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WAVE FINITE ELEMENT METHOD FOR ELECTROMECHANICAL PERIODIC WAVEGUIDES

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Abstract. A novel Finite Element formulation is proposed for cases involving the coupling of mechanical and electrical waveguides. This allows using the Transfer Matrix formalism proposed in the Wave Finite Element method. Electrical variables are added to the state vectors in order to compute propagation constants for electromechanical waves. Frequency response fonctions of coupled structures can also be obtained for various mechanical and electrical boundary conditions.

1 INTRODUCTION

A Finite Element method for structures covered with piezoelectric elements was proposed by Thomas et al. [1] who focused on thin piezoelectric patches shunted with independent electrical circuits. The model is based on a condensation of the electrical degrees of freedom in order to recast the system into a standard mechanical formulation. However, this method is not applicable when considering interconnections of several patches through an electrical network. In this case, an electromechanical waveguide is created and a wave can propagate simultaneously in the mechanical and electrical domains. Because there are electrical nodes that interconnect successive elements, the electrical degrees of freedom cannot be condensed in the mechanical problem. For example, this arises when considering a structure coupled to its electrical analogue for a multimodal vibration damping purpose [2, 3]. Based on a periodic distribution, a Transfer Matrix Method can be implemented but it requires the use of external electrical degrees of freedom. A novel Wave Finite Element method dedicated to electromechanical periodic waveguides is then required.

In this study, electric charge displacement and voltage vectors are defined by analogy with displacement and force vectors. The equivalent of a dynamic stiffness matrix is obtained from the constitutive equations of the electromechanical problem. The main difference with a purely mechanical formulation [4, 5, 6, 7] or a problem involving independent piezoelectric shunts [8, 9] is that the state vectors include both mechanical and electrical variables [10]. The global "dynamic stiffness matrix" is then rearranged to bring together the left and right variables.

With this partitioning, the Wave Finite Element method can be applied and a Transfer Matrix is defined after condensation of the internal mechanical degrees of freedoms. This method offers a convenient numerical model for the analysis of wave propagation in periodic structures involving an interconnected array of piezoelectric patches. Finally, we show the example of a rod coupled to a line of inductors. The propagation constants are extracted from the Transfer Matrix and we point out the different waves propagating through the electromechanical waveguide as well as the band-gap effects.

2 FINITE ELEMENT MODEL

2.1 Piezoelectric coupling

In this paper, we consider an electromechanical waveguide such as the one in Fig. 1. An electrical network is coupled to a mechanical structure through an array of piezoelectric elements. Each unit cell involves a pair of piezoelectric patches connected in parallel. Other geometries or electrical connections could be considered but we restrain the analysis to a single piezoelectric degree of freedom per unit cell. The variables V_I and $\dot{q}_\mathrm{I}$ represent the voltage across the pair of piezoelectric patches and the electrical current flowing through it. A convenient finite element model was proposed by Thomas et al. [1] for problems involving thin piezoelectric patches and independent shunts. If $\boldsymbol{q_m}$ represents the mechanical displacements and $\boldsymbol{F_m}$ corresponds to the external forces, the coupled problem is given as

$$\begin{bmatrix} \boldsymbol{M_m} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_{\boldsymbol{m}} \\ \ddot{V}_\mathrm{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K_m} & \boldsymbol{K_c} \\ -\boldsymbol{K_c}^{\boldsymbol{T}} & C^\varepsilon \end{bmatrix} \begin{bmatrix} \boldsymbol{q_m} \\ V_\mathrm{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_m} \\ q_\mathrm{I} \end{bmatrix}, \tag{1}$$

where $\boldsymbol{M_m}$, $\boldsymbol{K_m}$ and $\boldsymbol{K_c}$ are respectively the mass, stiffness and coupling matrices. The constant C^ε is the piezoelectric capacitance when no motion is allowed, i.e. $\boldsymbol{q_m} = \boldsymbol{0}$. Considering a shunt made of an impedance Z in parallel with the piezoelectric elements at angular frequency ω, the relation $V_\mathrm{I} = -\mathrm{j}\omega Z q_\mathrm{I}$ allows condensing the electrical degrees of freedom. Indeed, from Eq. (1) we obtain

$$\boldsymbol{M_m}\ddot{\boldsymbol{q}}_{\boldsymbol{m}} + \left[\boldsymbol{K_m} + \frac{1}{C^\varepsilon + \dfrac{1}{\mathrm{j}\omega Z}} \boldsymbol{K_c}\boldsymbol{K_c}^{\boldsymbol{T}} \right] \boldsymbol{q_m} = \boldsymbol{F_m}, \tag{2}$$

which represents a standard mechanical problem with an added-stiffness term due to piezoelectric coupling [1, 11].

