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On the determination of missing boundary data for solids with nonlinear material behaviors, using displacement fields measured on a part of their boundaries

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The paper is devoted to the derivation of a numerical method for expanding available mechanical fields (stress vector and displacements) on a part of the boundary of a solid into its interior and up to unreachable parts of its boundary (with possibly internal surfaces). This expansion enables various identification or inverse problems to be solved in mechanics. The method is based on the solution of a nonlinear elliptic Cauchy problem because the mechanical behavior of the solid is considered as nonlinear (hyperelastic or elastoplastic medium). Advantage is taken of the assumption of convexity of the potentials used for modeling the constitutive equation, encompassing previous work by the authors for linear elastic solids, in order to derive an appropriate error functional. Two illustrations are given in order to evaluate the overall efficiency of the proposed method within the framework of small strains and isothermal transformation.

1. Introduction

The problem of using overspecified measured boundary data on a part of a solid (displacement and stress vector fields) in order to extend the mechanical fields within the solid, or to identify missing or unknown boundary conditions, is still partially open despite potential applications being extremely numerous in mechanical and material sciences as well as in industry. Advances in the development of digital cameras and image correlation techniques (DIC) now make it possible to have measurement means for full field surface displacements that are cheap and easy to manage and, more importantly, leading to very large amounts of information (see for example, [Avril et al., 2008](#); [Sutton et al., 2009](#)). This information obtained at the surface has been used in the literature to define a large number of inverse or identification problems with various applications (see [Avril et al., 2008](#); [Bonnet and Constantinescu, 2005](#); [Grediac and Hild, 2013](#)) and references therein. In the same spirit, [Moireau et al. \(2009\)](#) make use of alternative measurement techniques, such as Tagged Magnetic Resonance Imaging with the Harp algorithm ([Osman et al., 2000](#)). Nevertheless, the use of these surface data is still largely restricted either to qualitative estimation or to quantitative analysis based on a plane mechanical state or on homogeneous-through-the-thickness assumptions. The development of accurate and efficient methods for the 3D extension of the displacement field measured on a stress free surface would lead to a lot of new applications in mechanics. For instance, in analyzing complex mock-ups or experiments (3D numerical imaging,

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identification of boundary conditions in [Andrieux and Baranger, 2008a](#)) or physical parameters entering into their description, such as the friction coefficient, in identifying geometrical defects ([Baranger and Andrieux, 2009](#)) or contact surfaces ([Andrieux and Baranger, 2012](#)), in designing new non-destructive analysis or monitoring techniques, in computing linear fracture mechanics parameters ([Andrieux and Baranger, 2013](#)), etc.

One promising approach in dealing with this problem is to first reformulate it within the continuous framework, taking advantage of the fact that the amount and spatial density of the information obtained using the digital image correlation techniques makes it possible to consider that the complete displacement field is available on a part of the boundary and is not reduced on it to pinpoint data only, and second to then reformulate it as a Cauchy or Data Completion Problem, taking into account the fact that an overspecified data pair is given on a part of the boundary. Cauchy problems or Data Completion Problems belong to the class of inverse problems and are usually ill-posed in the sense of [Hadamard \(1923\)](#).

Linear elliptic Cauchy problems have been extensively studied since the 1920s ([Hadamard, 1923](#)) and compatibility conditions between the Cauchy data are known to be met, in order to ensure existence ([Fursikov, 2000](#)). Theoretical results for existence and data compatibility conditions for nonlinear elliptic Cauchy problems have been addressed by [Klüger and Leitao \(2003\)](#). Numerous numerical approaches are available in the literature for linear elliptic Cauchy problems, although the complexity of the algorithms and the large amount of computation needed limit the applications in almost all of the papers to two-dimensional problems: fixed point algorithms ([Kozlov et al., 1992](#); [Baumeister and Leitao, 2001](#); [Marin and Lesnic, 2004](#)); variational approaches based on Steklov–Poincaré operators ([Ben Belgacem and El Fekih, 2005](#); [Mejdi et al., 2006](#)); least-squares method with vanishing regularization ([Cimetière et al., 2001](#)); methods using fundamental solutions ([Marin and Lesnic, 2004](#); [Young et al., 2008](#)); or boundary integral techniques ([Marin and Lesnic, 2002](#)); moment methods associated with Backus–Gilbert techniques ([Hon and Wei, 2001](#)); quasi-reversibility methods ([Bourgeois, 2005, 2006](#)); and lastly energy error based methods ([Andrieux and Baranger, 2008a, 2012, 2013](#); [Andrieux et al., 2006](#); [Baranger and Andrieux, 2008, 2009, 2011](#); [Escriva et al., 2007](#)) on the spirit of which this paper is based.

Very few papers can be found for nonlinear operators. [Baumeister and Leitao \(2001\)](#) proposed to solve a nonlinear Cauchy problem for a nonlinear scalar conduction equation by a change of variable and the solution of a linear Cauchy problem, but this method cannot be extended generally to other operators. The existence of a solution for a nonlinear Cauchy problem has also been studied by [Egger and Leitão \(2009\)](#) and [Klüger and Leitao \(2003\)](#), by a constructive method using a fixed point algorithm similar to the one designed by [Kozlov et al. \(1992\)](#).

In this paper, nonlinear solids are addressed and an extension of the variational method previously designed by the authors for linear elasticity is developed for convex hyperelastic solids and for dissipative solids governed by an elastoplastic constitutive relation described in the Generalized Standard Materials format ([Halphen and Nguyen, 1975](#)). For this last application, the derivation of the class of the error in a constitutive equation is based on previous work within the context of material parameter identification by [Hadj-Sassi \(2007\)](#), [Hadj-Sassi and Andrieux \(2006\)](#).

The following part is devoted to the definition of the Cauchy problem for nonlinear solids and the derivation of the variational methods. Parts 3 and 4 address the question of building an error that can ground the functional to be minimized in order to obtain the solution of the Data Completion problem, respectively for hyperelastic materials and elastoplastic ones.

2. Reformulation as a Cauchy problem

Let Ω be given a regular domain, the boundary of which is decomposed into three non-overlapping parts Γ_m , Γ_b , and Γ_u , for instance see [Fig. 1](#). The usual boundary conditions are given on Γ_b (combination of normal stress vector and displacement vector components). Γ_m (the subscript m stands for measurements) is the part where, using DIC acquisition, both displacement and stress vector components (usually zero for the latter) are available, and make up an overspecified boundary data pair. Lastly, Γ_u is the remaining part of the boundary, where no boundary data is known:

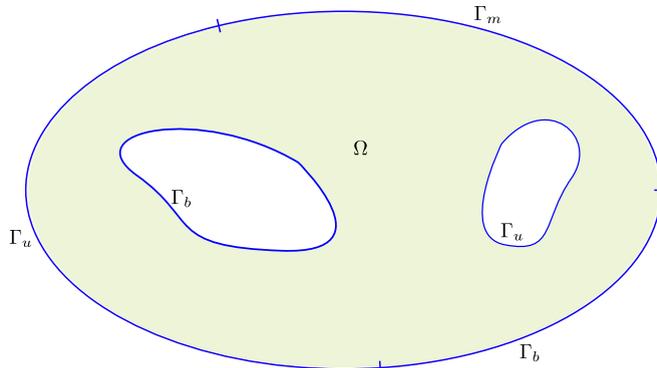


Fig. 1. The domain Ω and its boundaries.

$$\partial\Omega = \Gamma_b \cup \Gamma_m \cup \Gamma_u, \quad \Gamma_i \cap \Gamma_j = \emptyset, \quad i, j = m, b, u \quad (1)$$

The data on Γ_m will be denoted by $(\mathbf{F}_m, \mathbf{U}_m)$ and are functions of space and time. For the sake of simplicity, from now we will assume that Γ_b is empty, but the extension of the method to situations with non-empty Γ_b is straightforward. The boundary Γ_u is generally non-connected and can possibly contain internal surfaces such as cracks or boundaries of cavities and inclusions. Within the framework of linearized strain and for isothermal transformation, the constitutive equation for the solid is written in an abstract format as a relation between the Cauchy stress tensor σ and the linearized strain tensor ϵ , and can also involve internal variables α :

$$f(\sigma, \epsilon, \alpha) = 0 \quad (2)$$

This equation must be complemented with an evolution equation for the internal variables α :

$$g(\sigma, \epsilon, \dot{\alpha}) = 0 \quad (3)$$

All these equations can also, and generally will, be inclusion equations within the framework of convex analysis, using the notion of sub-differential.

The problem that we want to address is the reconstruction of the displacement, stress and internal variable fields within the solid over a time interval $[0, D]$ provided the data pair $(\mathbf{F}^m, \mathbf{U}^m)$ is given during this period on the part Γ_m under the assumption of a quasi-static evolution. The initial state of the solid is given by the fields $(\mathbf{u}_0, \sigma_0, \alpha_0)$ and is assumed to be known as well. The associated Cauchy problem is the following: *to find the fields $(\mathbf{u}, \sigma, \alpha)$ in the domain Ω and over the time interval $[0, D]$, such that:*

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma = 0 & \text{in } \Omega \\ \epsilon(\mathbf{u}) = [\nabla \mathbf{u}]^{\text{sym}} & \text{in } \Omega \\ f(\sigma, \epsilon, \alpha) = 0 & \text{in } \Omega \\ g(\sigma, \epsilon, \dot{\alpha}) = 0 & \text{in } \Omega \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \sigma|_{t=0} = \sigma_0, \quad \alpha|_{t=0} = \alpha_0 & \text{in } \Omega \end{array} \right. \quad (4)$$

and:

$$\mathbf{u} = \mathbf{U}^m, \quad \sigma \cdot \mathbf{n} = \mathbf{F}^m \quad \text{on } \Gamma_m \quad (5)$$

The Data Completion Problem consists in determining the missing boundary data pair (τ, η) on the remaining part Γ_u of the boundary of the solid: *to find (τ, η) on Γ_u and over the time interval $[0, D]$, such that there exists $(\mathbf{u}, \sigma, \alpha)$ in the domain Ω and over the time interval $[0, D]$ fulfilling the following:*

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma = 0 & \text{in } \Omega \\ \epsilon(\mathbf{u}) = [\nabla \mathbf{u}]^{\text{sym}} & \text{in } \Omega \\ f(\sigma, \epsilon, \alpha) = 0 & \text{in } \Omega \\ g(\sigma, \epsilon, \dot{\alpha}) = 0 & \text{in } \Omega \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \sigma|_{t=0} = \sigma_0, \quad \alpha|_{t=0} = \alpha_0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{U}^m, \quad \sigma \cdot \mathbf{n} = \mathbf{F}^m & \text{on } \Gamma_m \end{array} \right. \quad (6)$$

and:

$$\mathbf{u} = \tau, \quad \sigma \cdot \mathbf{n} = \eta \quad \text{on } \Gamma_u \quad (7)$$

The Cauchy Problem (CP) and the Data Completion Problem (DCP) are very similar and differ only with regard to the unknowns on which they are formulated (fields in the solid for CP, boundary data on Γ_u for DCP). Nevertheless, due to the fact that the fields $(\mathbf{u}, \sigma, \alpha)$ appear in both problems, we shall from now on use one or the other denomination for the problem at hand. Similar Cauchy problems or Data Completion Problems can be defined with only a part of the displacement fields being given on Γ_m the typical case being the situation where only the tangential part of the displacement is available. The proposed method can be formally adapted without difficulties although the results are clearly affected by the smaller quantity of data used (see [Andrieux and Baranger, 2008a](#) for an analysis and examples in the case of linear elasticity).

