the composition $C_1 \circ_i C_2$ is obtained by gluing the last marked point of C_2 to the i-th marked point of C_1 (see (2.2.3) and Figure 4 for an example of $\circ_1$ with stable curves). Notice that here $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are empty. A standard way of completing this is to put $\mathcal{O}(0) = \mathcal{O}(1) = \mathrm{pt}$ so that $\mathcal{O}(1)$ is the unit.
(2) Another example of operads that we will use is $\mathcal{O}_\beta(n) = \mathfrak{M}_{0,n+1,\beta}$. This is a graded operad that is in the composition (2.3.1), we sum the grading :

$$\circ_i : \mathcal{O}_\beta(a) \times \mathcal{O}_{\beta'}(b) \to \mathcal{O}_{\beta+\beta'}(a+b-1)$$

The composition morphism for this operad is by gluing the curves as in the previous example.
(3) Let V a vector space. Put $\mathcal{O}(n) = \mathrm{End}(V)[n] := \mathrm{Hom}(V \times \cdots \times V, V)$. This is called the endomorphism operad in vector spaces. The composition is given by

$$(f \circ_i g)(v_1, \ldots, v_{a+b-1}) = f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+b-1}), v_{i+b} \ldots, v_{a+b-1})$$

Let $\mathcal{O} := \{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ and $\mathcal{E} := \{\mathcal{E}(n)\}_{n\in\mathbb{N}}$ be two operads. A morphism of operads from $f : \mathcal{O} \to \mathcal{E}$ is a family of morphism $f_n : \mathcal{O}(n) \to \mathcal{E}(n)$ such that the following diagram is commutative

(2.3.3)
$$\begin{array}{ccc} \mathcal{O}(a) \times \mathcal{O}(b) & \xrightarrow{f_a, f_b} & \mathcal{E}(a) \times \mathcal{E}(b) \\ \downarrow{\scriptstyle \circ_i} & & \downarrow{\scriptstyle \circ_i} \\ \mathcal{O}(a+b-1) & \xrightarrow{f_{a+b-1}} & \mathcal{E}(a+b-1) \end{array}$$

3. Lax algebra structure on X

In Corollary 2.2.5, we have a collection of morphisms $H_*(\overline{\mathcal{M}}_{0,n+1}) \to \mathrm{End}(H^*(X))[n]$ that form a morphism of operads. The idea is to remove the (co)homology from this statement, that is, to construct in a purely geometrical way, a collection morphisms $\overline{\mathcal{M}}_{0,n+1} \to \mathrm{End}(X)[n]$ in an appropriate category and then to see if these morphisms form a morphism of operads. The correct category is the $(\infty, 1)$-category of derived stacks and the morphism is only a lax morphism of ∞-operads (see Theorem 3.1.2).

3.1. **Main result.** Denote by $\mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X, \beta)$ the derived enhancement of $\overline{\mathcal{M}}_{0,n+1}(X, \beta)$ (see subsection 3.3). From the two natural morphisms of Remark 2.1.3, we get the following diagram

(3.1.1)
$$\begin{array}{ccc} & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X, \beta) & \\ {\scriptstyle p, e_1, \ldots, e_n}\swarrow & & \searrow{\scriptstyle p, e_{n+1}} \\ \overline{\mathcal{M}}_{0,n+1} \times X^n & & \overline{\mathcal{M}}_{0,n+1} \times X \end{array}$$

We prefer to state our theorem and then give explanations about it.

Theorem 3.1.2. *Let X be a smooth projective variety. The diagram* (3.1.1) *give a family of morphisms*

$$\varphi_n : \overline{\mathcal{M}}_{0,n+1} \to \underline{\mathrm{End}}^{\mathrm{cor}}(X)[n] := \underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)$$

that forms a lax morphism of ∞-operads in the category of derived stacks.

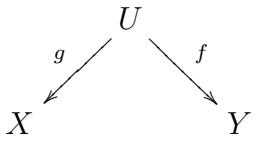
Remark 3.1.3. In more conceptual terms, X is lax $\{\overline{\mathcal{M}}_{0,n+1}\}_n$-algebra in the category of correspondence in derived stack.

In the next sections, we will explain the contents of this theorem, namely

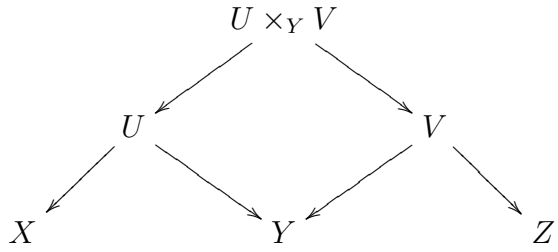
- In §3.2, we define the notion of correspondances in a cateogry.
- In §3.3, we define the natural derived enhancement of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and in 3.3.2, we explain the underlying notation $\underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)$.
- In §3.4, we explain what is a lax morphism between ∞-operads.
- Th notion of ∞-operads is a bit delicat and it is explain in In §4.1.1.

3.2. **Category of correspondances.** Let $\mathbf{dSt_C}$ be the ∞-category of derived stacks. We denote $\mathbf{dSt}_{\mathbf{C}}^{\mathrm{cor}}$ the $(\infty, 2)$-category of correspondences in derived stack which is defined informally as follows (See §10 in [DK12]). To have a formal definition, we refer to the notion of span in the website nLab.

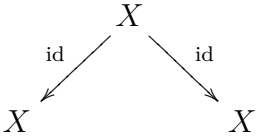
(1) Object of $\mathbf{dSt}_{\mathbf{C}}^{\mathrm{cor}}$ are objects of $\mathbf{dSt_C}$.
(2) The 1-morphism of $\mathbf{dSt}_{\mathbf{C}}^{\mathrm{cor}}$ between X and Y, denoted by $X \dashrightarrow Y$, is a diagram



There is no condition on f or g. The composition is given by fiber product



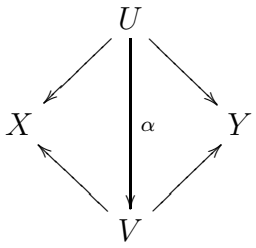
Notice that a morphism from X to Y is also a morphism from Y to X but the composition is not the identity which is



Hence a morphism of scheme $f : X \to Y$ induces a morphism $X \dashrightarrow Y$ in correspondances given by $\mathrm{id}_X : X \to X$ and $f : X \to Y$. This morphism

$X \dashrightarrow Y$ is an isomorphism if and only if we have $X = X \times_Y X$ i.e., f is a monomorphism.

(3) The 2-morphisms are not necessarily isomorphisms, they are $\alpha : U \to V$ that make the diagram commutative.



The diagram

$$\begin{array}{ccc} & \mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X,\beta) & \\ {\scriptstyle e_1,\dots,e_n}\swarrow & & \searrow{\scriptstyle e_{n+1}} \\ X^n & & X \end{array} \tag{3.2.1}$$

is by definition a morphism in $\mathbf{dSt}^{\mathrm{cor}}_{\mathbf{C}}$ between $X^n \dashrightarrow X$. Notice that the object that makes the correspondence is a derived stack so we need to be in the category of $\mathbf{dSt}^{\mathrm{cor}}_{\mathbf{C}}$ and not in the category of correspondence in schemes (or Deligne-Mumford stacks).

3.3. **Derived enhancement.**

3.3.1. *Derived enhancement of* $\mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X,\beta)$. Here we follow the idea of Schürg-Toën-Vezzosi [STV15] with a small modification. Let $g, n \in \mathbb{N}$ and $\beta \in H_2(X,\beta)$. Recall the definition of $\mathfrak{M}_{g,n,\beta}$ the moduli space defined before Theorem 2.1.6. We denote the relative internal hom in derived stacks by

$$\mathbb{R}\,\mathrm{Hom}_{\mathrm{dst}_k/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta}) \tag{3.3.1}$$

As $\mathfrak{M}_{g,n+1,\beta} \to \mathfrak{M}_{g,n,\beta}$ is the universal curve, a point in $\mathbb{R}\,\mathrm{Hom}_{\mathrm{dst}_k/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta})$ is by definition a morphism from $f : C \to X$ where $[C] \in \mathfrak{M}_{g,n,\beta}$. Notice that the degree f is not related for the moment to β. The truncation of (3.3.1) is

$$\mathrm{Hom}_{\mathrm{dst}_k/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta})$$

and inside it, we have an immersion

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \hookrightarrow \mathrm{Hom}_{\mathrm{dst}_k/\mathfrak{M}_{g,n,\beta}}(\mathfrak{M}_{g,n+1,\beta}, X \times \mathfrak{M}_{g,n,\beta}) \tag{3.3.2}$$

given by stable maps $(\mathcal{C}, \underline{\sigma}, f : \mathcal{C} \to X)$ such that the degree of f on each irreducible component $\mathcal{C}_i$ of $\mathcal{C}$, the degree of $f\mid_{\mathcal{C}_i}$ is β_i i.e., we have the equality $(f\mid_{\mathcal{C}_i})_*[\mathcal{C}_i] = \beta_i$. This immersion is open because the degree is discrete.

Using the following result of Schuïg-Toën-Vezzosi, we have

Proposition 3.3.3 (Proposition 2.1 in [STV15]). *Let X be in $\mathbf{dSt_C}$ and an open immersion of $Y \hookrightarrow t_0(X)$ where $t_0(X)$ is the truncation of X. Then there exists a unique derived enhancement of Y, denoted by $\widehat{Y}$, such that the following diagram is cartesian*

$$\begin{array}{ccc} Y & \overset{open}{\hookrightarrow} & t_0(X) \\ {\scriptstyle closed}\downarrow & & \downarrow{\scriptstyle closed} \\ \widehat{Y} & \overset{open}{\hookrightarrow} & X \end{array}$$

Taking $Y = \overline{\mathcal{M}}_{g,n}(X,\beta)$ and the open immersion (3.3.2), we get a derived enhancement, which we denote by $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Remark 3.3.4. To define the derived enhancement of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, Schürg-Toën-Vezzosi (see [STV15]) used the moduli space of prestable curve denoted by $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ instead of the moduli space of Costello $\mathfrak{M}_{g,n,\beta}$ in (3.3.1). So they use the universal curve of $\mathfrak{M}^{\mathrm{pre}}_{g,n}$ in (3.3.1) instead of $\mathfrak{M}_{g,n+1,\beta}$. As we will see in the proof (see section 4), the fact that $\mathfrak{M}_{g,n+1,\beta}$ is the universal curve is fundamental, that is the reason why we made this little change.

