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► **To cite this version:**

Jean-Louis Krivine. On the Structure of Classical Realizability Models of ZF. 20th International Conference on Types for Proofs and Programs (TYPES 2014), May 2014, Paris, France. hal-01737111

**HAL Id: hal-01737111**

**<https://hal.science/hal-01737111>**

Submitted on 19 Mar 2018

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# On the structure of classical realizability models of ZF

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March 19, 2018

**Abstract.** In [4, 5, 6], we have introduced the technique of *classical realizability*, which permits to extend the Curry-Howard correspondence between proofs and programs, to Zermelo-Fraenkel set theory. The models of ZF we obtain in this way, are called *realizability models* ; this technique is an extension of the method of forcing, in which the ordered sets (sets of *conditions*) are replaced with more complex first order structures called *realizability algebras*.

We show here that every realizability model  $\mathcal{N}$  of ZF contains a transitive submodel, which has the same ordinals as  $\mathcal{N}$ , and which is an elementary extension of the ground model. It follows that the constructible universe of a realizability model is an elementary extension of the constructible universe of the ground model.

We obtain this result by showing the existence of an ultrafilter on the *characteristic Boolean algebra*  $\mathfrak{B}$  of the realizability model, which is defined in [5, 6].

## Introduction

We use here the basic notions and notations of the theory of *classical realizability*, which was developed in [4, 5, 6].

We consider a model  $\mathcal{M}$  of  $\text{ZF} + \text{V} = \text{L}$ , which we call the *ground model*<sup>1</sup> and, in  $\mathcal{M}$ , a *realizability algebra*  $\mathcal{A} = (\Lambda, \Pi, \Lambda \star \Pi, \text{QP}, \perp\!\!\!\perp)$ .

$\Lambda$  is the set of *terms*,  $\Pi$  is the set of *stacks*,  $\Lambda \star \Pi$  is the set of *processes*,  $\text{QP} \subset \Lambda$  is the set of *proof-like terms*, and  $\perp\!\!\!\perp$  is a distinguished subset of  $\Lambda \star \Pi$ .

They satisfy the axioms of *realizability algebra*, which are given in [4] or [6].

In the model  $\mathcal{M}$ , we use the language of ZF with the binary relation symbols  $\notin, \subset$  and function symbols, which we shall define when needed, by means of formulas of ZF.

We can now build (see [4]) the *realizability model*  $\mathcal{N}$ , which has the same set of individuals as  $\mathcal{M}$ , the truth value set of which is  $\mathcal{P}(\Pi)$ , endowed with a suitable Boolean algebra structure.

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<sup>1</sup>In fact, it suffices that  $\mathcal{M}$  satisfy the *choice principle CP*, which is written as follows, in the language of ZF with a new binary relation symbol  $\triangleleft$ : “ $\triangleleft$  is a well ordering relation on  $\mathcal{M}$ ”.

It is well known that, in every countable model of ZFC, we can define such a binary symbol, so as to get a model of  $\text{ZF} + \text{CP}$ . Thus,  $\text{ZF} + \text{CP}$  is a *conservative extension* of ZFC.

The language of this model has three binary relation symbols  $\notin, \in, \subset$ , and the same function symbols as the model  $\mathcal{M}$ , with the same interpretation.

The *formulas* are built as usual, from atomic formulas, *with the only logical symbols*  $\perp, \rightarrow, \forall$ . We shall use the notations :

$\neg F$  for  $F \rightarrow \perp$  ;  $F_1, \dots, F_n \rightarrow F$  for  $(F_1 \rightarrow (\dots \rightarrow F_n)) \rightarrow F$  ;  
 $\exists x F$  for  $\neg \forall x \neg F$  ;  $\exists x \{F_1, \dots, F_n\}$  for  $\neg \forall x (F_1, \dots, F_n \rightarrow \perp)$ .

**Notation.** We shall often use the notation  $\vec{x}$  for a finite sequence  $x_1, \dots, x_n$  ; for instance, we shall write  $F[\vec{x}]$  for  $F[x_1, \dots, x_n]$ .

By means of the completeness theorem, we obtain from  $\mathcal{N}$  an ordinary model  $\mathcal{N}'$ , with truth values in  $\{0, 1\}$ . The set of individuals of  $\mathcal{N}'$  generally *strictly contains*  $\mathcal{N}$ .

The elements of  $\mathcal{N}'$  (resp.  $\mathcal{M}$ ) are called *individuals of  $\mathcal{N}$*  (resp.  $\mathcal{M}$ ). The individuals are generally denoted by  $a, b, c, \dots, a_0, a_1, \dots$ .

In [4] or [5], we define a theory  $ZF_\varepsilon$ , written in this language. We show that it is a *conservative extension* of ZF, and that the model  $\mathcal{N}$  *satisfies the axioms of  $ZF_\varepsilon$* , which means that each of these axioms is *realized by a proof-like term*.

