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Probabilistic μ -analysis for system performances assessment. ^{*}

Alexandre Falcoz ^{*} Daniel Alazard ^{**} Christelle Pittet ^{***}

^{*} Airbus Defence and Space, 31 rue des cosmonautes, 31402 Toulouse Cedex, France (e-mail: alexandre.falcoz@airbus.com).

^{**} ISAE-SUPAERO, 10 Avenue Edouard Belin, BP-54032, 31055 Toulouse Cedex 4, France (e-mail: daniel.alazard@isae.fr)

^{***} CNES, 18 avenue E. Belin 31401 Toulouse cedex 9, France (e-mail: christelle.pittet@cnes.fr)

Abstract: \mathcal{H}_∞/μ methods are commonly used in Airbus Defence and Space for the design and validation of control solutions. Formulated in a worst-case paradigm, these methods necessarily lead to overly conservative solutions since sized on the extreme cases. However, the acceptance for relaxed control performances requires mastering the risk associated to the detected unlikely events calling for probabilistic performances metrics in the validation process. A probabilistic μ -analysis method is presented in this paper to exhaustively explore the uncertain parametric domain while evaluating the cumulative probability density function of the performance index. Recent μ -analysis tools implemented in the ONERA's SMAC toolbox are coupled with a dichotomic search algorithm in order to delimit the safe parametric domain while incrementing the probability of success of criteria. The proposed algorithm is applied to a didactic second order system to demonstrate the performances of the method.

Keywords: Probabilistic μ -analysis, randomly distributed parameters, Branch & Bound algorithm.

1. INTRODUCTION

\mathcal{H}_∞/μ methods are the preferred design tools commonly used in Airbus Defence and Space development programs with many successful applications (Falcoz et al. (2015); Preda et al. (2015)). Main relevance of \mathcal{H}_∞ control methods relies on the Bode Sensitivity Integral Theorem (Ruth et al. (2010)) which provides to the control engineers a fundamental law of conservation as in Physics the conservation of Energy or Momentum (Stein and Doyle (1991); Doyle (1979)). It intuitively expresses the trade-off between quantities related to a generalized system performance and stability robustness which is generalized by the structured singular value μ introduced by Doyle et al. (1982) and Safonov (1982). The fundamental picture of the μ -paradigm is illustrated in Fig.1 for both the robust stability and performance problem. $M(s)$ is a stable real-valued linear time-invariant model representing the nominal closed-loop system and Δ is a block-diagonal operator belonging to the structure Δ with $\mathcal{B}_\Delta = \{\Delta \in \Delta : \bar{\sigma}(\Delta) < 1\}$. If only parametric uncertainties are considered, Δ is described by the following structure:

$$\Delta := \{diag\{\delta_1 I_{n_1}, \dots, \delta_n I_{n_n}\} : -1 \leq \delta_i \leq 1\} \quad (1)$$

where δ_i are normalized real scalars (i.e $|\delta_i| < 1$) repeated if $n_i > 1$. Then, giving the set of allowable $\Delta = \Delta(\delta)$, fundamental question in the μ -paradigm relies on the existence of any nontrivial solutions in the loop equations of systems presented in Fig.1. Specifically, assuming that

the problem is normalized and well-posed, if $\mu_\Delta < 1$ (i.e, evaluation of the function μ over the uncertainty block Δ), stability and performance are guaranteed for all values of the model uncertainty. If $\mu_\Delta > 1$ then there exists combinations for which the stability and performance objectives are violated. μ -framework is a

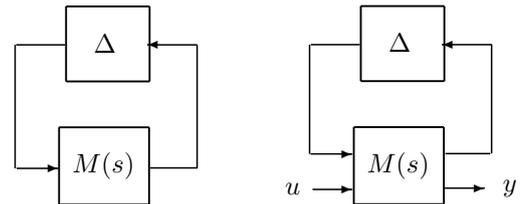


Fig. 1. Standard interconnections for robust stability (left) and worst-case performance (right) analysis.

worst-case paradigm which generates hard bound from the analysis (Khatri and Parrilo (1998)). The risk associated to bad events is not quantified and systematically lead in practice to a new controller tuning as long as $\mu_\Delta > 1$. Sized on unlikely worst-case combinations, it necessarily results to overly robust control solutions at the expense of the final control performances.