However, this method is not applicable when considering interconnections of successive unit cells with an electrical network. There are electrical charges that are flowing in and out of the unit cell. This means that there is no direct relation between the internal variables V_I and q_I. In the end, the electrical variables cannot be condensed in the mechanical problem. A novel formulation that is able to take into account external electrical degrees of freedom is thus required.

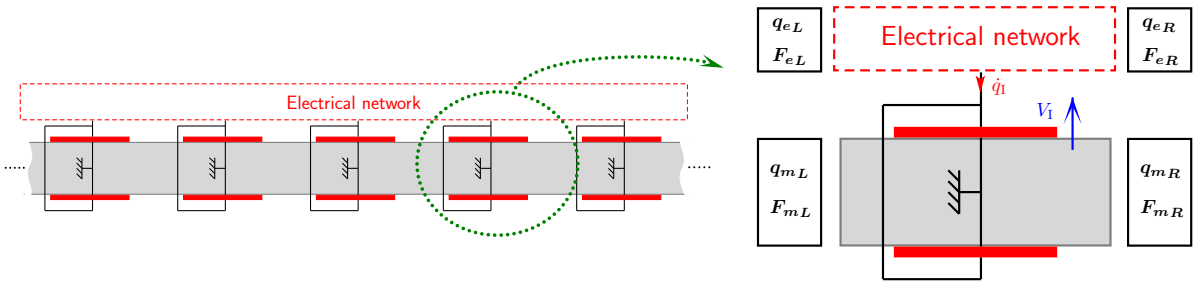

Figure 1: Interconnected array of piezoelectric patches and corresponding electromechanical unit cell highlighting left and right degrees of freedom.

2.2 Electrical degrees of freedom

From the topology of the electrical network, the electric charge q_I flowing through the pair of piezoelectric patches can be defined relatively to a charge vector $\boldsymbol{q_e}$ as

$$q_\text{I} = \boldsymbol{S^T q_e}. \tag{3}$$

If we focus on a one-dimensional unit cell, the vector $\boldsymbol{q_e} = \left[\begin{array}{cc} \boldsymbol{q_{eL}} & \boldsymbol{q_{eR}} \end{array} \right]^T$ refer to the electrical charges that are exchanged with the neighboring left and right unit cells. The matrix $\boldsymbol{S}$ depends on the internal connections of the considered portion of electrical network. By analogy with the force vector $\boldsymbol{F_m}$, we define $\boldsymbol{F_e} = \left[\begin{array}{cc} \boldsymbol{F_{eL}} & \boldsymbol{F_{eR}} \end{array} \right]^T$ that contains the voltage contributions on both sides of the electrical part of the unit cell. In the following, we consider that the total energy of the purely electrical problem (when $\boldsymbol{q_m} = \boldsymbol{0}$) can be written under the form

$$E_e = \frac{1}{2}\boldsymbol{q_e^T K_e q_e} + \frac{1}{2}\boldsymbol{\dot{q}_e^T M_e \dot{q}_e}, \tag{4}$$

where $\boldsymbol{K_e}$ and $\boldsymbol{M_e}$ are symmetric matrices that are the analogues of the mechanical stiffness and mass matrices. Then, the energy in the electrical network excluding the piezoelectric capacitor is $E_e - \dfrac{q_\text{I}^2}{2C^\varepsilon}$. Furthermore, the electrical power flowing into the electrical network in Fig. 1 is equal to

$$P_e = \boldsymbol{F_e^T \dot{q}_e} - V_\text{I}\dot{q}_\text{I} \tag{5}$$

If the electrical network is a conservative system, a power balance induces that the input power is equal to the time derivative of the energy as

$$P_e = \frac{d\left(E_e - \dfrac{q_\text{I}^2}{2C^\varepsilon}\right)}{dt}. \tag{6}$$