Given a term  $\xi \in \Lambda$  and a closed formula  $F[a_1, \dots, a_n]$  in the language of  $ZF_\varepsilon$ , with parameters  $a_1, \dots, a_n$  in  $\mathcal{N}$  (or, which is the same, in  $\mathcal{M}$ ), we shall write :

$\xi \Vdash F[a_1, \dots, a_n]$  in order to say that the term  $\xi$  realizes  $F[a_1, \dots, a_n]$ .

The truth value of this formula is a subset of  $\Pi$ , denoted by  $\|F[a_1, \dots, a_n]\|$ .

We write  $\Vdash F[a_1, \dots, a_n]$  in order to say that  $F[a_1, \dots, a_n]$  is realized by some proof-like term.

Thus, the model  $\mathcal{N}'$  satisfies  $ZF_\varepsilon$  ; therefore, in  $\mathcal{N}'$ , we can define a model of ZF, denoted  $\mathcal{N}'_\varepsilon$ , in which the equality is interpreted by *the extensional equivalence* denoted by  $x \simeq y$  (that is  $x \subset y \wedge y \subset x$ ).

The general properties of the realizability models are described in [6] ; we shall use the definitions and notations of this paper.

In what follows, unless otherwise stated, each formula of  $ZF_\varepsilon$  must be interpreted in  $\mathcal{N}$  (its truth value is a subset of  $\Pi$ ) or, if one prefers, in  $\mathcal{N}'$  (then its truth value is 0 or 1). If the formula must be interpreted in  $\mathcal{M}$ , (in that case, it does not contains the symbol  $\notin$ ) it will be explicitly stated.

## Function symbols

**Notations.** The formula  $\forall z (z \notin y \rightarrow z \notin x)$  is denoted by  $x \subseteq y$  (*strong inclusion*) ; the formula  $x \subseteq y \wedge y \subseteq x$  is denoted by  $x \cong y$  (*strong extensional equivalence*).

We recall that  $\subset$  and  $\simeq$  are the symbols of inclusion and of extensional equivalence of ZF :  $x \subset y \equiv \forall z (z \notin y \rightarrow z \notin x)$  ;  $x \simeq y \equiv (x \subset y \wedge y \subset x)$ .

## Function symbols associated with axioms of $ZF_\varepsilon$

In this section, we define a function symbol for each of the following axioms of  $ZF_\varepsilon$  : comprehension, pairing, union, power set and collection.

### Comprehension.

For each formula  $F[y, \vec{z}]$  of  $ZF_\varepsilon$ , (where  $\vec{z}$  is a finite sequence of variables  $z_1, \dots, z_n$ ) we define, in  $\mathcal{M}$ , a symbol of function of arity  $n + 1$ , denoted provisionally by  $\text{Compr}_F(x, \vec{z})$ , ( $\text{Compr}$  is an abbreviation for *Comprehension*) by setting :

$$\text{Compr}_F(a, \vec{c}) = \{(b, \xi \cdot \pi) ; (b, \pi) \in a, \xi \Vdash F[b, \vec{c}]\}.$$

It was shown in [6] (and it is easily checked) that we have :

$$\|b \notin \text{Compr}_F(a, \vec{c})\| = \|F[b, \vec{c}] \rightarrow b \notin a\|. \text{ Thus, we have :}$$

$$\text{I } \Vdash \forall x \forall y \forall \vec{z} (y \notin \text{Compr}_F(x, \vec{z}) \rightarrow (F[y, \vec{z}] \rightarrow y \notin x)) ;$$

$$\text{I } \Vdash \forall x \forall y \forall \vec{z} ((F[y, \vec{z}] \rightarrow y \notin x) \rightarrow y \notin \text{Compr}_F(x, \vec{z})).$$

Therefore, instead of  $\text{Compr}_F(x, \vec{z})$ , we shall use for this function symbol, the more intuitive notation  $\{y \varepsilon x ; F[y, \vec{z}]\}$ , in which  $y$  is a *bounded variable*.

### Pairing.

We define the following binary function symbol :

$$\text{pair}(x, y) = \{z \varepsilon \{x, y\} \times \Pi ; (z = x) \vee (z = y)\}.$$

It is easily checked that we have the desired property :

$$\Vdash \forall x \forall y \forall z (z \varepsilon \text{pair}(x, y) \leftrightarrow z = x \vee z = y).$$

**Remark.** We could also define a symbol  $\text{pair}(x, y)$ , with this property, directly in  $\mathcal{M}$ , as follows :

$$\text{pair}(x, y) = \{(x, \underline{1} \cdot \pi) ; \pi \in \Pi\} \cup \{(y, \underline{0} \cdot \pi) ; \pi \in \Pi\}.$$

In the sequel, when working in  $\mathcal{N}$ , we shall use the (natural) abbreviations :

$\{x, y\}$  for  $\text{pair}(x, y)$  ;  $(x, y)$  for  $\text{pair}(\text{pair}(x, x), \text{pair}(x, y))$ .