The general variational method for solving these problems is based on two steps. First, two auxiliary usual well-posed problems \mathcal{P}_i , $i = 1, 2$ are defined, each one using one only of the overspecified boundary data on Γ_m and a given normal stress vector field η over $[0, D]$ on Γ_u :

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma_i = 0 & \text{in } \Omega \\ \epsilon(\mathbf{u}_i) = [\nabla \mathbf{u}_i]^{\text{sym}} & \text{in } \Omega \\ f(\sigma_i, \epsilon_i, \alpha_i) = 0 & \text{in } \Omega \\ g(\sigma_i, \epsilon_i, \dot{\alpha}_i) = 0 & \text{in } \Omega \\ \mathbf{u}_i|_{t=0} = \mathbf{u}_0, \quad \sigma_i|_{t=0} = \sigma_0, \quad \alpha_i|_{t=0} = \alpha_0 & \text{in } \Omega \end{array} \right. \quad (8)$$

And respectively for \mathcal{P}_1 and \mathcal{P}_2 :

$$(\mathcal{P}_1) \begin{cases} \mathbf{u}_1 = \mathbf{U}^m & \text{on } \Gamma_m \\ \boldsymbol{\sigma}_1 \mathbf{n} = \boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad \text{and} \quad (\mathcal{P}_2) \begin{cases} \boldsymbol{\sigma}_2 \mathbf{n} = \mathbf{F}^m & \text{on } \Gamma_m \\ \boldsymbol{\sigma}_2 \mathbf{n} = \boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad \text{with} \quad \int_{\Gamma_m} \mathbf{F}^m d\Gamma + \int_{\Gamma_u} \boldsymbol{\eta} d\Gamma = 0 \quad (9)$$

It is clear that if a surface traction field $\boldsymbol{\eta}_{\text{opt}}$ on Γ_u is such that $\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{RBM}$, where **RBM** is a Rigid Body Motion, the two problems \mathcal{P}_1 and \mathcal{P}_2 will have the same solution $(\boldsymbol{\sigma}, \boldsymbol{\alpha})$. Therefore the Cauchy problem is solved with \mathbf{u}_1 and the solution of the Data Completion Problem is the pair $(\mathbf{u}_1, \boldsymbol{\eta}_{\text{opt}})$ on Γ_u . A general variational solution method can thus be derived by a second step consisting in building an error function \mathcal{E} between the states $e_1 = (\mathbf{u}_1, \boldsymbol{\sigma}_1, \boldsymbol{\alpha}_1)$ and $e_2 = (\mathbf{u}_2, \boldsymbol{\sigma}_2, \boldsymbol{\alpha}_2)$ as a functional of $\boldsymbol{\eta}$ and by minimizing it over all the possible surface traction fields defined on Γ_u :

$$\boldsymbol{\eta}_{\text{opt}} = \text{Arg min}_{\boldsymbol{\eta}} \mathcal{J}(\boldsymbol{\eta}) \equiv \int_{\Omega} \mathcal{E}(e_1(\boldsymbol{\eta}), e_2(\boldsymbol{\eta})) d\Omega \quad (10)$$

The proposed approach is very general and can be applied within frameworks other than nonlinear mechanics. The first step is very similar in each situation although the design of the error functional is very different. Attention must be paid to the choice of the error function \mathcal{E} in order to design an efficient solution method.

For linear elastic media, where the state e_i reduces to the displacement \mathbf{u}_i , a natural choice is the elastic energy density of the difference $\boldsymbol{\epsilon}(\mathbf{u}_1) - \boldsymbol{\epsilon}(\mathbf{u}_2)$ or equivalently $\boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2)$. Then the stress gap is $(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) = \mathbb{H} : \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2)$ and the elastic energy density is:

$$\mathcal{E}(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2} \mathbb{H} : \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) : \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \quad (11)$$

where \mathbb{H} is Hooke's tensor.

[Baranger and Andrieux \(2011\)](#) showed that the corresponding functional $\mathcal{J}(\boldsymbol{\eta})$ has the following properties:

- \mathcal{J} is positive, quadratic and therefore convex.
- If the data $(\mathbf{U}^m, \mathbf{F}^m)$ are compatible, then: $\mathcal{J}(\boldsymbol{\eta}_{\text{opt}}) = 0 \Leftrightarrow \mathbf{u}_1(\boldsymbol{\eta}_{\text{opt}})$ solves the Cauchy problem.
- \mathcal{J} can be computed by a surface integral over the boundary part Γ_m only.

This choice for \mathcal{E} is somehow optimal because the prob [lem \(10\)](#) reduces to the minimization of a convex quadratic functional for solving a linear problem (as the Cauchy problem turns out to be for the Lamé operator). Furthermore, it can be generally established that, for linear elliptic problems, the fixed point algorithm of [Kozlov et al. \(1992\)](#) can be interpreted as a special descent method for the above functional, namely the alternating direction descent method. This feature explains the far smaller number of iterations needed for convergence for the proposed method when conjugate gradient algorithms are used in order to perform the minimization of the functional \mathcal{J} ([Baranger and Andrieux, 2011](#)). Lastly, the surface integral equivalent expression of \mathcal{J} leads to a very simple form for the adjoint problems that are used to compute its gradients.

To deal with elastodynamics, in [Andrieux and Baranger \(2008b\)](#) some supplementary terms must be added, involving the kinetic energy difference and the elastic energy of the difference of \mathbf{u}_1 and \mathbf{u}_2 at the final time $t=D$.

For nonlinear solids, another approach must be followed for building the error. This will be the purpose of the following parts for convex hyperelasticity and for some Generalized Standard (dissipative) materials describing the elastoplastic behavior.

3. The error gap in constitutive equations for nonlinear convex hyperelastic materials

In this part, we assume that strains are infinitesimal and that the material has a hyperelastic constitutive law. Therefore, there exists an elastic potential $\varphi(\boldsymbol{\epsilon})$ representing the stored strain energy, such that the stress tensor can be derived by

$$\boldsymbol{\sigma} = \frac{\partial \varphi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} \quad (12)$$

The potential $\varphi(\boldsymbol{\epsilon})$ is a convex and lower semi-continuous differentiable real function. If it is not quadratic, the material has a nonlinear elastic behavior, and the second order operator on \mathbf{u} defined by the first equation of [\(4\)](#) turns out to be nonlinear elliptic PDE ([Taylor, 2011](#)). Therefore, the prob [lem \(4\)](#) is a nonlinear elliptic boundary value one and its associated Cauchy problem is also a nonlinear one.

In order to derive an error function with similar properties as those for linear elliptic Cauchy problems, advantage must be taken of the convexity of the potential φ . The error will no longer be an *error in energy* but rather an *error in the constitutive equation*. The following lemma, is the key-point to build the new error functional; for more details see [Ekeland and Témam \(1999\)](#).

Lemma 1. *Fenchel's inequality or conjugacy formula. Let ψ be a lower semi-continuous, convex function from \mathbb{R}^n into $\bar{\mathbb{R}}$; then, for any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$:*

1. $\psi(x) + \psi^*(y) - x \cdot y \geq 0$
2. $\psi(x) + \psi^*(y) - x \cdot y = 0 \Leftrightarrow y \in \partial \psi(x) \Leftrightarrow x \in \partial \psi^*(y)$

where $\partial\psi(t)$ is the subdifferential set of ψ at t :

$$\partial\psi(t) = \{g \in R^n | \psi(t) - \psi(z) \geq g \cdot (t - z) \quad \forall z \in R^n\} \quad (13)$$

and ψ^* is the Legendre–Fenchel conjugate function of ψ :

$$\psi^*(y) = \sup_x [x \cdot y - \psi(x)] \quad (14)$$

For any pair (σ, ϵ) , the error in the constitutive equation for the hyperelastic material with convex potential φ is then defined by the scalar function e_φ :

$$e_\varphi(\sigma, \epsilon) \equiv \varphi(\epsilon) + \varphi^*(\sigma) - \sigma : \epsilon \quad (15)$$

and has then the following properties:

$$e_\varphi(\sigma, \epsilon) \geq 0 \quad \forall (\sigma, \epsilon) \quad (16)$$

$$e_\varphi(\sigma, \epsilon) = 0 \Leftrightarrow \sigma \in \partial\varphi(\epsilon) \Leftrightarrow \epsilon \in \partial\varphi^*(\sigma) \quad (17)$$

We then define the positive error in the constitutive equation for the two states (ϵ_1, σ_1) and (ϵ_2, σ_2) by:

$$\mathcal{E}_{\text{ECE}}(\epsilon_1, \epsilon_2) = e_\varphi(\sigma_1, \epsilon_2) + e_\varphi(\sigma_2, \epsilon_1) \quad (18)$$

It is straightforward to see that the following equivalent expression holds:

$$\mathcal{E}_{\text{ECE}}(\epsilon_1, \epsilon_2) = (\sigma_1 - \sigma_2) : (\epsilon_1 - \epsilon_2) \quad (19)$$