Notice that their derived enhancement is the same as ours as the morphism $\mathfrak{M}_{g,n,\beta} \to \mathfrak{M}_{g,n}$ is étale (See [Cos06]).

3.3.2. *Definition of* $\underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)$. The underling notation means the internal hom $\underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)$. To be more precise, it is the sheaf

$$\underline{\mathrm{Hom}}^{\mathrm{cor}}(X^n, X)(U) := \mathrm{Hom}^{\mathrm{cor}}(X^n \times U, X \times U)$$

It turns out that this is a derived stack because $\mathrm{Hom}^{\mathrm{cor}}(X^n \times U, X \times U)$ is the same as the category of derived stack over $X^{n+1} \times U$.

By Yoneda's lemma, the morphism φ_n of Theorem 3.1.2 is exactly given by an object in $\mathrm{Hom}^{\mathrm{cor}}(X^n \times \overline{\mathcal{M}}_{0,n+1}, X \times \overline{\mathcal{M}}_{0,n+1})$ which is the diagram (3.1.1).

3.4. **Lax morphism.** Recall that a classical morphism of operad is a commutative diagram (2.3.3). A *lax morphism* is given by a collection of 2-morphisms $(\alpha_{a,b})_{a,b\in\mathbb{N}}$ which are not an isomorphism.

$$\begin{array}{ccc} \mathcal{O}(a) \times \mathcal{O}(b) & \xrightarrow{(f_a, f_b)} & \mathcal{E}(a) \times \mathcal{E}(b) \\ {\scriptstyle \circ_i}\downarrow & \swarrow{\scriptstyle \alpha_{a,b}} & \downarrow{\scriptstyle \circ_i} \\ \mathcal{O}(a+b-1) & \xrightarrow{f_{a+b-1}} & \mathcal{E}(a+b-1) \end{array}$$

In the following, we will explain why the Theorem 3.1.2 is lax in geometrical term . Let $\sigma \in \overline{\mathcal{M}}_{0,a+1}$ and $\tau \in \overline{\mathcal{M}}_{0,b+1}$. Denote by $\mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta)$ (resp. $\mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,a+1}(X,\beta)$) the inverse image of $p^{-1}(\sigma)$ (resp. $p^{-1}(\tau)$).

The composition is $f_{a+b-1} \circ \circ_i$ given by

(3.4.1)
$$\begin{array}{ccc} & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}_{0,a+b+1}(X,\beta) & \\ \swarrow & & \searrow \\ X^{a+b} & & X \end{array}$$

The second composition morphism $\circ_1 \circ (f_a, f_b)$ is given by the following fibered product. Let β', β'' such that $\beta' + \beta'' = \beta$.

(3.4.2)
$$\begin{array}{ccccc} & & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta) \times_X \coprod_\beta \mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,b+1}(X,\beta) & & \\ & \swarrow & & \searrow & \\ & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta) \times X^b & & \coprod_\beta \mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,a+1}(X,\beta) & \\ {\scriptstyle e_1,\dots,e_a,\mathrm{id}_{X^a}} \swarrow & & \searrow {\scriptstyle e_{a+1},\mathrm{id}_{X^b}} & {\scriptstyle e_1,\dots,e_b} \swarrow & & \searrow {\scriptstyle e_{b+1}} \\ X^a \times X^b & & X \times X^b & & X \end{array}$$

Let fix β. Finally, the 2-morphism α is given by the gluing morphism

$$\alpha : \coprod_{\substack{\beta',\beta'' \\ \beta'+\beta''=\beta}} \mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta') \times_X \mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,b+1}(X,\beta'') \to \mathbb{R}\overline{\mathcal{M}}^{\sigma\circ\tau}_{0,a+b+1}(X,\beta) \tag{3.4.3}$$

Notice that we can glue the stable maps denoted by $(C, x_1, \dots, x_{a+1}, f)$ and $(\widetilde{C}, \widetilde{x}_1, \dots, \widetilde{x}_{b+1}, \widetilde{f})$ because the fiber product is over X which means that $f(x_{a+1}) = \widetilde{f}(\widetilde{x}_1)$. This morphism α is surjective but not injective on points. To see the non injectivity, consider Figure 3.4, then the gluing curves are the same. Notice that by stability condition, we have $\beta_2 \neq 0$. The two couple of curves $(C_1 \circ C_2, C_3)$ and $(C_1, C_2 \circ C_3)$ are in two different connected components of

$$\coprod_{\substack{\beta',\beta'' \\ \beta'+\beta''=\beta}} \mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta') \times_X \mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,b+1}(X,\beta'').$$

4. Proof of our main result

4.1. **Brane action.** In this section, we explain the main theorem of [Toë13]. This theorem has a lot of prerequisites (like ∞-operads, unital and coherent operads) that are too complicated for this survey. We refer to the definition of ∞-operads by Lurie [Lur14, Definition 2.1.1.8] and to the Definition 3.3.1.4 for the notion of coherent ∞-operad.

Theorem 4.1.1 (see Theorem [Toë13])**.** *Let $\mathcal{O}^\otimes$ be an ∞-operad in the ∞-category of spaces such that*

(1) $\mathcal{O}^\otimes(0) = \mathcal{O}^\otimes(1)$ are contractible.
(2) the operad is unital and coherent

Then $\mathcal{O}(2)$ is a $\mathcal{O}^\otimes$-algebra in the ∞-category of co-correspondence.

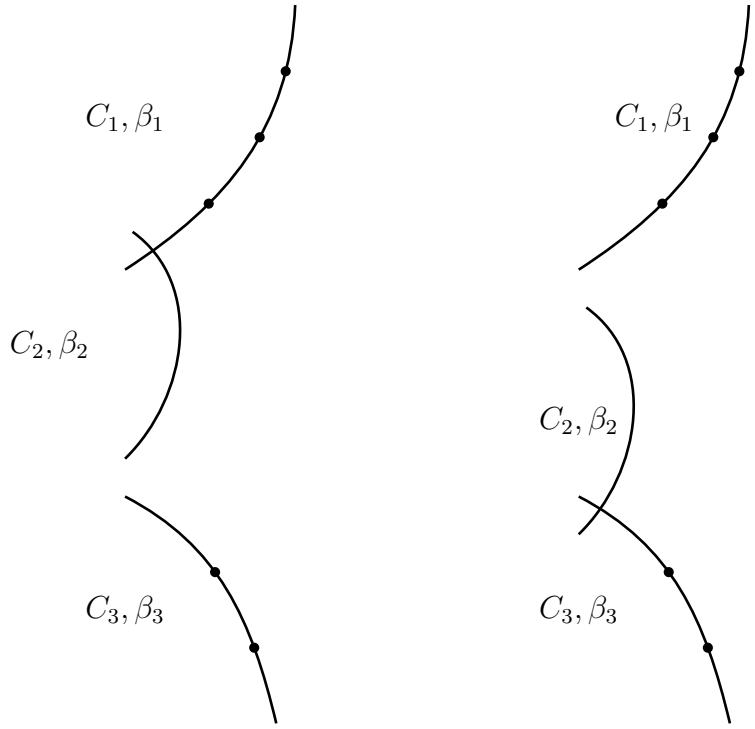


FIGURE 5. Geometric reason of the lax action

Example 4.1.2. We will illustrate the hypothesis and the conclusion of this theorem for the operad $\mathcal{O}(n) := \overline{\mathcal{M}}_{0,n+1}$. We choose this example because it is a well-known operad and it is easier to explain. Notice that to prove (see §4.2.1) our main theorem, we need to apply to an other operad which is $\coprod_\beta \mathfrak{M}_{0,n+1,\beta}$ but the main ideas are the same. Notice that we set $\overline{\mathcal{M}}_{0,1} = \overline{\mathcal{M}}_{0,2} := \mathrm{pt}$ (with the usual definition they are empty). By definition, we impose that $\mathcal{O}(1)$ is the unit. For the operad $\mathcal{O}$, the following diagram is cartesian (See below for an explanation).

$$\begin{array}{ccc} \mathcal{O}(n) \times \mathcal{O}(m+1) \coprod_{\mathcal{O}(2)\times\mathcal{O}(n)\times\mathcal{O}(m)} \mathcal{O}(n+1) \times \mathcal{O}(m) & \longrightarrow & \mathcal{O}(n+m) \\ \downarrow q & & \downarrow p \\ \mathcal{O}(n) \times \mathcal{O}(m) & \xrightarrow{\circ} & \mathcal{O}(n+m-1) \end{array}$$

This property was called of "configuration type" in [Toë13]. Notice that in the context of [Lur14, Definition 3.3.1.4], this notion was called "coherent". As p is flat, we need to prove that it is a cartesian diagram in the stack category. Let $(C_1, x_1, \ldots, x_{n+1})$ be in $\mathcal{O}(n)$ and $(C_2, y_1, \ldots, y_{m+1})$ be in $\mathcal{O}(m)$. As $\mathcal{O}(n+1) \to \mathcal{O}(n)$ is the universal curve, we deduce that $q^{-1}(C_1, C_2) = C_1 \coprod_{\mathrm{pt}} C_2$ which is exactly $C_1 \circ C_2$. This implies that the diagram above is cartesian.