### Union and power set.

We define below two unary function symbols  $\overline{\cup}x$  and  $\overline{\mathcal{P}}(x)$ , such that :

$$\Vdash \forall x \forall z (z \varepsilon \overline{\cup}x \leftrightarrow (\exists y \varepsilon x) z \varepsilon y).$$

$$\Vdash \forall x (\forall y \varepsilon \overline{\mathcal{P}}(x)) (\forall z \varepsilon y) (z \varepsilon x) ; \quad \Vdash \forall x \forall y (\exists y' \varepsilon \overline{\mathcal{P}}(x)) \forall z (z \varepsilon y' \leftrightarrow z \varepsilon x \wedge z \varepsilon y).$$

**Theorem 1.** *Let  $\mathcal{V}, \mathcal{Q}$  be the unary function symbols defined in  $\mathcal{M}$  as follows :*

$$\mathcal{V}(a) = \text{Cl}(a) \times \Pi \quad \text{and} \quad \mathcal{Q}(a) = \mathcal{P}(\text{Cl}(a) \times \Pi) \times \Pi$$

where  $\text{Cl}(a)$  is the transitive closure of  $a$ . Then, we have :

$$\text{i) } \text{I } \Vdash \forall x \forall y \forall z (z \varepsilon y, z \notin \mathcal{V}(x) \rightarrow y \notin x).$$

$$\text{ii) } \text{I } \Vdash \forall x \forall \vec{z} (\{y \varepsilon x ; F[y, \vec{z}]\} \varepsilon \mathcal{Q}(x)) \quad \text{for every formula } F[x, \vec{z}] \text{ of } ZF_\varepsilon.$$

i) Let  $a, b, c$  be individuals in  $\mathcal{M}$ ,  $\xi, \eta \in \Lambda$  and  $\pi \in \Pi$  such that :

$$\xi \Vdash c \varepsilon b, \eta \Vdash c \notin \mathcal{V}(a) \text{ and } \pi \in \|b \notin a\| ; \text{ we have therefore } (b, \pi) \in a.$$

We must show  $\xi \star \eta \cdot \pi \in \perp$ .

We show that  $\|c \notin b\| \subset \|c \notin \mathcal{V}(a)\|$  : indeed, if  $\rho \in \|c \notin b\|$ , then we have  $(c, \rho) \in b$ . But we have  $(b, \pi) \in a$  and thus  $c \in \text{Cl}(a)$  and it follows that  $\|c \notin \mathcal{V}(a)\| = \Pi$ .

Therefore,  $\eta \Vdash c \notin b$  ; by hypothesis on  $\xi$ , we have  $\xi \star \eta \cdot \pi \in \perp$ .

ii) Let  $a, \vec{c}$  be individuals in  $\mathcal{M}$  ; we must show  $\text{I } \Vdash A \varepsilon \mathcal{Q}(a)$ , where  $A = \{y \varepsilon a ; F[y, \vec{c}]\}$ .

We have  $A = \{(b, \xi \cdot \pi) ; (b, \pi) \in a, \xi \Vdash F[b, \vec{c}]\}$  and therefore  $A \subset \text{Cl}(a) \times \Pi$ . But we have :

$$\|A \notin \mathcal{Q}(a)\| = \{\pi \in \Pi ; (A, \pi) \in \mathcal{Q}(a)\} = \Pi \quad \text{and therefore } \text{I } \Vdash A \varepsilon \mathcal{Q}(a).$$

Q.E.D.

We can now define the function symbols  $\overline{\cup}$  and  $\overline{\mathcal{P}}$  by setting :

$$\overline{\cup}x = \{z \varepsilon \mathcal{V}(x) ; (\exists y \varepsilon x) z \varepsilon y\} ; \quad \overline{\mathcal{P}}(x) = \{y \varepsilon \mathcal{Q}(x) ; y \subseteq x\}.$$

**Collection.**

We shall use in the following, function symbols associated with a strong form of the *collection scheme*.

In order to define these function symbols, it is convenient to decompose them, which is done in theorems 2, 3 and 4.

**Theorem 2.** *For each formula  $F(x, \vec{z})$  of  $ZF_\varepsilon$ , we have :*

$$\Vdash \forall \vec{z} (\exists x F(x, \vec{z}) \rightarrow (\exists x \varepsilon \phi_F(\vec{z})) F(x, \vec{z})) ; \quad \Vdash \forall \vec{z} (\forall x \varepsilon \phi_F(\vec{z})) F(x, \vec{z})$$

where  $\phi_F$  is a function symbol defined in  $\mathcal{M}$ .

