Some properties of orbital varieties in extremal nilpotent orbits

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SOME PROPERTIES OF ORBITAL VARIETIES IN EXTREMAL NILPOTENT ORBITS

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Dedicated to Professor Anthony Joseph on the occasion of his 75th birthday

Abstract. The intersection between a nilpotent orbit of a simple Lie algebra and a Borel subalgebra is always equidimensional. Its irreducible components are called orbital varieties. Orbital varieties belonging to different nilpotent orbits may have quite different behaviours. The orbital varieties of the subregular nilpotent orbit are always smooth but they have in general infinitely many B-orbits. At the opposite, the minimal nilpotent orbit is spherical but its orbital varieties may have singularities. In this paper, we characterize the orbital varieties of the subregular nilpotent orbit which have a finite number of B-orbits and we give a smoothness criterion for the orbital varieties of the minimal nilpotent orbit.

1. Introduction

1.1. **Nilpotent orbits.** Let G be a connected simple algebraic group over $\mathbb{K}$ (an algebraically closed field of characteristic zero). By $\mathfrak{g}$ we denote the Lie algebra of G, by $(g, x) \mapsto g \cdot x$ we denote the adjoint action. Let $B \subset G$ be a Borel subgroup and let $\mathfrak{n} \subset \mathfrak{g}$ be the nilpotent radical of the Lie algebra of B.

An adjoint orbit $\mathcal{O} = G \cdot x := \{g \cdot x : g \in G\}$ is called nilpotent if the intersection $\mathcal{O} \cap \mathfrak{n}$ is nonempty. The set $\mathcal{N} := G \cdot \mathfrak{n}$ is the nilpotent cone. It consists of a finite number of nilpotent orbits. We emphasize four of them:

- The regular nilpotent elements form a single orbit $\mathcal{O}_{\mathrm{reg}}$, called the regular nilpotent orbit, which is dense in $\mathcal{N}$.
- There is a single orbit $\mathcal{O}_{\mathrm{subreg}}$, called the subregular nilpotent orbit, which is dense in $\mathcal{N} \setminus \mathcal{O}_{\mathrm{reg}}$.
- There is a single nontrivial nilpotent orbit $\mathcal{O}_{\mathrm{min}}$ of minimal dimension, called the minimal nilpotent orbit; it lies in the closure of every nontrivial nilpotent orbit.
- The only closed nilpotent orbit is the trivial orbit $\mathcal{O}_{\mathrm{triv}} = \{0\}$.

1.2. **Orbital varieties.** Every nilpotent orbit $\mathcal{O} \subset \mathcal{N}$ has a structure of symplectic variety, in particular its dimension $\dim \mathcal{O}$ is even. The intersection $\mathcal{O} \cap \mathfrak{n}$ is a quasi-affine variety, which is in fact equidimensional of dimension $\frac{1}{2} \dim \mathcal{O}$ (see [10]). The irreducible components of $\mathcal{O} \cap \mathfrak{n}$ are called orbital varieties. They are

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B-stable, Lagrangian subvarieties of $\mathcal{O}$. Orbital varieties arise in geometric representation theory, in relation with associated varieties of simple highest weight modules. We refer to the works of A. Joseph [8, 9] and references therein. In [9], the orbital varieties of the minimal nilpotent orbit are studied with respect to their quantization properties.

Orbital varieties may be singular and may have an infinite number of B-orbits, and they have a complicated intersection pattern. There is no general classification of orbital varieties with respect to their geometrical or topological properties. We refer to [7] for some partial classifications, mainly in type A.

In this paper we study some properties of orbital varieties for an arbitrary simple algebraic group G, but in the case of the particular nilpotent orbits mentioned above.

There is not much to say about the trivial nilpotent orbit $\mathcal{O}_{\mathrm{triv}} = \{0\}$ and its sole orbital variety $\mathcal{O}_{\mathrm{triv}} \cap \mathfrak{n} = \{0\}$. In the case of the regular nilpotent orbit, the intersection $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{n}$ is a single B-orbit, hence a single B-homogeneous (and therefore smooth) orbital variety. For the remaining two extremal nilpotent orbits $\mathcal{O}_{\mathrm{min}}$ and $\mathcal{O}_{\mathrm{subreg}}$, the situation is not so straightforward. We stress the following facts:

- The minimal nilpotent orbit $\mathcal{O}_{\mathrm{min}}$ is spherical, hence every orbital variety of $\mathcal{O}_{\mathrm{min}}$ has a finite number of B-orbits. Moreover $\mathcal{O}_{\mathrm{min}} \cap \mathfrak{n}$ contains a unique closed B-orbit, which therefore lies in every orbital variety. However the orbital varieties of $\mathcal{O}_{\mathrm{min}}$ may be singular.

  In this paper, we characterize the singular orbital varieties of $\mathcal{O}_{\mathrm{min}}$.

- At the opposite, in the subregular nilpotent orbit $\mathcal{O}_{\mathrm{subreg}}$, every orbital variety is smooth; in fact it is open in the nilradical of some minimal parabolic subalgebra. However an orbital variety of $\mathcal{O}_{\mathrm{subreg}}$ does not always contain a dense B-orbit, and two orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$ rarely intersect.

  In this paper, we characterize the orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$ *which have a dense* B*-orbit (resp. a finite number of* B*-orbits), and we characterize the pairs of orbital varieties of* $\mathcal{O}_{\mathrm{subreg}}$ *which intersect.*

In particular, the results shown in this paper illustrate how orbital varieties belonging to different nilpotent orbits may have different properties. Our main results are stated in terms of roots, simple roots and biggest root: see Section 2.2. The orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$ can indeed be parameterized by the simple roots whereas the orbital varieties of $\mathcal{O}_{\mathrm{min}}$ can be parameterized by the simple long roots. These parameterizations are explained in Section 2.1.

2. Main results

2.1. **Parameterization of orbital varieties.** Hereafter we fix a maximal torus $T \subset B$ and let $\mathfrak{h} \subset \mathfrak{g}$ denote the corresponding Cartan subalgebra. We then consider the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$, the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

and the subset of positive roots Φ^+ corresponding to the choice of B and $\mathfrak{n}$, i.e., such that $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Let $\Pi \subset \Phi^+$ be the set of simple roots.

Let $W = W(G,T)$ be the Weyl group. By [12] there is a surjective map from W onto the set of orbital varieties of $\mathcal{N}$: for every $w \in W$, there is a unique nilpotent orbit $\mathcal{O}_w$ which intersects the linear space $\mathfrak{n} \cap (w \cdot \mathfrak{n})$ densely; then, the set $\mathcal{V}_w := \overline{B \cdot (\mathfrak{n} \cap (w \cdot \mathfrak{n}))} \cap \mathcal{O}_w$ is an orbital variety, and every orbital variety is obtained in this way. In particular $\mathcal{O}_e = \mathcal{O}_{\mathrm{reg}}$, $\mathcal{V}_e = \mathcal{O}_{\mathrm{reg}} \cap \mathfrak{n}$, $\mathcal{V}_{w_0} = \mathcal{O}_{w_0} = \{0\}$, where $e, w_0 \in W$ respectively stand for the neutral element and the longest element.

The orbital varieties contained in the nilpotent orbits $\mathcal{O}_{\min}$ and $\mathcal{O}_{\mathrm{subreg}}$ have an alternative, handy parameterization, obtained as follows.

Every simple root $\alpha \in \Pi$ gives rise to a minimal parabolic subgroup P_α and a nilradical $\mathfrak{n}_\alpha = \bigoplus_{\gamma \in \Phi^+ \setminus \{\alpha\}} \mathfrak{g}_\gamma$. By [5, §4.1], we have

$$\mathfrak{n} \setminus \mathcal{O}_{\mathrm{reg}} = \bigcup_{\alpha \in \Pi} \mathfrak{n}_\alpha, \quad \text{hence} \quad \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n} = \bigcup_{\alpha \in \Pi} \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha.$$

By [5, Theorem 7.1.1], for every $\alpha \in \Pi$, the intersection $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ is P_α-homogeneous (thus irreducible, smooth) and dense in $\mathfrak{n}_\alpha$. This yields:

Proposition 1. (a) *The subsets $\mathcal{V}_{\mathrm{subreg}}(\alpha) := \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$, for $\alpha \in \Pi$, are exactly the irreducible components of $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}$, i.e., the orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$.*

(b) *$\mathcal{V}_{\mathrm{subreg}}(\alpha)$ is P_α-homogeneous (thus smooth) and its closure is the linear space $\mathfrak{n}_\alpha$ (thus this closure is also smooth).*

(c) *For $\alpha, \alpha' \in \Pi$, the orbital varieties $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ and $\mathcal{V}_{\mathrm{subreg}}(\alpha')$ intersect if and only if the roots α, α' are not orthogonal.*

(d) *$\mathcal{V}_{\mathrm{subreg}}(\alpha) = \mathcal{V}_{s_\alpha}$, where $s_\alpha \in W$ is the simple reflection attached to the root α.*

Proof. Parts (a) and (b) follow from the previous discussion. Part (d) follows from the definitions of $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ and $\mathcal{V}_{s_\alpha}$. Let us show part (c). If the simple roots α and α' are orthogonal, then the intersection $\mathfrak{n}_\alpha \cap \mathfrak{n}_{\alpha'}$ is the nilradical of a parabolic subalgebra. By [5, Theorem 7.1.1], the maximal dimension of a nilpotent orbit intersecting $\mathfrak{n}_\alpha \cap \mathfrak{n}_{\alpha'}$ is $2 \dim \mathfrak{n}_\alpha \cap \mathfrak{n}_{\alpha'} < 2 \dim \mathfrak{n}_\alpha = \dim \mathcal{O}_{\mathrm{subreg}}$. Hence $\mathcal{V}_{\mathrm{subreg}}(\alpha) \cap \mathcal{V}_{\mathrm{subreg}}(\alpha') = \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha \cap \mathfrak{n}_{\alpha'} = \emptyset$. We have shown that two orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$ have an empty intersection if they correspond to simple roots which are orthogonal, i.e., which are not connected by an edge in the Dynkin diagram. By [10], the variety $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}$ is connected. This implies that $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ and $\mathcal{V}_{\mathrm{subreg}}(\alpha')$ must intersect if there is an edge between α and α' in the Dynkin diagram. □

For every root $\alpha \in \Phi^+$ we fix a root vector $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. In the simply laced cases, we say that all the roots are long. In general, let Φ_ℓ (resp., Φ_ℓ^+) stand for the set of long roots (resp., positive long roots) and let $\Pi_\ell \subset \Pi$ be the subset of simple long roots. Let $\preceq$ be the usual partial order on the root system Φ determined by the choice of the set of positive roots Φ^+. Let $\beta_{\max} \in \Phi^+$ be the biggest root, i.e., the biggest element of Φ with respect to the order $\preceq$. It is always a long root, and the root vector $e_{\beta_{\max}}$ is a representative of the minimal nilpotent orbit $\mathcal{O}_{\min}$. Note that

$$B \cdot e_{\beta_{\max}} = \mathfrak{g}_{\beta_{\max}} \setminus \{0\}, \quad \text{hence} \quad \mathcal{O}_{\min} = G \cdot e_{\beta_{\max}} = \bigcup_{w \in W} B \cdot e_{w(\beta_{\max})},$$

where the last equality follows from the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, whence

$$\mathcal{O}_{\min} = \bigcup_{\alpha \in \Phi_\ell} B \cdot e_\alpha \quad \text{and} \quad \mathcal{O}_{\min} \cap \mathfrak{n} = \bigcup_{\alpha \in \Phi_\ell^+} B \cdot e_\alpha \tag{1}$$

since the Weyl group W acts transitively on the set of long roots. The next statement follows from [4, 8] and [3, §6.1].

Proposition 2. (a) *For every $\alpha \in \Phi_\ell^+$, we have*

$$\overline{B \cdot e_\alpha} \cap \mathcal{O}_{\min} = \bigcup_{\gamma \succeq \alpha} B \cdot e_\gamma$$

where the union is taken over all long roots $\gamma \in \Phi_\ell^+$ satisfying $\gamma \succeq \alpha$.

(b) *Thus, the subsets $\mathcal{V}_{\min}(\alpha) := \overline{B \cdot e_\alpha} \cap \mathcal{O}_{\min}$, for $\alpha \in \Pi_\ell$, are exactly the irreducible components of $\mathcal{O}_{\min} \cap \mathfrak{n}$, i.e., the orbital varieties of $\mathcal{O}_{\min}$. Every orbital variety contains in particular the orbit $B \cdot e_{\beta_{\max}}$.*

(c) *Every orbital variety $\mathcal{V}_{\min}(\alpha)$ (for $\alpha \in \Pi_\ell$) is normal, Cohen–Macaulay, and has rational singularities.*

(d) $\mathcal{V}_{\min}(\alpha) = \mathcal{V}_{s_\alpha w_0}$.

2.2. **Statement of main results.** As in Section 2.1, $\beta_{\max}$ stands for the biggest root. It decomposes as a sum of simple roots

$$\beta_{\max} = \sum_{\alpha \in \Pi} n(\alpha)\alpha,$$

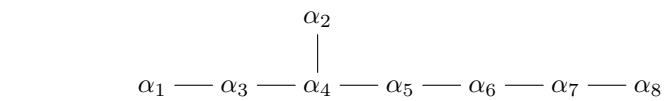
where the coefficients $n(\alpha)$ are positive integers.

Our first main result is a smoothness criterion for the orbital varieties of the minimal nilpotent orbit. The proof is given in Section 3.3.