Let us explain now the conclusion of this theorem. Notice that $\mathcal{O}(2) = \overline{\mathcal{M}}_{0,3}$ is a point. The statement means that we have a morphism of ∞-operad that is a family of

morphisms

$$\varphi_n : \mathcal{O}(n) \to \underline{\mathrm{Hom}}^{\mathbf{CoCorr}}(\coprod_{i=1}^{n} \mathcal{O}(2), \mathcal{O}(2))$$

where the morphism (φ_n) are compatible with the composition law. The $\underline{\mathrm{Hom}}$ is the same meaning that in §3.3.2. The category of co-correspondances is in the same spirit as correspondance (See §3.2) but with the arrows in the other directions. The morphism φ_n is given by the following diagram

$$\begin{array}{ccccc} \mathcal{O}(n) \times \coprod_{i=1}^{n} \mathcal{O}(2) & \xrightarrow{\circ} & \mathcal{O}(n+1) & \xleftarrow{\circ'} & \mathcal{O}(2) \times \mathcal{O}(n) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{O}(n) & & \end{array} \tag{4.1.3}$$

Let explain this diagram with $\mathcal{O}(n) = \overline{\mathcal{M}}_{0,n+1}$. We have

(1) The morphism $\mathcal{O}(n+1) \to \mathcal{O}(n)$ is to forget the last marked point.
(2) The map $\circ : \mathcal{O}(n) \times \coprod_{i=1}^{n} \mathcal{O}(2) \to \mathcal{O}(n+1)$ is given by the n possible gluings of the third marked point of $\mathcal{O}(2) = \overline{\mathcal{M}}_{0,3}$ with one of the marked points x_i for $i \in \{1, \dots, n\}$ in $\mathcal{O}(n)$.
(3) The $\circ'$ is the gluing of last marked point x_{n+1} of $\mathcal{O}(n)$ with the third of $\mathcal{O}(2)$.

4.2. **Sketch of proof of Theorem 3.1.2.** In this section, we explain how to apply Theorem 4.1.1 to get our main theorem.

Here we take $\mathcal{O}(n) = \coprod_\beta \mathfrak{M}_{0,n+1,\beta}$. This is an operad in algebraic stack. One can check that all we said before in the previous section for $\overline{\mathcal{M}}_{0,n+1}$ works as well for $\coprod_\beta \mathfrak{M}_{0,n+1,\beta}$.

Let X be a smooth projective variety. We apply the functor $\mathbb{R}\,\mathrm{Hom}_{/\mathfrak{M}_{0,n+1,\beta}}(-, X \times \mathfrak{M}_{0,n+1,\beta})$ to Theorem 4.1.1. As the source curve of a stable map may not be a stable curve, we need to use Theorem 4.1.1 with an other operad than $\overline{\mathcal{M}}_{0,n+1}$. That's why we use $\coprod_\beta \mathfrak{M}_{0,n+1,\beta}$. We deduce the following result.

Theorem 4.2.1. *The variety X is an $\mathfrak{M}^{\otimes}$-algebra in the category of correspondances in derived stacks. The algebra structure is given by the*

$$\begin{array}{ccc} & \mathbb{R}\overline{\mathcal{M}}_{0,n+1}(X, \beta) & \\ \swarrow & & \searrow \\ X^n \times \mathfrak{M}_{0,n+1,\beta} & & X \times \mathfrak{M}_{0,n+1,\beta} \end{array}$$

Remark 4.2.2. To apply Theorem 4.1.1, we need to do several modifications

(1) Notice that in this statement, the action is strong that means that the lax morphisms are equivalences (See §3.4). The geometrical reason is the following. We can repeat the construction of §3.4 replacing $\overline{\mathcal{M}}_{0,n+1}$ by $\mathfrak{M}_{0,n,\beta}$. The difference is that the forgetting morphism $q : \overline{\mathcal{M}}_{0,n+1}(X, \beta) \to \overline{\mathfrak{M}}_{0,n+1,\beta}$ does not contract

any component of the curve. More precisely, let $\sigma \in \mathfrak{M}_{0,a+1,\beta}$ and $\tau \in \mathfrak{M}_{0,b+1,\beta'}$. Denote by

$$\mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta') = q^{-1}(\sigma).$$

Take care that in §3.4, we use $\mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta') = p^{-1}(\sigma)$ where $p : \overline{\mathcal{M}}_{0,n+1}(X,\beta) \to \overline{\mathcal{M}}_{0,n+1}$. Writing the same kind of diagram as (3.4.2) we get the corresponding α given by

$$\widetilde{\alpha} : \mathbb{R}\overline{\mathcal{M}}^{\sigma}_{0,a+1}(X,\beta') \times_X \mathbb{R}\overline{\mathcal{M}}^{\tau}_{0,b+1}(X,\beta'') \to \mathbb{R}\overline{\mathcal{M}}^{\sigma\circ\tau}_{0,a+b+1}(X,\beta) \tag{4.2.3}$$

which is now an isomorphism because from the glued curve, there is a unique possibility to cut it with respect to σ and τ.

(2) First, Theorem 4.1.1 apply only to operads in spaces and here we have operads in derived stacks. This can be done using non-planar rooted trees and dendroidal sets. More precisely, one can enrich ∞-operads using Segal functor from the nerve of Ω^{op} to derived stacks. Thanks to the work of [CHH16] and [HHM13] these two definitions coincide on topological spaces.

(3) Second, the condition $\mathcal{O}(0) = \mathcal{O}(1) = \mathrm{pt}$ is not satisfied by $\mathfrak{M}_{0,n,\beta}$. So we impose that for any $\beta \neq 0$, $\mathfrak{M}^{\mathrm{fake}}_{0,1,\beta} = \mathfrak{M}^{\mathrm{fake}}_{0,2,\beta} = \emptyset$ and that $\mathfrak{M}^{\mathrm{fake}}_{0,1,0} = \mathfrak{M}^{\mathrm{fake}}_{0,2,0} = \mathrm{pt}$ is with $\mathfrak{M}^{\mathrm{fake}}_{0,2,0}$ being the neutral element.

(4) An other issue is that $\mathfrak{M}_{0,n,\beta}$ is not a coherent operad because the inclusion of schemes in derived stacks does not commute with pushouts even along closed immersion. We only have a canonical morphism

$$\theta : C_1 \overset{dst}{\coprod_{\mathrm{pt}}} C_2 \to C_1 \overset{sch}{\coprod_{\mathrm{pt}}} C_2$$

Nevertheless, most of the proof of Theorem 4.1.1 is still valid and we know that the functor $\mathbb{R}\,\mathrm{Hom}(-,X)$ will see θ as an equivalence.

The next step in order to prove Theorem 3.1.2 is to understand the morphism of operads

$$\coprod_{\beta} \mathfrak{M}_{0,n+1,\beta} \to \overline{\mathcal{M}}_{0,n+1}.$$

Embedding this morphism in the ∞-operads, it turns out that this morphism is a lax morphism of operads. This is the reason why the final action in Theorem 3.1.2 is lax.

5. Comparison with other definition

5.1. **Quantum product in cohomology and in G_0-theory.** In this section, we review the definition of the quantum product in cohomology and in G_0-theory. Recall that X is a smooth projective variety. Givental-Lee defined in [Lee04] the Gromov-Witten invariants in G_0-theory. For that they defined a virtual structure sheaf, denoted by $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$, on the moduli space of stable maps. Recall the morphism $e_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to$

X are the evaluation morphism at the i-th marked point. For any $E_1, \dots, E_n \in G_0(X)$, the Gromov-Witten invariants in G_0-theory are

$$\langle E_1, \dots, E_n \rangle_{0,n,\beta}^{G_0} := \chi\left(\bigotimes_{i=1}^{n} e_i^* E_i \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,n}(X,\beta)}\right) \in \mathbb{Z}$$

where $\chi(.)$ is the Euler characteristic.

Let $\mathrm{NE}(X)$ be the Neron-Severi group of X that is the subset of $H_2(X,\mathbb{Z})$ generated by image of curves in X.

Definition 5.1.1. *Let $\gamma_1, \gamma_2 \in H^*(X)$. The quantum product in $H^*(X)$ is defined by*

$$\gamma_1 \bullet^{H^*} \gamma_2 = \sum_{\beta \in \mathrm{NE}(X)} Q^\beta \mathrm{ev}_{3*}\left(\mathrm{ev}_1^* \gamma_1 \cup \mathrm{ev}_2^* \gamma_2 \cap [\overline{\mathcal{M}}_{0,3}(X,\beta)]^{\mathrm{vir}}\right). \tag{5.1.2}$$

One can see this product as a formal power series in Q. Hence, the quantum product lies in $H^*(X) \otimes \Lambda$ where Λ is the Novikov ring i.e., it is the algebra generated by Q^β for $\beta \in \mathrm{NE}(X)$.

We will recall the definition of the virtual class $\left[\overline{\mathcal{M}}_{0,n}(X,\beta)\right]^{\mathrm{vir}}$ (defined by Behrend-Fantechi) and the virtual sheaf $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ (defined by Lee [Lee04]) in §5.4 and §5.5.

In G_0-theory, we define the quantum product with the following formula.

Definition 5.1.3. *Let $F_1, F_2 \in G_0(X)$. The quantum product in G_0-theory is defined to be the element in $G_0(X) \otimes \Lambda$*

$$F_1 \bullet^{G_0} F_2 = \sum_{\beta \in \mathrm{NE}(X)} Q^\beta \mathrm{ev}_{3*}\left(\mathrm{ev}_1^* F_1 \otimes \mathrm{ev}_2^* F_2 \otimes \sum_{r \in \mathbb{N}} \sum_{\substack{(\beta_0,\dots,\beta_r)| \\ \sum \beta_i = \beta}} (-1)^r \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,3}(X,\beta_0)} \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta_1)} \cdots \otimes \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta_r)}\right)$$

The term $r = 0$ in the formula in Definition 5.1.3 is of the same shape (5.1.2). One has to understand the other terms, i.e. $r > 0$, are "corrections terms".

5.2. **About the associativity.** The most important property of these two products is the associativity. It is proved by Kontsevich-Manin [KM96] (See also [FP97]) that the quantum product in cohomology is associative. Notice that the key formula for the associativity is given in Theorem 5.3.4 which states that virtual classes behave with respect to the morphisms α's and the gluing morphisms. Recall that the morphisms α's are the one that appear in the lax action (3.4.3).

Later, when Givental and Lee (See [Lee04]) try to define a quantum product in G_0-theory they want an associative product. If one put the same kind of formula as in (5.1.2), the product is not associative. Hence the key observation of Givental and Lee is Theorem 5.3.9 which is the analogue of Theorem 5.3.4 in G_0-theory that is how the virtual sheaves behave with respect to the morphisms α's and the gluing morphisms.