In this expression, the potential φ no longer appears nor does φ_{**} , so these potentials do not need to be explicitly calculated. The knowledge of the constitutive equation (12) is sufficient for the applications. Nevertheless, attention must be paid to the fact that the positiveness of the last expression (19) of \mathcal{E}_{ECE} is ensured only if such convex potentials exist. Finally the variational form of the Data Completion Problem is the following:

$$\eta_{\text{opt}} = \text{Arg min}_\eta \mathcal{J}_{\text{ECE}}(\eta) \equiv \int_\Omega (\sigma_1 - \sigma_2) : (\epsilon_1 - \epsilon_2) d\Omega \quad (20)$$

with $(\sigma_1, \epsilon_1, \sigma_2, \epsilon_2)$ solutions of:

$$\begin{cases} \text{div } \sigma_i = 0 & \text{in } \Omega \\ \epsilon(\mathbf{u}_i) = [\nabla \mathbf{u}_i]^{\text{sym}} & \text{in } \Omega \\ \sigma_i = \frac{\partial}{\partial \epsilon} \varphi(\epsilon(\mathbf{u}_i)) & \text{in } \Omega \end{cases} \quad \text{for } i = 1, 2 \quad (21)$$

and respectively for \mathcal{P}_1 and \mathcal{P}_2 :

$$(\mathcal{P}_1) \begin{cases} \mathbf{u}_1 = \mathbf{U}_m & \text{on } \Gamma_m \\ \sigma_1 \cdot \mathbf{n} = \boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad \text{and} \quad (\mathcal{P}_2) \begin{cases} \sigma_2 \cdot \mathbf{n} = \mathbf{F}_m & \text{on } \Gamma_m \\ \sigma_2 \cdot \mathbf{n} = \boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad (22)$$

For the hyperelastic case, knowledge of the initial state is not necessary. The functional \mathcal{J}_{ECE} is positive and its minimum is zero if the data are compatible, with the following characterization and relation with the solution of the Cauchy problem for the corresponding fields \mathbf{u}_1 and \mathbf{u}_2 .

Theorem 1. *If the potential φ is lower semi-continuous, strictly convex and differentiable, and if the data $(\mathbf{U}_m, \mathbf{F}_m)$ are compatible, then the following equivalence holds for \mathbf{u}_1 and \mathbf{u}_2 satisfying (21) and (22): (6)*

$$\mathcal{J}_{\text{ECE}}(\eta) = 0 \Leftrightarrow \begin{cases} \exists (t, \omega) \in R^3 \times R^3, & \mathbf{u}_2(x; \eta) = \mathbf{u}_1(x; \eta) + t + \omega \wedge x \\ \mathbf{u}_1(\eta) & \text{is the solution of the Cauchy problem, } \eta \text{ is the solution of the Data Completion Problem} \end{cases} \quad (23)$$

Proof. The implication leading to $\mathcal{J}_{\text{ECE}} = 0$ is obvious because if \mathbf{u}_1 and \mathbf{u}_2 differ from a rigid body displacement field $t + \omega \wedge x$, then $\epsilon(\mathbf{u}_1) = \epsilon(\mathbf{u}_2)$ a.e. in Ω , so that by the constitutive equation $\sigma_1 = \sigma_2$. The field \mathbf{u}_1 is then a solution of (21) and (22); furthermore $\mathcal{E}_{\text{ECE}} = 0$ almost everywhere.

The converse implication is straightforward too. Note that because of the differentiability of φ , the subdifferential $\partial\varphi(\epsilon)$ reduces at every point ϵ to the set $\{\frac{\partial\varphi}{\partial\epsilon}\}$; the same property holds for φ^* . If \mathcal{J}_{ECE} vanishes for a couple of fields $(\mathbf{u}_1, \mathbf{u}_2)$, then \mathcal{E}_{ECE} is zero a.e. so that the two positive errors $e_\varphi(\sigma_2, \epsilon(\mathbf{u}_1))$ and $e_\varphi(\sigma_1, \epsilon(\mathbf{u}_2))$ vanish simultaneously in the domain Ω . Using Lemma 1 for $(\sigma_1, \epsilon(\mathbf{u}_2))$ and the constitutive equation for $(\sigma_2, \epsilon(\mathbf{u}_2))$, we then have:

$$\left. \begin{aligned} \boldsymbol{\epsilon}(\mathbf{u}_1) &= \frac{\partial \varphi^*(\boldsymbol{\sigma}_2)}{\partial \boldsymbol{\sigma}} \\ \boldsymbol{\epsilon}(\mathbf{u}_2) &= \frac{\partial \varphi^*(\boldsymbol{\sigma}_2)}{\partial \boldsymbol{\sigma}} \end{aligned} \right\} \Leftrightarrow \boldsymbol{\epsilon}(\mathbf{u}_1) = \boldsymbol{\epsilon}(\mathbf{u}_2) \quad \forall x \in \Omega \quad (24)$$

so that \mathbf{u}_1 and \mathbf{u}_2 differ only by a rigid body displacement field. The surface traction fields $\boldsymbol{\sigma}_1 \cdot \mathbf{n}$ and $\boldsymbol{\sigma}_2 \cdot \mathbf{n}$ are then equal on Γ_m and \mathbf{u}_1 solves the Cauchy problem (6). \square

The formulation (20) is similar to the error (10) in the energy approach used for linear elasticity, except apparently for the form of the error, which is no longer the elastic energy of the difference field $(\mathbf{u}_1 - \mathbf{u}_2)$. In fact, for linear problems, i.e., problems in which the constitutive potential φ is quadratic, the following proposition shows that the actual formulation via an error in the constitutive equation encompasses the error in energy.

Proposition 1. *If the potential φ is quadratic, then the error in the constitutive equation $\mathcal{E}_{\text{ECE}}[(\boldsymbol{\epsilon}(\mathbf{u}_1), \boldsymbol{\sigma}_1), (\boldsymbol{\epsilon}(\mathbf{u}_2), \boldsymbol{\sigma}_2)]$ is exactly twice the energy of the difference of the fields \mathbf{u}_1 and \mathbf{u}_2 :*

$$\varphi(\boldsymbol{\epsilon}) \text{ quadratic} \Rightarrow \mathcal{E}_{\text{ECE}}[(\boldsymbol{\epsilon}(\mathbf{u}_1), \boldsymbol{\sigma}_1), (\boldsymbol{\epsilon}(\mathbf{u}_2), \boldsymbol{\sigma}_2)] \equiv (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) : (\boldsymbol{\epsilon}(\mathbf{u}_1) - \boldsymbol{\epsilon}(\mathbf{u}_2)) = 2\varphi(\boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2)) \quad (25)$$

Remark 1. A remark about Bregman distances.

Interestingly, the equivalent form (19) of the error \mathcal{E}_{ECE} can also be obtained with reference to the concept of Bregman distance introduced within the context of convex optimization (Bregman, 1967; Kiwiel, 1997).

Let J be a convex function, the generalized Bregman distance between \mathbf{u} and \mathbf{v} with respect to J is the non-negative scalar:

$$D_J^p(\mathbf{u}, \mathbf{v}) = J(\mathbf{u}) - J(\mathbf{v}) - (p, \mathbf{u} - \mathbf{v}) \quad \text{for } p \in \partial J(\mathbf{v}) \quad (26)$$

It is readily seen that the error in the constitutive equation is exactly the symmetrized generalized Bregman distance between $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ with respect to the hyperelastic potential φ , for any pairs $(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}_1)$ and $(\boldsymbol{\sigma}_2, \boldsymbol{\epsilon}_2)$ satisfying the constitutive equation:

$$e_\varphi(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}_2) + e_\varphi(\boldsymbol{\sigma}_2, \boldsymbol{\epsilon}_1) = D_\varphi^{\boldsymbol{\sigma}_1}(\boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_1) + D_\varphi^{\boldsymbol{\sigma}_2}(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2) \quad (27)$$

The general energy error approach developed here can be linked with pioneering works, such as those of Knowles (1998) who identified distributed parameters in an elliptical equation (Knowles, 1998; Kohn and McKenney, 1990), and in a similar framework and more generally with the development of constitutive equation errors for various applications (Ladeveze, 1999; Ladevèze and Chouaki, 1999; Ladeveze and Leguillon, 1983). Barbu and Kunisch (1996) also use also the Fenchel inequality in a penalized least-square method for identification in nonlinear state dependent problems, in order to express the mathematical constraint implied by the convex constitutive equation between the variables involved in the problem.

The equivalent expression (19) for the error \mathcal{E}_{ECE} leads to another alternative equivalent form of the functional \mathcal{J}_{ECE} involving only a boundary integral over Γ_m :

$$\mathcal{J}_{\text{ECE}}(\boldsymbol{\eta}) = \int_{\Gamma_m} (\boldsymbol{\sigma}_1(\boldsymbol{\eta}) \cdot \mathbf{n} - T_m) \cdot (\mathbf{U}_m - \mathbf{u}_2(\boldsymbol{\eta})) \, dS \quad (28)$$

Using (19) the above expression is simply obtained by taking into account the definition of the fields \mathbf{u}_i and using each one as a virtual field in the principle of virtual power expressed for the other one. This expression will be used in numerical applications because of its efficiency from the computational standpoint.

The variational formulation of the nonlinear elliptic Cauchy problem derived here necessitates the use of minimization algorithms. Due to the ill-posedness of the Cauchy problem and for efficiency of the method, it is important to use robust and efficient algorithms that take advantage of the precise knowledge of the gradient of the objective functional \mathcal{J}_{ECE} . The gradient must be computed by an adjoint method, because of the implicit dependence of the fields \mathbf{u}_1 and \mathbf{u}_2 with respect to the variable $\boldsymbol{\eta}$ and the relatively high cost of evaluation of the function \mathcal{J}_{ECE} itself (Chavent, 1991; Griewank, 1993), involving the solution of problems \mathcal{P}_1 and \mathcal{P}_2 . The following property gives the expression of the gradient of the functional $\mathcal{J}_{\text{ECE}}(\boldsymbol{\eta})$, for twice-differentiable hyperelastic potentials φ .