Theorem 1. *Let $\alpha \in \Pi_\ell$ be a simple long root and let $\mathcal{V}_{\min}(\alpha) = \overline{B \cdot e_\alpha} \cap \mathcal{O}_{\min}$ be the corresponding orbital variety of the minimal nilpotent orbit $\mathcal{O}_{\min}$. Then:*

$$\mathcal{V}_{\min}(\alpha) \text{ is smooth if and only if } n(\alpha) = 1.$$

Our second main result is a criterion of finiteness of number of B-orbits / existence of dense B-orbit for the orbital varieties of the subregular nilpotent orbit. In the result below we say that a simple root α is extremal if it belongs to only one (possibly multiple) edge of the Dynkin diagram (i.e., there is only one simple root which is not orthogonal to α). In types E_6, E_7, E_8, we consider the numbering of the simple roots determined by the following diagram (of type E_8)



and its subdiagrams $\{\alpha_1, \ldots, \alpha_7\}$ (of type E_7) and $\{\alpha_1, \ldots, \alpha_6\}$ (of type E_6).

Theorem 2. *Let $\alpha \in \Pi$ be a simple root and let $\mathcal{V}_{\mathrm{subreg}}(\alpha) = \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ be the corresponding orbital variety of the subregular nilpotent orbit $\mathcal{O}_{\mathrm{subreg}}$. The following conditions are equivalent:*

(i) *$\mathcal{V}_{\mathrm{subreg}}(\alpha)$ has a finite number of B-orbits;*

(ii) *$\mathcal{V}_{\mathrm{subreg}}(\alpha)$ has a dense B-orbit;*

(iii) *One of the following conditions is satisfied:*

(1) *the group G is of type A or B;*
(2) *the group G is of type C or D, and α is an extremal root of the Dynkin diagram;*
(3) *the group G is of type G_2 or F_4, and α is long and extremal;*
(4) *the group G is of type E_6 (resp., E_7) and $\alpha \in \{\alpha_1, \alpha_6\}$ (resp., $\alpha = \alpha_7$).*

The proof is given in Section 4.

Corollary 1. (a) *In types A and C, every orbital variety of $\mathcal{O}_{\min}$ is smooth.*
(b) *In types G_2, F_4, and E_8, every orbital variety of $\mathcal{O}_{\min}$ is singular.*
(c) *In types A and B, every orbital variety of $\mathcal{O}_{\mathrm{subreg}}$ has a finite number of B-orbits.*
(d) *In type E_8, every orbital variety of $\mathcal{O}_{\mathrm{subreg}}$ has an infinite number of B-orbits.*

3. Proof of Theorem 1

3.1. **Notation.** Recall that the root system Φ is endowed with the partial order $\preceq$ defined by letting $\alpha \preceq \beta$ if $\beta - \alpha$ is a sum of simple roots, and $\beta_{\max}$ stands for the biggest element of Φ relatively to this order.

Given two positive roots α, β, we write:

- $\alpha \lessdot \beta$ if $\beta - \alpha \in \Phi^+$;
- $\alpha \lll \beta$ if $\beta = \alpha + k\eta$ for some $\eta \in \Phi^+$ and some positive integer k.

Moreover we consider the set

$$M_{\max} := \{\alpha \in \Phi^+ : \alpha \lessdot \beta_{\max}\}.$$

The following technical lemmas can be checked case by case. The first lemma is immediate in the simply laced case (where by convention we say that all the roots are long, i.e., there is no short root).

Lemma 1. *Let γ be a short positive root such that the set*

$$\{\beta \in \Phi^+ \text{ long} : \beta \preceq \gamma\}$$

is nonempty. Then, this set contains a biggest element β_0 relatively to the order $\preceq$, and we have $\beta_0 \lll \gamma$.

Moreover, let us suppose that $\gamma \in M_{\max}$, so that $\beta_{\max} - \gamma \in \Phi^+$. Then, the following alternative holds:

(i) *Either $\gamma - \beta_0 = \beta_{\max} - \gamma$,*
(ii) *Or, for all root β such that $\beta_0 \preceq \beta \prec \gamma$, we have $\beta + \beta_{\max} - \gamma \notin \Phi$.*

Lemma 2. *Let α be a simple long root such that $n(\alpha) \geq 2$. Then there is a couple (γ, γ') of positive roots such that $\alpha \preceq \gamma$, $\alpha \preceq \gamma'$, and $\gamma + \gamma' = \beta_{\max}$.*

For every root α, we fix a morphism of algebraic groups $u_\alpha : \mathbb{K} \to G$ such that $hu_\alpha(s)h^{-1} = u_\alpha(\alpha(h)s)$ for all $h \in T$ and $\operatorname{Im} du_\alpha = \mathfrak{g}_\alpha$ (see [11, Lemma 7.3.3]). Note that there is a nonzero root vector $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ such that

$$\operatorname{Ad} u_\alpha(s) = \exp(s \operatorname{ad} x_\alpha) \quad \text{for all } s \in \mathbb{K} \tag{2}$$

where $\operatorname{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ stand for the adjoint representations.

3.2. **Tangent space of the minimal nilpotent orbit at the biggest root vector.** Recall that for each positive root α we consider a root vector $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. The biggest root vector $e_{\beta_{\max}}$ is a representative of the minimal nilpotent orbit $\mathcal{O}_{\min}$. By $T_{e_{\beta_{\max}}}\mathcal{O}_{\min}$ we denote the tangent space of $\mathcal{O}_{\min}$ at $e_{\beta_{\max}}$.

Proposition 3. $T_{e_{\beta_{\max}}}\mathcal{O}_{\min} = \mathfrak{g}_{\beta_{\max}} \oplus [\mathfrak{g}_{\beta_{\max}}, \mathfrak{g}_{-\beta_{\max}}] \oplus \bigoplus_{\gamma \in M_{\max}} \mathfrak{g}_\gamma$.

Proof. We first claim that

$$\dim T_{e_{\beta_{\max}}}\mathcal{O}_{\min} = \dim \mathcal{O}_{\min} = |M_{\max}| + 2. \tag{3}$$

The first equality in (3) follows from the fact that $\mathcal{O}_{\min}$ is smooth. For showing the second equality, we compute the stabilizer $\mathfrak{z}_{\mathfrak{g}}(e_{\beta_{\max}}) := \{v \in \mathfrak{g} : [v, e_{\beta_{\max}}] = 0\}$. Let $v \in \mathfrak{g}$ and let us write $v = h + \sum_{\alpha \in \Phi} v_\alpha$ where $h \in \mathfrak{h}$ and $v_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. We get

$$[v, e_{\beta_{\max}}] = \beta_{\max}(h) e_{\beta_{\max}} + [v_{-\beta_{\max}}, e_{\beta_{\max}}] + \sum_{\alpha \in \Phi \setminus \{-\beta_{\max}\}} [v_\alpha, e_{\beta_{\max}}],$$

and the equality $[v, e_{\beta_{\max}}] = 0$ holds if and only if $h \in \ker \beta_{\max}$, $v_{-\beta_{\max}} = 0$, and $v_\alpha = 0$ whenever $\alpha + \beta_{\max}$ is a root. The last fact is equivalent to having $-\alpha \in M_{\max}$. Altogether, this yields

$$\dim \mathfrak{z}_{\mathfrak{g}}(e_{\beta_{\max}}) = \dim \mathfrak{g} - 2 - |M_{\max}|.$$

Since $\dim \mathcal{O}_{\min} = \dim \mathfrak{g} - \dim \mathfrak{z}_{\mathfrak{g}}(e_{\beta_{\max}})$, the verification of (3) is complete.

In view of (3), for showing the proposition, it suffices to show the inclusion

$$\mathfrak{g}_{\beta_{\max}} \oplus [\mathfrak{g}_{\beta_{\max}}, \mathfrak{g}_{-\beta_{\max}}] \oplus \bigoplus_{\gamma \in M_{\max}} \mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}}\mathcal{O}_{\min}. \tag{4}$$

There is a cocharacter $\lambda : \mathbb{K}^* \to T$ such that $\lambda(\mathbb{K}^*) \cdot e_{\beta_{\max}} = \mathbb{K}^* e_{\beta_{\max}}$. Hence $\mathbb{K}^* e_{\beta_{\max}} \subset \mathcal{O}_{\min}$, which yields the inclusion $\mathfrak{g}_{\beta_{\max}} = T_{e_{\beta_{\max}}}(\mathbb{K}^* e_{\beta_{\max}}) \subset T_{e_{\beta_{\max}}}\mathcal{O}_{\min}$. By (2), the inclusion

$$[\mathfrak{g}_{-\beta_{\max}}, \mathfrak{g}_{\beta_{\max}}] = \mathbb{K}[x_{-\beta_{\max}}, e_{\beta_{\max}}] = T_{e_{\beta_{\max}}}\big(u_{-\beta_{\max}}(\mathbb{K}) \cdot e_{\beta_{\max}}\big) \subset T_{e_{\beta_{\max}}}\mathcal{O}_{\min}$$

holds. Similarly, for every $\gamma \in M_{\max}$, letting $\gamma' := \beta_{\max} - \gamma$ (which is a positive root), the inclusion

$$\mathfrak{g}_\gamma = \mathbb{K}[x_{-\gamma'}, e_{\beta_{\max}}] = T_{e_{\beta_{\max}}}\big(u_{-\gamma'}(\mathbb{K}) \cdot e_{\beta_{\max}}\big) \subset T_{e_{\beta_{\max}}}\mathcal{O}_{\min}$$

holds. Altogether we get (4). The proof is complete. □

3.3. **Tangent space of orbital varieties of the minimal nilpotent orbit at the biggest root vector.** Theorem 1 is implied by Proposition 4 (b) below. We need two preparatory lemmas.

Lemma 3. *For every $x \in \mathcal{O}_{\min}$, we have $\mathbb{K}^* x \subset B \cdot x$.*

Proof. By (1), there is a long root vector e_β and an element $b \in B$ such that $x = b \cdot e_\beta$. For every $s \in \mathbb{K}^*$, we can find $h \in T$ such that $h \cdot e_\beta = s e_\beta$, whence $sx = (bh) \cdot e_\beta \in B \cdot e_\beta = B \cdot x$. □

Recall that for every positive long root α, the root vector e_α belongs to the minimal nilpotent orbit $\mathcal{O}_{\min}$ and the biggest long root vector $e_{\beta_{\max}}$ belongs to the closure of $B \cdot e_\alpha$ (see Proposition 2 (a)).

Lemma 4. *Let $\gamma \in M_{\max}$.*

(a) *Assume that the root γ is long. Then $\mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}}\overline{B \cdot e_\gamma}$.*
(b) *Assume that γ is short and such that the set $\{\beta \in \Phi^+$ long $: \beta \preceq \gamma\}$ is nonempty, hence contains a biggest element β_0 (see Lemma 1). Then $\mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}}\overline{B \cdot e_{\beta_0}}$.*

Proof. Let $\gamma' := \beta_{\max} - \gamma$, which is a positive root. First we show part (a) of the lemma. In view of (2), there is an element $x_{\gamma'} \in \mathfrak{g}_{\gamma'} \setminus \{0\}$ such that

$$e_\gamma + s^{-1}[x_{\gamma'}, e_\gamma] = u_{\gamma'}(s^{-1}) \cdot e_\gamma \in B \cdot e_\gamma$$

for all $s \in \mathbb{K}^*$. Note that $[x_{\gamma'}, e_\gamma] \in \mathfrak{g}_{\beta_{\max}} \setminus \{0\} = \mathbb{K}^* e_{\beta_{\max}}$. By Lemma 3, we get $e_{\beta_{\max}} + s e_\gamma \in B \cdot e_\gamma$ for all $s \in \mathbb{K}^*$. Whence the inclusion $\mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}}\overline{B \cdot e_\gamma}$.

Next let us show part (b) of the lemma. First assume that condition (i) of Lemma 1 holds, so that $\gamma = \beta_0 + \gamma'$ and $\beta_{\max} = \beta_0 + 2\gamma'$. In view of (2), we have in this case

$$e_{\beta_0} + s^{-1}[x_{\gamma'}, e_{\beta_0}] + \frac{1}{2}\, s^{-2}[x_{\gamma'}, [x_{\gamma'}, e_{\beta_0}]] = u_{\gamma'}(s^{-1}) \cdot e_{\beta_0} \in B \cdot e_{\beta_0} \quad \text{for all } s \in \mathbb{K}^*$$

for some $x_{\gamma'} \in \mathfrak{g}_{\gamma'} \setminus \{0\}$. Note that $[x_{\gamma'}, e_{\beta_0}] \in \mathfrak{g}_\gamma \setminus \{0\}$ while $[x_{\gamma'}, [x_{\gamma'}, e_{\beta_0}]] \in \mathfrak{g}_{\beta_{\max}} \setminus \{0\}$, hence $[x_{\gamma'}, [x_{\gamma'}, e_{\beta_0}]] = s_0^{-1} e_{\beta_{\max}}$ for some $s_0 \in \mathbb{K}^*$. Invoking also Lemma 3, this yields

$$e_{\beta_{\max}} + 2s s_0 [x_{\gamma'}, e_{\beta_0}] + 2s^2 s_0 e_{\beta_0} \in B \cdot e_{\beta_0} \quad \text{for all } s \in \mathbb{K}^*.$$

Whence the inclusion $\mathfrak{g}_\gamma = \mathbb{K}[x_{\gamma'}, e_{\beta_0}] \subset T_{e_{\beta_{\max}}}\overline{B \cdot e_{\beta_0}}$.