Probability-based approaches are then generally preferred in the majority of industrial standards. The probability density function of the performance index is experimentally approximated based on Monte Carlo methods (Stengel and Ryan (1991)). Then, if $(\Omega, \mathcal{F}, P_r)$ is a probability space, $\Delta(\delta)$, with $\delta = [\delta_1, \dots, \delta_n]^T$, relies on a random

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block diagonal matrix where $\delta_i : \Omega \rightarrow \mathfrak{R}$ are independent randomly distributed variables with specific probability density functions $f_i(\delta_i)$. That is denoted $\delta_i \sim f_i$ in the general case and, for instance: $\delta_i \sim \mathcal{U}[a_i, b_i]$ in the case of a uniform distribution where a_i and b_i are the minimal and the maximal values or $\delta_i \sim \mathcal{N}(m_i, \sigma_i)$ in the case of a normal distribution where m_i and σ_i are the nominal value and standard deviation of the i^{th} parameter. Given a performance index $\gamma > 0$, the analysis problem consists in approximating the probability of success of the performance index $J(\Delta(\delta))$, i.e.

$$Pr(J(\Delta(\delta)) \leq \gamma). \quad (2)$$

In the case of the robust performance problem depicted in Fig. 1:

$$J(\Delta(\delta)) = \|F_u(M, \Delta(\delta))\|_\infty$$

where $F_u(M, \Delta)$ is the upper linear fractional transformation of M and Δ .

$Pr(J(\Delta(\delta)) \leq \gamma)$ is estimated based on N samples random draws of the uncertain parameters δ^i , $i = 1, \dots, n$ according to their respective probability density functions $f_i(\delta_i)$. If K corresponds to the number of tests for which $J(\Delta(\delta)) \leq \gamma$, immediately one can write the statistical estimation of $Pr(J(\Delta(\delta)) \leq \gamma)$, i.e.,

$$\widehat{P}_r(J(\Delta(\delta)) \leq \gamma) = \frac{K}{N} \quad (3)$$

However, when a fine quantification of rare events is required, (i.e., located in the distribution tails), Monte Carlo methods become intractable and the number of draws is a priori fixed according to both the accuracy of the statistical estimation $\epsilon \in (0, 1)$ and the confidence level $\varphi \in (0, 1)$ to guarantee that:

$$P_r \left(\left| Pr(J(\Delta(\delta))) \leq \gamma - \widehat{P}_r(J(\Delta(\delta)) \leq \gamma) \right| \leq \epsilon \right) \geq 1 - \varphi \quad (4)$$

Eq. 4 allows a priori to know how good the estimate $\widehat{P}_r(J(\Delta(\delta)))$ of $P_r(J(\Delta(\delta)))$ is when a finite number of samples is employed. Such an assessment is obtained by the Chernoff bound (Chernoff (1952)) if ϵ and φ are fixed a priori, i.e.,

$$N \geq \frac{1}{2\epsilon^2} \log \frac{2}{\varphi} \quad (5)$$

Chernoff's condition means that if the number of employed samples N satisfies Eq.5, then \widehat{P}_r will be ϵ -close to P_r , except for very unlikely events, that may happen with probability smaller than φ . Efficient to evaluate *quite likely* events, Monte Carlo methods becomes intractable for rare events evaluation since the statistical estimation error $\sigma(\widehat{P}_r)$ decreases asymptotically as $\mathcal{O}(N^{-1/2})$. Variance reduction techniques (Fishman (2005)) like importance sampling, Latin Hypercube or stratified sampling, among others, are then commonly employed intending to speed up the convergence properties with a limited numbers of sampling. The scenario approach is introduced in Calafiore and Campi (2006) for robust control design with *a-priori* specified levels of probabilistic guarantee of robustness.

In contrast to stochastic methods, probabilistic μ -analysis intends to save the deterministic and exhaustive exploration of parametric space offered by the μ framework. However, instead of computing the conservative worst-case μ value, we are interested in the probabilistic distribution

of μ ; given a probability distribution on the set of uncertainties $f_i(\delta_i)$ and a dichotomic search algorithm. Inspired by works reported in Zhu et al. (1996); Zhu (2000); Balas et al. (2012) and taking benefit from recent enhanced μ -analysis tools (Roos et al. (2011)), this work investigates probabilistic robust performances analysis method to provide to system engineers systematic validation methods associated to probabilistic decision making metrics.