We show  $\lambda x(x) \Vdash \forall x (x \varepsilon \Phi_F(\vec{z}) \rightarrow F(x, \vec{z})) \rightarrow \forall x F(x, \vec{z})$  where the function symbol  $\Phi_F$  is defined as follows :

By means of the collection scheme in  $\mathcal{M}$ , we define a function symbol  $\Psi(\vec{z})$  such that :

$$\|\forall x F(x, \vec{z})\| = \bigcup_{x \in \Psi(\vec{z})} \|F(x, \vec{z})\| \quad \text{and we set} \quad \Phi_F(\vec{z}) = \Psi(\vec{z}) \times \Pi.$$

Let  $\xi \Vdash \forall x (x \varepsilon \Phi_F(\vec{z}) \rightarrow F(x, \vec{z}))$  and  $\pi \in \|\forall x F(x, \vec{z})\|$ .

Then  $\pi \in \|F(x, \vec{z})\|$  for some  $x \in \Psi(\vec{z})$ , and therefore  $\Vdash x \varepsilon \Phi_F(\vec{z})$  and  $\xi \star \Vdash \pi \in \perp$ .

Therefore, by replacing  $F$  with  $\neg F$ , we have  $\Vdash \exists x F(x, \vec{z}) \rightarrow (\exists x \varepsilon \Phi_{\neg F}(\vec{z})) F(x, \vec{z})$ .

Thus, we only need to set  $\phi_F(\vec{z}) = \{x \varepsilon \Phi_{\neg F}(\vec{z}) ; F(x, \vec{z})\}$ .

Q.E.D.

**Theorem 3.** *For every formula  $F(y, \vec{z})$  of  $ZF_\varepsilon$ , we have :*

$$\Vdash \forall \vec{z} (\exists x \forall y (F(y, \vec{z}) \rightarrow y \varepsilon x) \rightarrow \forall y (F(y, \vec{z}) \leftrightarrow y \varepsilon \gamma_F(\vec{z})))$$

where  $\gamma_F$  is a function symbol defined in  $\mathcal{M}$ .

By theorem 2, we have :

$$\Vdash \forall \vec{z} (\exists x \forall y (F(y, \vec{z}) \rightarrow y \varepsilon x) \rightarrow (\exists x \varepsilon \phi(\vec{z})) \forall y (F(y, \vec{z}) \rightarrow y \varepsilon x))$$

where  $\phi$  is a function symbol. Therefore we have, by definition of  $\overline{\bigcup} \phi(\vec{z})$  :

$$\Vdash \forall \vec{z} \left( \exists x \forall y (F(y, \vec{z}) \rightarrow y \varepsilon x) \rightarrow \forall y (F(y, \vec{z}) \rightarrow y \varepsilon \overline{\bigcup} \phi(\vec{z})) \right).$$

Now, we only need to set  $\gamma_F(\vec{z}) = \{y \varepsilon \overline{\bigcup} \phi(\vec{z}) ; F(y, \vec{z})\}$  (comprehension scheme).

Q.E.D.

When the hypothesis  $\exists x \forall y (F(y, \vec{z}) \rightarrow y \varepsilon x)$  is satisfied, we say that *the formula  $F(y, \vec{z})$  defines a set*.

For the function symbol  $\gamma_F(\vec{z})$ , we shall use the more intuitive notation  $\{y ; F(y, \vec{z})\}$ , where  $y$  is a bounded variable.

**Theorem 4.**

*Let  $f(x, \vec{z})$  be a  $(n+1)$ -ary function symbol (defined in  $\mathcal{M}$ ). Then, we have :*

$$\Vdash \forall a \forall y \forall \vec{z} (y \varepsilon \phi_f(a, \vec{z}) \leftrightarrow (\exists x \varepsilon a) (y = f(x, \vec{z})))$$

where  $\phi_f$  is a  $(n+1)$ -ary function symbol.

We define, in  $\mathcal{M}$ , the symbol  $\phi_f$  as follows :

Let  $a_0, y_0, \vec{z}_0$  be fixed individuals in  $\mathcal{M}$  ; we set  $\phi_f(a_0, \vec{z}_0) = \{(f(x, \vec{z}_0), \pi) ; (x, \pi) \in a_0\}$ .

Then, we have immediately  $\|y_0 \notin \phi_f(a_0, \vec{z}_0)\| = \|\forall x (y_0 = f(x, \vec{z}_0) \leftrightarrow x \notin a_0)\|$ . Therefore :

$\Vdash \forall x (y_0 = f(x, \vec{z}_0) \leftrightarrow x \notin a_0) \leftrightarrow y_0 \notin \phi_f(a_0, \vec{z}_0)$  which gives the desired result.