Finally assume that condition (ii) of Lemma 1 holds. By Lemma 1, there is a positive root η and a positive integer k such that $\gamma = \beta_0 + k\eta$. Let $r \geq k$ be the integer such that $\gamma_\ell := \beta_0 + \ell\eta$ is a root for all $\ell \in \{0, 1, \ldots, r\}$ and is not a root whenever $\ell > r$. By (2), there are root vectors $e'_{\gamma_\ell} \in \mathfrak{g}_{\gamma_\ell} \setminus \{0\}$ (for $\ell = 0, 1, \ldots, r$) such that

$$u_\eta(t) \cdot e_{\beta_0} = \sum_{\ell=0}^{r} t^\ell e'_{\gamma_\ell} \quad \text{for all } t \in \mathbb{K}.$$

By assumption, $\gamma_\ell + \gamma'$ is a root if and only if $\ell = k$, and $\gamma_k + \gamma' = \gamma + \gamma' = \beta_{\max}$. Applying again (2), we get

$$t^k s^{-1} s_0^{-1} e_{\beta_{\max}} + \sum_{\ell=0}^{r} t^\ell e'_{\gamma_\ell} = u_{\gamma'}(s^{-1}) u_\eta(t) \cdot e_{\beta_0} \text{ for all } t \in \mathbb{K}, \text{ all } s \in \mathbb{K}^*,$$

for some $s_0 \in \mathbb{K}^*$. Whence (by Lemma 3)

$$e_{\beta_{\max}} + s s_0 t^{-k} \sum_{\ell=0}^{r} t^\ell e'_{\gamma_\ell} \in B \cdot e_{\beta_0} \quad \text{for all } t \in \mathbb{K}, \text{ all } s \in \mathbb{K}^*.$$

We deduce that

$$\sum_{\ell=0}^{r} t^\ell e'_{\gamma_\ell} \in T_{e_{\beta_{\max}}}\overline{B \cdot e_{\beta_0}} \quad \text{for all } t \in \mathbb{K}$$

and therefore

$$\mathfrak{g}_\gamma \subset \bigoplus_{\ell=0}^{r} \mathfrak{g}_{\gamma_\ell} \subset T_{e_{\beta_{\max}}}\overline{B \cdot e_{\beta_0}}.$$

The proof of the lemma is complete. □

Proposition 4. *Let α be a simple long root, so that $\mathcal{V}_{\min}(\alpha) := \overline{B \cdot e_\alpha} \cap \mathcal{O}_{\min}$ is an orbital variety of $\mathcal{O}_{\min}$ and $B \cdot e_{\beta_{\max}}$ is the unique closed B-orbit of $\mathcal{V}_{\min}(\alpha)$ (see Proposition 2).*

(a) $T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha) = \mathfrak{g}_{\beta_{\max}} \oplus \bigoplus_{\gamma \in M_{\max},\ \gamma \succeq \alpha} \mathfrak{g}_\gamma$.

(b) *The following conditions are equivalent:*
  (i) *$\mathcal{V}_{\min}(\alpha)$ is singular;*
  (ii) *$|\{\gamma \in M_{\max} : \gamma \succeq \alpha\}| > \frac{1}{2}|M_{\max}|$;*
  (iii) *there exists a couple (γ, γ') of positive roots such that $\gamma \succeq \alpha$, $\gamma' \succeq \alpha$, $\gamma + \gamma' = \beta_{\max}$;*
  (iv) *$n(\alpha) \geq 2$.*

Proof. First we show the inclusion $\supset$ in (a). Since $\mathfrak{g}_{\beta_{\max}} \setminus \{0\} = B \cdot e_{\beta_{\max}} \subset \mathcal{V}_{\min}(\alpha)$, we have $\mathfrak{g}_{\beta_{\max}} \subset T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha)$. Next let $\gamma \in M_{\max}$ such that $\gamma \succeq \alpha$. If γ is a long root, then we have $B \cdot e_\gamma \subset \mathcal{V}_{\min}(\alpha)$ by Proposition 2, hence $\mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha)$ by Lemma 4 (a). Assume now that γ is a short root and let β_0 be as in Lemmas 1 and 4 (b). The maximality property of β_0 implies that $\beta_0 \succeq \alpha$, whence $B \cdot e_{\beta_0} \subset \mathcal{V}_{\min}(\alpha)$ (by Proposition 2). In view of Lemma 4 (b), this yields $\mathfrak{g}_\gamma \subset T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha)$. Altogether we get the inclusion $\supset$ in (a).

In view of the inclusion $\mathcal{V}_{\min}(\alpha) \subset \overline{B \cdot e_\alpha} \subset \bigoplus_{\gamma \succeq \alpha} \mathfrak{g}_\gamma$ and of Proposition 3, we also have

$$T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha) \subset \Big(\bigoplus_{\gamma \succeq \alpha} \mathfrak{g}_\gamma \Big) \cap T_{e_{\beta_{\max}}} \mathcal{O}_{\min} = \mathfrak{g}_{\beta_{\max}} \oplus \bigoplus_{\gamma \in M_{\max},\ \gamma \succeq \alpha} \mathfrak{g}_\gamma,$$

and this completes the proof of part (a).

On the one hand, part (a) and Proposition 3 yield

$$\dim T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha) = 1 + |\{\gamma \in M_{\max} : \gamma \succeq \alpha\}|$$

and

$$\dim \mathcal{V}_{\min}(\alpha) = \frac{1}{2} \dim \mathcal{O}_{\min} = 1 + \frac{1}{2}|M_{\max}|$$

(recall from Section 1.2 that we have $\dim \mathcal{V} = \frac{1}{2} \dim \mathcal{O}$ whenever $\mathcal{V}$ is an orbital variety of a nilpotent orbit $\mathcal{O}$). On the other hand, since $B \cdot e_{\beta_{\max}}$ is the unique closed B-orbit in $\mathcal{V}_{\min}(\alpha)$, we know that $\mathcal{V}_{\min}(\alpha)$ is singular if and only if $\dim T_{e_{\beta_{\max}}} \mathcal{V}_{\min}(\alpha) > \dim \mathcal{V}_{\min}(\alpha)$. The equivalence between conditions (i) and (ii) of part (b) ensues.

Since α necessarily occurs in the decomposition of $\beta_{\max}$ as a sum of simple roots, for every $\gamma \in M_{\max}$ we must have $\gamma \succeq \alpha$ or $\beta_{\max} - \gamma \succeq \alpha$. Whence $|\{\gamma \in M_{\max} : \gamma \succeq \alpha\}| \geq \frac{1}{2}|M_{\max}|$ with strict inequality if and only if there is an element $\gamma \in M_{\max}$ such that $\gamma \succeq \alpha$ and $\beta_{\max} - \gamma \succeq \alpha$, which is equivalent to the existence of a couple (γ, γ') as in (iii). Conditions (ii) and (iii) of part (b) are therefore equivalent.

The implication (iii)⇒(iv) is immediate while the inverse implication (iv)⇒(iii) follows from Lemma 2. The proof of part (b) of the statement is now complete. □

4. Proof of Theorem 2

As in Section 2.1, for every simple root α, we denote by P_α the corresponding standard minimal parabolic subgroup and by $\mathfrak{n}_\alpha$ the nilpotent radical of its Lie algebra.

As noted in Proposition 1 (b), the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha) := \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ of $\mathcal{O}_{\mathrm{subreg}}$ attached to α is P_α-homogeneous. In view of well-known properties of spherical varieties [2, 13] (see also [6, Lemma 1]), this fact already guarantees the equivalence between parts (i) and (ii) of Theorem 2. The purpose of this section is to prove the equivalence between parts (ii) and (iii) of Theorem 2.

4.1. **Criteria of existence of dense B-orbit.** For a simple root α, we also denote by $\mathfrak{p}_\alpha$ the Lie algebra of the minimal parabolic subgroup P_α and by $\mathrm{Rad}(\mathfrak{p}_\alpha)$ its radical, i.e., the intersection of the Borel subalgebras of $\mathfrak{p}_\alpha$; in other words

$$\mathrm{Rad}(\mathfrak{p}_\alpha) = \{h \in \mathfrak{h} : \alpha(h) = 0\} \oplus \mathfrak{n}_\alpha.$$

Proposition 5. *Let α be a simple root. Let $x \in \mathcal{V}_{\mathrm{subreg}}(\alpha) = \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$. The following conditions are equivalent:*

(i) *$\mathcal{V}_{\mathrm{subreg}}(\alpha)$ has a dense B-orbit;*
(ii) *$\{y \in \mathfrak{p}_\alpha : [y, x] = 0\} \not\subset \mathrm{Rad}(\mathfrak{p}_\alpha)$.*

Proof. The orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ (which coincides with the P_α-orbit of x) has a dense B-orbit if and only if there exists an element $g \in P_\alpha$ such that $\dim B\cdot(g\cdot x) = \dim \mathcal{V}_{\mathrm{subreg}}(\alpha)$. By $\mathfrak{b}$ we denote the Lie algebra of the Borel subgroup B. Note that

$$\begin{aligned} \dim B \cdot (g \cdot x) &= \dim B - \dim\{y \in \mathfrak{b} : [y, g \cdot x] = 0\} \\ &= \dim B - \dim\{y \in g^{-1} \cdot \mathfrak{b} : [y, x] = 0\} \end{aligned}$$

and

$$\dim \mathcal{V}_{\mathrm{subreg}}(\alpha) = \dim P_\alpha \cdot x = \dim P_\alpha - \dim\{y \in \mathfrak{p}_\alpha : [y, x] = 0\}$$

hence

$$\dim \mathcal{V}_{\mathrm{subreg}}(\alpha) - \dim B\cdot(g\cdot x) = 1 - \dim\{y \in \mathfrak{p}_\alpha : [y, x] = 0\}/\{y \in g^{-1}\cdot\mathfrak{b} : [y, x] = 0\}.$$

Therefore the existence of a dense B-orbit in $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ is equivalent to the existence of a Borel subalgebra $\mathfrak{b}' = g^{-1} \cdot \mathfrak{b} \subset \mathfrak{p}_\alpha$ such that $\{y \in \mathfrak{p}_\alpha : [y, x] = 0\} \not\subset \mathfrak{b}'$. This property is equivalent to condition (ii) of the statement. The proof is complete. □

Proposition 5 is an efficient criterion of existence of dense B-orbit once we know a representative x of the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha)$, i.e., an element x of the intersection $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$. Such an element x is called a Richardson element of the nilradical $\mathfrak{n}_\alpha$. In the classical cases, due to the combinatorial classification of nilpotent orbits in terms of Jordan forms, it is possible to construct Richardson elements for many parabolic subalgebras; see [1]. In the classical cases, our proof of Theorem 2 relies on Proposition 5. In particular we construct Richardson elements for nilradicals of the form $\mathfrak{n}_\alpha$ (the constructions made in [1] do not apply to all the nilradicals of this form).

In the exceptional cases, out of our knowledge, there is no construction of Richardson elements. For this reason, we cannot use Proposition 5 for proving Theorem 2 in the exceptional cases (however, as a byproduct of our proof, we provide Richardson elements for certain nilradicals $\mathfrak{n}_\alpha$; see also Remark 1 below). We rely on the construction of Chevalley bases and on the following criterion.

Proposition 6. *Let α be a simple root and let $\mathcal{V}_{\mathrm{subreg}}(\alpha) = \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ be the corresponding orbital variety of $\mathcal{O}_{\mathrm{subreg}}$. Let $x \in \mathfrak{n}_\alpha$ and let $M_\alpha(x)$ be the matrix of the linear transformation $\mathfrak{b} \to \mathfrak{n}_\alpha$, $y \mapsto [y, x]$ (relatively to some bases of $\mathfrak{b}$ and $\mathfrak{n}_\alpha$, e.g., subbases of a Chevalley basis of $\mathfrak{g}$). Then*

(a) $\dim B \cdot x = \operatorname{rank} M_\alpha(x)$.
(b) $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ *has a dense* B*-orbit if and only if for some* $x \in \mathfrak{n}_\alpha$ *the rows of* $M_\alpha(x)$ *are linearly independent.*

Proof. Part (a) is obtained as follows,

$$\dim B\cdot x = \dim B - \dim\{y \in \mathfrak{b} : [y,x]=0\} = \dim \mathfrak{b} - \dim\ker M_\alpha(x) = \operatorname{rank} M_\alpha(x).$$

From part (a), it follows that if the rows of $M_\alpha(x)$ are linearly independent, i.e., $\operatorname{rank} M_\alpha(x) = \dim \mathfrak{n}_\alpha$, then $B \cdot x$ is a dense, open subset of $\mathfrak{n}_\alpha$, which implies that its intersection with $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ is nonempty; in fact this ensures that x belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ (i.e., to $\mathcal{V}_{\mathrm{subreg}}(\alpha)$) since this set is B-stable. The equivalence in part (b) immediately follows from part (a) and this observation. □

Corollary 2. *We consider a connected subdiagram of the Dynkin diagram of* $\mathfrak{g}$*, which corresponds to a subset of simple roots* $\Pi' \subset \Pi$*. Let* $G' \subset G$ *be the connected simple algebraic subgroup corresponding to* Π'*, let* $\mathfrak{g}' \subset \mathfrak{g}$ *be its Lie algebra, and let* $B' \subset G'$ *be the standard Borel subgroup.*

In this way, in addition to the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ *(relative to* $\mathfrak{g}$*), a simple root* $\alpha \in \Pi'$ *determines an orbital variety* $\mathcal{V}'_{\mathrm{subreg}}(\alpha)$ *relative to* $\mathfrak{g}'$*, contained in the subregular nilpotent orbit of* $\mathfrak{g}'$*.*

If $\mathcal{V}'_{\mathrm{subreg}}(\alpha)$ *has no dense* B'*-orbit, then* $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ *has no dense* B*-orbit.*