2. PROBABILISTIC μ -ANALYSIS

To introduce the probabilistic μ problem, let the analysis problem be represented by the general interconnection scheme of Fig. 1 where the uncertain parameter vector δ is contained in the normalized hypercube:

$$\mathcal{C} = \{\delta : -1 \leq \delta_i \leq 1, i = 1, \dots, n\}$$

with $\delta_i \sim \mathcal{U}[-1, 1]$. The uncertainty block Δ belongs to structure $\mathbf{\Delta}$ as defined in Eq. (1). The definition of the structured singular value is given by:

$$\mu_{\mathbf{\Delta}}(M) = \left(\min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) : \det(I - M\Delta) = 0\} \right)^{-1} \quad (6)$$

where $\bar{\sigma}(A)$ denotes the largest singular value of A . Definition (6) characterizes the robust stability measure. More generally, on can also define the robust performance measure by:

$$\mu_{\mathbf{\Delta}, \gamma}(M) = \left(\min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) : \|F_u(M, \Delta)\|_\infty \leq \gamma\} \right)^{-1} \quad (7)$$

for a given performance level γ . Then, the robust stability measure is a particular case of the robust performance measure ($\gamma = \infty$):

$$\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}, \infty}(M).$$

It is well known that exact computation of $\mu_{\mathbf{\Delta}, \gamma}(M)$ is NP hard. $\mu_{\mathbf{\Delta}}(M)$ is then approximated by an upper bound ($\bar{\mu}_{\mathbf{\Delta}, \gamma}(M)$) and a lower bound ($\underline{\mu}_{\mathbf{\Delta}, \gamma}(M)$) using polynomial-time algorithms (Young et al. (1995); Young and Doyle (1997); Seiler et al. (2010)). $\bar{\mu}_{\mathbf{\Delta}, \gamma}(M)$ provides a guaranteed but conservative value of the robustness margin while $\underline{\mu}_{\mathbf{\Delta}, \gamma}(M)$ is associated to the worst-case parametric configuration. This work only considers $\bar{\mu}_{\mathbf{\Delta}, \gamma}(M)$ as the performance index for the probabilistic μ problem and recent development proposed by the authors of reference (Roos et al. (2011)) are exploited to compute a *guaranteed* upper bound of the μ function over the whole frequency range (see (Roos et al. (2011)) for more details).

Considering a parametric domain \mathcal{C} with a given uncertainty structure $\mathbf{\Delta}$ and a given set of probability density function f_i for each parameters δ_i , the main concern of the probabilistic μ -analysis consists in computing a lower bound $\underline{s}_{\mathbf{\Delta}, \gamma}(M)$ of the probability of success $s_{\mathbf{\Delta}, \gamma}(M)$, i.e. the value of the cumulative distribution function over the valid (or successful) parametric sub-domain $\mathcal{V}_{\mathbf{\Delta}, \gamma}(M)$, defined by:

$$\mathcal{V}_{\mathbf{\Delta}, \gamma}(M) = \{\delta \in \mathcal{C} : \|F_u(M, \Delta(\delta))\|_\infty \leq \gamma\} \quad (8)$$

and

$$s_{\mathbf{\Delta}, \gamma}(M) = Pr(\delta \in \mathcal{V}_{\mathbf{\Delta}, \gamma}(M) \mid \delta_i \sim f_i, \forall i). \quad (9)$$

One can also define the invalid parametric sub-domain $\mathcal{I}_{\mathbf{\Delta}, \gamma}(M)$ such that: $\mathcal{V}_{\mathbf{\Delta}, \gamma}(M) \cup \mathcal{I}_{\mathbf{\Delta}, \gamma}(M) = \mathcal{C}$

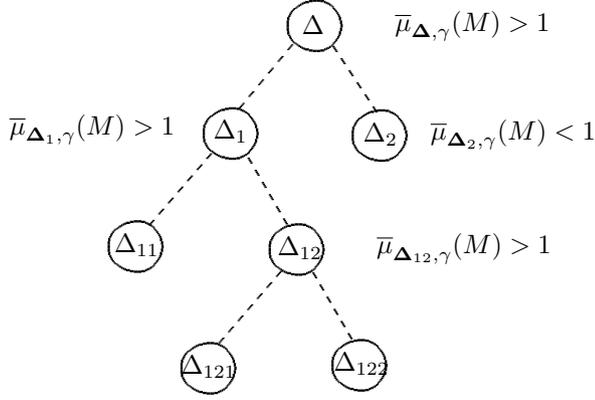


Fig. 2. Dichotomic parametric space exploration of the probabilistic μ -analysis procedure.