Proposition 2. *Provided that the hyperelastic potential is twice differentiable, the gradient of the functional \mathcal{J}_{ECE} can be computed by the following expression:*

$$\nabla \mathcal{J}_{\text{ECE}}(\boldsymbol{\eta}) = - (\mathbf{v}_1^*(\boldsymbol{\eta}) + \mathbf{v}_2^*(\boldsymbol{\eta}))|_{\Gamma_u} \quad (29)$$

where \mathbf{v}_1^* and \mathbf{v}_2^* are adjoint fields that are solution of:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}_1^* = -\nabla \cdot (\mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_1)]: \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2)) & \text{in } \Omega \\ \boldsymbol{\sigma}_1^* = \mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_1)]: \boldsymbol{\epsilon}(\mathbf{v}_1^*) & \text{in } \Omega \\ \mathbf{v}_1^* = 0 & \text{on } \Gamma_m \\ \boldsymbol{\sigma}_1^* \cdot \mathbf{n} = -\mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_1)]: \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} & \text{on } \Gamma_u \end{cases} \quad (30)$$

and:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}_2^* = \nabla \cdot (\mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_2)]: \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2)) & \text{in } \Omega \\ \boldsymbol{\sigma}_2^* = \mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_2)]: \boldsymbol{\epsilon}(\mathbf{v}_2^*) & \text{in } \Omega \\ \boldsymbol{\sigma}_2^* \cdot \mathbf{n} = \mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_2)]: \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} & \text{on } \Gamma_u \\ \boldsymbol{\sigma}_2^* \cdot \mathbf{n} = \mathcal{T}[\boldsymbol{\epsilon}(\mathbf{u}_2)]: \boldsymbol{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} + \boldsymbol{\sigma}_1^* \cdot \mathbf{n} - \mathbf{F}_m & \text{on } \Gamma_m \end{cases} \quad (31)$$

The fourth-order tensor \mathcal{T} is the second derivative of the hyperelastic potential φ or tangent stiffness tensor:

$$\mathcal{T}[\omega] = \frac{\partial^2 \varphi}{\partial \boldsymbol{\epsilon}^2}(\omega) \quad \text{or} \quad (T[\omega])_{ijkl} = \frac{\partial^2 \varphi}{\partial \epsilon_{ij} \partial \epsilon_{kl}}(\omega) \quad (32)$$

This result is classically obtained using the following Lagrangian:

$$\begin{aligned} L(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2; \boldsymbol{\eta}) &= \int_{\Omega} \mathcal{E}_{\text{ECE}}((\boldsymbol{\epsilon}(\mathbf{u}_1), \boldsymbol{\sigma}_1), (\boldsymbol{\epsilon}(\mathbf{u}_2), \boldsymbol{\sigma}_2)) d\Omega + \int_{\Omega} \left[\frac{\partial \varphi(\boldsymbol{\epsilon}(\mathbf{u}_1))}{\partial \boldsymbol{\epsilon}} : \boldsymbol{\epsilon}(\mathbf{v}_1) + \frac{\partial \varphi(\boldsymbol{\epsilon}(\mathbf{u}_2))}{\partial \boldsymbol{\epsilon}} : \boldsymbol{\epsilon}(\mathbf{v}_2) \right] d\Omega - \int_{\Gamma_m} \mathbf{F}_m \cdot \mathbf{v}_2 \, dS \\ &\quad - \int_{\Gamma_u} \boldsymbol{\eta} \cdot (\mathbf{v}_1 + \mathbf{v}_2) \, dS \end{aligned} \quad (33)$$

defined on the following space product where the functional regularity for the fields in the space $V(\Omega)$ of displacements on the domain Ω is not precisely specified, as it is beyond the scope of this paper:

$$\begin{aligned} (\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) &\in V_m \times V(\Omega) \times V_0 \times V(\Omega) \\ \text{with: } V_m &= \{\mathbf{u} \in V(\Omega), \mathbf{u}|_{\Gamma_m} = \mathbf{U}_m\} \\ V_0 &= \{\mathbf{u} \in V(\Omega), \mathbf{u}|_{\Gamma_m} = 0\} \end{aligned} \quad (34)$$

For linear elasticity, i.e., for quadratic hyperelastic potentials, the tensor \mathcal{T} is constant and reduces to the Hooke tensor. In this case, the adjoint problems defined by [Andrieux and Baranger \(2008a\)](#) for linear elasticity are recovered. They are far similar than in the truly nonlinear cases, especially because the source distribution term disappears and only boundary conditions involving the data and surface traction fields associated with \mathbf{u}_1 and \mathbf{u}_2 appear. Nevertheless, for the non-quadratic cases, the adjoint problems remain linearly elastic problems, with a space-dependent stiffness tensor.

The case of incompressible materials can be addressed in a very similar way ([Andrieux and Baranger, 2015](#)) and although the hyperelastic potential involves only the deviatoric part of the strain tensor, it remains possible to derive the same error form (19) involving the total strain and stress tensors.

Illustration: We illustrate the proposed approach for solving nonlinear Cauchy problems in hyperelasticity with the simple example of a pressurized hollow sphere with nonlinear compressibility, where because of the spherical symmetry, the Cauchy data reduce to the pair (external radial displacement, external pressure) and the unknown $\boldsymbol{\eta}$ reduces to the inner pressure. The homogeneous domain Ω , lies between two concentric spheres with internal radius a and external radius b . Due to symmetry, the stress vector and the prescribed displacement on the spherical boundaries have only radial and constant components in the spherical coordinate system centered on the common center of the two limiting spheres:

$$u(a, \varphi, \theta) = U_a \mathbf{e}_r, \quad u(b, \varphi, \theta) = U_b \mathbf{e}_r, \quad \boldsymbol{\sigma}(a, \varphi, \theta) \cdot \mathbf{n} = P_a \mathbf{e}_r, \quad \boldsymbol{\sigma}(b, \varphi, \theta) \cdot \mathbf{n} = P_b \mathbf{e}_r$$

where U_a, U_b, P_a and P_b are four scalars. The boundary Γ_m where the Cauchy data are given is the external boundary $r=b$, the Cauchy data reduce to the pair of scalars $(U_b, P_b) \equiv (\mathbf{U}_m, \mathbf{F}_m)$. The unknowns of the data completion on Γ_u or Cauchy problem also reduce to the pair (U_a, P_a) on the internal boundary $r=a$.

The material constituting the hollow sphere is a nonlinear isotropic material with regular (polynomial) potential. A general form of convex potentials for an isotropic material can be built using only the two first invariants I_1 and I_2 of the strain tensor $\boldsymbol{\epsilon}$ and a convex function G on \mathbb{R}^2 as follows:

$$\varphi(\boldsymbol{\epsilon}) = G(I_1(\boldsymbol{\epsilon}), I_2(\boldsymbol{\epsilon})), \quad I_1(\boldsymbol{\epsilon}) = \operatorname{tr} \boldsymbol{\epsilon}, \quad I_2(\boldsymbol{\epsilon}) = I_2(\boldsymbol{\epsilon}) = \frac{1}{2} \boldsymbol{\epsilon}^d : \boldsymbol{\epsilon}^d, \quad \boldsymbol{\epsilon}^d = \boldsymbol{\epsilon} - \frac{1}{3} I_1 \mathbb{1}, \quad \mathbb{1} \text{ is the identity tensor.}$$

Given that, unlike the third one, the first and the second invariant are convex functions of the strain tensor, the potential φ is necessarily convex when G has this property. Here, the function G is chosen as a polynomial convex function (φ is then lower semi-continuous):

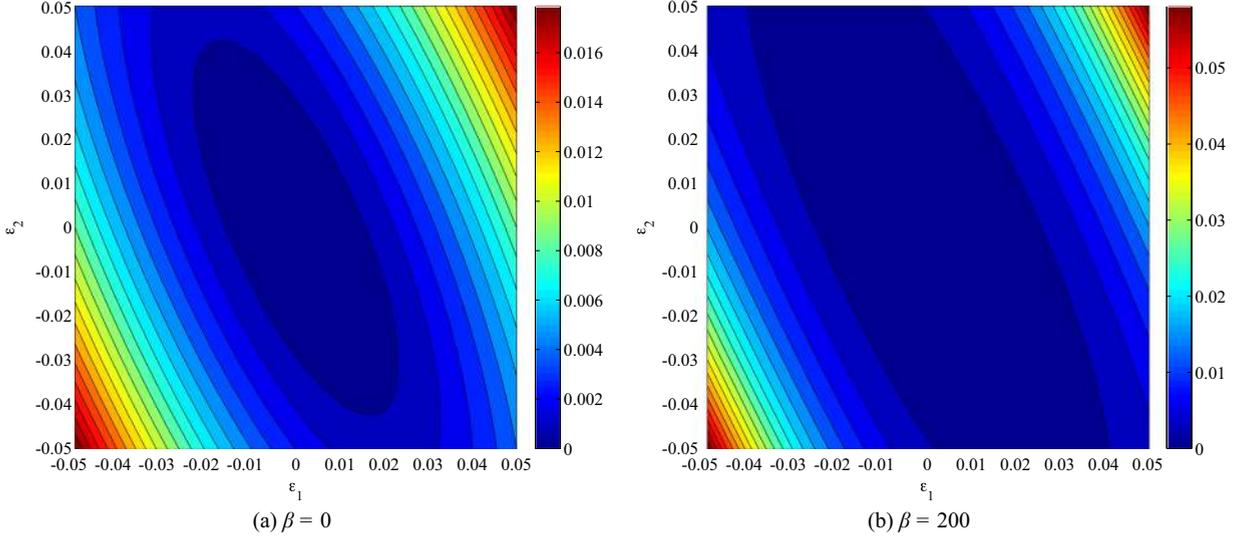


Fig. 2. Isolines of the hyperelastic potential φ for $\beta=0$ (left, linear elasticity) and $\beta=200$ (right).

$$G(X, Y) = \frac{1}{2}\kappa X^2 + \frac{1}{4}\beta\kappa X^4 + 2\mu Y, \quad \kappa, \mu, \beta > 0 \quad (35)$$

where $X = I_1(\epsilon)$ and $Y = I_2(\epsilon^d)$. The material constants κ and μ can be related to the compressibility and the shear or Coulomb modulus encountered in isotropic linear elasticity via the Young modulus E and Poisson ratio ν , $\mu = \frac{E}{2(1+\nu)}$ and $\kappa = \frac{E}{2(1-2\nu)(1+\nu)}$, whereas the non-dimensional coefficient β is a measure of the nonlinearity in the compressibility behavior of the material ($\beta = 0$ corresponds to a linear isotropic elastic material). Fig. 2 shows the level lines of the potential φ in the plane of principal strains in two dimensions ($\epsilon_1, \epsilon_2 = \epsilon_3$) for two values of this parameter ($\beta=0$ i.e. linear elasticity, and $\beta=200$). The stress-strain relation is derived by the chain rule and is written as:

$$\sigma = \kappa \operatorname{tr} \epsilon \mathbf{I} + \beta\kappa(\operatorname{tr} \epsilon)^3 \mathbf{I} + 2\mu \epsilon^d \quad (36)$$