The symmetry property of both matrices $\boldsymbol{K_e}$ and $\boldsymbol{M_e}$ thus gives

$$\boldsymbol{F_e^T \dot{q}_e} - V_\text{I}\dot{q}_\text{I} = \boldsymbol{q_e^T K_e \dot{q}_e} + \boldsymbol{\dot{q}_e^T M_e \ddot{q}_e} - \frac{q_\text{I}}{C^\varepsilon}\dot{q}_\text{I}, \tag{7}$$

which is equivalent to

$$\boldsymbol{\dot{q}_e^T}\left(\boldsymbol{F_e} - \boldsymbol{S}V_\text{I} - \boldsymbol{K_e q_e} - \boldsymbol{M_e \ddot{q}_e} + \frac{1}{C^\varepsilon}\boldsymbol{S S^T q_e}\right) = 0 \tag{8}$$

This true for any electric current vector $\dot{\boldsymbol{q}}_{\boldsymbol{e}}$ so

$$\boldsymbol{F_e} = \boldsymbol{K_e q_e} + \boldsymbol{M_e}\ddot{\boldsymbol{q}}_{\boldsymbol{e}} + \boldsymbol{S}V_{\mathrm{I}} - \frac{1}{C^{\varepsilon}}\boldsymbol{S}\boldsymbol{S}^{\boldsymbol{T}}\boldsymbol{q_e}. \tag{9}$$

In the end, as we also know from Eq. (1) that

$$V_{\mathrm{I}} = \frac{1}{C^{\varepsilon}}\left(\boldsymbol{S}^{\boldsymbol{T}}\boldsymbol{q_e} + \boldsymbol{K_c}^{\boldsymbol{T}}\boldsymbol{q_m}\right), \tag{10}$$

we get a relation between the voltage vector $\boldsymbol{F_e}$ and the displacement vectors:

$$\boldsymbol{F_e} = \boldsymbol{K_e q_e} + \boldsymbol{M_e}\ddot{\boldsymbol{q}}_{\boldsymbol{e}} + \frac{1}{C^{\varepsilon}}\boldsymbol{S}\boldsymbol{K_c^T}\boldsymbol{q_m}. \tag{11}$$

Note that a principle of superposition applies because $\boldsymbol{F_e}$ is a sum of two contributions. The first one corresponds to the voltage vector when $\boldsymbol{q_m} = \boldsymbol{0}$, i.e. when no mechanical displacement is allowed. This is a purely electrical contribution that only depends on the electrical network and the piezoelectric capacitance. On the other hand, the second contribution, observed when $\boldsymbol{q_e} = \boldsymbol{0}$, is directly related to piezoelectric coupling.

2.3 Electromechanical formulation

Now that all the mechanical and electrical vectors have been defined for both displacements and forcing terms, they can be organized into a single matrix formulation. Under harmonic excitation at angular frequency ω, Eqs. (1) and (10) lead to a force vector

$$\boldsymbol{F_m} = \left[\boldsymbol{K_m} + \frac{1}{C^{\varepsilon}}\boldsymbol{K_c}\boldsymbol{K_c}^{\boldsymbol{T}} - \omega^2\boldsymbol{M_m}\right]\boldsymbol{q_m} + \frac{1}{C^{\varepsilon}}\boldsymbol{K_c}\boldsymbol{S}^{\boldsymbol{T}}\boldsymbol{q_e}. \tag{12}$$

Similarly, Eq. (11) gives a voltage vector

$$\boldsymbol{F_e} = \left[\boldsymbol{K_e} - \omega^2\boldsymbol{M_e}\right]\boldsymbol{q_e} + \frac{1}{C^{\varepsilon}}\boldsymbol{S}\boldsymbol{K_c^T}\boldsymbol{q_m} \tag{13}$$

This leads to the following dynamic stiffness matrix formulation involving a combination of mechanical and electrical degrees of freedom:

$$\left[\begin{bmatrix} \boldsymbol{K_m} + \frac{1}{C^{\varepsilon}}\boldsymbol{K_c}\boldsymbol{K_c}^{\boldsymbol{T}} & \frac{1}{C^{\varepsilon}}\boldsymbol{K_c}\boldsymbol{S}^{\boldsymbol{T}} \\ \frac{1}{C^{\varepsilon}}\boldsymbol{S}\boldsymbol{K_c^T} & \boldsymbol{K_e} \end{bmatrix} - \omega^2\begin{bmatrix} \boldsymbol{M_m} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M_e} \end{bmatrix}\right]\begin{bmatrix} \boldsymbol{q_m} \\ \boldsymbol{q_e} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_m} \\ \boldsymbol{F_e} \end{bmatrix}. \tag{14}$$