Q.E.D.

**Remark.** The *connective*  $\leftrightarrow$  is defined in [5, 6].

For the function symbol  $\phi_f(a, \vec{z})$ , we shall use the more intuitive notation  $\{f(x, \vec{z}) ; x \varepsilon a\}$ , where  $x$  is a bounded variable. We call it *image of a by the function*  $f(x)$ .

## Symbols for characteristic functions

Let  $R(x_1, \dots, x_n)$  be a  $n$ -ary relation defined in  $\mathcal{M}$ . Its *characteristic function*, with values in  $\{0, 1\}$ , will be denoted by  $\langle R(x_1, \dots, x_n) \rangle$ . Therefore, we have :  $\mathcal{M} \models \forall \vec{x} (R(\vec{x}) \leftrightarrow \langle R(\vec{x}) \rangle = 1)$ .

Therefore, in the realizability model  $\mathcal{N}$ , the function symbol  $\langle R(\vec{x}) \rangle$  takes its values in  $\mathbb{I}2$ . The theorem 8 below, shows that, if a binary relation  $y \prec x$  is well founded in  $\mathcal{M}$ , then the relation  $\langle y \prec x \rangle = 1$  is well founded in  $\mathcal{N}$ .

## Miscellaneous symbols

In the following, we shall use some function symbols, the definition and properties of which are given in [6]. We simply recall their definition below.

- The unary function symbol  $\mathbb{I}$ , defined in  $\mathcal{M}$  by  $\mathbb{I}x = x \times \Pi$ .  
For any individual  $E$  of  $\mathcal{M}$ , the *restricted quantifier*  $\forall x^{\mathbb{I}E}$  is defined in [5] or [6] by :  $\|\forall x^{\mathbb{I}E} F[x]\| = \bigcup_{x \in E} \|F[x]\|$  and we have  $\Vdash \forall x^{\mathbb{I}E} F[x] \leftrightarrow \forall x (x \varepsilon \mathbb{I}E \rightarrow F[x])$ .  
In the realizability model  $\mathcal{N}$ , the formula  $x \varepsilon \mathbb{I}E$  may be intuitively understood as “ $x$  is of type  $E$ ”. For instance,  $\mathbb{I}2$  may be considered as *the type of booleans* and  $\mathbb{I}\mathbb{N}$  as *the type of integers*.
- the function symbols  $\wedge, \vee, \neg$ , with domains  $\{0, 1\} \times \{0, 1\}$  and  $\{0, 1\}$ , and values in  $\{0, 1\}$ , are defined in  $\mathcal{M}$  by means of the usual truth tables.  
These functions define, in  $\mathcal{N}$ , a structure of Boolean algebra on  $\mathbb{I}2$ .  
We call it the *characteristic Boolean algebra* of the realizability model  $\mathcal{N}$ .
- a binary function symbol with domain  $\{0, 1\} \times \mathcal{M}$ , denoted by  $(\alpha, x) \mapsto \alpha x$ , by setting :  
$$0x = \emptyset ; 1x = x.$$
  
In the model  $\mathcal{N}$ , the domain of this function is  $\mathbb{I}2 \times \mathcal{N}$ .
- a binary function symbol  $\sqcup$  with domain  $\mathcal{M} \times \mathcal{M}$ , by setting  $x \sqcup y = x \cup y$ .  
**Remark.** The extension of this function to the model  $\mathcal{N}$  is *not* the union  $\cup$ , which explains the use of another symbol.

**Lemma 5** (Linearity).

Let  $f$  be a binary function symbol, defined in  $\mathcal{M}$ . Then, we have :

i)  $\Vdash \forall \alpha^{\mathbb{I}2} \forall x \forall y (\alpha f(x, y) = \alpha f(\alpha x, y))$ .

ii) Moreover, if  $f(\emptyset, \emptyset) = \emptyset$ , then :

$\Vdash \forall \alpha^{\mathbb{I}2} \forall \alpha'^{\mathbb{I}2} \forall x \forall y \forall x' \forall y' (\alpha \wedge \alpha' = 0 \leftrightarrow f(\alpha x \sqcup \alpha' x', \alpha y \sqcup \alpha' y') = \alpha f(x, y) \sqcup \alpha' f(x', y'))$ .

**Remark.** The *connective*  $\leftrightarrow$  is defined in [5, 6].

It suffices to check :

for (i) the two cases  $\alpha = 0, 1$  ;

for (ii) the three cases  $(\alpha, \alpha') = (0, 0), (0, 1), (1, 0)$  ;

which is trivial.

Q.E.D.