More precisely, if $\mathcal{B} := B \cdot x$ *is a dense orbit of* $\mathcal{V}_{\mathrm{subreg}}(\alpha)$*, denoting by* x' *the natural projection of* x *onto* $\mathfrak{g}'$*, we have that* $\mathcal{B}' := B' \cdot x'$ *is a dense orbit of* $\mathcal{V}_{\mathrm{subreg}}(\alpha')$*. Moreover, the map* $\mathcal{B} \to \mathcal{B}'$*,* $x \mapsto x'$ *is then surjective.*

Proof. Let $\Phi' \subset \Phi$ be the root system generated by Π', i.e., the subset of roots which are linear combinations of the elements of Π'. Let $\Phi'^+ := \Phi' \cap \Phi^+$ be the subset of positive roots. Let $\mathfrak{b}'$ be the Lie algebra of B', let $\mathfrak{n}'_\alpha \subset \mathfrak{b}'$ be the nilradical corresponding to α, let $\mathfrak{h}'$ be the standard Cartan subalgebra of $\mathfrak{g}'$. Thus

$$\mathfrak{b}' = \mathfrak{h}' \oplus \bigoplus_{\gamma \in \Phi'^+} \mathfrak{g}_\gamma \quad \text{and} \quad \mathfrak{n}'_\alpha = \bigoplus_{\gamma \in \Phi'^+ \setminus \{\alpha\}} \mathfrak{g}_\gamma.$$

Recall that for each root $\gamma \in \Phi^+$ we consider a root vector $e_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$. Let $\{\lambda'_\gamma : \gamma \in \Pi'\} \subset \mathfrak{h}'$ and $\{\lambda_\gamma : \gamma \in \Pi\} \subset \mathfrak{h}$ be the dual bases of $\Pi' \subset \mathfrak{h}'^*$ and $\Pi \subset \mathfrak{h}^*$, respectively. Let an element $x \in \mathfrak{n}_\alpha$ and let $x' \in \mathfrak{n}'_\alpha$ be its image by the projection relative to the decomposition $\mathfrak{n}_\alpha = \mathfrak{n}'_\alpha \oplus \bigoplus_{\gamma \in \Phi^+ \setminus \Phi'^+} \mathfrak{g}_\gamma$. Let $M_\alpha(x)$ be the matrix of the linear map $\mathfrak{b} \to \mathfrak{n}_\alpha$, $y \mapsto [y,x]$ in the bases $\{\lambda_\gamma : \gamma \in \Pi\} \cup \{e_\gamma : \gamma \in \Phi^+\}$ (of $\mathfrak{b}$) and $\{e_\gamma : \gamma \in \Phi^+ \setminus \{\alpha\}\}$ (of $\mathfrak{n}_\alpha$). Let $M'_\alpha(x')$ be the matrix of the map $\mathfrak{b}' \to \mathfrak{n}'_\alpha$, $y \mapsto [y,x']$ in the bases $\{\lambda'_\gamma : \gamma \in \Pi'\} \cup \{e_\gamma : \gamma \in \Phi'^+\}$ (of $\mathfrak{b}'$) and $\{e_\gamma : \gamma \in \Phi'^+ \setminus \{\alpha\}\}$ (of $\mathfrak{n}'_\alpha$). Then (up to adding columns of zeros) the matrix $M'_\alpha(x')$ coincides with the submatrix of $M_\alpha(x)$ formed by the rows corresponding to the basis vectors e_γ for $\gamma \in \Phi'^+ \setminus \{\alpha\}$. Therefore, if the rows of $M'_\alpha(x')$ are linearly dependent, then so are the rows of $M_\alpha(x)$. The corollary now follows from Proposition 6 (b) (the last claim follows from the fact that the map $x \mapsto x'$ is B'-equivariant). □

4.2. **Proof of Theorem 2 in classical cases.** We rely on a technical lemma:

Lemma 5. *Let* α *be a simple root. Assume that there is an element* $x \in \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_\alpha$ *of the form*

$$x = \sum_{\gamma \in I} x_\gamma \quad \textit{with } x_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$$

where I is a subset of $\Phi^+ \setminus \{\alpha\}$ satisfying the following conditions:

(A) *There is $\gamma \in I$ such that $\gamma - \alpha \in \Phi \setminus (I \cup I')$ where $I' = I + \Phi^+$;*
(B) *There is $\gamma \in I$ such that $\gamma + \alpha \in \Phi \setminus (I \cup \hat{I}')$ where $\hat{I}' = I + (\Phi^+ \setminus \{\alpha\})$;*
(C) *For every $\delta \in \{\gamma' - \gamma : \gamma, \gamma' \in I\} \cap \Phi^+ \setminus \{\alpha\}$, there is $\beta \in I$ such that $\beta + \delta \in \Phi \setminus I$ and $(\beta + \delta) - \beta' \notin \Phi^+ \setminus \{\alpha\}$ for all $\beta' \in I \setminus \{\beta\}$;*
(D) *Φ is contained in the linear space spanned by I.*

Then $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ has no dense B-orbit.

Proof. In view of Proposition 5, it suffices to show the inclusion

$$\{y \in \mathfrak{p}_\alpha : [y, x] = 0\} \subset \mathfrak{n}_\alpha. \tag{5}$$

So, let $y \in \mathfrak{p}_\alpha$ such that $[y, x] = 0$, write

$$y = h + y_\alpha + y_{-\alpha} + y' \quad \text{with } h \in \mathfrak{h},\ y_\alpha \in \mathfrak{g}_\alpha,\ y_{-\alpha} \in \mathfrak{g}_{-\alpha},\ y' \in \mathfrak{n}_\alpha,$$

and let us show that $h = y_\alpha = y_{-\alpha} = 0$. Let $\gamma_1 \in I$ and $\gamma_2 \in I$ be the elements provided by conditions (A) and (B), respectively. First we see that

$$[y_{-\alpha}, x] = [-h - y_\alpha - y', x] \in \bigoplus_{\gamma \in I \cup I'} \mathfrak{g}_\gamma.$$

The vector $[y_{-\alpha}, x_{\gamma_1}]$ is the component of $[y_{-\alpha}, x]$ in the root space $\mathfrak{g}_{\gamma_1 - \alpha}$. Since $\gamma_1 - \alpha \notin I \cup I'$, we must have $[y_{-\alpha}, x_{\gamma_1}] = 0$, hence $y_{-\alpha} = 0$. Next we see that

$$[y_\alpha, x] = [-h - y', x] \in \bigoplus_{\gamma \in I \cup \hat{I}'} \mathfrak{g}_\gamma$$

and the condition $\gamma_2 + \alpha \notin I \cup \hat{I}'$ implies that $[y_\alpha, x_{\gamma_2}] = 0$; since $\gamma_2 + \alpha$ is a root, this forces $y_\alpha = 0$. Thus the relation

$$[y', x] = [-h, x] \in \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma \tag{6}$$

holds. We claim that $[y', x] = 0$. Arguing by contradiction, say $[y', x] \neq 0$. Hence there are roots $\gamma \in I$ and $\delta \in \Phi^+ \setminus \{\alpha\}$ such that $[y'_\delta, x_\gamma] \neq 0$, where y'_δ is the component of y' in the root space $\mathfrak{g}_\delta$. By (6), this yields $\gamma' := \gamma + \delta \in I$. Then, let $\beta \in I$ be as in condition (C). Condition (C) implies that $[y'_\delta, x_\beta]$ is the component of $[y', x]$ in the root space $\mathfrak{g}_{\beta+\delta}$ and that it is nonzero. Since $\beta + \delta \notin I$, this contradicts (6). Therefore $[y', x] = 0$ and in turn (again by (6)) $[h, x] = 0$. The last relation implies that $h \in \bigcap_{\gamma \in I} \ker \gamma$, which, in view of condition (D), yields $h = 0$. The proof of the lemma is complete. □

Hereafter we denote by $\{\lambda_\alpha : \alpha \in \Pi\}$ the basis of the Cartan subalgebra $\mathfrak{h}$ which is dual to the basis of $\mathfrak{h}^*$ formed by the simple roots.

By $E_{i,j}^{(n)}$ we denote the elementary $n \times n$ matrix with 1 in position (i, j) and zeros elsewhere. By ${}^t a$ we denote the transpose of a matrix a. For each classical Lie algebra $\mathfrak{g}$ considered below, we consider the root datum (Φ, Φ^+) corresponding to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ formed by diagonal matrices and the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ formed by upper triangular matrices.

Proof of Theorem 2 in type A. Assume that G is a simple group of type A_{n-1}, i.e., $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{K})$ is the space of $n \times n$ matrices of trace zero. The subregular orbit $\mathcal{O}_{\mathrm{subreg}}$ consists of all nilpotent matrices $x \in \mathfrak{sl}_n(\mathbb{K})$ with Jordan form $(n-1, 1)$. Let $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$ and $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}$. The

matrix $e_{\varepsilon_i-\varepsilon_j} := E^{(n)}_{i,j}$ is a root vector for the root $\varepsilon_i - \varepsilon_j$. Let $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, \ldots, n-1)$ be the simple roots.

Let a simple root $\alpha = \alpha_i$. Up to automorphism of the Dynkin diagram, we may suppose that $i < n-1$. The matrix $x := e_{\alpha_i+\alpha_{i+1}} + \sum_{j \notin \{i,i+1\}} e_{\alpha_j}$ is an element of $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_i}$ and the matrix $y := \lambda_{\alpha_i} - \lambda_{\alpha_{i+1}} \in \mathfrak{p}_{\alpha_i} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_i})$ is such that $[y, x] = 0$. From Proposition 5, we conclude that $\mathcal{V}_{\mathrm{subreg}}(\alpha_i)$ has a dense B-orbit. □

Proof of Theorem 2 in type B. Assume that G is a simple group of type B_m with $m \geq 2$, i.e., $\mathfrak{g} = \mathfrak{so}_n(\mathbb{K})$ with $n = 2m+1$, seen as the subalgebra of $\mathfrak{sl}_n(\mathbb{K})$ of matrices which are skew symmetric with respect to the skew diagonal. The roots are $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\} \cup \{\pm\varepsilon_i : 1 \leq i \leq m\}$ and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\} \cup \{\varepsilon_i : 1 \leq i \leq m\}$ and corresponding root vectors are

$$e_{\varepsilon_i-\varepsilon_j} = E^{(n)}_{i,j} - E^{(n)}_{n+1-j,n+1-i},\ e_{\varepsilon_i+\varepsilon_j} = E^{(n)}_{i,n+1-j} - E^{(n)}_{j,n+1-i} \text{ for } 1 \leq i < j \leq m,$$
$$e_{\varepsilon_i} = E^{(n)}_{i,m+1} - E^{(n)}_{m+1,n+1-i} \quad \text{for } 1 \leq i \leq m, \quad e_{-\alpha} = {}^t e_\alpha \quad \text{for } \alpha \in \Phi^+.$$

The simple roots are $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, m-1$, and $\alpha_m := \varepsilon_m$.

The subregular orbit $\mathcal{O}_{\mathrm{subreg}}$ consists of nilpotent matrices $x \in \mathfrak{so}_n(\mathbb{K})$ of Jordan form $(n-2, 1, 1)$. For $i \in \{1, \ldots, m-1\}$, the matrix

$$x_i := e_{\alpha_i+\alpha_{i+1}} + \sum_{j \notin \{i,i+1\}} e_{\alpha_j}$$

belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_i}$. Moreover the matrix $y_i := \lambda_{\alpha_i} - \lambda_{\alpha_{i+1}}$ is an element of $\mathfrak{p}_{\alpha_i} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_i})$ such that $[y_i, x_i] = 0$. From Proposition 5, it follows that $\mathcal{V}_{\mathrm{subreg}}(\alpha_i)$ has a dense B-orbit. Note that x_{m-1} also belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_m}$ and y_{m-1} also belongs to $\mathfrak{p}_{\alpha_m} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_m})$, hence the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha_m)$ has also a dense B-orbit. We have shown that, in type B, all the orbital varieties of $\mathcal{O}_{\mathrm{subreg}}$ have a dense B-orbit. □

Proof of Theorem 2 in type C. Assume G of type C_m with $m \geq 3$. We deal with the following realization of $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{K})$ with $n = 2m$:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & -a^* \end{pmatrix} : a, b, c \text{ are } n \times n \text{ matrices},\ b = b^*,\ c = c^* \right\}$$

where x^* stands for the transpose of x by the skew diagonal. In this case, we have the roots $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq m\}$ and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\} \cup \{2\varepsilon_i : 1 \leq i \leq m\}$, and we consider the following root vectors:

$$e_{\varepsilon_i-\varepsilon_j} = E^{(n)}_{i,j} - E^{(n)}_{n+1-j,n+1-i},\ e_{\varepsilon_i+\varepsilon_j} = E^{(n)}_{i,n+1-j} + E^{(n)}_{j,n+1-i} \text{ for } 1 \leq i < j \leq m,$$
$$e_{2\varepsilon_i} = E^{(n)}_{i,n+1-i} \quad \text{for } 1 \leq i \leq m, \quad e_{-\alpha} = {}^t e_\alpha \quad \text{for } \alpha \in \Phi^+.$$

The simple roots are $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, m-1$, and $\alpha_m := 2\varepsilon_m$.

The subregular orbit $\mathcal{O}_{\mathrm{subreg}} \subset \mathfrak{sp}_n(\mathbb{K})$ consists of all nilpotent matrices $x \in \mathfrak{sp}_n(\mathbb{K})$ of Jordan form $(n-2, 2)$.