The coupled problem is thus organized like any purely mechanical problem involving mass and stiffness matrices. The only difference is that the global displacement and force vectors contain both mechanical and electrical contributions. With such an electromechanical formulation, classical methods such as the assembly of element matrices or Transfer Matrix methods can be applied.

3 WAVE FINITE ELEMENT METHOD

3.1 Transfer matrix

The dynamic stiffness matrix given in Eq. (14) is rearranged to bring together the mechanical and electrical left and right degrees of freedom [10], as

$$\begin{bmatrix} \tilde{\boldsymbol{D}}_{\mathrm{LL}} & \tilde{\boldsymbol{D}}_{\mathrm{LI}} & \tilde{\boldsymbol{D}}_{\mathrm{LR}} \\ \tilde{\boldsymbol{D}}_{\mathrm{IL}} & \tilde{\boldsymbol{D}}_{\mathrm{II}} & \tilde{\boldsymbol{D}}_{\mathrm{IR}} \\ \tilde{\boldsymbol{D}}_{\mathrm{RL}} & \tilde{\boldsymbol{D}}_{\mathrm{RI}} & \tilde{\boldsymbol{D}}_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} \\ \boldsymbol{q}_{m\mathrm{I}} \\ \boldsymbol{q}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{\mathrm{L}} \\ \boldsymbol{0} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix} \tag{15}$$

where $\boldsymbol{q}_{\mathrm{L}} = \begin{bmatrix} \boldsymbol{q}_{m\mathrm{L}} & \boldsymbol{q}_{e\mathrm{L}} \end{bmatrix}^T$, $\boldsymbol{q}_{\mathrm{R}} = \begin{bmatrix} \boldsymbol{q}_{m\mathrm{R}} & \boldsymbol{q}_{e\mathrm{R}} \end{bmatrix}^T$, $\boldsymbol{F}_{\mathrm{L}} = \begin{bmatrix} \boldsymbol{F}_{m\mathrm{L}} & \boldsymbol{F}_{e\mathrm{L}} \end{bmatrix}^T$ and $\boldsymbol{F}_{\mathrm{R}} = \begin{bmatrix} \boldsymbol{F}_{m\mathrm{R}} & \boldsymbol{F}_{e\mathrm{R}} \end{bmatrix}^T$. This expression is based on the decomposition of the mechanical vectors $\boldsymbol{F}_{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{F}_{m\mathrm{L}} & \boldsymbol{0} & \boldsymbol{F}_{m\mathrm{R}} \end{bmatrix}^T$ and $\boldsymbol{q}_{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{q}_{m\mathrm{L}} & \boldsymbol{q}_{m\mathrm{I}} & \boldsymbol{q}_{m\mathrm{R}} \end{bmatrix}^T$ where $\boldsymbol{q}_{m\mathrm{I}}$ is the mechanical displacement vector of the internal nodes of the unit cell. With this partitioning, the procedures of the Wave Finite Element method [4, 5, 6, 7, **?**, 9, 11] can be implemented. This starts with an elimination of the internal degrees of freedoms through