Our contribution to this question is Theorem 5.3.11 which is the geometric explanation that explains the two Theorems 5.3.9 and 5.3.9.

Notice that Givental-Lee packed the complicated formula of 5.1.3 in a very clever way. Notice that $\overline{\mathcal{M}}_{0,2}(X,\beta)=\overline{\mathcal{M}}_{0,2}\times X$ is empty if $\beta=0$. As before put $\overline{\mathcal{M}}_{0,2}=\mathrm{pt}$. Then we put

$$\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}}:=\mathcal{O}_X+\sum_{\substack{\beta\in \mathrm{NE}(X)\\ \beta\neq 0}} Q^{\beta}\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta)}\in G_0(X)\otimes\Lambda \tag{5.2.1}$$

Let invert the Formula above formally in $G_0(X)\otimes\Lambda$. The terms in front of Q^{β} is

$$\sum_{r\in\mathbb{N}}\sum_{\substack{(\beta_0,\dots,\beta_r)|\\ \sum\beta_i=\beta}}(-1)^r\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta_0)}\otimes\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta_1)}\cdots\otimes\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{0,2}(X,\beta_r)} \tag{5.2.2}$$

The Formula (5.2.1) and (5.2.2) are the reason of the "metric" (See Formula (16) in [Lee04] for more details) because one can express in a compact form the Formula (5.1.3) using the inverse of the metric.

5.3. **Key diagram.** Let us consider the following homotopical fiber product. Let $n_1,n_2\in\mathbb{N}_{\geq 2}$. Put $n=n_1+n_2$.

$$\begin{array}{ccc} Z_{\beta} & \longrightarrow & \mathbb{R}\overline{\mathcal{M}}_{0,n}(X,\beta)\\ \downarrow & & \downarrow p\\ \overline{\mathcal{M}}_{0,n_1+1}\times\overline{\mathcal{M}}_{0,n_2+1} & \xrightarrow{g} & \overline{\mathcal{M}}_{0,n}\end{array} \tag{5.3.1}$$

The fiber over a point (σ,τ) is denoted by $\overline{\mathcal{M}}^{\sigma\circ\tau}(X,\beta)$ in § 3.4 that is stable maps where the curve stabilise to $\sigma\circ\tau$. In Figure 6, we have an example of a fiber over $\sigma\circ\tau$ where we have a tree of $\mathbb{P}^1$ in the middle.

Using the universal property of the fiber product we get the morphism (see (3.4.3))

$$\alpha:\coprod_{\beta'+\beta''=\beta}\mathbb{R}\overline{\mathcal{M}}_{0,n_1+1}(X,\beta')\times_X\mathbb{R}\overline{\mathcal{M}}_{0,n_2+1}(X,\beta'')\to Z_{\beta} \tag{5.3.2}$$

where the left hand side is defined by the following homotopical fiber product

(5.3.3)

$$\begin{array}{ccc} \mathbb{R}\overline{\mathcal{M}}_{0,n_1+1}(X,\beta')\times_X\mathbb{R}\overline{\mathcal{M}}_{0,n_2+1}(X,\beta'') & \longrightarrow & \mathbb{R}\overline{\mathcal{M}}_{0,n_1+1}(X,\beta')\times\mathbb{R}\overline{\mathcal{M}}_{0,n_2+1}(X,\beta'')\\ \downarrow & & \downarrow e_1,e_{n_2+1}\\ X & \xrightarrow{\Delta} & X\times X\end{array}$$

The heart of the associativity of the quantum products in cohomology (see Theorem 5.3.9 for G_0-theory) is the following statement.

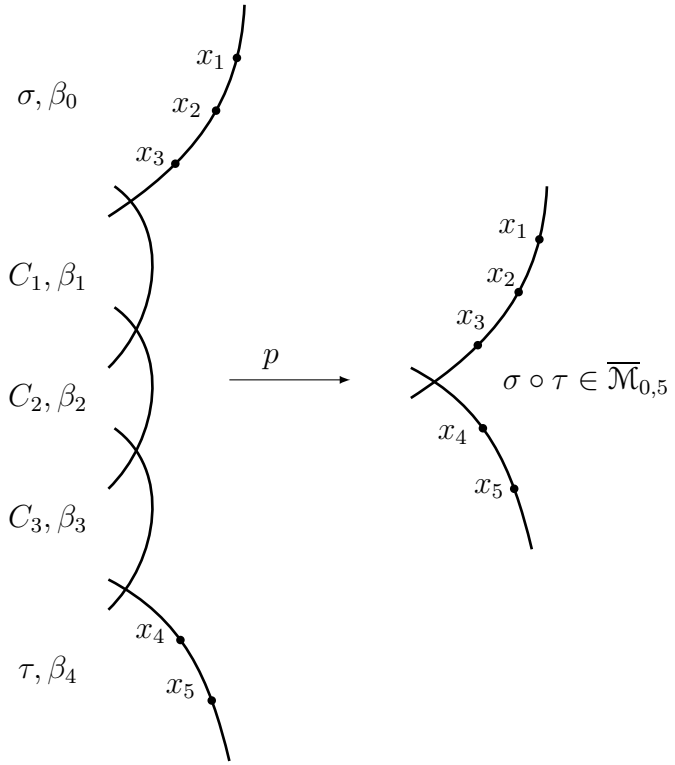


FIGURE 6. Example of a stable map above $\sigma \circ \tau$ with a tree of $\mathbb{P}^1$ in the middle. The tree $C_1 \circ C_2 \circ C_3$ is contracting by p to the node of $\sigma \circ \tau$.

Theorem 5.3.4 (Theorem 5.2 [LT98])**.** *We have the following equality in the Chow ring of the truncation of Z_β.*

(5.3.5)

$$\alpha_* \left(\sum_{\beta'+\beta''=\beta} \Delta^! \left([\overline{\mathcal{M}}_{0,n_1+1}(X,\beta')]^{\mathrm{vir}} \otimes [\overline{\mathcal{M}}_{0,n_2+1}(X,\beta'')]^{\mathrm{vir}} \right) \right) = g^! [\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}$$

Remark 5.3.6. In [Beh97], Behrend proves that the virtual class satisfies five properties, called *orientation* (see §7 in [BM96b]), namely: mapping to a point, products, cutting edges, forgetting tails and isogenies. The formula (5.3.5) is a combination of cutting tails and isogenies.

The analogue statement in G_0-theory need a bit more of notations. We denote

$$\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta) := \mathbb{R}X_{g,n,\beta}.$$

Let r, n_1, n_2 be in $\mathbb{N}$ with $n_1 + n_2 = n$ and let β be in $\mathrm{NE}(X)$. Let $\underline{\beta} = (\beta_0, \ldots, \beta_r)$ be a partition of β. Notice that there is only a finite number of partition.

We denote by

$$\mathbb{R}X_{0,n_1,n_2,\underline{\beta}} := \mathbb{R}X_{0,n_1+1,\beta_0} \times_X \mathbb{R}X_{0,2,\beta_1} \times_X \cdots \times_X \mathbb{R}X_{0,2,\beta_{r-1}} \times_X \mathbb{R}X_{0,n_2+1,\beta_r}$$

We generalize the situation of (5.3.5) by the following homotopical cartesian diagram

$$\begin{array}{ccc} \mathbb{R}X_{0,n_1,n_2,\underline{\beta}} & \longrightarrow & \mathbb{R}X_{0,n_1+1,\beta_0} \times \left(\prod_{k=1}^{r-1} \mathbb{R}X_{0,2,\beta_i}\right) \times \mathbb{R}X_{0,n_2+1,\beta_r} \\ \downarrow & & \downarrow \\ X^r & \xrightarrow{\Delta^r} & (X \times X)^r \end{array} \tag{5.3.7}$$

Gluing all the stable maps and using the universal property of Z_β, we have a morphism

$$\alpha_r : \coprod_{\beta=\sum_{i=0}^{r}\beta_i} \mathbb{R}X_{0,n_1,n_2,\underline{\beta}} \to Z_\beta \tag{5.3.8}$$

Notice that α_1 is the α of (3.4.3)

Finally, we can state the analogue of Theorem 5.3.4 in G_0-theory.

Theorem 5.3.9 (Proposition 11 in [Lee04])**.** *We have the following equality in the G_0-group of the truncation of Z_β.*

$$\sum_{r\in\mathbb{N}}(-1)^r\alpha_{r*}\left(\sum_{\sum_{i=0}^{r}\beta_i=\beta}(\Delta^r)^!\left(\mathcal{O}^{\mathrm{vir}}_{X_{0,n_1+1\beta_0}}\otimes\mathcal{O}^{\mathrm{vir}}_{X_{0,2,\beta_1}}\otimes\cdots\otimes\mathcal{O}^{\mathrm{vir}}_{X_{0,2,\beta_{r-1}}}\otimes\mathcal{O}^{\mathrm{vir}}_{X_{0,n_2+1,\beta_r}}\right)\right)=g^!\mathcal{O}^{\mathrm{vir}}_{X_{0,n,\beta}}$$

Remark 5.3.10.
(1) Comparing Theorem 5.3.4 with Theorem 5.3.9, we see that the formulas are more complicated in G_0-theory. We see that moduli spaces of the kind $\overline{\mathcal{M}}_{0,2}(X,\beta)$ appears in G_0-theory. This corresponds to stable curve with tree of $\mathbb{P}^1$ in the middle (see Figure 6). Notice that this is the same reason why the action of the main Theorem 3.1.2 is lax.
(2) Also in G_0-theory, there are 5 axioms, called orientation (see Remark 5.3.6), for the virtual sheaf $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$. They are proved by Lee in [Lee04].

Denote by

$$X_{r,\beta} := \coprod_{\sum\beta_i=\beta} \mathbb{R}X_{0,n_1+1,\beta_0} \times_X \mathbb{R}X_{0,2,\beta_1} \times_X \cdots \times_X \mathbb{R}X_{0,2,\beta_{r-1}} \times_X \mathbb{R}X_{0,n_2+1,\beta_r}$$

We deduce a semi-simplicial object in the category of derived stacks where the $r+1$-morphisms from $X_{r+1,\beta} \to X_{r,\beta}$ are given by gluing two stable maps together. We have

$$X_{0,\beta} \;\substack{\longleftarrow\\ \longleftarrow}\; X_{1,\beta} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X_{2,\beta} \;\substack{\longleftarrow\\ \vdots\\ \longleftarrow}\; \cdots$$

Moreover, for any r we have a morphism of gluing all stable maps from $X_{r,\beta} \to Z_\beta$ hence a morphism $\operatorname{colim} X_{\bullet,\beta} \to Z_\beta$.