The stress strain law is depicted in Fig. 3: (ϵ_1, σ_1) and (ϵ_1, σ_2) for $\beta=200$ and three values of the orthogonal strains $\epsilon_2 = \epsilon_3$. The general form of the function \mathcal{J}_{ECE} is

$$\mathcal{J}_{\text{ECE}}(p) = \int_{\Omega} (\sigma_1 - \sigma_2) : (\epsilon_1 - \epsilon_2) \quad (37)$$

by substituting σ_1 and σ_2 by their expression derived from (36), it can be written as function of the strains and of the parameters β, κ and μ :

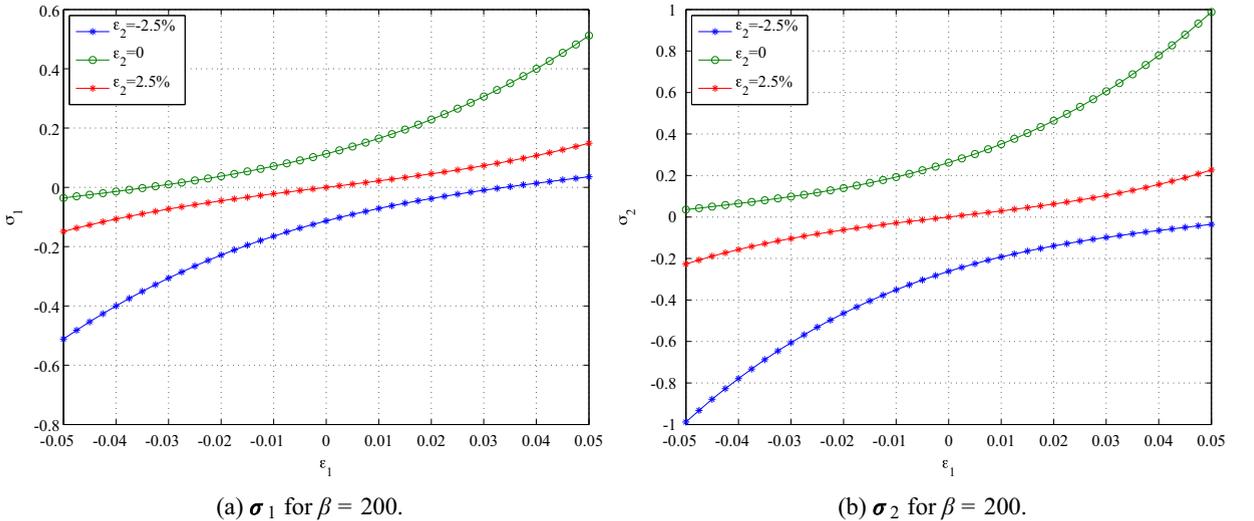


Fig. 3. Stress-strain curves for the hyperelastic material.

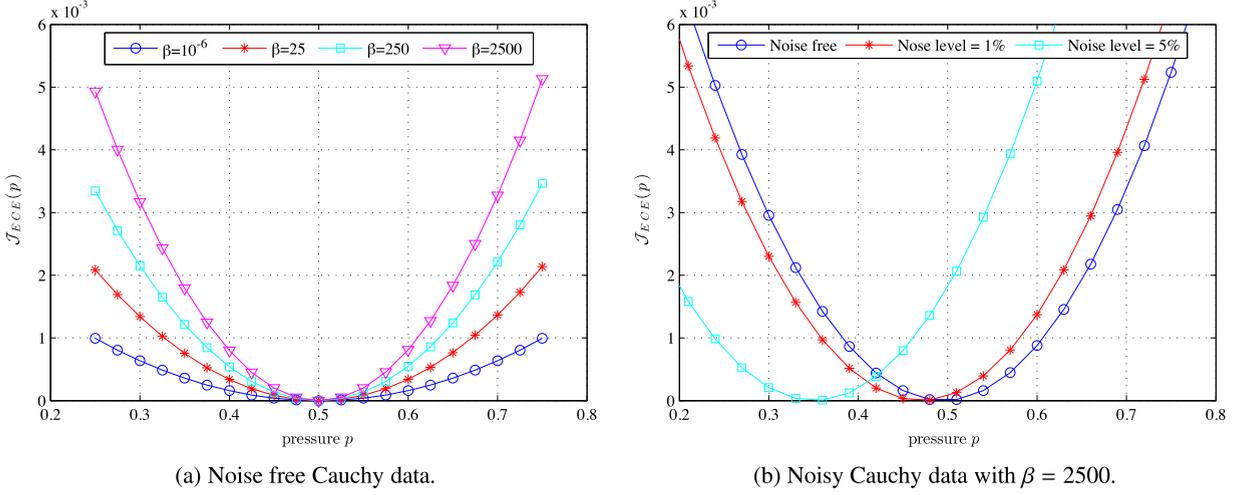


Fig. 4. \mathcal{J}_{ECE} for different values of β and noise levels.

$$\mathcal{J}_{ECE}(p) = \int_{\Omega} \kappa \operatorname{tr}(\epsilon_1 - \epsilon_2)^2 + 2\mu(\epsilon_1^d - \epsilon_2^d)^2 + \beta \kappa (\operatorname{tr}(\epsilon_1)^3 - \operatorname{tr}(\epsilon_2)^3) \operatorname{tr}(\epsilon_1 - \epsilon_2) \quad (38)$$

The energy error gap reduces to a function of a scalar variable (the internal pressure p) and has a very simple expression when using the expression (28) for the functional \mathcal{J}_{ECE} :

$$\mathcal{J}_{ECE}(p) = 4\pi b^2 (\sigma_1^{\text{tr}}(b; p) - P_b)(U_b - \mathbf{u}_2(b; p))$$

It can be seen in Fig. 4 that the error function $\mathcal{J}_{ECE}(p)$ remains convex, even for strong nonlinear behavior and flattens as the non-linearity decreases (with the decrease of the parameter β).

Another illustration with a bi-dimensional problem and a non-smooth (i.e., non-twice-differentiable) potential, modeling the nonsymmetric elasticity in the traction and compression ranges observed for cracked geomaterials, can be found in [Andrieux and Baranger \(2015\)](#).

4. The error gap in constitutive equations for Generalized Standard Elastoplastic materials

This part addresses another nonlinear problem in mechanics, the elastoplastic case, adopting the framework of Generalized Standard materials of [Halphen and Nguyen \(1975\)](#). More specifically, the general constitutive equation studied hereafter is obtained by involving the following ingredients:

- a set of state variables $(\epsilon, \epsilon^p, \alpha)$, where ϵ is the linearized strain tensor, ϵ_p is the (additive) plastic strain and α is a set of additional internal variables, possibly empty for perfect plasticity;
- a convex, lower semi-continuous, differentiable free (or stored internal) energy density: $W(\epsilon - \epsilon^p, \alpha)$;
- a convex, lower semi-continuous, positively 1-homogeneous dissipation pseudo-potential: $\Psi(\dot{\epsilon}^p, \dot{\alpha}; \epsilon^p, \alpha)$, that is $\Psi(\lambda \dot{\epsilon}^p, \lambda \dot{\alpha}; \epsilon^p, \alpha) = \lambda \Psi(\dot{\epsilon}^p, \dot{\alpha}; \epsilon^p, \alpha) \forall \lambda \geq 0$.

it is written as:

$$\begin{cases} \sigma = \frac{\partial W}{\partial \epsilon} = - \frac{\partial W}{\partial \epsilon^p}, & \mathbf{A} = - \frac{\partial W}{\partial \alpha} \\ \sigma \in \partial_{\epsilon^p} \Psi, & \mathbf{A} \in \partial_{\alpha} \Psi \end{cases} \quad (39)$$

where σ is the stress tensor and $\partial_x \Psi$ stands for the sub-differential of Ψ with respect to x . As Ψ is positively 1-homogeneous, (39) is equivalent to the normality rule for $(\dot{\epsilon}^p, \dot{\alpha})$ with respect to a convex yield function $f(\sigma, \mathbf{A}) \leq 0$. This framework encompasses a large amount of associated plasticity laws and ensures that the Clausius–Duhem inequality is fulfilled. In applications, the time interval is discretized into finite time increments and the incremental form of the preceding equation is chosen as the total implicit one. This choice maintains the existence of a global incremental convex variational form of the evolution equation (for positive hardening behavior), see [Mialon \(1986\)](#) and [Simo and Hughes \(1998\)](#). The implicit incremental form is then:

$$\begin{cases} \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p - \Delta\boldsymbol{\epsilon}^p, \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}), \\ \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} \in \partial_{\boldsymbol{\epsilon}^p} \Psi(\Delta\boldsymbol{\epsilon}^p, \Delta\boldsymbol{\alpha}; \boldsymbol{\epsilon}^p + \Delta\boldsymbol{\epsilon}^p, \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}) \\ \mathbf{A} + \Delta\mathbf{A} = -\frac{\partial W}{\partial \boldsymbol{\alpha}}(\boldsymbol{\epsilon} + \Delta\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p - \Delta\boldsymbol{\epsilon}^p, \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}), \\ \mathbf{A} + \Delta\mathbf{A} \in \partial_{\boldsymbol{\alpha}} \Psi(\Delta\boldsymbol{\epsilon}^p, \Delta\boldsymbol{\alpha}; \boldsymbol{\epsilon}^p + \Delta\boldsymbol{\epsilon}^p, \boldsymbol{\alpha} + \Delta\boldsymbol{\alpha}) \end{cases} \quad (40)$$