The element $x_1 := e_{2\varepsilon_1} + \sum_{j=2}^{m} e_{\alpha_j}$ belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_1}$ and the element $y_1 := e_{-\alpha_1} + e_{\varepsilon_1+\varepsilon_3}$ belongs to $\mathfrak{p}_{\alpha_1} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_1})$ and satisfies $[y_1, x_1] = 0$. The element $x_m := e_{2\varepsilon_{m-1}} + \sum_{j=1}^{m-1} e_{\alpha_j}$ belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_m}$ and satisfies $[e_{-\alpha_m}, x_m] = 0$. By applying Proposition 5, we get that $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ and $\mathcal{V}_{\mathrm{subreg}}(\alpha_m)$ contain a dense B-orbit.

Finally let us show that $\mathcal{V}_{\mathrm{subreg}}(\alpha_i)$ has no dense B-orbit whenever $i \in \{2, \dots, m-1\}$. In view of Corollary 2, arguing by induction on $m \geq 3$, we may assume that $i = 2$. Let $I = \{\alpha_j : j \notin \{2,3\}\} \cup \{\alpha_2 + \alpha_3, 2\varepsilon_3\}$ and set $x_2 = \sum_{\alpha \in I} e_\alpha$. Note that $x_2 \in \mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_2}$. Moreover, it is easy to see that the set I fulfills the conditions (A)–(D) of Lemma 5. It follows that $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has no dense B-orbit. The proof of the theorem is complete in type C. □

Proof of Theorem 2 in type D. Assume G of type D_m for $m \geq 4$. Hence $\mathfrak{g} = \mathfrak{so}_n(\mathbb{K})$ with $n = 2m$, seen as the subalgebra of $\mathfrak{sl}_n(\mathbb{K})$ formed by matrices which are skew symmetric by the skew diagonal. The roots are $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\}$ and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq m\}$ and we consider the root vectors

$$e_{\varepsilon_i - \varepsilon_j} = E^{(n)}_{i,j} - E^{(n)}_{n+1-j,n+1-i} \quad \text{and} \quad e_{\varepsilon_i + \varepsilon_j} = E^{(n)}_{i,n+1-j} - E^{(n)}_{j,n+1-i}$$

for $1 \leq i < j \leq m$, and $e_{-\alpha} = {}^t e_\alpha$ for all $\alpha \in \Phi^+$. The simple roots are $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, m-1$, and $\alpha_m := \varepsilon_{m-1} + \varepsilon_m$. The subregular orbit $\mathcal{O}_{\mathrm{subreg}}$ is the set of nilpotent elements $x \in \mathfrak{so}_n(\mathbb{K})$ of Jordan form $(n-3, 3)$.

The element $x_1 := e_{\varepsilon_1 + \varepsilon_m} + \sum_{j=2}^m e_{\alpha_j}$ belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_1}$. The matrix $y_1 := 2e_{-\alpha_1} - e_{\varepsilon_1+\varepsilon_4} - e_{\varepsilon_2-\varepsilon_{m-1}} - e_{\varepsilon_3-\varepsilon_m} + e_{\varepsilon_3+\varepsilon_m}$ belongs to $\mathfrak{p}_{\alpha_1} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_1})$ and commutes with x_1. By Proposition 5, we deduce that $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ has a dense B-orbit. The element $x_{m-1} := e_{\varepsilon_{m-2}+\varepsilon_{m-1}} + \sum_{j \neq m-1} e_{\alpha_j}$ belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_{m-1}}$ and it commutes with $y_{m-1} := e_{-\alpha_{m-1}} + e_{\alpha_m} \in \mathfrak{p}_{\alpha_{m-1}} \setminus \mathrm{Rad}(\mathfrak{p}_{\alpha_{m-1}})$, hence $\mathcal{V}_{\mathrm{subreg}}(\alpha_{m-1})$ contains a dense B-orbit. The symmetry of the Dynkin diagram guarantees that $\mathcal{V}_{\mathrm{subreg}}(\alpha_m)$ also contains a dense B-orbit.

Next we show that $\mathcal{V}_{\mathrm{subreg}}(\alpha_{m-2})$ has no dense B-orbit. In view of Corollary 2, we may assume that $m = 4$. In this case $x := e_{\alpha_1} + e_{\alpha_1+\alpha_2} + e_{\alpha_2+\alpha_3} + e_{\alpha_4}$ is an element of $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_2}$. Then, Lemma 5 shows that $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has no dense B-orbit.

Finally assume that $m \geq 5$ and let us show that the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha_i)$ has no dense B-orbit for $i \in \{2, \dots, m-3\}$. Invoking again Corollary 2, we may assume that $i = 2$. Letting $I = \{\alpha_j : j \notin \{2,3\}\} \cup \{\alpha_2 + \alpha_3, \alpha_3 + \ldots + \alpha_{m-2} + \alpha_m\}$, it is easy to check that the set I fulfills conditions (A)–(D) of Lemma 5 and that the element $x := \sum_{\alpha \in I} e_\alpha$ belongs to $\mathcal{O}_{\mathrm{subreg}} \cap \mathfrak{n}_{\alpha_2}$. Hence, by Lemma 5, $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ contains no dense B-orbit. The proof is complete in type D. □

4.3. **Proof of Theorem 2 in exceptional cases.** In this section, G is a simple algebraic group of exceptional type. As in Section 4.2, we denote by $\{\lambda_\alpha : \alpha \in \Pi\}$ the basis of the Cartan subalgebra $\mathfrak{h}$ which is dual to the basis of $\mathfrak{h}^*$ formed by the simple roots. Moreover for each exceptional type we have determined a Chevalley basis of $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, i.e., the subbasis of a Chevalley basis of $\mathfrak{g}$. To this end, we consider the following total ordering of the positive roots:

- In types G_2 and F_4, the simple roots are ordered according to the following Dynkin diagrams:

  $$G_2 : \ \alpha_1 \Longleftarrow \alpha_2 \qquad F_4 : \ \alpha_1 \text{ —— } \alpha_2 \Longrightarrow \alpha_3 \text{ —— } \alpha_4$$

  In type E_8 (and in type E_6, resp., E_7), the simple roots are ordered according to the numbering of the Dynkin diagram drawn above Theorem 2 (and its subdiagram of vertices $\alpha_1, \dots, \alpha_6$, resp., $\alpha_1, \dots, \alpha_7$)
- Next each positive root is identified with the tuple of its coordinates in the basis Π. We consider the partial order determined by the height (i.e.,

the sum of the coordinates) and we order the roots of same height by the lexicographic order of the coordinates.

For instance here is the ordered list of the positive roots in type G_2, identified with the couples of their coordinates in the basis (α_1, α_2):

$$\alpha_1 = [1,0],\ \alpha_2 = [0,1],\ \alpha_3 = [1,1],\ \alpha_4 = [2,1],\ \alpha_5 = [3,1],\ \alpha_6 = [3,2].$$

In each case let r be the number of simple roots, so $\alpha_1, \ldots, \alpha_r$ are the simple roots, and let n denote the number of positive roots. For $i \in \{1, \ldots, r\}$, we set $\lambda_i := \lambda_{\alpha_i}$. Finally let $(e_1, \ldots, e_n)$ be the Chevalley basis of $\mathfrak{n}$, numbered according to the total ordering of the positive roots. We fix an element

$$x = \sum_{j=1}^{n} x_j e_j \in \mathfrak{n}.$$

We consider the linear map $\mathfrak{b} \to \mathfrak{n}$, $y \mapsto [y, x]$, and we denote by $A(x)$ the matrix of this map between the bases $(\lambda_1, \ldots, \lambda_r, e_1, \ldots, e_n)$ and $(e_1, \ldots, e_n)$ of $\mathfrak{b}$ and $\mathfrak{n}$, respectively. Note that for every simple root $\alpha = \alpha_i$ (with $i \in \{1, \ldots, r\}$), the matrix $M_\alpha(x)$ of Proposition 6 is obtained from the matrix $A(x)$ by deleting the i-th row of the matrix (i.e, the row of the matrix corresponding to e_i) and replacing all the coefficients x_i by zeros. The explicit matrices $A(x)$ corresponding to the different exceptional cases are given in the Appendix.

Proof of Theorem 2 in type G_2. On the one hand, for $x \in \mathfrak{n}_{\alpha_1}$, it is easy to see that the matrix $M_{\alpha_1}(x)$ has rank at most 4 whereas it has five rows. By Proposition 6, $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ has no dense B-orbit. On the other hand, the matrix $M_{\alpha_2}(e_1 + e_6)$ has five linearly independent rows. It follows from Proposition 6 that $B \cdot (e_1 + e_6)$ is a dense B-orbit of $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$. □

Proof of Theorem 2 in type F_4. The roots $\alpha_2, \alpha_3, \alpha_4$ generate a root system of type C_3. Hence it follows from Corollary 2 and the proof of the theorem in type C that $\mathcal{V}_{\mathrm{subreg}}(\alpha_3)$ has no dense B-orbit.

For every $x \in \mathfrak{n}_{\alpha_2}$, the rows of the matrix $M_{\alpha_2}(x)$ corresponding to the root vectors $e_1, e_3, e_4, \ldots, e_{10}$ are linearly dependent. By Proposition 6, it follows that $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has no dense B-orbit.

For $x \in \mathfrak{n}_{\alpha_4}$, the rows of the matrix $M_{\alpha_4}(x)$ corresponding to the root vectors e_j for $j \in \{1, \ldots, 16\} \setminus \{4\}$ are linearly dependent, and this shows that $\mathcal{V}_{\mathrm{subreg}}(\alpha_4)$ has no dense B-orbit.

Finally, it can be seen that the matrix $M_{\alpha_1}(e_2 + e_3 + e_4 + e_{12})$ has linearly independent rows, and this shows that $B \cdot (e_2 + e_3 + e_4 + e_{12})$ is dense in $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$. The proof is complete in type F_4. □

Proof of Theorem 2 in types E_6, E_7, E_8. In type E_8, the roots $\alpha_1, \ldots, \alpha_5$ generate a root system of type D_5 while the roots $\alpha_2, \ldots, \alpha_8$ generate a root system of type D_7. By comparing Corollary 2 and the proof of the theorem in type D, we deduce that the orbital varieties of type E_8 corresponding to $\alpha_3, \ldots, \alpha_7$ have no dense B-orbit. Arguing in the same way shows that the orbital varieties of type E_7 (resp., E_6) attached to the roots $\alpha_3, \ldots, \alpha_6$ (resp., $\alpha_3, \ldots, \alpha_5$) have no dense B-orbit.

In type E_6, letting $x_1 := \sum_{j=2}^{6} e_{\alpha_j} + e_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}$, it can be seen that the matrix $M_{\alpha_1}(x_1)$ has linearly independent rows. Therefore, Proposition 6 implies that the element x_1 belongs to a dense B-orbit of $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$. In view of the symmetry of the Dynkin diagram of type E_6, the orbital variety $\mathcal{V}_{\mathrm{subreg}}(\alpha_6)$ has also

a dense B-orbit whose representative is $\sum_{j=1}^{5} e_{\alpha_j} + e_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}$. By combining Corollary 2 and the proof of Theorem 2 in type D, we obtain that, if $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has a dense B-orbit, then this orbit contains an element $x \in \mathfrak{n}_{\alpha_2}$ whose natural projection on the subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of type D_5 corresponding to the set of simple roots $\Pi' = \{\alpha_1, \ldots, \alpha_5\}$ is $x' = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_2+\alpha_4+\alpha_5}$. For such an element x, on can see that the matrix $M_{\alpha_2}(x)$ has linearly dependent rows (the row corresponding to the root vector $e_{\alpha_2+\alpha_4+\alpha_5+\alpha_6}$ is a linear combination of the rows above it). By Proposition 6, this implies that $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has no dense B-orbit. This completes the proof of the theorem in type E_6.

In type E_7, the above proof of the theorem in type E_6 and Corollary 2 imply that $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ has no dense B-orbit. It can be seen that the matrix $M_{\alpha_7}(x_7)$ corresponding to the element $x_7 := \sum_{j=1}^{6} e_{\alpha_j} + e_{\alpha_6+\alpha_7} + e_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ has linearly independent rows, hence Proposition 6 implies that $B \cdot x_7$ is a dense B-orbit of $\mathcal{V}_{\mathrm{subreg}}(\alpha_7)$. Finally, invoking again Corollary 2, we obtain that a dense B-orbit of $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ (if it exists) must contain an element $x \in \mathfrak{n}_{\alpha_1}$ whose natural projection on the Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of type E_6 corresponding to the set of simple roots $\Pi' = \{\alpha_1, \ldots, \alpha_6\}$ is the element x_1 written above. However, for such an element x, one can check that the row of the matrix $M_{\alpha_1}(x)$ corresponding to the root vector $e_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}$ is a linear combination of the rows above it. This implies that $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ has no dense B-orbit. This completes the proof of the theorem in type E_7.

In type E_8, comparing Corollary 2 with the proof of the theorem in type E_7 given above, we already deduce that $\mathcal{V}_{\mathrm{subreg}}(\alpha_1)$ and $\mathcal{V}_{\mathrm{subreg}}(\alpha_2)$ have no dense B-orbit. Invoking again Corollary 2 and the proof of the theorem in type D, we obtain that a dense B-orbit of $\mathcal{V}_{\mathrm{subreg}}(\alpha_8)$ (if it exists) should have a representative $x \in \mathfrak{n}_{\alpha_8}$ whose natural projection x' on the subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of type D_7 corresponding to the simple roots $\Pi' = \{\alpha_2, \ldots, \alpha_8\}$ is given by $x' = \sum_{j=2}^{7} e_j + e_{43}$. However, a careful calculation shows that for any such element x, the row of $M_{\alpha_8}(x)$ corresponding to the root vector e_{68} is a linear combination of the rows above it. This implies that $\mathcal{V}_{\mathrm{subreg}}(\alpha_8)$ has no dense B-orbit. The proof of the theorem is complete. □

Remark 1. Note that, in the proofs done in this section, in each case where $\mathcal{V}_{\mathrm{subreg}}(\alpha)$ has a dense B-orbit, we provide a representative of this orbit. This element is in particular a Richardson element of the nilradical $\mathfrak{n}_\alpha$.