The following theorem was not proved in [MR15]. We will prove it in the appendix.

Theorem 5.3.11. *We have that* $\operatorname{colim} X_{\bullet,\beta} = Z_\beta$.

5.4. **Virtual object from derived algebraic geometry.** In this section, we explain how derived algebraic geometry will provide a sheaf in $G_0(\overline{\mathcal{M}}_{g,n}(X,\beta))$ that we will compare to the virtual sheaf of Lee.

Lemma 5.4.1 (See for example [Toë14] p.192-193)**.** *Let X be a derived algebraic stack. Denote by $t_0(X)$ its truncation. Denote by $\iota : t_0(X) \hookrightarrow X$ be the closed embedding. The morphism $\iota_* : G_0(t_0(X)) \to G_0(X)$ is an isomorphism. Moreover we have that*

$$(\iota_*)^{-1}[\mathcal{F}] = \sum_i (-1)^i [\pi_i(\mathcal{F})]$$

Applying this lemma to the situation where $X = \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$, we put

$$\left[\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}\right] := \iota_*^{-1}[\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}].$$

where the DAG means Derived Algebraic Geometry. Notice that the sheaf $\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ depends on the derived structure that we put on the moduli space of stable maps.

The following theorem was not stated in [MR15].

Theorem 5.4.2. *The DAG-virtual sheaf $\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ satisfies the orientation axiom in G_0-theory. That is*

(1) Mapping to a point. Let $\beta = 0$, we have

$$\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,0)} = \sum_i (-1)^i \wedge^i (R^1\pi_*\mathcal{O}_{\mathcal{C}} \boxtimes T_X)^\vee$$

where $\mathcal{C}$ is the universal curve of $\overline{\mathcal{M}}_{g,n}$ and $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n}$.

(2) Product. We have

$$\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g_1,n_1}(X,\beta_1)\times\overline{\mathcal{M}}_{g_2,n_2}(X,\beta_2)} = \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g_1,n_1}(X,\beta_1)} \boxtimes \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g_2,n_2}(X,\beta_2)}$$

(3) Cutting edges. With the notation of Diagram (5.3.3)*, we have*

$$\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g_1,n_1}(X,\beta_1)\times_X\overline{\mathcal{M}}_{g_2,n_2}(X,\beta_2)} = \Delta^! \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g_1,n_1}(X,\beta_1)\times\overline{\mathcal{M}}_{g_2,n_2}(X,\beta_2)}$$

(4) Forgetting tails. Forgetting the last marked point marked points, we get a morphism $\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$. We have the following equality.

$$\pi^* \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)} = \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n+1}(X,\beta)}.$$

(5) Isogenies. The are two formulas. The morphism π above induces a morphism $\psi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(X,\beta)$. With notation of Diagram (5.3.1)*, we have*

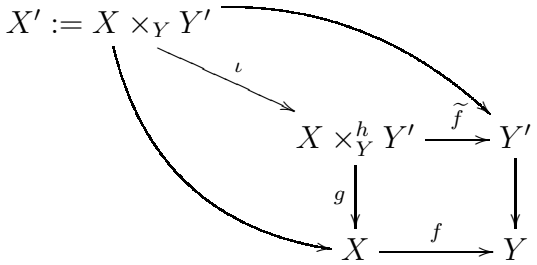
$$\psi_* \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n+1}(X,\beta)} = g^! \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}.$$

The second formula is

$$\sum_{r\in\mathbb{N}} (-1)^r \alpha_{r*} \sum_{\sum_{i=0}^r \beta_i = \beta} \mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,n_1+1}(X,\beta_0)\times_X X_{0,2}(X,\beta_1)\times_X \cdots \times_X X_{0,2}(X,\beta_{r-1})\times_X X_{0,n_2+1}(X,\beta_r)} = g^! \mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,n}(X,\beta)}$$

where g is defined in the key diagram (5.3.1)*.*

Before proving this theorem, we need a preliminary result. Consider a homotopical cartesian morphisms of schemes

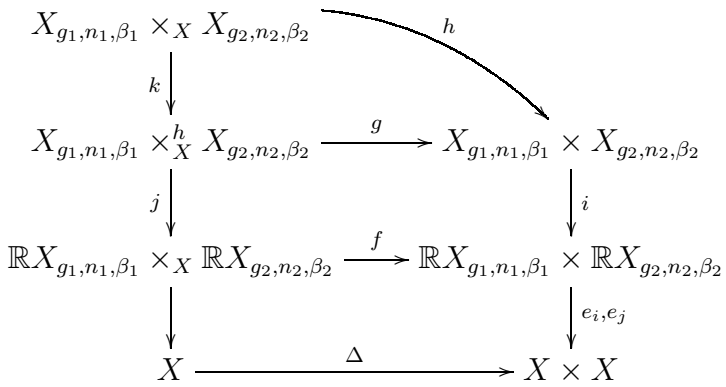


Denote by $X \times^h_Y Y'$ the homotopical pullback so that we have the closed immersion $\iota : X' \to X \times^h_Y Y'$. Assume that f is a regular closed immersion. We have a rafined Gysin morphism (see [Lee04, p.4], [Ful98, ex.18.3.16] or chapter 6 in [FL85]) which turns to be

$$f^! : G(Y') \to G(X') \qquad [\mathcal{F}_{Y'}] \mapsto (\iota_*)^{-1} \circ \widetilde{f}^*[\mathcal{F}_{Y'}]. \tag{5.4.3}$$

Proof of Theorem 5.4.2. (1). Strangely this proof is not easy and we postpone to the Appendix B. (2). This follows from the Künneth formula.

(3). We have the following diagram.



We deduce the following equalities

$$\begin{aligned}
\Delta^! \mathcal{O}^{\mathrm{vir,DAG}}_{X_{g_1,n_1,\beta_1} \times X_{g_2,n_2,\beta_2}} &= \Delta^!(i_*)^{-1}\mathcal{O}_{\mathbb{R}X_{g_1,n_1,\beta_1} \times \mathbb{R}X_{g_2,n_2,\beta_2}} \\
&= (k_*)^{-1} g^*(i_*)^{-1}\mathcal{O}_{\mathbb{R}X_{g_1,n_1,\beta_1} \times \mathbb{R}X_{g_2,n_2,\beta_2}} && \text{by definition of rafined Gysin morphism} \\
&= (k_*)^{-1}(j_*)^{-1} f^* \mathcal{O}_{\mathbb{R}X_{g_1,n_1,\beta_1} \times \mathbb{R}X_{g_2,n_2,\beta_2}} && \text{by derived base change} \\
&= (k_*)^{-1}(j_*)^{-1}\mathcal{O}_{\mathbb{R}X_{g_1,n_1,\beta_1} \times_X \mathbb{R}X_{g_2,n_2,\beta_2}} \\
&= \mathcal{O}^{\mathrm{vir,DAG}}_{X_{g_1,n_1,\beta_1} \times_X X_{g_2,n_2,\beta_2}}
\end{aligned}$$

(4). As $\widetilde{\pi} : \mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ is the universal curve (hence, it is flat) and π is the truncation of $\widetilde{\pi}$. The derived base change formula implies the equality.

(5). We have the following diagram

$$\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n+1}(X,\beta) & \xrightarrow{\psi} & \overline{\mathcal{M}}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{a} & \overline{\mathcal{M}}_{g,n}(X,\beta) \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\
\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta) & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{b} & \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta) \\
 & \searrow & \downarrow & & \downarrow \\
 & & \overline{\mathcal{M}}_{g,n+1} & \xrightarrow{c} & \overline{\mathcal{M}}_{g,n}
\end{array}$$

Notice that as c is flat, the upper right square is also h-cartesian. We have

$$\begin{aligned}
c^! \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)} &= c^!(i_*)^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \\
&= a^*(i_*)^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \\
&= (j_*)^{-1} b^* \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} && \text{by derived base change} \\
&= (j_*)^{-1}\mathcal{O}_{\overline{\mathcal{M}}_{g,n+1}\times_{\overline{\mathcal{M}}_{g,n}}\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\psi_* \mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)} &= \psi_*(k_*)^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta)} \\
&= (j_*)^{-1}\varphi_*\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta)}
\end{aligned}$$

The formula follows from the equality below which is a consequence of the proof of Proposition 9 in [Lee04].

$$\varphi_*\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n+1}(X,\beta)} = \mathcal{O}_{\overline{\mathcal{M}}_{g,n}\times_{\overline{\mathcal{M}}_{g,n}}\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}$$

To prove the second formula of (5), we use the key Diagram (5.3.1)) with Theorem 5.3.11. Let g_1, g_2, n_1, n_2 be integers. Put $g = g_1 + g_2$ and $n = n_1 + n_2$ and denote $\overline{\mathcal{M}}_i := \overline{\mathcal{M}}_{g_i,n_i+1}$.

$$\begin{array}{ccc}
t_0(Z_\beta) & \xrightarrow{a} & \\
\downarrow{\scriptstyle k} & & \\
\left(\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2\right) \times^h \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{b} & \overline{\mathcal{M}}_{g,n}(X,\beta) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\
Z_\beta & \xrightarrow{c} & \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta) \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2 & \xrightarrow{g} & \overline{\mathcal{M}}_{g,n}
\end{array}$$

We have

$$\begin{aligned} g^!\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)} &= g^!(i_*)^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \\ &= (k_*)^{-1}b^*(i_*)^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \\ &= (k_*^{-1})(j_*)^{-1}c^*\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \qquad \text{by derived base change} \\ &= (j\circ k)^{-1}_*\mathcal{O}_{Z_\beta} \end{aligned}$$

We deduce the formula by observing that Z_β is the colimit of $X_{\bullet,\beta}$ (see Theorem 5.3.11) and that the structure sheaf of a co-limit is the alternating sum of $\mathcal{O}_{X_{r,\beta}}$.