For the sake of simplicity, we will drop the arguments of the potentials. An Incremental Cauchy problem can be formulated as follows, *in order to determine* $(\Delta\mathbf{u}, \Delta\boldsymbol{\sigma}, \Delta\boldsymbol{\epsilon}^p, \Delta\boldsymbol{\alpha})$ *satisfying:*

$$\begin{cases} \operatorname{div}[\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}] = 0, & \boldsymbol{\epsilon}(\mathbf{u} + \Delta\mathbf{u}) = [\nabla(\mathbf{u} + \Delta\mathbf{u})]^{\text{sym}} \\ \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}, & \mathbf{A} + \Delta\mathbf{A} = -\frac{\partial W}{\partial \boldsymbol{\alpha}} \\ \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma} \in \partial_{\boldsymbol{\epsilon}^p} \Psi(\Delta\boldsymbol{\epsilon}^p, \Delta\boldsymbol{\alpha}), & \mathbf{A} + \Delta\mathbf{A} \in \partial_{\boldsymbol{\alpha}} \Psi(\Delta\boldsymbol{\epsilon}^p, \Delta\boldsymbol{\alpha}) \\ \Delta\mathbf{u} = \Delta\mathbf{U}_m, & \Delta\boldsymbol{\sigma} \cdot \mathbf{n} = \Delta\mathbf{F}_m \quad \text{on } \Gamma_m \end{cases} \quad (41)$$

As previously, the first step of the solution method is to split this problem into two well-posed incremental problems $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$:

$$\begin{cases} \operatorname{div}[\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_i] = 0, & \boldsymbol{\epsilon}(\mathbf{u} + \Delta\mathbf{u}_i) = [\nabla(\mathbf{u} + \Delta\mathbf{u}_i)]^{\text{sym}} \\ \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_i = \frac{\partial W}{\partial \boldsymbol{\epsilon}}, & \mathbf{A} + \Delta\mathbf{A}_i = -\frac{\partial W}{\partial \boldsymbol{\alpha}} \\ \boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_i \in \partial_{\boldsymbol{\epsilon}^p} \Psi(\Delta\boldsymbol{\epsilon}_i^p, \Delta\boldsymbol{\alpha}_i) \end{cases} \quad \text{for } i = 1, 2 \quad (42)$$

and respectively for $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$:

$$(\Delta\mathcal{P}_1) \begin{cases} \Delta\mathbf{u}_1 = \Delta\mathbf{U}_m & \text{on } \Gamma_m \\ \Delta\boldsymbol{\sigma}_1 \cdot \mathbf{n} = \Delta\boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad (\Delta\mathcal{P}_2) \begin{cases} \Delta\boldsymbol{\sigma}_2 \cdot \mathbf{n} = \Delta\mathbf{F}_m & \text{on } \Gamma_m \\ \Delta\boldsymbol{\sigma}_2 \cdot \mathbf{n} = \Delta\boldsymbol{\eta} & \text{on } \Gamma_u \end{cases} \quad (43)$$

The building of errors between the two states $(\Delta\mathbf{u}_1, \Delta\boldsymbol{\sigma}_1, \Delta\boldsymbol{\epsilon}_1^p, \Delta\boldsymbol{\alpha}_1)$ and $(\Delta\mathbf{u}_2, \Delta\boldsymbol{\sigma}_2, \Delta\boldsymbol{\epsilon}_2^p, \Delta\boldsymbol{\alpha}_2)$ is now based on the convexity property of the free energy and dissipation potential, by using the following lemma:

Lemma 2. *Let $f(x)$ be a real convex lower semi-continuous function on \mathbb{R}^n , then for any quadruplet (x_1, y_1, x_2, y_2) such that $y_1 \in \partial f(x_1)$, $y_2 \in \partial f(x_2)$, one has:*

- (i) $(y_1 - y_2) \cdot (x_1 - x_2) \geq 0$.
- (ii) $(y_1 - y_2) \cdot (x_1 - x_2) = 0 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$, if f and its convex conjugate f^* are truly nonlinear.

Proof. The proof is straightforward when using the Legendre–Fenchel inequality in [Ekeland and T  mam \(1999\)](#) involving the conjugate function f^* of the function f :

$$\begin{cases} f(x) + f^*(y) - x \cdot y \geq 0 & \forall x, y \\ f(x) + f^*(y) - x \cdot y = 0 & \Leftrightarrow y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y) \end{cases}$$

Indeed, the inequality (i) is obtained by summing the Fenchel inequalities for (x_1, y_2) and (x_2, y_1) and then by subtracting the Fenchel equalities for (x_1, y_1) and (x_2, y_2) . It is shown that (i) vanishes when

$$y_1 \in \partial f(x_2) \quad \text{and} \quad y_2 \in \partial f(x_1)$$

Let us take the first result and use the definition of the sub-differential at x_2 and at x_1 with y_1 :

$$\left. \begin{aligned} f(x) - f(x_2) &\geq y_1 \cdot (x - x_2) \\ f(x) - f(x_1) &\geq y_1 \cdot (x - x_1) \end{aligned} \right\} \Rightarrow f(x_1) - f(x_2) \geq y_1 \cdot (x_1 - x_2) \geq f(x_1) - f(x_2)$$

so that if f is truly nonlinear, one must have $x_2 = x_1$. By the same reasoning, interchanging the dual quantities x and y , and using the conjugate function f^* , one obtains $y_2 = y_1$. \square

Owing to the general form of the constitutive equation, we can then derive several errors with suitable properties. They are positive quantities and, whenever they vanish the distance between the two state variable increments vanishes together with the distance of their dual counterparts.

$$\begin{cases} \mathcal{E}_W(\Delta\boldsymbol{\sigma}_1, \Delta\boldsymbol{\epsilon}_1; \Delta\boldsymbol{\sigma}_2, \Delta\boldsymbol{\epsilon}_2) = (\Delta\boldsymbol{\sigma}_1 - \Delta\boldsymbol{\sigma}_2) : (\Delta\boldsymbol{\epsilon}_1^e - \Delta\boldsymbol{\epsilon}_2^e) - (\mathbf{A}_1 - \mathbf{A}_2) \cdot (\Delta\boldsymbol{\alpha}_1 - \Delta\boldsymbol{\alpha}_2) \\ \mathcal{E}_\Psi(\Delta\boldsymbol{\sigma}_1, \Delta\boldsymbol{\epsilon}_1^p; \Delta\boldsymbol{\sigma}_2, \Delta\boldsymbol{\epsilon}_2^p) = (\Delta\boldsymbol{\sigma}_1 - \Delta\boldsymbol{\sigma}_2) : (\Delta\boldsymbol{\epsilon}_1^p - \Delta\boldsymbol{\epsilon}_2^p) + (\mathbf{A}_1 - \mathbf{A}_2) \cdot (\Delta\boldsymbol{\alpha}_1 - \Delta\boldsymbol{\alpha}_2) \end{cases} \quad (44)$$

The first error is naturally called the error in free energy, whereas the second one is called the error in dissipation. They can be combined in order to define a general function via a parameter $0 \leq \chi \leq 1$, as shown in a different framework by [Hadj-Sassi \(2007\)](#) and [Hadj-Sassi and Andrieux \(2006\)](#),

$$\mathcal{E}_\chi = (1 - \chi)\mathcal{E}_W + \chi\mathcal{E}_\Psi \quad (45)$$

The previous errors (42) are recovered for extreme values of the parameter χ . The parameterization enables different weights to be placed on the errors in the stored energy and the dissipated one, but outstandingly the middle value of the parameter, $\chi = 1/2$, which balances the free energy error and dissipated one exactly, leads to what can be called the Drucker error ([Drucker, 1959](#)). It involves only the stress and strain tensors and is then an error in mechanical energy:

$$\mathcal{E}_{\frac{1}{2}} = \frac{1}{2}(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1 - \Delta\epsilon_2). \quad (46)$$

It is worth noting that, due to [Lemma 2](#), the extreme value $\chi=1$, which is the use of an error in dissipation only, must be discarded because the pseudo-potential of dissipation Ψ , as a 1-homogeneous potential, is not a truly nonlinear one.

Expression of the error \mathcal{E}_χ for perfect plasticity and isotropic hardening plasticity.

- **Perfect elastoplastic material:**

The free energy is $W(\epsilon, \epsilon^p) = \frac{1}{2}(\epsilon - \epsilon^p) : \mathbb{C} : (\epsilon - \epsilon^p)$, where $\epsilon - \epsilon^p = \epsilon^e$ is the elastic strain tensor and $\epsilon^p : I = 0$ and $\Psi(\dot{\epsilon}^p) = \sigma_0 \|\dot{\epsilon}^p\|$. σ_0 and \mathbb{C} denote the yielding stress and Hooke's tensor, respectively. The yield function is $f(\sigma) = \sigma_{\text{eq}} - \sigma_0$, where σ_{eq} is the Von Mises stress. Therefore, the error functional is:

$$\mathcal{E}_\chi = (1 - \chi)(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1^e - \Delta\epsilon_2^e) + \chi(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1^p - \Delta\epsilon_2^p) \quad (47)$$

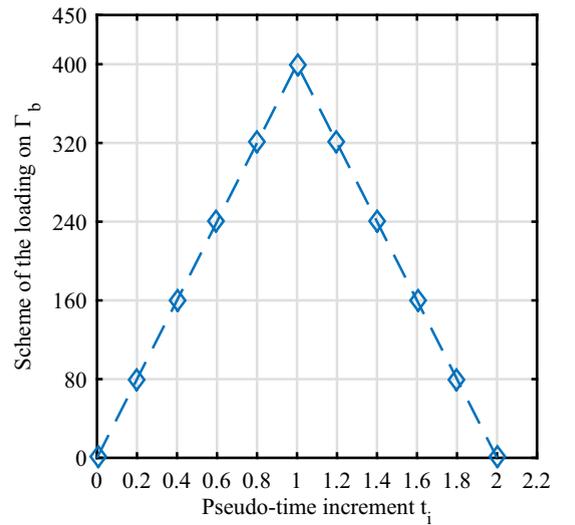
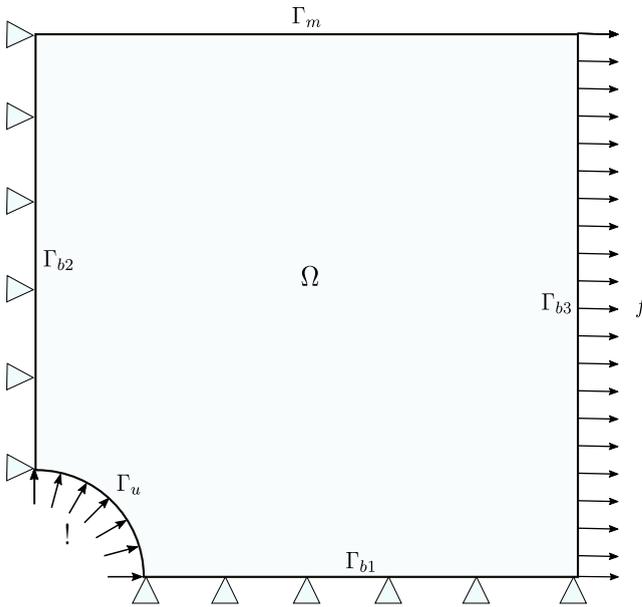
- **Elastoplastic material with isotropic hardening:**