$$\begin{bmatrix} \boldsymbol{D}_{\mathrm{LL}} & \boldsymbol{D}_{\mathrm{LR}} \\ \boldsymbol{D}_{\mathrm{RL}} & \boldsymbol{D}_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} \\ \boldsymbol{q}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{\mathrm{L}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} \boldsymbol{D}_{\mathrm{LL}} = \tilde{\boldsymbol{D}}_{\mathrm{LL}} - \tilde{\boldsymbol{D}}_{\mathrm{LI}} \tilde{\boldsymbol{D}}_{\mathrm{II}}^{-1} \tilde{\boldsymbol{D}}_{\mathrm{IL}} \\ \boldsymbol{D}_{\mathrm{LR}} = \tilde{\boldsymbol{D}}_{\mathrm{LR}} - \tilde{\boldsymbol{D}}_{\mathrm{LI}} \tilde{\boldsymbol{D}}_{\mathrm{II}}^{-1} \tilde{\boldsymbol{D}}_{\mathrm{IR}} \\ \boldsymbol{D}_{\mathrm{RL}} = \tilde{\boldsymbol{D}}_{\mathrm{RL}} - \tilde{\boldsymbol{D}}_{\mathrm{RI}} \tilde{\boldsymbol{D}}_{\mathrm{II}}^{-1} \tilde{\boldsymbol{D}}_{\mathrm{IL}} \\ \boldsymbol{D}_{\mathrm{RR}} = \tilde{\boldsymbol{D}}_{\mathrm{RR}} - \tilde{\boldsymbol{D}}_{\mathrm{RI}} \tilde{\boldsymbol{D}}_{\mathrm{II}}^{-1} \tilde{\boldsymbol{D}}_{\mathrm{IR}} \end{array}, \tag{16}$$

and the condensed dynamic stiffness matrix is then transformed into a transfer matrix:

$$\begin{bmatrix} \boldsymbol{q}_{\mathrm{R}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{D}_{\mathrm{LR}}^{-1} \boldsymbol{D}_{\mathrm{LL}} & -\boldsymbol{D}_{\mathrm{LR}}^{-1} \\ \boldsymbol{D}_{\mathrm{RL}} - \boldsymbol{D}_{\boldsymbol{RR}} \boldsymbol{D}_{\mathrm{LR}}^{-1} \boldsymbol{D}_{\mathrm{LL}} & -\boldsymbol{D}_{\mathrm{RR}} \boldsymbol{D}_{\mathrm{LR}}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} \\ -\boldsymbol{F}_{\mathrm{L}} \end{bmatrix}. \tag{17}$$

As both force vectors $\boldsymbol{F}_{\mathrm{L}}$ and $\boldsymbol{F}_{\mathrm{R}}$ corresponds to external forcing, the use of $-\boldsymbol{F}_{\mathrm{L}}$ is required in order to ensure the continuity of the state vector [4, 5].

3.2 Propagation constants

A one-dimensional periodic structure can be defined as a series of identical unit cells. As in Eq. (17), the relation between the mechanical states at the right and left ends of a unit cell is described by a transfer matrix $\boldsymbol{T}$ through

$$\begin{bmatrix} \boldsymbol{q}_{\mathrm{R}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} \\ -\boldsymbol{F}_{\mathrm{L}} \end{bmatrix}. \tag{18}$$

The Floquet-Bloch theory induces that, for a wave propagating in the one-dimensional medium, the state on the right of a unit cell is equal to $e^{\pm\mu}$ times the state on the left [7, 12], the sign depending on the direction of the wave. Here, μ is the propagation constant, which is usually defined as $\mu = \delta + \mathrm{j}\theta$, where δ is the attenuation constant, j is the imaginary unit and θ is the phase constant [8]. Alternatively, the propagation constant can be written $\mu = \alpha a + \mathrm{j}ka$, where α is the attenuation rate, k is the wavenumber and a is the length of the unit cell. From Eq. (18), if $\begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} & -\boldsymbol{F}_{\mathrm{L}} \end{bmatrix}^T$ is an eigenvector of the transfer matrix $\boldsymbol{T}$,

$$\begin{bmatrix} \boldsymbol{q}_{\mathrm{R}} \\ \boldsymbol{F}_{\mathrm{R}} \end{bmatrix} = e^{\pm\mu} \begin{bmatrix} \boldsymbol{q}_{\mathrm{L}} \\ -\boldsymbol{F}_{\mathrm{L}} \end{bmatrix}. \tag{19}$$

Consequently, $\mu = \ln(\lambda)$, where λ represents an eigenvalue of the transfer matrix $\boldsymbol{T}$. Considering state vectors including both mechanical and electrical variables, we obtain phase and attenuation constants related to the propagation in an electromechanical waveguide. Generally speaking, we can not distinguish mechanical waves from electrical waves because of the coupling between both physical domains.