Appendix

In this appendix, $\mathfrak{g}$ is a simple Lie algebra of exceptional type, $(\lambda_1, \ldots, \lambda_r)$ is the basis of the Cartan subalgebra $\mathfrak{h}$ which is dual to the basis of $\mathfrak{h}^*$ formed by the simple roots $\alpha_1, \ldots, \alpha_r$, $(e_1, \ldots, e_n)$ is a Chevalley basis of the maximal nilpotent subalgebra $\mathfrak{n}$ (the numbering of the vectors corresponds to the total ordering of the positive roots determined by the height, roots with the same height being ordered according to the lexicographic order of their coordinates).

Given $x = \sum_{j=1}^{n} x_j e_j$, we denote by $A(x)$ the matrix of the linear map $\mathfrak{b} \to \mathfrak{n}$, $y \mapsto [y, x]$ with respect to the bases $(\lambda_1, \ldots, \lambda_r, e_1, \ldots, e_n)$ of $\mathfrak{b}$ and $(e_1, \ldots, e_n)$ of $\mathfrak{n}$. In this appendix, we describe the matrix $A(x)$ in the different exceptional cases.

In Figures 1 and 2, we give the matrix $A(x)$ in types G_2 and F_4, respectively. For clarity, the zero coefficients are replaced by dots. In type F_4, we write the matrix

in the form $A(x) = \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix}$ where $A_{1,1}$, $A_{2,1}$, and $A_{2,2}$ are the 12×14 matrices described in Figure 2.

$$\begin{pmatrix}
x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_3 & x_3 & x_2 & -x_1 & \cdot & \cdot & \cdot & \cdot \\
2x_4 & x_4 & 2x_3 & \cdot & -2x_1 & \cdot & \cdot & \cdot \\
3x_5 & x_5 & -3x_4 & \cdot & \cdot & 3x_1 & \cdot & \cdot \\
3x_6 & 2x_6 & \cdot & -x_5 & -3x_4 & 3x_3 & x_2 & \cdot
\end{pmatrix}$$

FIGURE 1. The matrix $A(x)$ in type G_2

$$A_{1,1} = \begin{pmatrix}
x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & x_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & x_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_5 & x_5 & \cdot & \cdot & x_2 & -x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_6 & x_6 & \cdot & \cdot & x_3 & -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & x_7 & x_7 & \cdot & \cdot & x_4 & -x_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_8 & x_8 & x_8 & \cdot & x_6 & \cdot & -x_5 & \cdot & x_3 & -x_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_9 & 2x_9 & \cdot & \cdot & \cdot & 2x_6 & \cdot & \cdot & -2x_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_{10} & x_{10} & x_{10} & \cdot & x_7 & \cdot & -x_6 & \cdot & x_4 & -x_2 & \cdot & \cdot & \cdot \\
x_{11} & x_{11} & 2x_{11} & \cdot & x_9 & \cdot & 2x_8 & \cdot & \cdot & \cdot & \cdot & -2x_3 & -x_1 & \cdot \\
x_{12} & x_{12} & x_{12} & x_{12} & x_{10} & \cdot & \cdot & -x_8 & x_7 & \cdot & -x_5 & x_4 & \cdot & -x_1
\end{pmatrix}$$

$$A_{2,1} = \begin{pmatrix}
\cdot & x_{13} & 2x_{13} & x_{13} & \cdot & \cdot & x_{10} & -x_9 & \cdot & -x_7 & x_6 & \cdot & x_4 & -x_3 \\
x_{14} & 2x_{14} & 2x_{14} & \cdot & \cdot & x_{11} & \cdot & \cdot & -x_9 & 2x_8 & \cdot & -2x_6 & x_5 & \cdot \\
x_{15} & x_{15} & 2x_{15} & x_{15} & x_{13} & \cdot & x_{12} & -x_{11} & \cdot & \cdot & x_8 & -x_7 & \cdot & \cdot \\
\cdot & x_{16} & 2x_{16} & 2x_{16} & \cdot & \cdot & \cdot & 2x_{13} & \cdot & \cdot & -2x_{10} & \cdot & \cdot & 2x_7 \\
x_{17} & 2x_{17} & 2x_{17} & x_{17} & \cdot & x_{15} & \cdot & -x_{14} & -x_{13} & x_{12} & \cdot & -x_{10} & \cdot & x_8 \\
x_{18} & x_{18} & 2x_{18} & 2x_{18} & x_{16} & \cdot & \cdot & 2x_{15} & \cdot & \cdot & -2x_{12} & \cdot & \cdot & \cdot \\
x_{19} & 2x_{19} & 3x_{19} & x_{19} & \cdot & \cdot & x_{17} & \cdot & \cdot & -x_{15} & -x_{14} & x_{13} & x_{12} & x_{11} \\
x_{20} & 2x_{20} & 2x_{20} & 2x_{20} & x_{18} & \cdot & \cdot & 2x_{17} & -x_{16} & \cdot & \cdot & \cdot & \cdot & -2x_{12} \\
x_{21} & 2x_{21} & 3x_{21} & 2x_{21} & \cdot & \cdot & x_{20} & x_{19} & \cdot & -x_{18} & x_{17} & x_{16} & \cdot & -x_{15} \\
x_{22} & 2x_{22} & 4x_{22} & 2x_{22} & \cdot & \cdot & 2x_{21} & \cdot & \cdot & \cdot & 2x_{19} & \cdot & -x_{18} & \cdot \\
x_{23} & 3x_{23} & 4x_{23} & 2x_{23} & \cdot & x_{22} & \cdot & \cdot & \cdot & 2x_{21} & \cdot & \cdot & -x_{20} & 2x_{19} \\
2x_{24} & 3x_{24} & 4x_{24} & 2x_{24} & x_{23} & \cdot & \cdot & \cdot & x_{22} & \cdot & \cdot & 2x_{21} & \cdot & \cdot
\end{pmatrix}$$

$$A_{2,2} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_4 & -x_3 & -x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -2x_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -x_6 & x_5 & x_4 & -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2x_7 & \cdot & \cdot & -2x_4 & -x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-x_{10} & -x_9 & -x_8 & x_7 & x_6 & \cdot & -x_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 2x_{10} & \cdot & \cdot & \cdot & x_5 & -2x_4 & -x_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & x_{13} & -x_{12} & \cdot & x_{10} & -x_8 & -x_7 & x_6 & -x_4 & -x_3 & \cdot & \cdot & \cdot & \cdot \\
x_{16} & \cdot & -2x_{15} & \cdot & 2x_{13} & -x_{11} & \cdot & x_9 & -2x_7 & \cdot & -2x_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -2x_{17} & x_{16} & \cdot & -x_{14} & 2x_{13} & \cdot & -2x_{10} & x_9 & -2x_6 & -x_2 & \cdot & \cdot \\
-x_{20} & 2x_{19} & \cdot & x_{18} & -2x_{17} & \cdot & 2x_{15} & -x_{14} & -2x_{12} & x_{11} & -2x_8 & -x_5 & -x_1 & \cdot
\end{pmatrix}$$

FIGURE 2. The blocks $A_{1,1}$, $A_{2,1}$, $A_{2,2}$ of the matrix $A(x)$ in type F_4

In type E_8, instead of drawing the matrix $A(x)$, we give the list of the roots α_j, for $j = 1, \ldots, 120$. For each root α_j, we indicate its coordinates $[\epsilon_{j,1}, \ldots, \epsilon_{j,8}]$ in the basis $(\alpha_1, \ldots, \alpha_8)$ and the couples (a, b) such that $e_j = +[e_a, e_b] = -[e_b, e_a]$. This information is sufficient for characterizing the matrix $A(x)$: the row of the matrix corresponding to e_j contains the coefficient $x_j \epsilon_{j,i}$ in the column corresponding to λ_i for all $i \in \{1, \ldots, 8\}$, and for each one of the listed couples (a, b) it contains x_b in the column corresponding to e_a and $-x_a$ in the column corresponding to e_b, and these are all the nonzero coefficients in the j-th row of the matrix.

In type E_6 (resp. E_7) the matrix $A(x)$ is obtained from the matrix $A(x)$ of type E_8 by deleting the columns corresponding to λ_7 and λ_8 (resp., the column

corresponding to λ_8) and by deleting the rows and the columns corresponding to e_j whenever $(\epsilon_{j,7}, \epsilon_{j,8}) \neq (0,0)$ (resp., whenever $\epsilon_{j,8} \neq 0$).

$\alpha_1 = [1,0,0,0,0,0,0,0]$.
$\alpha_2 = [0,1,0,0,0,0,0,0]$.
$\alpha_3 = [0,0,1,0,0,0,0,0]$.
$\alpha_4 = [0,0,0,1,0,0,0,0]$.
$\alpha_5 = [0,0,0,0,1,0,0,0]$.
$\alpha_6 = [0,0,0,0,0,1,0,0]$.
$\alpha_7 = [0,0,0,0,0,0,1,0]$.
$\alpha_8 = [0,0,0,0,0,0,0,1]$.
$\alpha_9 = [1,0,1,0,0,0,0,0]$: $(1,3)$.
$\alpha_{10} = [0,1,0,1,0,0,0,0]$: $(2,4)$.
$\alpha_{11} = [0,0,1,1,0,0,0,0]$: $(3,4)$.
$\alpha_{12} = [0,0,0,1,1,0,0,0]$: $(4,5)$.
$\alpha_{13} = [0,0,0,0,1,1,0,0]$: $(5,6)$.
$\alpha_{14} = [0,0,0,0,0,1,1,0]$: $(6,7)$.
$\alpha_{15} = [0,0,0,0,0,0,1,1]$: $(7,8)$.

$\alpha_{16} = [1,0,1,1,0,0,0,0]$: $(1,11)$, $(9,4)$.
$\alpha_{17} = [0,1,1,1,0,0,0,0]$: $(2,11)$, $(3,10)$.
$\alpha_{18} = [0,1,0,1,1,0,0,0]$: $(2,12)$, $(10,5)$.
$\alpha_{19} = [0,0,1,1,1,0,0,0]$: $(3,12)$, $(11,5)$.
$\alpha_{20} = [0,0,0,1,1,1,0,0]$: $(4,13)$, $(12,6)$.
$\alpha_{21} = [0,0,0,0,1,1,1,0]$: $(5,14)$, $(13,7)$.
$\alpha_{22} = [0,0,0,0,0,1,1,1]$: $(6,15)$, $(14,8)$.
$\alpha_{23} = [1,1,1,1,0,0,0,0]$: $(1,17)$, $(2,16)$, $(9,10)$.
$\alpha_{24} = [1,0,1,1,1,0,0,0]$: $(1,19)$, $(9,12)$, $(16,5)$.
$\alpha_{25} = [0,1,1,1,1,0,0,0]$: $(2,19)$, $(3,18)$, $(17,5)$.
$\alpha_{26} = [0,1,0,1,1,1,0,0]$: $(2,20)$, $(10,13)$, $(18,6)$.
$\alpha_{27} = [0,0,1,1,1,1,0,0]$: $(3,20)$, $(11,13)$, $(19,6)$.
$\alpha_{28} = [0,0,0,1,1,1,1,0]$: $(4,21)$, $(12,14)$, $(20,7)$.
$\alpha_{29} = [0,0,0,0,1,1,1,1]$: $(5,22)$, $(13,15)$, $(21,8)$.

$\alpha_{30} = [1,1,1,1,1,0,0,0]$: $(1,25)$, $(2,24)$, $(9,18)$, $(23,5)$.
$\alpha_{31} = [1,0,1,1,1,1,0,0]$: $(1,27)$, $(9,20)$, $(16,13)$, $(24,6)$.
$\alpha_{32} = [0,1,1,2,1,0,0,0]$: $(4,25)$, $(17,12)$, $(18,11)$, $(19,10)$.
$\alpha_{33} = [0,1,1,1,1,1,0,0]$: $(2,27)$, $(3,26)$, $(17,13)$, $(25,6)$.
$\alpha_{34} = [0,1,0,1,1,1,1,0]$: $(2,28)$, $(10,21)$, $(18,14)$, $(26,7)$.
$\alpha_{35} = [0,0,1,1,1,1,1,0]$: $(3,28)$, $(11,21)$, $(19,14)$, $(27,7)$.
$\alpha_{36} = [0,0,0,1,1,1,1,1]$: $(4,29)$, $(12,22)$, $(20,15)$, $(28,8)$.

$\alpha_{37} = [1,1,1,2,1,0,0,0]$: $(1,32)$, $(4,30)$, $(18,16)$, $(23,12)$, $(24,10)$.
$\alpha_{38} = [1,1,1,1,1,1,0,0]$: $(1,33)$, $(2,31)$, $(9,26)$, $(23,13)$, $(30,6)$.
$\alpha_{39} = [1,0,1,1,1,1,1,0]$: $(1,35)$, $(9,28)$, $(16,21)$, $(24,14)$, $(31,7)$.
$\alpha_{40} = [0,1,1,2,1,1,0,0]$: $(4,33)$, $(17,20)$, $(26,11)$, $(27,10)$, $(32,6)$.
$\alpha_{41} = [0,1,1,1,1,1,1,0]$: $(2,35)$, $(3,34)$, $(17,21)$, $(25,14)$, $(33,7)$.
$\alpha_{42} = [0,1,0,1,1,1,1,1]$: $(2,36)$, $(10,29)$, $(18,22)$, $(26,15)$, $(34,8)$.
$\alpha_{43} = [0,0,1,1,1,1,1,1]$: $(3,36)$, $(11,29)$, $(19,22)$, $(27,15)$, $(35,8)$.