Basically, the μ -analysis procedure consists in evaluating $\bar{\mu}_{\Delta,\gamma}(M)$ over the entire parametric domain \mathcal{C} associated to the uncertainty structure Δ . In the normalized parametric space, $\frac{1}{\bar{\mu}_{\Delta,\gamma}(M)}$ is a lower bound of the half-side of the largest centered hypercube inside the valid parametric sub-domain. If $\bar{\mu}_{\Delta,\gamma}(M) \leq 1$, this means that the probability of success of the evaluated performance index is equal to 1. If $\bar{\mu}_{\Delta,\gamma}(M) > 1$, one can only conclude, assuming that $\delta_i \sim \mathcal{U}[-1, 1]$, $\forall i$, that the probability of success is greater than:

$$\underline{s}_{\Delta,\gamma}(M) = \left(\frac{1}{\bar{\mu}_{\Delta,\gamma}(M)} \right)^n. \quad (10)$$

This lower bound is very conservative for two reasons:

- the valid parametric domain may be larger than the hypercube identified by $\bar{\mu}_{\Delta,\gamma}(M)$,
- the assumption $\delta_i \sim \mathcal{U}[-1, 1]$, $\forall i$ is very restrictive from a practical point of view. Truncated normal distributions $\mathcal{N}_{[-1,1]}(0, \sigma)$ (see appendix A for more details) are more representative and commonly adopted to express that the probability for δ to be around a corner of \mathcal{C} is lower than the probability to be around the center of \mathcal{C} .

However, if $\bar{\mu}_{\Delta,\gamma}(M) > 1$, \mathcal{C} can be divided into 2 hyperrectangles \mathcal{C}_1 and \mathcal{C}_2 , associated to uncertainty structures Δ_1 and Δ_2 , by cutting in the middle of the longest edge of \mathcal{C} . The μ upper bound $\bar{\mu}_{\Delta_k,\gamma}(M)$ is then successively evaluated over each hyperrectangle \mathcal{C}_k (associated to the uncertainty structure Δ_k). If $\bar{\mu}_{\Delta_k,\gamma}(M) < 1$, then the probability of the parameters to evolve in the current hyperrectangle \mathcal{C}_k is computed by the cumulative distribution function (assuming stochastic independence of the parameters δ_i):

$$Pr(\mathcal{C}_k) = Pr(\delta \in \mathcal{C}_k(\underline{\delta}^k, \bar{\delta}^k)) = \prod_{i=1}^n \int_{\underline{\delta}_i^k}^{\bar{\delta}_i^k} f_i(\delta_i) d\delta_i \quad (11)$$

where $\mathcal{C}_k(\underline{\delta}^k, \bar{\delta}^k)$ refers to the k^{th} hyperrectangle delimited by the un-normalized bounds $\underline{\delta}^k$ and $\bar{\delta}^k$ satisfying:

$$-1 \leq \underline{\delta}_i^k \leq \bar{\delta}_i^k \leq 1, \quad \forall i = 1, 2, \dots, n.$$

The probabilistic μ procedure is then inherently recursive and is implemented in a dichotomic algorithm for automatic space exploration and classification (see Fig.

2 for illustration). It relies on successive evaluations of $Pr(\mathcal{C}_k : \bar{\mu}_{\Delta_k,\gamma}(M) \leq 1)$ on a collection of hyperrectangles such that $Pr(\mathcal{C}_k \cap \mathcal{C}_l) = 0$, $k \neq l$.

According to the definition of $\mathcal{V}_{\Delta,\gamma}(M)$ and $s_{\Delta,\gamma}(M)$ given in Eq. (8) and (9), if $\bar{\mu}_{\Delta_k,\gamma}(M) \leq 1$, then $\mathcal{C}_k \subseteq \mathcal{V}_{\Delta,\gamma}(M)$, consequently $s_{\Delta,\gamma}(M) \geq Pr(\mathcal{C}_k)$. Extending this to include all hyperrectangles satisfying $\bar{\mu}_{\Delta_k,\gamma}(M) < 1$ leads to:

$$s_{\Delta,\gamma}(M) \geq \underline{s}_{\Delta,\gamma}(M) = \sum_{k: \bar{\mu}_{\Delta_k,\gamma}(M) < 1} Pr(\mathcal{C}_k) \quad (12)$$

From a practical point of view, $\underline{s}_{\Delta,\gamma}(M)$ is computed simply by testing the upper bound of the μ -function associated with the k^{th} hyperrectangle \mathcal{C}_k . Then, a cumulative sum on the probabilities of the associated hyperrectangles is computed. Algorithm is stopped when the probability to be inside the hyperrectangle to be explored is below a given value β ; i.e., when this hyperrectangle does not influence significantly the final value of the cumulative distribution $s_{\Delta,\gamma}(M)$. The probabilistic μ -analysis procedure is summarized in the following pseudo-code:

Algorithm 1 Probabilistic μ

```

1: Problem data:
2:  $M(s)$ ,  $\Delta$ ,  $\gamma$ ,  $\underline{\delta}$ ,  $\bar{\delta}$ ,  $f_i(\delta_i)$  ( $i = 1, \dots, n$ ),  $\beta$ 
3: Initialization:
4:  $c = 0$ ,  $\underline{\delta}^k = \underline{\delta}$ ,  $\bar{\delta}^k = \bar{\delta}$ 
5: procedure  $\mu_{prob}$ 
6:   Fct:  $c \leftarrow \mu_{prob}(M(s), \Delta, \gamma, \underline{\delta}^k, \bar{\delta}^k, c, f_i, \beta)$ 
7:   Compute:  $\bar{\mu}_{\Delta_k,\gamma}(M)$ 
8:   if  $\bar{\mu}_{\Delta_k,\gamma}(M) < 1$  then
9:      $p(k) = Pr(\delta \in \mathcal{C}_k(\underline{\delta}^k, \bar{\delta}^k))$ 
10:     $c = c + p(k)$ 
11:   else
12:     Sub-division w.r.t. the longest edge:
13:      $\mathcal{C}_k(\underline{\delta}^k, \bar{\delta}^k) \rightarrow \{\mathcal{C}_{k_1}(\underline{\delta}^{k_1}, \bar{\delta}^{k_1}); \mathcal{C}_{k_2}(\underline{\delta}^{k_2}, \bar{\delta}^{k_2})\}$ 
14:     if  $Pr(\mathcal{C}_{k_1}) > \beta$  then
15:       Fct:  $c \leftarrow \mu_{prob}(M(s), \Delta, \gamma, \underline{\delta}^{k_1}, \bar{\delta}^{k_1}, c, f_i, \beta)$ 
16:     end if
17:     if  $Pr(\mathcal{C}_{k_2}) > \beta$  then
18:       Fct:  $c \leftarrow \mu_{prob}(M(s), \Delta, \gamma, \underline{\delta}^{k_2}, \bar{\delta}^{k_2}, c, f_i, \beta)$ 
19:     end if
20:   end if
21: end procedure
22:  $\underline{s}_{\Delta,\gamma}(M) = c$ 

```

Remark: Normalization operations required to compute $\bar{\mu}_{\Delta_k,\gamma}(M)$ are not represented in this algorithm and are summarized in appendix B.

3. REFERENCE VALIDATION CASE

To motivate subsequent developments, let us consider the following second order system $G(s, q)$ affected by uncertain parameters:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad y = [1 \ 0] x \quad (13)$$

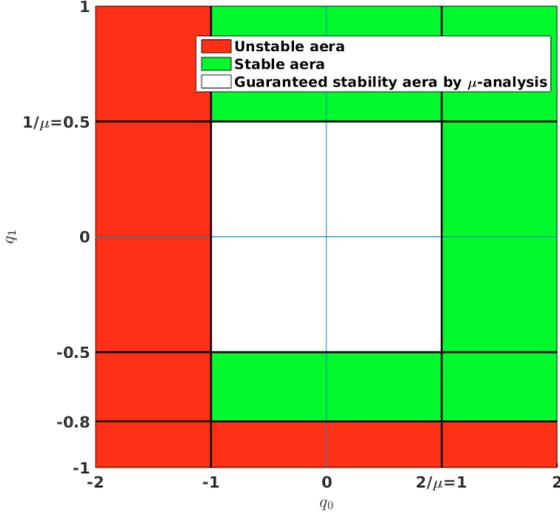


Fig. 3. Stability areas in the un-normalized parametric space (q_0, q_1) .

The real parameters $a_0 = a_0^0 + q_0$ and $a_1 = a_1^0 + q_1$ are uncertain with the following nominal values : $a_0^0 = 1; a_1^0 = 0.8$ and the additive uncertainties q_0 and q_1 such that: $|q_0| \leq 2, |q_1| \leq 1$. Suppose now that we are interesting in evaluating the stability domain of system given by Eq. (13). Application of the Routh's theorem directly gives the conditions for $G(s, q)$ to be stable. Then, $G(s, q)$ is stable i.f.f $q_0 \in]-1; +\infty[$ and $q_1 \in]-0.8; +\infty[$. Unstable (red) and stable (green) domains are depicted in Fig. 3.