3.3 Frequency response functions

Apart from propagation constants, a transfer matrix model also allows to compute frequency response functions [5, 6]. According to the definition of a periodic structure, each unit cell is described by an identical transfer matrix $\boldsymbol{T}$. Consequently, a basic solution to obtain the mechanical state on the right of the n^{th} unit cell consists in raising the corresponding transfer matrix to the power of n,

$$\begin{bmatrix} \boldsymbol{q_n} \\ \boldsymbol{F_n} \end{bmatrix} = \boldsymbol{T}^{\boldsymbol{n}} \begin{bmatrix} \boldsymbol{q_0} \\ -\boldsymbol{F_0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{T_n^{qq}} & \boldsymbol{T_n^{qF}} \\ \boldsymbol{T_n^{Fq}} & \boldsymbol{T_n^{FF}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q_0} \\ -\boldsymbol{F_0} \end{bmatrix}, \tag{20}$$

where the subscript 0 refers to the left end of the first unit cell and the forcing vectors corresponds to forces or voltages applied to the considered electromechanical structure. Then, when considering a finite number of n unit cells, the boundary conditions need to be introduced. For example, with a prescribed force vector $\boldsymbol{F_0}$ applied to the left end of a free-free periodic structure, as $\boldsymbol{F_n} = \boldsymbol{0}$, the displacement $\boldsymbol{q_n}$ at the right end is defined from Eq. (20) as

$$\boldsymbol{q_n} = (\boldsymbol{T_n^{qq}}\boldsymbol{T_n^{Fq}}^{-1}\boldsymbol{T_n^{FF}} - \boldsymbol{T_n^{qF}})\boldsymbol{F_0}. \tag{21}$$

Similar methods can be used for other boundary conditions such as free or blocked for the mechanical part and short-cicuited or open-circuited for the electrical part. In any case, it becomes possible to define frequency response functions between any variables at both ends of the electromechanical structure.

3.4 Application to longitudinal propagation

The case of a rod subjected to longitudinal propagation is considered. As show in Fig. 2, the rod is covered with an array of periodic patches which are interconnected through a line of inductors. This build an electrical network that is the discrete analogue of a rod [2, 13]. For a

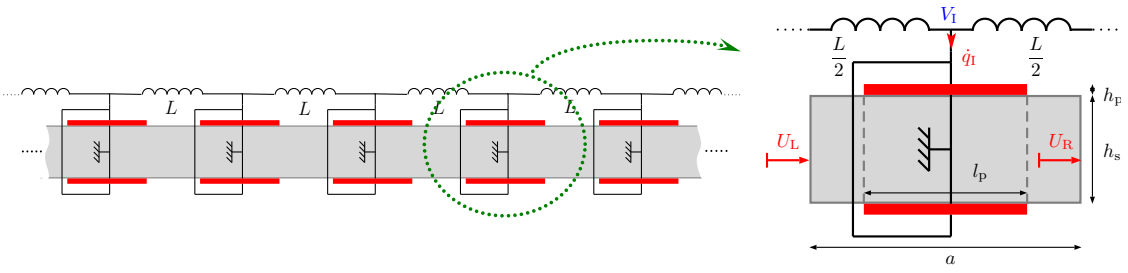

Figure 2: Rod covered with an array of interconnected piezoelectric patches.

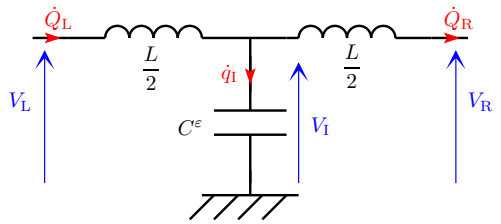

Figure 3: Electrical unit cell without piezoelectric coupling.

rod, the element stiffness matrix, mass matrix and coupling matrix can be defined as

$$\boldsymbol{K}_{\boldsymbol{m}}^{\mathrm{e}} = \frac{YS}{a_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad \boldsymbol{M}_{\boldsymbol{m}}^{\mathrm{e}} = \frac{\rho S a_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{K}_{\boldsymbol{c}}^{\mathrm{e}} = e\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{22}$$

where Y is the equivalent Young's modulus of the considered element, ρ is its density, a_e is its length, S is its cross-sectional area and e is the piezoelectric coupling coefficient [1, 2]. The global matrices $\boldsymbol{K_m}$, $\boldsymbol{M_m}$ and $\boldsymbol{K_c}$ for the unit cell are then obtained from an assembly of the element matrices. The corresponding displacement and force vectors are