$\alpha_{44} = [1,1,2,2,1,0,0,0]$: $(3,37)$, $(11,30)$, $(23,19)$, $(24,17)$, $(25,16)$, $(32,9)$.
$\alpha_{45} = [1,1,1,2,1,1,0,0]$: $(1,40)$, $(4,38)$, $(23,20)$, $(26,16)$, $(31,10)$, $(37,6)$.
$\alpha_{46} = [1,1,1,1,1,1,1,0]$: $(1,41)$, $(2,39)$, $(9,34)$, $(23,21)$, $(30,14)$, $(38,7)$.
$\alpha_{47} = [1,0,1,1,1,1,1,1]$: $(1,43)$, $(9,36)$, $(16,29)$, $(24,22)$, $(31,15)$, $(39,8)$.
$\alpha_{48} = [0,1,1,2,2,1,0,0]$: $(5,40)$, $(18,27)$, $(19,26)$, $(20,25)$, $(32,13)$, $(33,12)$.
$\alpha_{49} = [0,1,1,2,1,1,1,0]$: $(4,41)$, $(17,28)$, $(32,14)$, $(34,11)$, $(35,10)$, $(40,7)$.
$\alpha_{50} = [0,1,1,1,1,1,1,1]$: $(2,43)$, $(3,42)$, $(17,29)$, $(25,22)$, $(33,15)$, $(41,8)$.

$\alpha_{51} = [1,1,2,2,1,1,0,0]$: $(3,45)$, $(11,38)$, $(23,27)$, $(31,17)$, $(33,16)$, $(40,9)$, $(44,6)$.
$\alpha_{52} = [1,1,1,2,2,1,0,0]$: $(1,48)$, $(5,45)$, $(18,31)$, $(20,30)$, $(24,26)$, $(37,13)$, $(38,12)$.
$\alpha_{53} = [1,1,1,2,1,1,1,0]$: $(1,49)$, $(4,46)$, $(23,28)$, $(34,16)$, $(37,14)$, $(39,10)$, $(45,7)$.
$\alpha_{54} = [1,1,1,1,1,1,1,1]$: $(1,50)$, $(2,47)$, $(9,42)$, $(23,29)$, $(30,22)$, $(38,15)$, $(46,8)$.
$\alpha_{55} = [0,1,1,2,2,1,1,0]$: $(5,49)$, $(18,35)$, $(19,34)$, $(28,25)$, $(32,21)$, $(41,12)$, $(48,7)$.
$\alpha_{56} = [0,1,1,2,1,1,1,1]$: $(4,50)$, $(17,36)$, $(32,22)$, $(40,15)$, $(42,11)$, $(43,10)$, $(49,8)$.

$\alpha_{57} = [1,1,2,2,2,1,0,0]$: $(3,52)$, $(5,51)$, $(24,33)$, $(25,31)$, $(27,30)$, $(38,19)$, $(44,13)$, $(48,9)$.
$\alpha_{58} = [1,1,2,2,1,1,1,0]$: $(3,53)$, $(11,46)$, $(23,35)$, $(39,17)$, $(41,16)$, $(44,14)$, $(49,9)$, $(51,7)$.
$\alpha_{59} = [1,1,1,2,2,1,1,0]$: $(1,55)$, $(5,53)$, $(18,39)$, $(24,34)$, $(28,30)$, $(37,21)$, $(46,12)$, $(52,7)$.
$\alpha_{60} = [1,1,1,2,1,1,1,1]$: $(1,56)$, $(4,54)$, $(23,36)$, $(37,22)$, $(42,16)$, $(45,15)$, $(47,10)$, $(53,8)$.
$\alpha_{61} = [0,1,1,2,2,2,1,0]$: $(6,55)$, $(20,41)$, $(21,40)$, $(33,28)$, $(34,27)$, $(35,26)$, $(48,14)$, $(49,13)$.
$\alpha_{62} = [0,1,1,2,2,1,1,1]$: $(5,56)$, $(18,43)$, $(19,42)$, $(32,29)$, $(36,25)$, $(48,15)$, $(50,12)$, $(55,8)$.

$\alpha_{63} = [1,1,2,3,2,1,0,0]$: $(4,57)$, $(12,51)$, $(16,48)$, $(24,40)$, $(27,37)$, $(32,31)$, $(44,20)$, $(45,19)$, $(52,11)$.
$\alpha_{64} = [1,1,2,2,2,1,1,0]$: $(3,59)$, $(5,58)$, $(24,41)$, $(25,39)$, $(35,30)$, $(44,21)$, $(46,19)$, $(55,9)$, $(57,7)$.
$\alpha_{65} = [1,1,2,2,1,1,1,1]$: $(3,60)$, $(11,54)$, $(23,43)$, $(44,22)$, $(47,17)$, $(50,16)$, $(51,15)$, $(56,9)$, $(58,8)$.
$\alpha_{66} = [1,1,1,2,2,2,1,0]$: $(1,61)$, $(6,59)$, $(20,46)$, $(21,45)$, $(34,31)$, $(38,28)$, $(39,26)$, $(52,14)$, $(53,13)$.
$\alpha_{67} = [1,1,1,2,2,1,1,1]$: $(1,62)$, $(5,60)$, $(18,47)$, $(24,42)$, $(36,30)$, $(37,29)$, $(52,15)$, $(54,12)$, $(59,8)$.

$\alpha_{68} = [0, 1, 1, 2, 2, 2, 1, 1]$: $(6, 62)$, $(20, 50)$, $(29, 40)$, $(33, 36)$, $(42, 27)$, $(43, 26)$, $(48, 22)$, $(56, 13)$, $(61, 8)$.

$\alpha_{69} = [1, 2, 2, 3, 2, 1, 0, 0]$: $(2, 63)$, $(10, 57)$, $(18, 51)$, $(23, 48)$, $(30, 40)$, $(32, 38)$, $(33, 37)$, $(44, 26)$, $(45, 25)$, $(52, 17)$.
$\alpha_{70} = [1, 1, 2, 3, 2, 1, 1, 0]$: $(4, 64)$, $(12, 58)$, $(16, 55)$, $(24, 49)$, $(32, 39)$, $(35, 37)$, $(44, 28)$, $(53, 19)$, $(59, 11)$, $(63, 7)$.
$\alpha_{71} = [1, 1, 2, 2, 2, 2, 1, 0]$: $(3, 66)$, $(6, 64)$, $(21, 51)$, $(27, 46)$, $(38, 35)$, $(39, 33)$, $(41, 31)$, $(57, 14)$, $(58, 13)$, $(61, 9)$.
$\alpha_{72} = [1, 1, 2, 2, 2, 1, 1, 1]$: $(3, 67)$, $(5, 65)$, $(24, 50)$, $(25, 47)$, $(43, 30)$, $(44, 29)$, $(54, 19)$, $(57, 15)$, $(62, 9)$, $(64, 8)$.
$\alpha_{73} = [1, 1, 1, 2, 2, 2, 1, 1]$: $(1, 68)$, $(6, 67)$, $(20, 54)$, $(29, 45)$, $(38, 36)$, $(42, 31)$, $(47, 26)$, $(52, 22)$, $(60, 13)$, $(66, 8)$.
$\alpha_{74} = [0, 1, 1, 2, 2, 2, 2, 1]$: $(7, 68)$, $(21, 56)$, $(22, 55)$, $(34, 43)$, $(35, 42)$, $(36, 41)$, $(49, 29)$, $(50, 28)$, $(61, 15)$, $(62, 14)$.

$\alpha_{75} = [1, 2, 2, 3, 2, 1, 1, 0]$: $(2, 70)$, $(10, 64)$, $(18, 58)$, $(23, 55)$, $(30, 49)$, $(32, 46)$, $(41, 37)$, $(44, 34)$, $(53, 25)$, $(59, 17)$, $(69, 7)$.
$\alpha_{76} = [1, 1, 2, 3, 2, 2, 1, 0]$: $(4, 71)$, $(6, 70)$, $(16, 61)$, $(27, 53)$, $(28, 51)$, $(39, 40)$, $(45, 35)$, $(49, 31)$, $(58, 20)$, $(63, 14)$, $(66, 11)$.
$\alpha_{77} = [1, 1, 2, 3, 2, 1, 1, 1]$: $(4, 72)$, $(12, 65)$, $(16, 62)$, $(24, 56)$, $(32, 47)$, $(43, 37)$, $(44, 36)$, $(60, 19)$, $(63, 15)$, $(67, 11)$, $(70, 8)$.
$\alpha_{78} = [1, 1, 2, 2, 2, 2, 1, 1]$: $(3, 73)$, $(6, 72)$, $(27, 54)$, $(29, 51)$, $(38, 43)$, $(47, 33)$, $(50, 31)$, $(57, 22)$, $(65, 13)$, $(68, 9)$, $(71, 8)$.
$\alpha_{79} = [1, 1, 1, 2, 2, 2, 2, 1]$: $(1, 74)$, $(7, 73)$, $(21, 60)$, $(22, 59)$, $(34, 47)$, $(36, 46)$, $(39, 42)$, $(53, 29)$, $(54, 28)$, $(66, 15)$, $(67, 14)$.

$\alpha_{80} = [1, 2, 2, 3, 2, 2, 1, 0]$: $(2, 76)$, $(6, 75)$, $(10, 71)$, $(23, 61)$, $(33, 53)$, $(34, 51)$, $(45, 41)$, $(46, 40)$, $(49, 38)$, $(58, 26)$, $(66, 17)$, $(69, 14)$.
$\alpha_{81} = [1, 2, 2, 3, 2, 1, 1, 1]$: $(2, 77)$, $(10, 72)$, $(18, 65)$, $(23, 62)$, $(30, 56)$, $(32, 54)$, $(44, 42)$, $(50, 37)$, $(60, 25)$, $(67, 17)$, $(69, 15)$, $(75, 8)$.
$\alpha_{82} = [1, 1, 2, 3, 3, 2, 1, 0]$: $(5, 76)$, $(13, 70)$, $(19, 66)$, $(27, 59)$, $(28, 57)$, $(39, 48)$, $(52, 35)$, $(55, 31)$, $(61, 24)$, $(63, 21)$, $(64, 20)$, $(71, 12)$.
$\alpha_{83} = [1, 1, 2, 3, 2, 2, 1, 1]$: $(4, 78)$, $(6, 77)$, $(16, 68)$, $(27, 60)$, $(36, 51)$, $(45, 43)$, $(47, 40)$, $(56, 31)$, $(63, 22)$, $(65, 20)$, $(73, 11)$, $(76, 8)$.
$\alpha_{84} = [1, 1, 2, 2, 2, 2, 2, 1]$: $(3, 79)$, $(7, 78)$, $(21, 65)$, $(22, 64)$, $(39, 50)$, $(41, 47)$, $(43, 46)$, $(54, 35)$, $(58, 29)$, $(71, 15)$, $(72, 14)$, $(74, 9)$.

$\alpha_{85} = [1, 2, 2, 3, 3, 2, 1, 0]$: $(2, 82)$, $(5, 80)$, $(13, 75)$, $(25, 66)$, $(33, 59)$, $(34, 57)$, $(46, 48)$, $(52, 41)$, $(55, 38)$, $(61, 30)$, $(64, 26)$, $(69, 21)$, $(71, 18)$.
$\alpha_{86} = [1, 2, 2, 3, 2, 2, 1, 1]$: $(2, 83)$, $(6, 81)$, $(10, 78)$, $(23, 68)$, $(33, 60)$, $(42, 51)$, $(45, 50)$, $(54, 40)$, $(56, 38)$, $(65, 26)$, $(69, 22)$, $(73, 17)$, $(80, 8)$.
$\alpha_{87} = [1, 1, 2, 3, 3, 2, 1, 1]$: $(5, 83)$, $(13, 77)$, $(19, 73)$, $(27, 67)$, $(36, 57)$, $(47, 48)$, $(52, 43)$, $(62, 31)$, $(63, 29)$, $(68, 24)$, $(72, 20)$, $(78, 12)$, $(82, 8)$.
$\alpha_{88} = [1, 1, 2, 3, 2, 2, 2, 1]$: $(4, 84)$, $(7, 83)$, $(16, 74)$, $(22, 70)$, $(28, 65)$, $(39, 56)$, $(43, 53)$, $(49, 47)$, $(58, 36)$, $(60, 35)$, $(76, 15)$, $(77, 14)$, $(79, 11)$.

$\alpha_{89} = [1, 2, 2, 4, 3, 2, 1, 0]$: $(4, 85)$, $(12, 80)$, $(20, 75)$, $(32, 66)$, $(34, 63)$, $(40, 59)$, $(52, 49)$, $(53, 48)$, $(55, 45)$, $(61, 37)$, $(69, 28)$, $(70, 26)$, $(76, 18)$, $(82, 10)$.
$\alpha_{90} = [1, 2, 2, 3, 3, 2, 1, 1]$: $(2, 87)$, $(5, 86)$, $(13, 81)$, $(25, 73)$, $(33, 67)$, $(42, 57)$, $(52, 50)$, $(54, 48)$, $(62, 38)$, $(68, 30)$, $(69, 29)$, $(72, 26)$, $(78, 18)$, $(85, 8)$.
$\alpha_{91} = [1, 2, 2, 3, 2, 2, 2, 1]$: $(2, 88)$, $(7, 86)$, $(10, 84)$, $(22, 75)$, $(23, 74)$, $(34, 65)$, $(46, 56)$, $(49, 54)$, $(50, 53)$, $(58, 42)$, $(60, 41)$, $(79, 17)$, $(80, 15)$, $(81, 14)$.
$\alpha_{92} = [1, 1, 2, 3, 3, 2, 2, 1]$: $(5, 88)$, $(7, 87)$, $(19, 79)$, $(28, 72)$, $(29, 70)$, $(39, 62)$, $(43, 59)$, $(55, 47)$, $(64, 36)$, $(67, 35)$, $(74, 24)$, $(77, 21)$, $(82, 15)$, $(84, 12)$.