The system of Eq. (13) can be easily written under its equivalent $(M(s), \Delta)$ standard representation (Fig. 1 right). Then, exact computation of $\mu_{\Delta}(M)$ in the normalized parametric space immediately leads to $\bar{\mu}_{\Delta, \infty}(M) = \mu_{\Delta}(M) = 2$ which means that $|q_i|$ must be less than $0.5 \bar{q}_i$ ($i = 0, 1, \bar{q}_i$ being the upper bound of q_i) to guarantee that $G(s, q)$ is stable. This guaranteed stability domain is represented in the un-normalized parametric space of Fig. 3 (white). Since the uncertain parameters are assumed to be uniformly distributed, the lower bound of the probability that $G(s, q)$ is stable is 25% (Eq: (10)). Referring to Fig. 3, it is straightforward to see that $Pr(G(s, q) \text{ is stable}) = 67.5\%$ highlighting the fact that the μ -analysis procedure leads to overly pessimistic conclusions.

Suppose now that $q_0 \sim \mathcal{N}_{[-2, 2]}(0, 2/3)$ and $q_1 \sim \mathcal{N}_{[-1, 1]}(0, 1/3)$ and are independent, the joint probability density function (see Fig. 4) reads:

$$f_{q_0 q_1}(q_0, q_1) = f(q_0)f(q_1)$$

where $f(q_i)$ refers to the truncated normal distribution function $\mathcal{N}_{[a_i, b_i]}(m_i, \sigma_i)$ of the i^{th} parameter and is defined in appendix A. Exact computation of $Pr(G(s, q) \text{ is stable})$ on the truncated domain given by the analytic Routh's solution is given by the cumulative distribution function:

$$F(q_0, q_1) = \int_{-1}^2 \int_{-0.8}^1 f(q_0)f(q_1)dq_0dq_1 = 92.79\% \quad (14)$$

Applying now algorithm 1 to this problem leads to:

$$Pr(G(s, q) \in \mathcal{D}_s) = 92.52\% \quad (15)$$

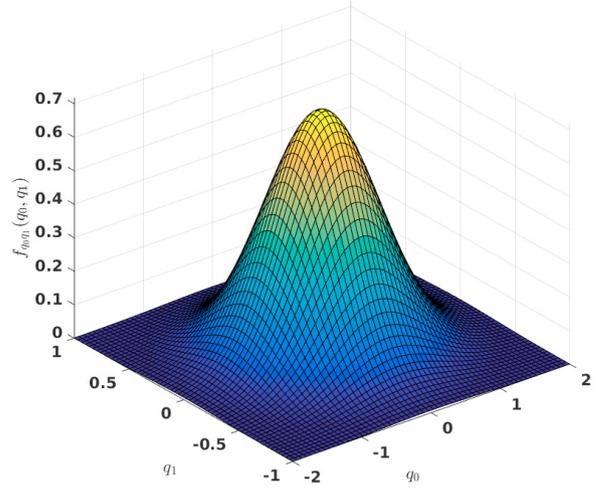


Fig. 4. Joint probability density function of parameters q_0 and q_1 .

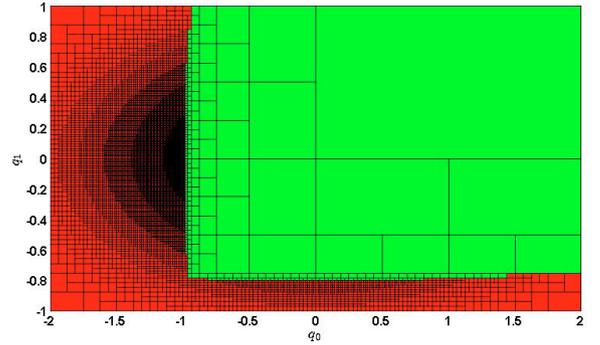


Fig. 5. Stability domain \mathcal{D}_s of $G(s, q)$.

which is very close to the analytic solution. The stability domain identified by the algorithm 1 is delimited by the green boxes in Fig. 5.

Suppose now that we are interesting in evaluating the probability of the system to be robustly stable while satisfying the performance criteria $\|G(s, q)\|_{\infty} < \gamma, \forall q_0 \in [-2, 2], q_1 \in [-1, 1]$ and $\gamma = \sqrt{2}$. A Monte Carlo simulation based on 26,500 samples has been performed to guarantee 1% of accuracy of the statistical estimation of $Pr(\|G(s, q)\|_{\infty} < \sqrt{2})$ with a confidence level of 1%. The estimated probability resulting to the post-processing of the experimental campaign is equal to 42.84% (see Fig. 6). In comparison, the probability given by the developed algorithm leads to

$$\underline{s}_{\Delta, \gamma}(M) = 41.24\%$$

which is a satisfactory lower bound of $Pr(\|G(s, q)\|_{\infty} < \sqrt{2})$. Fig 7 presents the convergence map of the dichotomic search algorithm where the green boxes correspond to the parametric hyperrectangles for which $\|G(s, q)\|_{\infty} < \sqrt{2}$ where the red ones correspond to $\|G(s, q)\|_{\infty} \geq \sqrt{2}$. Clearly, this simple example highlights the capability of the proposed algorithm to be used for probabilistic robust performance analysis.