## Well founded relations

In this section, we study properties of well founded relations in  $\mathcal{N}$ . All the results obtained here are, of course, trivial in ZF. The difficulties come from the fact that the relation  $\varepsilon$  of strong membership, does not satisfy extensionality.

Given a binary relation  $\prec$ , an individual  $a$  is said *minimal for*  $\prec$ , if we have  $\forall x \neg(x \prec a)$ . The binary relation  $\prec$  is called *well founded* if we have :

$$\forall X (\forall x (\forall y (y \prec x \rightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X)).$$

The intuitive meaning is that each non empty individual  $X$  has an  $\varepsilon$ -element minimal for  $\prec$ .

### Theorem 6.

If the relation  $x \prec y$  is well founded then, for every formula  $F[x, \vec{z}]$  of  $ZF_\varepsilon$ , we have :

$$\forall \vec{z} (\forall x (\forall y (y \prec x \rightarrow F[y, \vec{z}]) \rightarrow F[x, \vec{z}]) \rightarrow \forall x F[x, \vec{z}]).$$

Proof by contradiction ; we consider, in  $\mathcal{N}$ , an individual  $a$  and a formula  $G[x]$  such that :

$$(1) \quad G[a] ; \forall x (G[x] \rightarrow \exists y \{G[y], y \prec x\}).$$

We apply the axiom scheme of infinity of  $ZF_\varepsilon$  :

$$(2) \quad \exists c \{a \varepsilon c, (\forall x \varepsilon c) (\exists y F(x, y) \rightarrow (\exists y \varepsilon c) F(x, y))\}$$

by setting  $F(x, y) \equiv G[x] \wedge G[y] \wedge y \prec x$ . Let  $b = \{x \varepsilon c ; G(x)\}$  ; by (1) and (2), we get  $a \varepsilon b$ .

We obtain a contradiction with the hypothesis, by showing  $(\forall x \varepsilon b)(\exists y \varepsilon b)(y \prec x)$ .

Therefore, we suppose  $x \varepsilon c$  and  $G[x]$  ; by (2), we have :

$$\exists y \{G[x], G[y], y \prec x\} \rightarrow (\exists y \varepsilon c) \{G[x], G[y], y \prec x\}.$$

By  $G[x]$  and (1), we have  $\exists y \{G[x], G[y], y \prec x\}$ .

Therefore, we have  $(\exists y \varepsilon c) \{G[y], y \prec x\}$ , hence the result.

Q.E.D.

Therefore, in order to show  $\forall x F[x]$ , it suffices to show  $\forall x (\forall y (y \prec x \rightarrow F[y]) \rightarrow F[x])$ .

Then, we say that we have shown  $\forall x F[x]$  *by induction on  $x$ , following the well founded relation  $\prec$* .

### Theorem 7. The binary relation $x \varepsilon y$ is well founded.

We must show  $\forall x (\forall y (y \varepsilon x \rightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X)$ .

We apply theorem 6 to the well founded relation  $x \varepsilon y$  and the formula  $F[x] \equiv x \notin X$ .

This gives :  $\forall x (\forall y (y \varepsilon x \rightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X)$ .

Now, we have immediately  $\Vdash x \notin X \rightarrow x \notin X$ . Thus, it remains to show :

$$\Vdash \forall x (\forall y (y \varepsilon x \rightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (\forall y (y \varepsilon x \rightarrow y \notin X) \rightarrow x \notin X).$$

But we have  $x \notin X \equiv \forall x'(x' \simeq x \rightarrow x' \notin X)$ . Therefore, we need to show :  
 $\Vdash \forall x(\forall y(y \in x \rightarrow y \notin X) \rightarrow x \notin X), \forall y(y \varepsilon x \rightarrow y \notin X), x' \simeq x \rightarrow x' \notin X$  ; or else :  
 $\Vdash \forall y(y \varepsilon x \rightarrow y \notin X), x' \simeq x \rightarrow \forall y(y \in x' \rightarrow y \notin X)$ .  
Now, from  $x' \simeq x, y \in x'$ , we deduce  $y \in x$ . Thus, there is some  $y' \simeq y$  such that  $y' \varepsilon x$ .  
Then, from  $\forall y(y \varepsilon x \rightarrow y \notin X)$ , we deduce  $y' \notin X$ , and therefore  $y \notin X$ .

Q.E.D.

For instance, in the following, we shall use the fact that, if there is an ordinal  $\rho$  such that  $F[\rho]$ , then there exists a least such ordinal, for any formula  $F[\rho]$  written in the language of  $ZF_\varepsilon$ . This results from theorem 7.