The hardening function is defined by $R(\gamma) = -H\gamma + \sigma_0$, where γ is the accumulated plastic strain, so $\alpha = \gamma$ and $\mathbf{A} = -H\gamma$ and the free energy is defined by:

$$W(\epsilon, \epsilon^p, \alpha) = \frac{1}{2}(\epsilon - \epsilon^p) : \mathbb{C} : (\epsilon - \epsilon^p) + \int_0^\alpha (R(\beta) - \sigma_0) d\beta \quad (48)$$

and $\Psi(\dot{\epsilon}^p, \dot{\alpha}) = \sigma_0 \|\dot{\epsilon}^p\| + \mathbb{1}_{\{\dot{\alpha} = \sqrt{2/3} \|\dot{\epsilon}^p\|\}}$, where $\mathbb{1}_{\{\dot{\alpha} = \sqrt{2/3} \|\dot{\epsilon}^p\|\}}$ is the indicator function of the set $\{\dot{\alpha} = \sqrt{2/3} \|\dot{\epsilon}^p\|\}$ and the yield function is $f(\sigma, \mathbf{A}) = \sigma_{\text{eq}} + \mathbf{A} - \sigma_0$. The functional is then:

$$\begin{aligned} \mathcal{E}_\chi = & (1 - \chi)[(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1^e - \Delta\epsilon_2^e) - (R(\alpha + \Delta\alpha_1) - R(\alpha + \Delta\alpha_2)) \cdot (\Delta\alpha_1 - \Delta\alpha_2)] \\ & + \chi[(\Delta\sigma_1 - \Delta\sigma_2) : (\Delta\epsilon_1^p - \Delta\epsilon_2^p) + (R(\alpha + \Delta\alpha_1) - R(\alpha + \Delta\alpha_2)) \cdot (\Delta\alpha_1 - \Delta\alpha_2)] \end{aligned} \quad (49)$$



(a) Geometric and boundary details.

(b) Scheme of the loading f on Γ_b .

Fig. 5. Perforated plate and loading.

Equipped with this error in energy functional (47), which can be also called error in constitutive equations, we can then define the general error functional to be minimized in order to obtain the solution of the Data Completion Problem:

$$\mathcal{J}_x(\Delta\eta) = \int_{\Omega} \mathcal{E}_x(\Delta e_1(\Delta\eta), \Delta e_2(\Delta\eta)) d\Omega \quad (50)$$

The Drucker error $\mathcal{J}_{1/2}$ is the only one that can be computed by boundary integration over the entire external surface of the body (as it is the case in linear elasticity and hyperelasticity), using the virtual power principle. This feature has been widely used previously to improve the global performance of the solution algorithm for a linear Cauchy problem, see Baranger and Andrieux (2011).

Illustration: The proposed solution method has been implemented on the following two dimensional example under the assumption of plane strains. A perforated plate with unknown pressure applied on the hole boundary is loaded in its plane with a known distributed force f going from zero to a maximum value $f_{\max} = 400$ MPa on Γ_{b3} and unloaded back to zero, see Fig. 5. The plate is a square measuring 2×2 m² with a circular hole with radius $R=0.2$ m. The inner pressure is assumed to be symmetric, therefore only the quarter of the plate is studied as shown in Fig. 5a. The constitutive material of the plate is an elastoplastic material with linear isotropic hardening. It is characterized by the Young modulus $E=69\,000$ MPa, the Poisson ratio $\nu=0.33$, the yield stress $\sigma_0 = 500$ MPa and a strain-hardening modulus $H=0.2$ MPa.

The loading and unloading parts are divided into 5 increments. The displacements on the top and bottom parts of the boundary Γ_m (free of any loading) are available at various steps of the loading and unloading history. They are extracted from the solution of the forward problem, where the inner pressure is $p=100$ MPa. In order to avoid the inverse problem

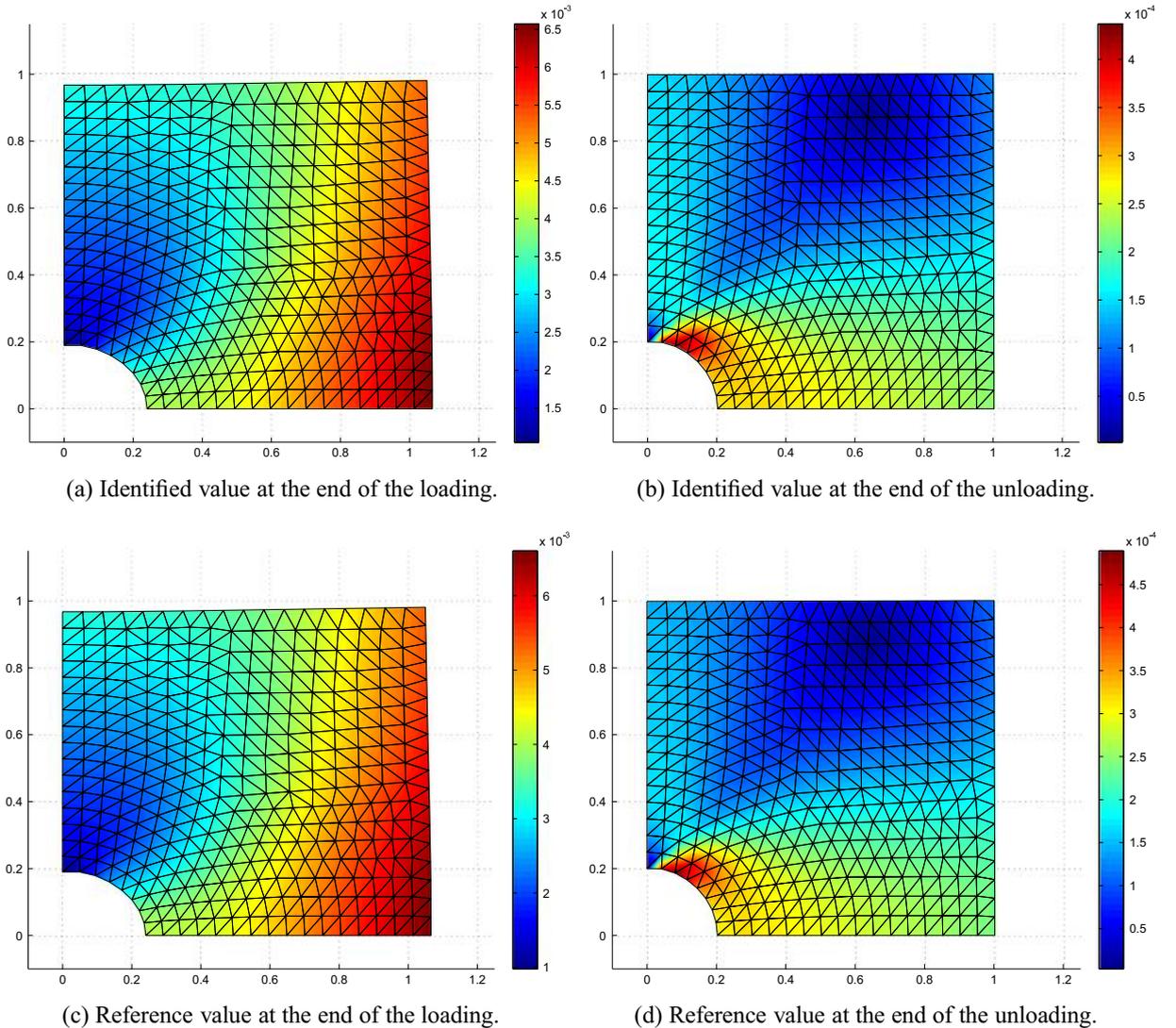


Fig. 6. Displacement modulus $\sqrt{u_x^2 + u_y^2}$.