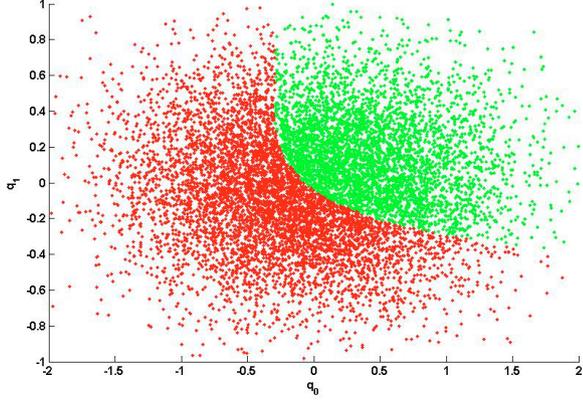


Fig. 6. Estimated domain for which the system is stable and $\|T_{yu}\|_\infty < \sqrt{2}$ based on 26,500 Monte Carlo runs.

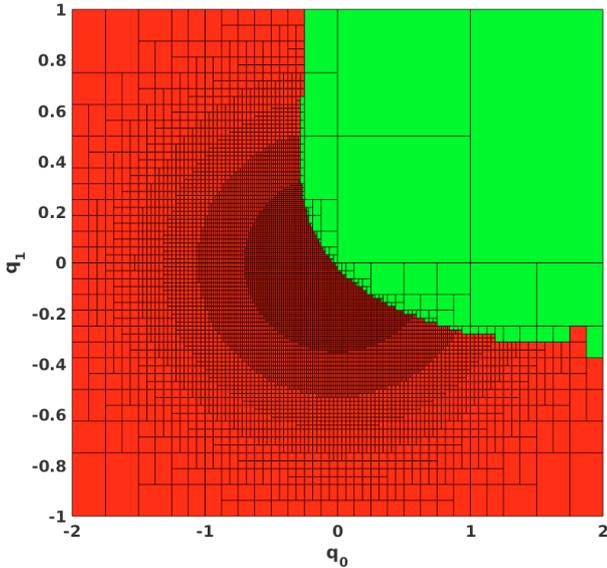


Fig. 7. Domain for which the system is stable and $\|T_{yu}\|_\infty < \sqrt{2}$ - **Algorithm 1** with $\beta = 0.0001$.

4. MODIFIED ALGORITHM

In fig. 7, one can notice that the exploration of the non-valid domain involves very tiny hyperrectangles \mathcal{C}_k as long as they satisfy the condition $Pr(\mathcal{C}_k) > \beta$. Although such an algorithm can detect any disjoint valid sub-domain, it is not efficient from a computational cost point of view. For analysis problem characterized by a single connected valid domain, another algorithm is proposed to stop the subdivision of the current hyperrectangle \mathcal{C}_k when it meets all of the following conditions:

$$\|F_u(M(s), \Delta(c_k))\|_\infty > \gamma \quad (16)$$

$$\|F_u(M(s), \Delta(c_{k-1}))\|_\infty > \gamma \quad (17)$$

$$Pr(\mathcal{C}_{k-1}) < \alpha \quad (18)$$

where c_k is the current hyperrectangle center, \mathcal{C}_{k-1} is the parent hyperrectangle of the current hyperrectangle \mathcal{C}_k , i.e. the one which the sub-division leads to \mathcal{C}_k , c_{k-1}