## Preservation of well-foundedness

**Theorem 8.** *Let  $\prec$  be a well founded binary relation, defined in the ground model  $\mathcal{M}$ . Then, the relation  $\langle y \prec x \rangle = 1$  is well founded in  $\mathcal{N}$ . In fact, we have :*

$$\mathbf{Y} \Vdash \forall X (\forall x(\forall y(\langle y \prec x \rangle = 1 \leftrightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x(x \notin X))$$

where  $\mathbf{Y} = (\lambda x \lambda f(f)(x) x f) \lambda x \lambda f(f)(x) x f$  (Turing fixpoint combinator).

Let  $\xi \in \Lambda$  be such that  $\xi \Vdash \forall x(\forall y(\langle y \prec x \rangle = 1 \leftrightarrow y \notin X_0) \rightarrow x \notin X_0)$ ,  $X_0$  being any individual in  $\mathcal{M}$ . We set  $F[x] \equiv (\forall \pi \in \|x \notin X_0\|)(\mathbf{Y} \star \xi \cdot \pi \in \perp)$ , and we have to show  $\forall x F[x]$ .

Since  $\prec$  is a well founded relation, it suffices to show  $\forall x(\forall y(y \prec x \rightarrow F[y]) \rightarrow F[x])$ , or else  $\neg F[x_0] \rightarrow (\exists y \prec x_0) \neg F[y]$ , for any individual  $x_0$ .

By the hypothesis  $\neg F[x_0]$ , there exists  $\pi_0 \in \|x_0 \notin X_0\|$  such that  $\mathbf{Y} \star \xi \cdot \pi_0 \notin \perp$  and therefore, we have  $\xi \star \mathbf{Y} \xi \cdot \pi_0 \notin \perp$ .

By hypothesis on  $\xi$ , we deduce  $\mathbf{Y} \xi \not\Vdash \forall y(\langle y \prec x_0 \rangle = 1 \leftrightarrow y \notin X_0)$ .

Thus, there exists  $y_0 \prec x_0$  such that  $\mathbf{Y} \xi \not\Vdash y_0 \notin X_0$ .

Therefore, we have  $(\exists \pi \in \|y_0 \notin X_0\|)(\mathbf{Y} \star \xi \cdot \pi \notin \perp)$ , that is  $\neg F[y_0]$ .

Q.E.D.

## Definition of a rank function

**Definition.** A function with domain  $D$  is an individual  $\phi$  such that :

$$(\forall z \varepsilon \phi)(\exists x \varepsilon D) \exists y(z = (x, y)) ; (\forall x \varepsilon D) \exists y((x, y) \varepsilon \phi) ;$$

$$\forall x \forall y \forall y'((x, y) \varepsilon \phi, (x, y') \varepsilon \phi \rightarrow y = y').$$

Let  $\phi$  be a function with domain  $D$  and  $F[y, \vec{z}]$  a formula of  $ZF_\varepsilon$ . Then, the formula :  $\exists y\{(x, y) \varepsilon \phi, F[y, \vec{z}]\}$  is denoted by  $F[\phi(x), \vec{z}]$ .

**Remark.** Beware, despite the same notation  $\phi(x)$ , it is not a function symbol.

By means of theorem 3, we define the binary function symbol  $\text{Im}$  by setting :

$$\text{Im}(\phi, D) = \{y ; (\exists x \varepsilon D) (x, y) \varepsilon \phi\}.$$

When  $\phi$  is a function with domain  $D$ , we shall use, for  $\text{Im}(\phi, D)$ , the more intuitive notation  $\{\phi(x) ; x \varepsilon D\}$ , which we call *image of the function  $\phi$* .

Let  $D' \subseteq D$ , that is  $\forall x(x \notin D \rightarrow x \notin D')$  ; a restriction of  $\phi$  to  $D'$  is, by definition, a function  $\phi'$  with domain  $D'$  such that  $\phi' \subseteq \phi$ .

For instance,  $\{z \varepsilon \phi ; (\exists x \varepsilon D') \exists y(z = (x, y))\}$  is a restriction of  $\phi$  to  $D'$ .

If  $\phi'_0, \phi'_1$  are both restrictions of  $\phi$  to  $D'$ , then  $\phi'_0 \cong \phi'_1$ .

**Definition.**

A binary relation  $\prec$  is called *ranked*, if we have  $\forall x \exists y \forall z (z \prec x \rightarrow z \varepsilon y)$ , in other words : the minorants of any individual form a set.

By theorem 3, if the relation  $\prec$  is ranked and defined by a formula  $P[x, y, \vec{u}]$  of  $ZF_\varepsilon$  with parameters  $\vec{u}$  in  $\mathcal{N}$ , we have :

$\mathcal{N} \models \forall x \forall y (x \prec y \leftrightarrow x \varepsilon f(y, \vec{u}))$ , for some symbol of function  $f$ , defined in  $\mathcal{M}$ .

In what follows, we suppose that  $\prec$  is a ranked *transitive* binary relation.