□

The last formula of Theorem 5.4.2 and the third one implies the following corollary.

Corollary 5.4.4. *We have the following equality in $G_0(t_0(Z_\beta))$.*

$$\sum_{r\in\mathbb{N}}(-1)^r\alpha_{r*}\left(\sum_{\sum_{i=0}^r\beta_i=\beta}(\Delta^r)^!\left(\mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,n_1+1}(X,\beta_0)}\otimes\mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,2}(X,\beta_1)}\otimes\cdots\otimes\mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,2}(X,\beta_{r-1})}\otimes\mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,n_2+1}(X,\beta_r)}\right)\right)=g^!\mathcal{O}^{\mathrm{vir,DAG}}_{X_{0,n}(X,\beta)}$$

5.5. **Virtual object from perfect obstruction theory.** Here we follow the approach of Behrend-Fantechi [BF97] to construct virtual object.

In the following, we denote by $\mathcal{M}$ a Deligne-Mumford stack. The reader can think of $\mathcal{M}$ being $\overline{\mathcal{M}}_{0,n}(X,\beta)$ as an example.

Definition 5.5.1. *Let $\mathcal{M}$ be a Deligne-Mumford stack. An element $E^\bullet$ in the derived category $D(\mathcal{M})$ in degree $(-1,0)$ is a perfect obstruction theory for $\mathcal{M}$ if we have a morphism $\varphi : E^\bullet \to \mathbb{L}_{\mathcal{M}}$ that satisfies*

(1) $h^0(\varphi)$ is an isomorphism,
(2) $h^{-1}(\varphi)$ is surjective.

Let $E^\bullet$ be a perfect obstruction theory. Following [BF97], we have the following morphisms.

(1) The morphism $a : C_{\mathcal{M}} \to h^1/h^0(E^\vee_\bullet)$, where $C_{\mathcal{M}}$ is the intrinsic normal cone and $h^1/h^0(E^\vee_\bullet)$ is the quotient stack $[E^\vee_{-1}/E^\vee_0]$. To understand how to construct this morphism, let us simplify the situation. Assume that $\mathcal{M}$ is embedded in something smooth, i.e $f : \mathcal{M} \hookrightarrow Y$ is a closed embedding with ideal sheaf $\mathcal{I}$. Then the intrinsic normal cone is the quotient stack $C_{\mathcal{M}} = [C_{\mathcal{M}}Y/f^*TY]$ where $C_{\mathcal{M}}Y := \operatorname{Spec}\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}$ is the normal cone of f. In this case, the intrinsic normal sheaf is $N_{\mathcal{M}} = [N_{\mathcal{M}}Y/f^*TY] = h^1/h^0(\mathbb{L}^\vee_{\mathcal{M}})$ where $N_{\mathcal{M}}Y := \operatorname{Spec}\operatorname{Sym}\mathcal{I}/\mathcal{I}^2$. As we have a morphism from the normal cone to the normal sheaf $C_{\mathcal{M}}Y \to N_{\mathcal{M}}Y$, we deduce a morphism from the intrinsic normal cone to the intrinsic normal sheaf i.e., a morphism

$$C_{\mathcal{M}} \to N_{\mathcal{M}} \tag{5.5.2}$$

Now the morphism of the perfect obstruction theory $\varphi : E^\bullet \to \mathbb{L}_{\mathcal{M}}$ induces a morphism from

$$N_{\mathcal{M}} \to [E_{-1}^\vee / E_0^\vee] \tag{5.5.3}$$

The morphism a is the composition of the two morphisms (5.5.2) and (5.5.3).

(2) We also have a natural morphism $b : \mathcal{M} \to h^1/h^0(E_\bullet^\vee)$ given by the zero section.

From these two morphisms, we can perform the homotopical fiber product

$$\begin{array}{ccc} \mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}} & \longrightarrow & C_{\mathcal{M}} \\ \downarrow{\scriptstyle r} & & \downarrow \\ \mathcal{M} & \longrightarrow & h^1/h^0(E_\bullet^\vee) \end{array} \tag{5.5.4}$$

As the standard fiber product is $\mathcal{M}$, we have that $\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}$ is a derived enhancement of $\mathcal{M}$ with $\widetilde{j} : \mathcal{M} \to \mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}$ the canonical closed embedding. Notice that in the case $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X, \beta)$, we get a derived enhancement which is different from $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, \beta)$ (see Remark 5.6.3). We will compare these two structures in § 5.6. Hence we can apply the Lemma 5.4.1 and we denote

$$[\mathcal{O}_{\mathcal{M}}^{\mathrm{vir,POT}}] := \widetilde{j}_*^{-1}[\mathcal{O}_{\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}}] \in G_0(\mathcal{M}) \tag{5.5.5}$$

where POT means Perfect Obstruction Theory. The definition of Lee for the virtual sheaf turns to be exactly this one. Indeed, Lee consider the following (not homotopical) la cartesian diagram

$$\begin{array}{ccccc} \mathcal{M} \times_{E_{-1}^\vee} C_1 & \longrightarrow & C_1 & \longrightarrow & C_{\mathcal{M}} \\ \downarrow{\scriptstyle r} & & \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & E_{-1}^\vee & \longrightarrow & h^1/h^0(E_\bullet^\vee) \end{array} \tag{5.5.6}$$

In [Lee04, p.8], Lee takes as a definition for the virtual sheaf

$$\mathcal{O}_{\mathcal{M}}^{\mathrm{vir}} := \sum_i (-1)^i \mathcal{T}or_i^{h^1/h^0}(\mathcal{O}_{\mathcal{M}}, \mathcal{O}_{C_1}) = \mathcal{O}_{\mathcal{M}} \otimes^{\mathbb{L}}_{h^1/h^0} \mathcal{O}_{C_1} = \mathcal{O}_{\mathcal{M}}^{\mathrm{vir,POT}}$$

where the last equality follows from Lemma 5.4.1.

5.6. **Comparison theorem of the two approachs.** Let $\mathcal{M} := \overline{\mathcal{M}}_{0,n}(X, \beta)$. In this section, we want to compare $\mathcal{O}_{\mathcal{M}}^{\mathrm{vir,DAG}}$ with $\mathcal{O}_{\mathcal{M}}^{\mathrm{vir,POT}}$. The first question is : what is the perfect obstruction theory we are choosing ?

This is given by the following result.

Proposition 5.6.1 ([STV11])**.** *Let $\mathbb{R}\mathcal{M}$ be a derived Deligne-Mumford stack. Denote by $\mathcal{M}$ its truncation and its truncation morphism by $j : \mathcal{M} \hookrightarrow \mathbb{R}\mathcal{M}$. Then $j^*\mathbb{L}_{\mathbb{R}\mathcal{M}} \to \mathbb{L}_{\mathcal{M}}$ is a perfect obstruction theory.*

Now the original question makes perfectly sense and we have the following result that says that they are the same sheaves.

Theorem 5.6.2 (See Proposition 4.3.2 in [MR15]). *In $G_0(\mathcal{M})$, we have*

$$[\mathcal{O}_{\mathcal{M}}^{\mathrm{vir,DAG}}] = [\mathcal{O}_{\mathcal{M}}^{\mathrm{vir,POT}}]$$

Remark 5.6.3. Notice that the two enhancements $\mathbb{R}\mathcal{M}$ or $\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}$ are not the same. Indeed, the second one has a retract $r : \mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}} \to \mathcal{M}$ given in the diagram (5.5.4) that is $r \circ \widetilde{j} = \mathrm{id}_{\mathcal{M}}$ where $\widetilde{j}$ is the closed immersion from $\mathcal{M}$ to $\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}$. From this we get the following exact triangle of cotangent complexes

$$\mathbb{L}_{\widetilde{j}}[-1] \to \widetilde{j}^* \mathbb{L}_{\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}} \to \mathbb{L}_{\mathcal{M}} \tag{5.6.4}$$

$$r^* \mathbb{L}_{\mathcal{M}} \to \mathbb{L}_{\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}} \to \mathbb{L}_r \tag{5.6.5}$$

Applying $\widetilde{j}^*$ to the second line, we get

$$\mathbb{L}_{\mathcal{M}} \to \widetilde{j}^* \mathbb{L}_{\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}} \to \widetilde{j}^* \mathbb{L}_r$$

This means that (5.6.4) has a splitting that is

$$\widetilde{j}^* \mathbb{L}_{\mathcal{M} \times^h_{h^1/h^0(E_\bullet^\vee)} C_{\mathcal{M}}} = \mathbb{L}_{\widetilde{j}}[-1] \oplus \mathbb{L}_{\mathcal{M}} \tag{5.6.6}$$

Comparing to the cotangent complex of $\mathbb{R}\mathcal{M}$ that has no reason to split, we get a priori two different derived enhancement of $\mathcal{M}$.

Notice that in the work of Fantechi-Göttsche [FG10, Lemma 3.5] (see also Roy Joshua [Jos10]), they prove that for a scheme X with a perfect obstruction theory $E^\bullet := [E^{-1} \to E^0]$, we have

$$\tau_X(\mathcal{O}_X^{\mathrm{vir,POT}}) = \mathrm{Td}(TX^{\mathrm{vir}}) \cap [X^{\mathrm{vir,POT}}] \tag{5.6.7}$$

where $TX^{\mathrm{vir}} \in G_0(X)$ is the class of $[E_0] - [E_1]$ where $[E_0 \to E_1]$ is the dual complex of $E^\bullet$ and $\tau_X : G_0(X) \to A_*(X)_{\mathbb{Q}}$.