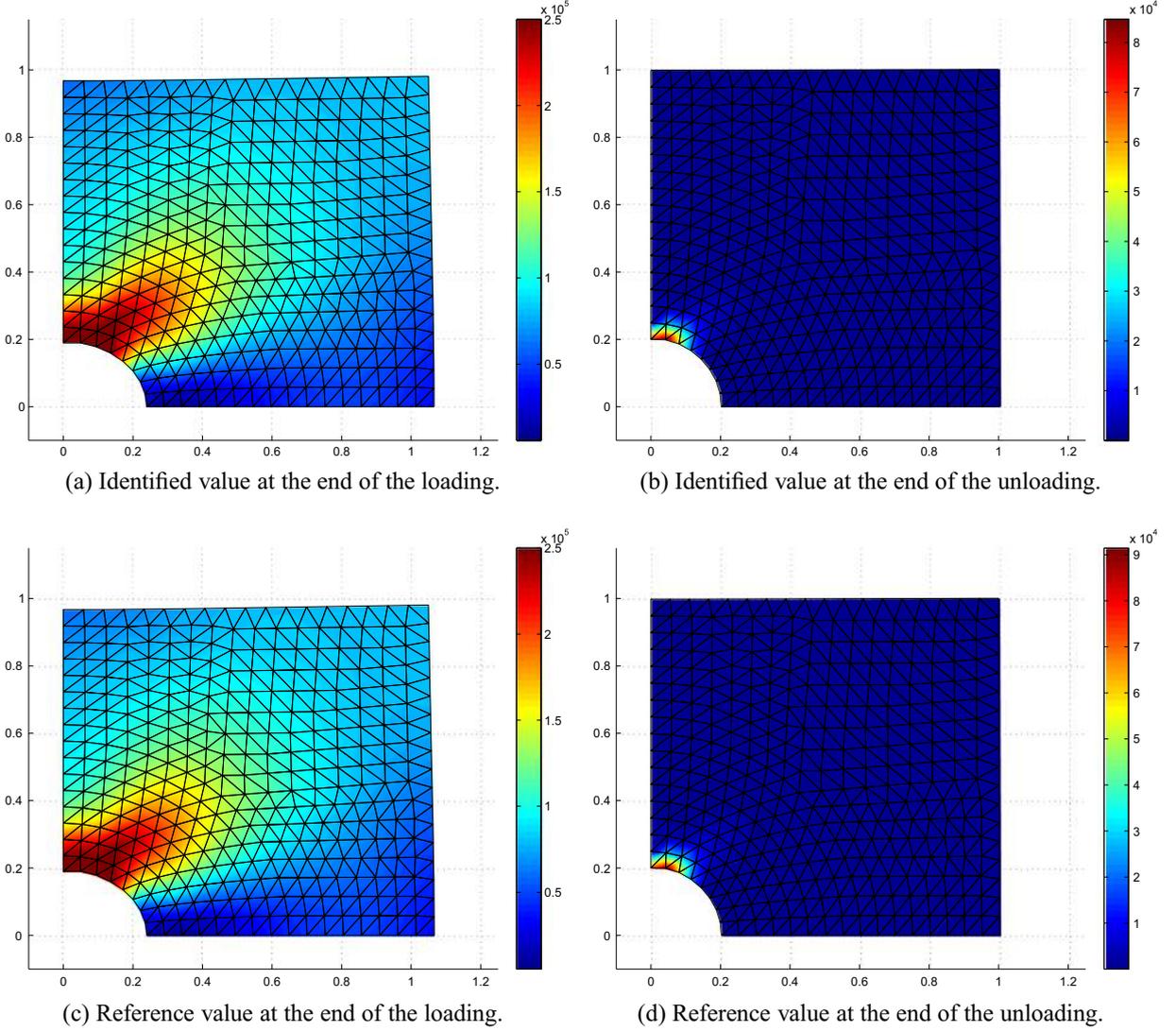


Fig. 7. Von Mises stress.

crime, the solution of this forward problem is obtained by the finite element method, with a different mesh than that used in the inverse problem. Consequently these data are slightly noisy.

For each step, the incremental Cauchy problem is solved by minimizing the Drucker error $\mathcal{J}_{1/2}$, starting from the mechanical state obtained at the end of the previous step. The two problems $\Delta\mathcal{P}_1$ and $\Delta\mathcal{P}_2$ are solved by the finite element method. The mesh data are 638 triangular elements, 358 nodes and 716 degrees of freedom. There are 18 unknowns nodal components on Γ_u corresponding to the discretized pressure p .

The gradient of the error discretized function $\mathcal{J}_{1/2}$ is computed by finite differences because the adjoint method cannot be directly applied here, since the dissipation pseudo-potential is not twice differentiable. The minimization was carried out by using the quasi-Newton algorithm of the Matlab optimization toolbox. The stopping criterion used at each increment is similar to that developed by [Rischette et al. \(2013, 2011\)](#). The average number of iterations for the minimization process varies from 100 to 200 iterations. In comparison to the alternating iterative method ([Kozlov et al., 1992](#); [Andrieux and Baranger, 2008a](#)), this number of iterations remains reasonable.

The results are illustrated in the following figures, where various identified mechanical fields within the solid are displayed. [Figs. 6a, b, c and d](#) show the exact and identified displacement moduli; [Figs. 7a, b, c and d](#) show the exact and identified Von Mises stresses and [Figs. 8a, b, c and d](#) show the exact and identified plastic strain tensor components at the end of the loading step and then at that of the unloading one. Comparing the identified data with the exact data, we observe good agreement for the reconstructed displacements, stresses and plastic strains, even “far away” from the boundary where the data are available.

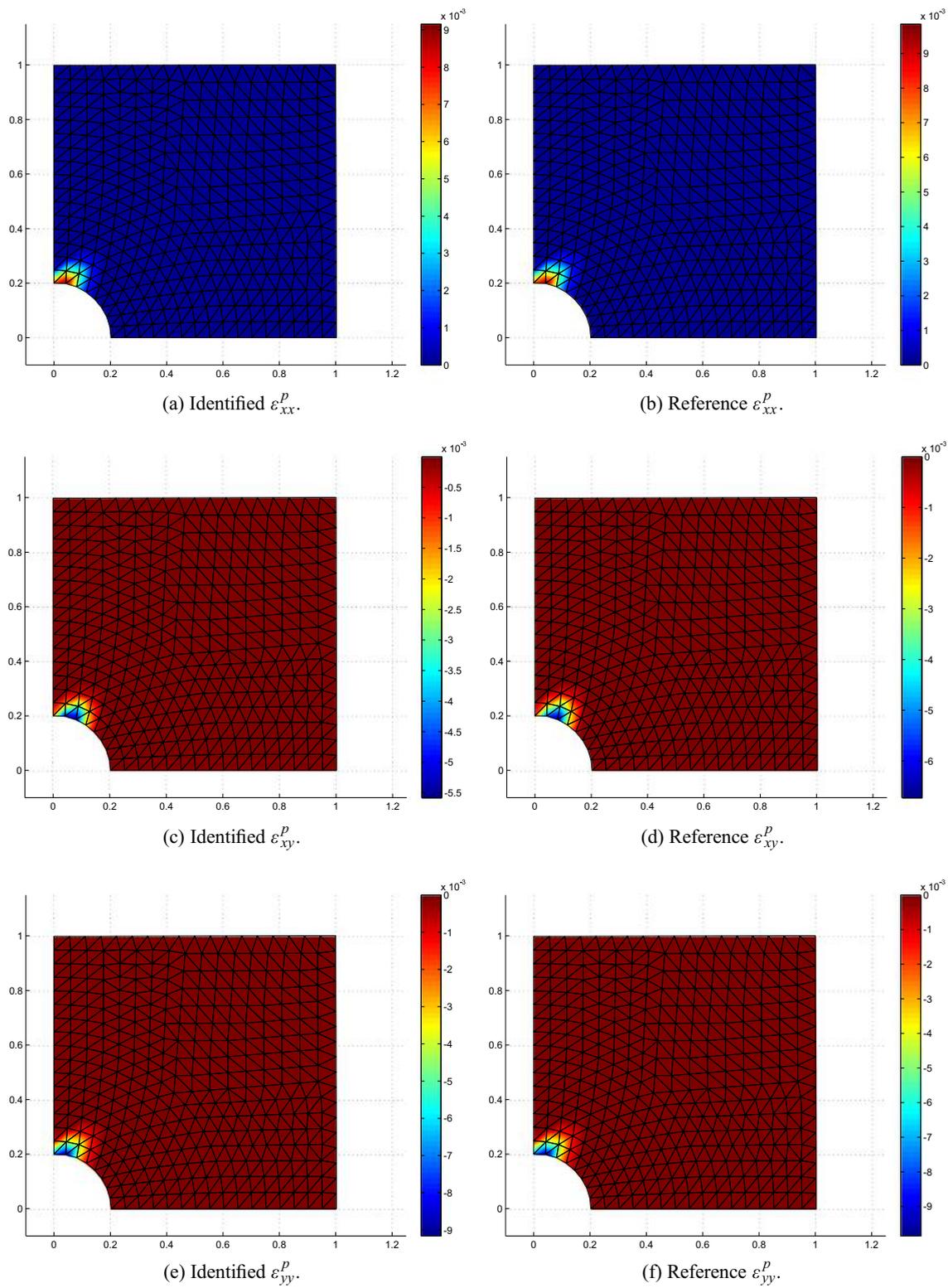


Fig. 8. Plastic strain components.

5. With regard to the issue of noisy measurements

Given that the Cauchy problem is known to be (severely) ill-posed (Ben Belgacem, 2007) and in the perspective of using measurements as input data in the identification process, attention must be paid to the effect of noise in the data on the results obtained with the proposed solution method, especially with regard to the reconstructed data on the boundary Γ_u . The generic situation encountered in mechanics is the following: the displacement field is estimated by digital image correlation techniques and the images are obtained on a stress free boundary part, so that the input data of the Cauchy problem are $(\mathbf{U}_m, \mathbf{F}_m) \equiv (\mathbf{U}_m^n, 0)$, where \mathbf{U}_m^n is a noisy measured version of the displacement field \mathbf{U}_m . It should first be noted that the positiveness of the error functional is not affected by noise in the data used, even though its minimum could no longer be zero because the compatibility of the data $(\mathbf{U}_m^n, 0)$ can be lost. In one-dimensional space, as in the illustration used for hyperelasticity, all of the data pairs are compatible so that the minimum of the error function remains zero, and the error function is just shifted and slightly deformed as illustrated in Fig. 4b, where the error in the constitutive equation function is plotted for various noise levels.

Detailed analyses of the effects of noise have been performed for linear Cauchy problems (elastostatics, stationary heat conduction Rischette et al., 2014, 2011) and have shown that a regularizing procedure consisting in stopping the minimization procedure with an ad hoc criterion related to the noise level, associated with a preliminary smoothing of the data by a Total Variation regularized projection are sufficient up to a noise level of some percents, details are given in Rischette et al. (2013) and Baranger et al. (2013).

6. Conclusion

In this paper, we derived a general method for using full-field surface displacement measurements, in order to obtain 3D information on mechanical quantities inside the solid, up to unreachable parts of its boundary. We focused on solids having nonlinear behavior, such as hyperelasticity or elastoplasticity. The method is based on the solution of a nonlinear Cauchy problem, solved by using a specially designed error in the constitutive equation between the solutions of two well-posed problems and minimizing it. This error is built by essentially using the Legendre–Fenchel inequality associated with the convexity properties of the potentials used for describing the constitutive equation of the material. The efficiency of the method now enables realistic 3D situations to be dealt with, as well as repeated expansions of surface fields, which are mandatory when addressing elastoplastic behavior with history and memory effects. It has been shown that the plastic zones, the plastic strain field and the residual stress field can be identified together with the missing boundary data. Thus, various identification problems can be addressed in the future with the solution of the Cauchy problem as a first step or as an ingredient.

Some improvements must be made in the computation of the gradient of the error functional for the case of non-twice-differentiable potentials, for which the general adjoint method cannot be directly applied. It is generically the case in elastoplasticity, for the pseudo-potential of dissipation, and can also appear in hyperelasticity, for example for potentials modeling the nonsymmetric elasticity in the traction and compression ranges observed for cracked geomaterials.

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