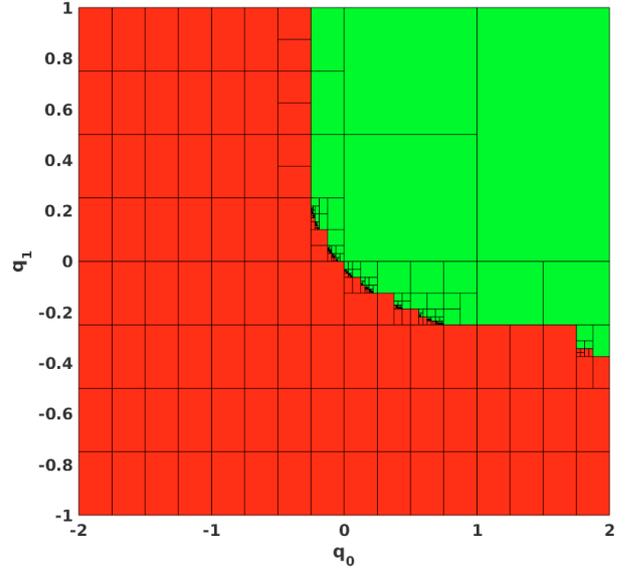


Fig. 8. Domain for which the system is stable and $\|T_{yu}\|_\infty < \sqrt{2}$ - **Algorithm 2** with $\beta = 0.00001$ and $\alpha = 0.25$.

is the center of \mathcal{C}_{k-1} and α is a given real number. α represents the size of the largest local valid sub-domain (in the normalized parametric space) which can be missed by this algorithm, named **Algorithm 2**.

The application of this algorithm on the previous robust performance analysis leads a probability of success is at least equal to 40.55%. Fig 8 presents the convergence map obtained with **Algorithm 2**. In comparison with Fig. 7, one can see that the parametric domain exploration is focused on the boundary between the valid sub-domain and the invalid sub-domain. The computational cost was divided by a factor 10 at the price of a very low degradation on the lower bound of the probability of success.

5. CONCLUSION

In this paper, a probabilistic μ -analysis procedure has been studied with the perspective of enhancing industrial validation process. The procedure is based on successive evaluation of the μ function on sub regions of the parametric space selected by a dichotomic search algorithm. In this work, μ upper bound is selected as the single performance metric providing a lower bound of the cumulative distribution of the evaluated performance metric, i.e. $s_{\Delta, \gamma}(M)$. The μ lower bound ($\underline{\mu}$), and more particularly undecided regions for which:

$$\bar{\mu}_{\Delta, \gamma}(M) > 1 \text{ and } \underline{\mu}_{\Delta, \gamma}(M) < 1,$$

have not been addressed and is currently under analysis. Additionally, improvement of the algorithm convergence properties exploiting the μ -sensitivity is under investigation and will be available in a next version. Finally, the proposed solution has been applied on a simple second order system for analytic validation. It is today under implementation on a real flexible satellite benchmark with many uncertainties to evaluate both the computational load and the convergence properties of the solution.

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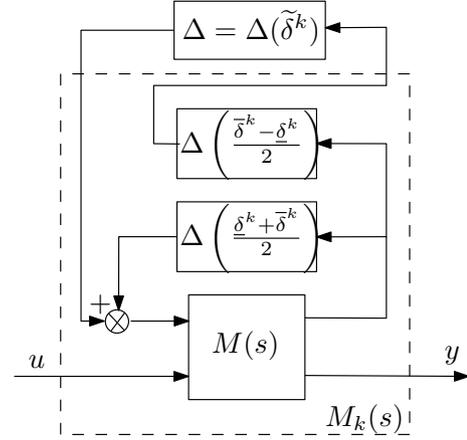


Fig. B.1. Normalization operation of $(M(s), \Delta_k)$.

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Appendix A. TRUNCATED NORMAL DISTRIBUTION

The truncated normal distribution for a real parameter δ is denoted $\delta \sim f(\delta) = \mathcal{N}_{[a,b]}(m, \sigma)$ ($a < b$) and is defined by:

$$f(\delta) = 0, \forall \delta < a,$$

$$f(\delta) = 0, \forall \delta > b,$$

$$f(\delta) = \frac{1}{\Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}, \forall a \leq \delta \leq b,$$

$$\text{with: } \Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

Such a cumulative distribution function can be easily computed using the Matlab Statistics and Machine Learning Toolbox.

Appendix B. NORMALIZATION OPERATION

To compute $\bar{\mu}_{\Delta_k, \gamma}(M)$, one can notice that the structure Δ_k , associated to the hyperrectangle $\mathcal{C}_k(\underline{\delta}^k, \bar{\delta}^k)$ is not normalized. One can define the normalized parameters $\tilde{\delta}^k \in [-1, 1]$ by:

$$\delta = \frac{\delta^k + \bar{\delta}^k}{2} + \frac{\bar{\delta}^k - \delta^k}{2} \tilde{\delta}^k.$$

Then $\bar{\mu}_{\Delta_k, \gamma}(M) = \bar{\mu}_{\Delta_k, \gamma}(M_k)$ where the normalized μ -problem $(M_k(s), \Delta)$ is depicted in Fig. B.1.