Notice that the Formula (5.6.7) with Theorem 5.6.2 implies that

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir,POT}} = \tau(\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}) \, \mathrm{Td}(T_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)})^{-1}$$

Appendix A. Proof of theorem 5.3.11

Theorem A.0.1. *The map*

$$f : \mathrm{colim}^{\mathrm{DM}} \, X_{\bullet,\beta} \to Z_\beta$$

of [MR15, (4.2.9)] *is an equivalence of derived Deligne-Mumford stacks.*

Proof. It follows from the discussion in the proof of [MR15, Prop. 4.2.1] that

$$\begin{array}{ccc} \mathrm{Perf}(Z_\beta) & \xrightarrow{\quad f^* \quad} & \mathrm{Perf}(\mathrm{colim}^{\mathrm{DM}} \, X_{\bullet,\beta}) \\ & {\scriptstyle h} \searrow \quad \swarrow {\scriptstyle g} & \\ & \lim_\Delta \mathrm{Perf}(X_{\bullet,\beta}) & \end{array}$$

commutes with the morphism h being an equivalence after h-descent for perfect complexes [HLP14, 4.12] and the morphism g being fully faithful after the result of gluing along closed immersions [Lur17, 16.2.0.1]. This immediately implies that the map f^* is an equivalence of categories because we have $g \circ f^* = h$ and g is conservative as it is fully faithful.

As both source and target of f are perfect stacks (the first being a colimit of perfect stacks along closed immersions and second being pullback of perfect stacks), f^* induces an equivalence

$$\mathrm{Qcoh}(Z_\beta) \xrightarrow{f^*} \mathrm{Qcoh}(\mathrm{colim}^{\mathrm{DM}}\ X_{\bullet,\beta})$$

We conclude that f is an equivalence using Tannakian duality [Lur17, 9.2.0.2].

□

Appendix B. Proof of Theorem 5.4.2.(1)

Let X be a derived stack. We will use the linear derived stacks $\mathbb{V}(\mathbb{E})$ (See [Toë14, p.200]) where $\mathbb{E}$ is a complex of quasi-coherent sheaf on X. We have a morphism $\mathbb{V}(\mathbb{E}) \to X$ and a zero section $s : X \to \mathbb{V}(\mathbb{E})$. One should understand that $\mathbb{V}(\mathbb{E})$ as a vector bundle where the fibers are $\mathbb{E}$.

It is a derived generalisation of $\operatorname{Spec}\operatorname{Sym}\mathcal{E}$ for a coherent sheaf $\mathcal{E}$. If $\mathbb{E}$ is a two terms complex with cohomology in degree 0 and 1, then we have that $t_0(\mathbb{V}(\mathbb{E}^\vee[-1])) = [h^1/h^0(\mathbb{E})]$ (See §2 in [BF97] for the definition of the quotient stacks).

Let recall some notation of §5.5 and §5.6. Let $g, n \in \mathbb{N}$ and $\beta \in H_2(X, \mathbb{Z})$. Denote by j the closed immersion $\overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$. To simplify the notation, put $\mathcal{M} = \overline{\mathcal{M}}_{g,n}(X,\beta)$ and $\mathbb{R}\mathcal{M} = \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$.

From the exact triangle

$$j^*\mathbb{L}_{\mathbb{R}\mathcal{M}} \to \mathbb{L}_{\mathcal{M}} \to \mathbb{L}_j$$

We deduce that following cartesian diagram

$$\begin{array}{ccc} \mathbb{V}(\mathbb{L}_j[-1]) & \longrightarrow & \mathbb{V}(\mathbb{L}_{\mathcal{M}}[-1]) \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbb{V}(j^*\mathbb{L}_{\mathbb{R}\mathcal{M}}[-1]) \end{array} \tag{B.0.1}$$

Recall that $j^*\mathbb{L}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)}$ is a two terms complex in degree -1 and 0 but in general it is not the case for $\mathbb{L}_j$ and $\mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$. Comparing with Behrend-Fantechi, we have $t_0(\mathbb{V}(\mathbb{L}_{\mathcal{M}}[-1]))$ is the intrinsic normal sheaf $N_{\mathcal{M}}$ (See §5.5) and we have the following cartesian diagram

(B.0.2)
$$\begin{array}{ccc} \mathcal{M} \times^h_{\mathbb{V}(j^*\mathbb{L}_{\mathbb{R}\mathcal{M}}[-1])} C_{\mathcal{M}} & \longrightarrow & C_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathbb{V}(\mathbb{L}_j[-1]) & \longrightarrow & \mathbb{V}(\mathbb{L}_{\mathcal{M}}[-1]) \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbb{V}(j^*\mathbb{L}_{\mathbb{R}\mathcal{M}}[-1]) \end{array}$$

Proposition B.0.3. *Let $g, n \in \mathbb{N}$ and $\beta \in H_2(X, \mathbb{Z})$. Denote by j the closed immersion $\overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ and by $s : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathbb{V}(\mathbb{L}_j[-1])$ be the zero section. We have the following equality in $G_0(\overline{\mathcal{M}}_{g,n}(X,\beta))$*

$$\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,\beta)} := j_*^{-1}\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} = s_*^{-1}(\mathcal{O}_{\mathbb{V}(\mathbb{L}_j)[-1]})$$

Proof. From Gaitsgory (see Proposition 2.3.6 p 18 Chapter IV.5 [Gai17]), we can construct an derived stack $\mathcal{Y}_{scaled}$ such that the following diagram has two homotopical fiber products

$$\begin{array}{ccccc} \mathbb{R}\mathcal{M} & \xrightarrow{h} & \mathcal{Y}_{scaled} & \xleftarrow{v} & \mathbb{V}(\mathbb{L}_j[-1]) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle s} \\ \mathcal{M}\times\{0\} & \xrightarrow{i_0} & \mathcal{M}\times\mathbb{A}^1 & \xleftarrow{i_1} & \mathcal{M}\times\{1\} \end{array}$$

We have

$$\begin{aligned} (s_*)^{-1}\mathcal{O}_{\mathbb{V}(\mathbb{L}_j[-1])} &= (s_*)^{-1}v^*\mathcal{O}_{\mathcal{Y}_{scaled}} \\ &= i_1^*(\sigma_*)^{-1}\mathcal{O}_{\mathcal{Y}_{scaled}} \\ &= i_0^*(\sigma_*)^{-1}\mathcal{O}_{\mathcal{Y}_{scaled}} \end{aligned}$$

The last equality follows from the $\mathbb{A}^1$- invariance of the G-theory. That is, we have that $G_0(\mathcal{M}\times\mathbb{A}^1) \to G_0(\mathcal{M})$ and $i_0^* = (\pi^*)^{-1} = i_1^*$ where π is the projection. Applying the same computation as above with the other homotopical fiber product, we get Formula. □

Remark B.0.4. This statement is a first step in proving Theorem 5.6.2. The last step is to prove that the inclusion $C_{\mathcal{M}} \to N_{\mathcal{M}}$ induces an equality of the structure sheaf in G_0-theory.

Corollary B.0.5. *For stable maps of degree 0, we have that*

$$\mathcal{O}^{\mathrm{vir,DAG}}_{\overline{\mathcal{M}}_{g,n}(X,0)} = \sum_i (-1)^i \wedge^i (TX \boxtimes R^1\pi_*\mathcal{O}_{\mathcal{C}})$$

Remark B.0.6. Notice that in the case of $\beta = 0$, we have that $\overline{\mathcal{M}}_{g,n}(X, \beta = 0) = \overline{\mathcal{M}}_{g,n} \times X$ which is smooth. Nevertheless, it has a derived enhancement, given by the $\mathbb{R}$ Map which has a retract given by the projection and the evaluation. For $\beta \neq 0$, this retract does not exist.

Proof. For $\beta = 0$, the smoothness of $\mathcal{M}$ implies that the intrinsic normal cone is the intrinsic normal sheaf that is we have the $C_{\mathcal{M}} = \mathbb{V}(\mathbb{L}_{\mathcal{M}}[-1])$ in the diagram (B.0.2). The second thing which is different is that $j : \mathcal{M} \to \mathbb{R}\mathcal{M}$ has a retract. This implies that $\mathbb{L}_j[-1] \simeq \mathbb{L}_{\mathcal{M}}[-1] \oplus j^*\mathbb{L}_{\mathbb{R}\mathcal{M}}$. Hence the Proposition B.0.3, implies that we need to compput $s_*^{-1}\mathcal{O}_{\mathbb{V}(\mathbb{L}_j[-1])}$ which is by standard computation $\sum_i (-1)^i \wedge^i (TX \boxtimes R^1\pi_*\mathcal{O}_{\mathcal{C}})$ where $\mathcal{C}$ is the universal curve of $\overline{\mathcal{M}}_{g,n}$.

□

From the proof, we see that the RHS of the formula is the structure sheaf of $\mathbb{V}(\mathbb{L}_j[-1])$. In fact, we think that $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X, 0)$ is isomorphic to $\mathbb{V}(\mathbb{L}_j[-1])$. This should follow from a general argument that we will detail in the next section for the affine case.

Appendix C. Alternative proof of Corollary B.0.5 in the affine case.

Proposition C.0.1. *Let $F = \operatorname{Spec} A$ be an affine quasi-smooth algebraic derived stack. Let $F_0 = \operatorname{Spec} \pi_0(A)$ its truncation and denote $j : F_0 \to F$ its closed immersion. Assume that F_0 is smooth and that F admit a retract $r : F \to F_0$. Then $F = \mathbb{V}(\mathbb{L}_j[-1])$.*

This proposition is a way of proving Corollary B.0.5 in the affine case without using the deformation argument of Gaitsgory. We believe that we can drop the affine assumption in the previous proposition.

Notice that we can drop the existence of the retract in the hypothesis because when $F0$ is smooth, there always exists a retract (see the Remark C.0.6).

Lemma C.0.2. *With the previous hypothesis, we have*

$$\pi_0(\mathbb{L}_j) = \pi_1(\mathbb{L}_j) = 0$$
$$\pi_2(\mathbb{L}_j) = \pi_1(j^*\mathbb{L}_F) = \pi_2(\mathbb{L}_{\pi_0(A)/\tau_{\leq 1}A}) = \pi_1(A)$$
$$\mathbb{L}_j[-1] \simeq \pi_1(A)[1]$$

Proof. We have the triangle

$$j^*\mathbb{L}_F \to \mathbb{L}_{F_0} \to \mathbb{L}_j.$$

Applying the hypothesis, we get

(1) As F is quasi-smooth, we have that $\pi_2(j^*\mathbb{L}_F) = 0$.
(2) As F_0 is smooth, we have that $\pi_2(\mathbb{L}_{F_0}) = \pi_1(\mathbb{L}_{F_0}) = 0$.
(3) As $j^*\mathbb{L}_F \to \mathbb{L}_{F_0}$ is a perfect obstruction theory, we deduce $\pi_0(j^*\mathbb{L}_F) \simeq \pi_0(\mathbb{L}_{F_0})$ and $\pi_1(j^*\mathbb{L}_F) \to \pi_1(\mathbb{L}_{F_0})$ is onto.