$$\boldsymbol{q_m} = \begin{bmatrix} U_{\mathrm{L}} \\ \boldsymbol{U}_{\mathrm{I}} \\ U_{\mathrm{R}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{F_m} = \begin{bmatrix} -N_{\mathrm{L}} \\ \boldsymbol{0} \\ N_{\mathrm{R}} \end{bmatrix}, \tag{23}$$

where the state variable U is related to longitudinal displacement, N corresponds to the normal force and $\boldsymbol{U}_{\mathrm{I}} = \boldsymbol{q}_{\boldsymbol{m}\mathrm{I}}$ is the mechanical displacement vector of the internal nodes of the unit cell. Considering the electrical part of the unit cell represented in Fig. 3, the line of inductors gives a single degree of freedom on each side of the unit cell which leads to

$$\boldsymbol{q_e} = \begin{bmatrix} Q_{\mathrm{L}} \\ Q_{\mathrm{R}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{F_e} = \begin{bmatrix} V_{\mathrm{L}} \\ -V_{\mathrm{R}} \end{bmatrix}, \tag{24}$$

where Q corresponds to electrical charge displacements and V is used for voltage variables. From the topology of the unit cell in Fig. 3 and the inductive and capacitive contributions to the total electrical energy, we get

$$q_{\mathrm{I}} = Q_{\mathrm{L}} - Q_{\mathrm{R}} \quad \text{and} \quad E_e = \frac{(Q_{\mathrm{L}} - Q_{\mathrm{R}})^2}{2C^{\varepsilon}} + \frac{L}{4}\dot{Q}_{\mathrm{L}} + \frac{L}{4}\dot{Q}_{\mathrm{R}}. \tag{25}$$

Then, Eqs. (3) and (4) give

$$\boldsymbol{S} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad , \quad \boldsymbol{K_e} = \frac{1}{C^{\varepsilon}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M_e} = \frac{L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{26}$$

In the end, the global dynamic stiffness matrix in Eq. (14) is computed and reorganized into a transfer matrix $\boldsymbol{T}$ by considering left and right state vectors

$$\boldsymbol{q}_{\mathrm{L}} = \begin{bmatrix} U_{\mathrm{L}} \\ Q_{\mathrm{L}} \end{bmatrix} \quad , \quad \boldsymbol{q}_{\mathrm{R}} = \begin{bmatrix} U_{\mathrm{R}} \\ Q_{\mathrm{R}} \end{bmatrix} \quad , \quad \boldsymbol{F}_{\mathrm{L}} = \begin{bmatrix} -N_{\mathrm{L}} \\ V_{\mathrm{L}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{F}_{\mathrm{R}} = \begin{bmatrix} N_{\mathrm{R}} \\ -V_{\mathrm{R}} \end{bmatrix}. \tag{27}$$

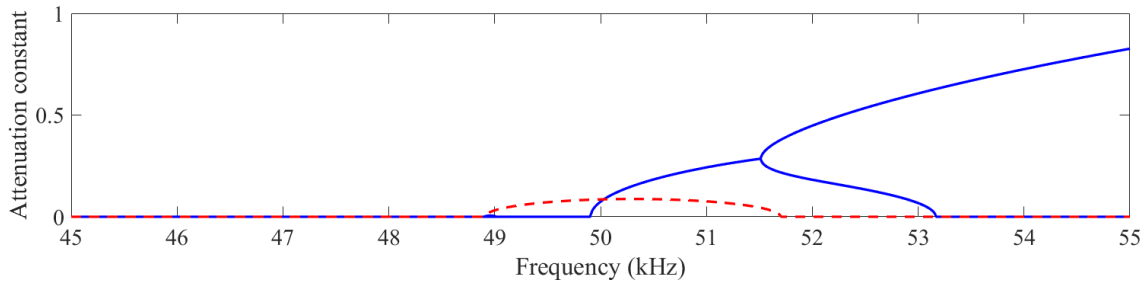

Figure 4: Attenuation constants - (−−) without network, (—) with $L = 1.2$ mH.