$\alpha_{93} = [1, 2, 3, 4, 3, 2, 1, 0]$: $(3, 89)$, $(11, 85)$, $(19, 80)$, $(27, 75)$, $(32, 71)$, $(40, 64)$, $(41, 63)$, $(55, 51)$, $(57, 49)$, $(58, 48)$, $(61, 44)$, $(69, 35)$, $(70, 33)$, $(76, 25)$, $(82, 17)$.
$\alpha_{94} = [1, 2, 2, 4, 3, 2, 1, 1]$: $(4, 90)$, $(12, 86)$, $(20, 81)$, $(32, 73)$, $(40, 67)$, $(42, 63)$, $(52, 56)$, $(60, 48)$, $(62, 45)$, $(68, 37)$, $(69, 36)$, $(77, 26)$, $(83, 18)$, $(87, 10)$, $(89, 8)$.
$\alpha_{95} = [1, 2, 2, 3, 3, 2, 2, 1]$: $(2, 92)$, $(5, 91)$, $(7, 90)$, $(25, 79)$, $(29, 75)$, $(34, 72)$, $(46, 62)$, $(50, 59)$, $(55, 54)$, $(64, 42)$, $(67, 41)$, $(74, 30)$, $(81, 21)$, $(84, 18)$, $(85, 15)$.
$\alpha_{96} = [1, 1, 2, 3, 3, 3, 2, 1]$: $(6, 92)$, $(14, 87)$, $(20, 84)$, $(28, 78)$, $(29, 76)$, $(31, 74)$, $(39, 68)$, $(43, 66)$, $(61, 47)$, $(71, 36)$, $(73, 35)$, $(79, 27)$, $(82, 22)$, $(83, 21)$, $(88, 13)$.

$\alpha_{97} = [2,2,3,4,3,2,1,0]$: $(1,93)$, $(9,89)$, $(16,85)$, $(24,80)$, $(31,75)$, $(37,71)$, $(45,64)$, $(46,63)$, $(57,53)$, $(58,52)$, $(59,51)$, $(66,44)$, $(69,39)$, $(70,38)$, $(76,30)$, $(82,23)$.

$\alpha_{98} = [1,2,3,4,3,2,1,1]$: $(3,94)$, $(11,90)$, $(19,86)$, $(27,81)$, $(32,78)$, $(40,72)$, $(50,63)$, $(57,56)$, $(62,51)$, $(65,48)$, $(68,44)$, $(69,43)$, $(77,33)$, $(83,25)$, $(87,17)$, $(93,8)$.

$\alpha_{99} = [1,2,2,4,3,2,2,1]$: $(4,95)$, $(7,94)$, $(12,91)$, $(32,79)$, $(34,77)$, $(36,75)$, $(53,62)$, $(55,60)$, $(56,59)$, $(67,49)$, $(70,42)$, $(74,37)$, $(81,28)$, $(88,18)$, $(89,15)$, $(92,10)$.

$\alpha_{100} = [1,2,2,3,3,3,2,1]$: $(2,96)$, $(6,95)$, $(14,90)$, $(26,84)$, $(29,80)$, $(34,78)$, $(38,74)$, $(46,68)$, $(50,66)$, $(61,54)$, $(71,42)$, $(73,41)$, $(79,33)$, $(85,22)$, $(86,21)$, $(91,13)$.

$\alpha_{101} = [2,2,3,4,3,2,1,1]$: $(1,98)$, $(9,94)$, $(16,90)$, $(24,86)$, $(31,81)$, $(37,78)$, $(45,72)$, $(54,63)$, $(57,60)$, $(65,52)$, $(67,51)$, $(69,47)$, $(73,44)$, $(77,38)$, $(83,30)$, $(87,23)$, $(97,8)$.

$\alpha_{102} = [1,2,3,4,3,2,2,1]$: $(3,99)$, $(7,98)$, $(11,95)$, $(19,91)$, $(32,84)$, $(41,77)$, $(43,75)$, $(55,65)$, $(56,64)$, $(58,62)$, $(70,50)$, $(72,49)$, $(74,44)$, $(81,35)$, $(88,25)$, $(92,17)$, $(93,15)$.

$\alpha_{103} = [1,2,2,4,3,3,2,1]$: $(4,100)$, $(6,99)$, $(14,94)$, $(26,88)$, $(34,83)$, $(36,80)$, $(45,74)$, $(53,68)$, $(56,66)$, $(61,60)$, $(73,49)$, $(76,42)$, $(79,40)$, $(86,28)$, $(89,22)$, $(91,20)$, $(96,10)$.

$\alpha_{104} = [2,2,3,4,3,2,2,1]$: $(1,102)$, $(7,101)$, $(9,99)$, $(16,95)$, $(24,91)$, $(37,84)$, $(46,77)$, $(47,75)$, $(58,67)$, $(59,65)$, $(60,64)$, $(70,54)$, $(72,53)$, $(79,44)$, $(81,39)$, $(88,30)$, $(92,23)$, $(97,15)$.

$\alpha_{105} = [1,2,3,4,3,3,2,1]$: $(3,103)$, $(6,102)$, $(11,100)$, $(14,98)$, $(33,88)$, $(41,83)$, $(43,80)$, $(51,74)$, $(56,71)$, $(58,68)$, $(61,65)$, $(76,50)$, $(78,49)$, $(84,40)$, $(86,35)$, $(91,27)$, $(93,22)$, $(96,17)$.

$\alpha_{106} = [1,2,2,4,4,3,2,1]$: $(5,103)$, $(13,99)$, $(18,96)$, $(21,94)$, $(26,92)$, $(34,87)$, $(36,85)$, $(52,74)$, $(59,68)$, $(61,67)$, $(62,66)$, $(73,55)$, $(79,48)$, $(82,42)$, $(89,29)$, $(90,28)$, $(95,20)$, $(100,12)$.

$\alpha_{107} = [2,2,3,4,3,3,2,1]$: $(1,105)$, $(6,104)$, $(9,103)$, $(14,101)$, $(16,100)$, $(38,88)$, $(46,83)$, $(47,80)$, $(51,79)$, $(58,73)$, $(60,71)$, $(66,65)$, $(76,54)$, $(78,53)$, $(84,45)$, $(86,39)$, $(91,31)$, $(96,23)$, $(97,22)$.

$\alpha_{108} = [1,2,3,4,4,3,2,1]$: $(3,106)$, $(5,105)$, $(13,102)$, $(21,98)$, $(25,96)$, $(33,92)$, $(41,87)$, $(43,85)$, $(57,74)$, $(61,72)$, $(62,71)$, $(64,68)$, $(78,55)$, $(82,50)$, $(84,48)$, $(90,35)$, $(93,29)$, $(95,27)$, $(100,19)$.

$\alpha_{109} = [2,2,3,4,4,3,2,1]$: $(1,108)$, $(5,107)$, $(9,106)$, $(13,104)$, $(21,101)$, $(30,96)$, $(38,92)$, $(46,87)$, $(47,85)$, $(57,79)$, $(64,73)$, $(66,72)$, $(67,71)$, $(78,59)$, $(82,54)$, $(84,52)$, $(90,39)$, $(95,31)$, $(97,29)$, $(100,24)$.

$\alpha_{110} = [1,2,3,5,4,3,2,1]$: $(4,108)$, $(12,105)$, $(20,102)$, $(28,98)$, $(32,96)$, $(40,92)$, $(43,89)$, $(49,87)$, $(61,77)$, $(62,76)$, $(63,74)$, $(70,68)$, $(82,56)$, $(83,55)$, $(88,48)$, $(93,36)$, $(94,35)$, $(99,27)$, $(103,19)$, $(106,11)$.

$\alpha_{111} = [2,2,3,5,4,3,2,1]$: $(1,110)$, $(4,109)$, $(12,107)$, $(20,104)$, $(28,101)$, $(37,96)$, $(45,92)$, $(47,89)$, $(53,87)$, $(63,79)$, $(66,77)$, $(67,76)$, $(70,73)$, $(82,60)$, $(83,59)$, $(88,52)$, $(94,39)$, $(97,36)$, $(99,31)$, $(103,24)$, $(106,16)$.

$\alpha_{112} = [1,3,3,5,4,3,2,1]$: $(2,110)$, $(10,108)$, $(18,105)$, $(26,102)$, $(32,100)$, $(34,98)$, $(40,95)$, $(49,90)$, $(50,89)$, $(61,81)$, $(62,80)$, $(69,74)$, $(75,68)$, $(85,56)$, $(86,55)$, $(91,48)$, $(93,42)$, $(94,41)$, $(99,33)$, $(103,25)$, $(106,17)$.

$\alpha_{113} = [2,3,3,5,4,3,2,1]$: $(1,112)$, $(2,111)$, $(10,109)$, $(18,107)$, $(26,104)$, $(34,101)$, $(37,100)$, $(45,95)$, $(53,90)$, $(54,89)$, $(66,81)$, $(67,80)$, $(69,79)$, $(75,73)$, $(85,60)$, $(86,59)$, $(91,52)$, $(94,46)$, $(97,42)$, $(99,38)$, $(103,30)$, $(106,23)$.

$\alpha_{114} = [2,2,4,5,4,3,2,1]$: $(3,111)$, $(11,109)$, $(19,107)$, $(27,104)$, $(35,101)$, $(44,96)$, $(47,93)$, $(51,92)$, $(58,87)$, $(63,84)$, $(70,78)$, $(71,77)$, $(72,76)$, $(82,65)$, $(83,64)$, $(88,57)$, $(97,43)$, $(98,39)$, $(102,31)$, $(105,24)$, $(108,16)$, $(110,9)$.

$\alpha_{115} = [2,3,4,5,4,3,2,1]$: $(2,114)$, $(3,113)$, $(17,109)$, $(25,107)$, $(33,104)$, $(41,101)$, $(44,100)$, $(51,95)$, $(54,93)$, $(58,90)$, $(69,84)$, $(71,81)$, $(72,80)$, $(75,78)$, $(85,65)$, $(86,64)$, $(91,57)$, $(97,50)$, $(98,46)$, $(102,38)$, $(105,30)$, $(108,23)$, $(112,9)$.

$\alpha_{116} = [2,3,4,6,4,3,2,1]$: $(4,115)$, $(16,112)$, $(17,111)$, $(32,107)$, $(40,104)$, $(44,103)$, $(49,101)$, $(51,99)$, $(58,94)$, $(60,93)$, $(69,88)$, $(75,83)$, $(76,81)$, $(77,80)$, $(86,70)$, $(89,65)$, $(91,63)$, $(97,56)$, $(98,53)$, $(102,45)$, $(105,37)$, $(110,23)$, $(113,11)$, $(114,10)$.

$\alpha_{117} = [2,3,4,6,5,3,2,1]$: $(5,116)$, $(18,114)$, $(19,113)$, $(30,110)$, $(32,109)$, $(44,106)$, $(48,104)$, $(55,101)$, $(57,99)$, $(64,94)$, $(67,93)$, $(69,92)$, $(75,87)$, $(77,85)$, $(82,81)$, $(89,72)$, $(90,70)$, $(95,63)$, $(97,62)$, $(98,59)$, $(102,52)$, $(108,37)$, $(111,25)$, $(112,24)$, $(115,12)$.

$\alpha_{118} = [2,3,4,6,5,4,2,1]$: $(6,117)$, $(20,115)$, $(31,112)$, $(33,111)$, $(45,108)$, $(48,107)$, $(57,103)$, $(61,101)$, $(69,96)$, $(71,94)$, $(73,93)$, $(80,87)$, $(82,86)$, $(83,85)$, $(89,78)$, $(90,76)$, $(97,68)$, $(98,66)$, $(100,63)$, $(105,52)$,

$(106, 51)$, $(109, 40)$, $(110, 38)$, $(113, 27)$, $(114, 26)$, $(116, 13)$.

$\alpha_{119} = [2, 3, 4, 6, 5, 4, 3, 1]$: $(7, 118)$, $(21, 116)$, $(34, 114)$, $(35, 113)$, $(46, 110)$, $(49, 109)$, $(58, 106)$, $(59, 105)$, $(61, 104)$, $(70, 100)$, $(71, 99)$, $(79, 93)$, $(80, 92)$, $(82, 91)$, $(88, 85)$, $(89, 84)$, $(95, 76)$, $(96, 75)$, $(97, 74)$, $(102, 66)$, $(103, 64)$, $(107, 55)$, $(108, 53)$, $(111, 41)$, $(112, 39)$, $(115, 28)$, $(117, 14)$.

$\alpha_{120} = [2, 3, 4, 6, 5, 4, 3, 2]$: $(8, 119)$, $(22, 117)$, $(36, 115)$, $(47, 112)$, $(50, 111)$, $(60, 108)$, $(62, 107)$, $(72, 103)$, $(73, 102)$, $(74, 101)$, $(81, 96)$, $(83, 95)$, $(84, 94)$, $(90, 88)$, $(91, 87)$, $(92, 86)$, $(98, 79)$, $(99, 78)$, $(100, 77)$, $(104, 68)$, $(105, 67)$, $(106, 65)$, $(109, 56)$, $(110, 54)$, $(113, 43)$, $(114, 42)$, $(116, 29)$, $(118, 15)$.

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