Applying the three properties above to the associated long exact sequence, we get

(1) As F is quasi-smooth, we have that $\pi_2(j^*\mathbb{L}_F) = 0$.
(2) As F_0 is smooth, we have that $\pi_2(\mathbb{L}_{F_0}) = \pi_1(\mathbb{L}_{F_0}) = 0$.
(3) As $j^*\mathbb{L}_F \to \mathbb{L}_{F_0}$ is a perfect obstruction theory, we deduce $\pi_0(j^*\mathbb{L}_F) \simeq \pi_0(\mathbb{L}_{F_0})$ and $\pi_1(j^*\mathbb{L}_F) \to \pi_1(\mathbb{L}_{F_0})$ is onto.

$$0 \longrightarrow 0 \longrightarrow \pi_2(\mathbb{L}_j) \to \pi_1(j^*\mathbb{L}_F) \longrightarrow 0 \longrightarrow 0 \to \pi_0(j^*\mathbb{L}_F) \to \pi_0(\mathbb{L}_{F_0}) \longrightarrow 0$$

We conclude that

(1) $\pi_2(\mathbb{L}_j) = \pi_1(j^*\mathbb{L}_F)$
(2) $\mathbb{L}_j$ is 2-connective.

To prove the second equality of the lemma, we use the Postnikov tower that is we consider the closed immersion $j_1 : F_0 \to F_1$ and $j_2 : F_1 \to F$ where F_1 is $\operatorname{Spec} \tau_{\leq 1}A$. We deduce the exact triangle

$$j_1^*\mathbb{L}_{j_2} \to \mathbb{L}_j \to \mathbb{L}_{j_1}$$

As we have j and j_1 are 1-connected and j_2 is 2-connected, we deduce from connective estimates that $\mathbb{L}_j$ and $\mathbb{L}_{j_1}$ are 2-connective and $\mathbb{L}_{j_2}$ is 3-connective (See Corollary 5.5 in [PV13]). We deduce from the long exact sequence that $\pi_2(\mathbb{L}_j) = \pi_2(\mathbb{L}_{j_1})$. How we apply Lemma 2.2.2.8 in [TV08] that implies that $\pi_2(\mathbb{L}_{j_1}) = \pi_1(A)$.

As we have that $\pi_k(\mathbb{L}_j) = 0$ for all $k \neq 2$ and $\pi_2(\mathbb{L}_j) = \pi_1(A)$, we deduce that $\mathbb{L}_j[-1] \simeq \pi_1(A)[1]$. □

Proof of Proposition C.0.1. To prove the proposition, we will show that

$$B := \operatorname{Sym}_{\pi_0(A)}(\pi_1(A)[1]) \simeq A \tag{C.0.3}$$

First, we will construct a morphism $f : B \to A$. Notice that $\pi_1(A)$ is a free $\pi_0(A)$ module by the last statement of Lemma C.0.2. Then we get an inclusion $\pi_1(A)[1] \to A$ of $\pi_0(A)$-modules which induces $f : B \to A$. Moreover f is an equivalence on π_0 and π_1 that is $\tau_{\leq 1}B \simeq \tau_{\leq 1}A$.

Then we construct an inverse from $A \to B$ using the Postnikov tower. We have $\varphi : A \to \tau_{\leq 1}A \simeq \tau_{\leq 1}B$. As B is the colimit of its Postnikov tower, we will proceed by induction on the Postnikov tower. First, we want to lift the morphism $\varphi : A \to \tau_{\leq 1}B$ to $A \to \tau_{\leq 2}B$. We use the following cartesian diagram (See Remark 4.3 in [PV13])

$$\begin{array}{ccc} \tau_{\leq 2}B & \longrightarrow & \tau_{\leq 1}B \\ \downarrow & & \downarrow d \\ \tau_{\leq 1}B & \xrightarrow{\mathrm{id},0} & \tau_{\leq 1}B \oplus \pi_2(B)[3] \end{array} \tag{C.0.4}$$

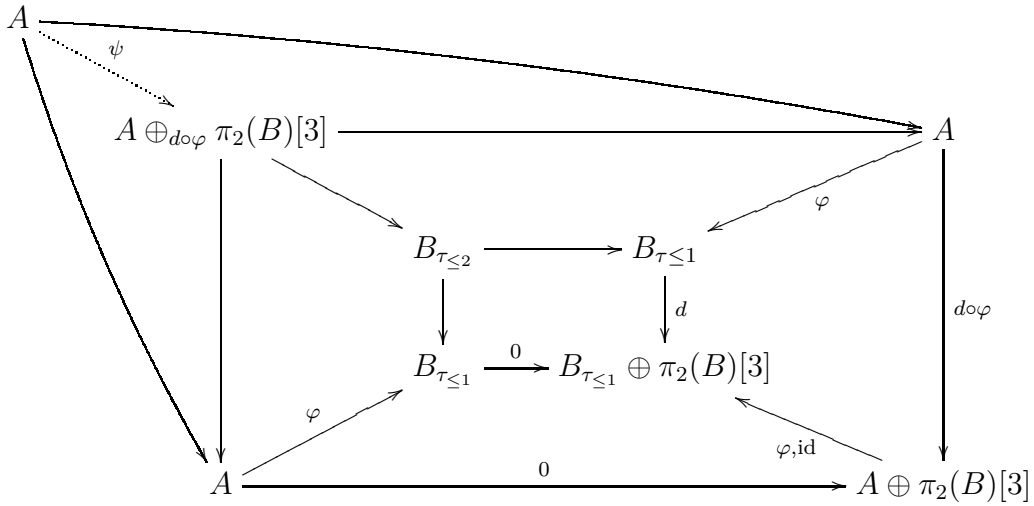
Hence, we need to construct a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \tau_{\leq 1}B \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle d} \\ \tau_{\leq 1}B & \xrightarrow{\mathrm{id},0} & \tau_{\leq 1}B \oplus \pi_2(B)[3] \end{array} \tag{C.0.5}$$

As $\mathbb{L}_A$ has a tor-amplitude in $[-1, 0]$, we have that

$$\pi_0(\mathrm{Map}(\mathbb{L}_A, \pi_2(B)[3])) = 0$$
$$\pi_1(\mathrm{Map}(\mathbb{L}_A, \pi_2(B)[3])) = 0$$

Hence we deduce a morphism from $\psi : A \to A \oplus_{d\circ\varphi} \pi_2(B)[3]$. Hence we get the morphism from $A \to B_{\tau_{\leq 2}}$.



Hence by induction, we get a morphism from $g : A \to B$. The composition $g \circ f : B \to A \to B$ is the identity on $\pi_1(B)$ and by the universal property of Sym, we deduce that $g \circ f = \mathrm{id}_B$. This implies that $\pi_i(B) = \wedge^i \pi_1(A) \to \pi_i(A)$ is injective. To finish the proof, we will prove that these morphisms are surjective.

For this purpose we use another characterization of afffine quasi-smooth derived scheme. Let us fix generators of $\pi_0(A)$. This choice is determined a surjective map of commutative k-algebras $k[x_1, .., x_n] \to \pi_0(A)$. As the polynomial ring is smooth, we proceed by induction on the Postnikov tower of A to construct a morphism from $k[x_1, \ldots, x_n] \to \tau_{\leq n}A$. We use the same idea as above for constructing the morphism $A \to B$. We get a map of cdga's $k[x_1, .., x_n] \to A$ which remains a closed immersion. Moreover, one can now choose generators for the kernel I of $k[x_1, \ldots, x_n] \to \pi_0(A)$, say, $f_1, .., f_m$ whose image in I/I^2 form a basis. The fact that $k[y_1, .., y_m]$ is smooth allows us to extend the zero composition map

$$k[y_1, ..., y_m] \to k[x_1, .., x_m] \to \pi_0(A)$$

to map

$$k[y_1, ..., y_m] \to k[x_1, .., x_m] \to A$$

together with a null-homotopy. This puts A in a commutative square of cdga's

$$\begin{array}{ccc} k[y_1, \dots, y_m] & \longrightarrow & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ k & \longrightarrow & A \end{array}$$

which we is a pushout square. Indeed, it suffices to show that the canonical map

$$k \otimes^{\mathbb{L}}_{k[y_1,..,y_m]} k[x_1, ..., x_n] \to A$$

induces an isomorphism between the cotangent complexes. But as $\mathrm{Spec}(A)$ is quasi-smooth, its cotangent complex is perfect in tor-amplitudes $-1, 0$, meaning that it can be written as

$$A^m \to A^n$$

and this identifies with the standard description of the cotangent complex of the derived tensor product $k \otimes^{\mathbb{L}}_{k[y_1,..,y_m]} k[x_1, ..., x_n]$. This implies that surjectivity of the morphisms $\pi_i(B) \to \pi_i(A)$. □

Remark C.0.6. As $F = \mathrm{Spec}\, A$ is a derived scheme (not necessarily quasi-smooth) and its truncation is F_0 is smooth, we have that $F_0 \to F$ admits a retract. We proceed by induction on the Postnikov tower of A to construct a lift

$$\begin{array}{ccc} & & A \\ & \nearrow & \downarrow \\ \pi_0(A) & \xrightarrow{\mathrm{id}} & \pi_0(A) \end{array}$$

We use the same kind of diagrams as (C.0.4) and (C.0.5) Indeed, as $\mathbb{L}_{F_0}$ is concentrated in degree 0, all the groups

$$\pi_0(\mathrm{Map}(\mathbb{L}_{F_0}, \pi_n(A)[n+1])) = \pi_1(\mathrm{Map}(\mathbb{L}_{F_0}, \pi_n(A)[n+1])) = 0$$

vanish for $n \geq 1$ saying that the liftings exist at each level of the Postnikov tower the space of choices of such liftings is connected.

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