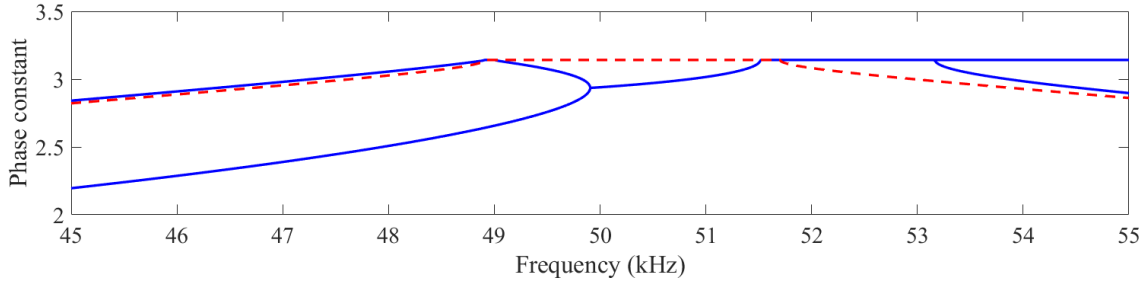

Figure 5: Phase constants - (−−) without network, (—) with $L = 1.2$ mH.

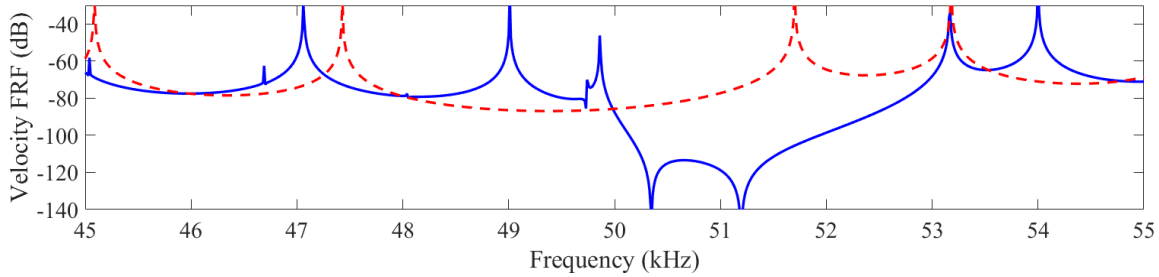

Figure 6: Frequency response function - (−−) without network, (—) with $L = 1.2$ mH.

This gives a relation between two state vectors including both mechanical and electrical variables:

$$\begin{bmatrix} U_{\mathrm{R}} \\ Q_{\mathrm{R}} \\ N_{\mathrm{R}} \\ -V_{\mathrm{R}} \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} U_{\mathrm{L}} \\ Q_{\mathrm{L}} \\ N_{\mathrm{L}} \\ -V_{\mathrm{L}} \end{bmatrix}. \tag{28}$$

Once the Transfer Matrix is obtained, its eigenvalues lead to the attenuation and phase constants. The following numerical calculations are based on the same geometry and material constants as in Ref. [2]. The objective is to show that propagation constants and frequency response functions can be computed for an electromechanical waveguide. The results are given in Figs. 4, 5 and 6. It is observed that the coupling to an electrical line strongly modify

the propagation constants. For this specific example, the original purely mechanical band-gap [8, 10] is replaced by an electromechanical band-gap corresponding to a larger attenuation constant. The effect on the frequency response function of a free-free rod made of 20 unit cells is illustrated in Fig. 6 where we see a strong reduction of vibration amplitude between 50 and 53 kHz. All of this shows that the proposed numerical model can be used for analyzing the coupling between mechanical and electrical band-gaps when considering propagation in electromechanical waveguides [14]. The proposed example is based on a simple geometry but there is no restriction regarding the complexity of the mechanical structure and the number of degrees of freedom as long as the stiffness, mass and coupling matrices have been defined.

4 CONCLUSIONS

- Considering electromechanical waveguides, energy considerations lead to the definition of a charge displacement vector and voltage vector that can be related to each other from the topology of the electrical network.
- The introduction of electrical degrees of freedom allows the use of a formulation based on the equivalent of a dynamic stiffness matrix that represents the whole electromechanical problem for a macro element which is the unit cell in the present paper.
- The Wave Finite Element method can be directly applied by considering electromechanical state vectors instead of purely mechanical ones.
- This gives us a way to analyze the coupling between mechanical and electrical waveguides from the computation of propagation constants and frequency response functions.
- It is shown that couplings between mechanical and electrical band-gaps occur, which can be of interest for the control of wave propagation in periodic structures.

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