A function  $\phi$  with domain  $\{x ; x \prec a\}$  will be called *a-inductive for  $\prec$* , if we have :

$\phi(x) \simeq \{\phi(y) ; y \prec x\}$  for every  $x \prec a$ . In other words :

$(\forall x \prec a)(\forall y \prec x) \phi(y) \in \phi(x) ; (\forall x \prec a)(\forall z \varepsilon \phi(x))(\exists y \prec x) z \simeq \phi(y)$ .

If  $\phi$  is *a-inductive for  $\prec$* , we set  $O(\phi, a) = \{\phi(x) ; x \prec a\}$  (image of  $\phi$ ).

**Lemma 9.** *Let  $\phi, \phi'$  be two functions, a-inductive for  $\prec$ . Then :*

i)  $\phi(x) \simeq \phi'(x)$  for every  $x \prec a$ .

ii)  $O(\phi, a) \simeq O(\phi', a)$ .

iii)  $(\forall x \prec a) On(\phi(x)) ; O(\phi, a)$  is an ordinal, called ordinal of  $\phi$ .

i) Proof by induction on  $\phi(x)$ , following  $\in$  : if  $u \varepsilon \phi(x)$ , then  $u \simeq \phi(y)$  with  $y \prec x$ .

Since  $\phi(y) \in \phi(x)$ , we have  $\phi(y) \simeq \phi'(y)$  by the induction hypothesis ;

therefore  $\phi(y) \in \phi'(x)$  and  $\phi(x) \subset \phi'(x)$ .

Conversely, if  $u \varepsilon \phi'(x)$ , then  $u \simeq \phi'(y)$  with  $y \prec x$ . Thus, we have  $\phi(y) \in \phi(x)$ , and therefore  $\phi(y) \simeq \phi'(y)$  by the induction hypothesis ; therefore  $u \in \phi(x)$  and  $\phi'(x) \subset \phi(x)$ .

ii) Immediate, by (i).

iii) We show  $On(\phi(x))$  by induction on  $\phi(x)$ , for the well founded relation  $\in$  :

If  $u \varepsilon \phi(x)$ , we have  $u \simeq \phi(y)$  with  $y \prec x$  ; therefore, we have  $On(u)$  by the induction hypothesis. If  $v \varepsilon u$ , then  $v \varepsilon \phi(y)$ , therefore  $v \simeq \phi(z)$  with  $z \prec y$  ; therefore  $v \in \phi(x)$ .

It follows that  $\phi(x)$  is a transitive set of ordinals, thus an ordinal.

Then,  $O(\phi, a)$  is also a transitive set of ordinals, and therefore an ordinal.

Q.E.D.

**Lemma 10.** *If  $\phi$  is a-inductive for  $\prec$ , and if  $b \prec a$ , then every restriction  $\psi$  of  $\phi$  to the domain  $\{x ; x \prec b\}$  is a b-inductive function for  $\prec$ .*

Indeed, we have,  $\psi(x) = \phi(x) \simeq \{\phi(y) ; y \prec x\} \simeq \{\psi(y) ; y \prec x\}$ .

Q.E.D.

By means of theorem 2, we define a unary function symbol  $\Phi$ , such that :

$\forall x (\forall f \varepsilon \Phi(x))(f \text{ is a } x\text{-inductive function}) ;$

$\forall x \forall f (f \text{ is a } x\text{-inductive function} \rightarrow \exists f (f \varepsilon \Phi(x)))$ .

In other words,  $\Phi(x)$  is a set of *x-inductive functions*, which is non void if there exists at least one such function.

Finally, we define the unary function symbol  $Rk$ , using theorem 4, by setting :

$$Rk(x) = \overline{\bigcup} \{O(f, x) ; f \varepsilon \Phi(x)\}$$

(the symbol  $\overline{\bigcup}$  is defined after theorem 1).

Therefore,  $Rk(x)$  is the union of the ordinals of the *x-inductive functions* in the set  $\Phi(x)$ .

Since all these ordinals are extensionally equivalent, by lemma 9(ii), their union  $\text{Rk}(x)$  is also an equivalent ordinal.

**Remarks.**

If there exists no  $x$ -inductive function, then  $\text{Rk}(x)$  is void.

The function symbols  $\text{O}, \Phi, \text{Rk}$  have additional arguments, which are the parameters  $\vec{u}$  of the formula  $P[x, y, \vec{u}]$  which defines the relation  $y \prec x$ .

We suppose now that  $\prec$  is a ranked transitive relation, which is *well founded*. It is therefore a *strict ordering*.

**Lemma 11.** *Every restriction of  $\text{Rk}$  to the domain  $\{x ; x \prec a\}$  is an  $a$ -inductive function for  $\prec$ .*

Proof by induction on  $a$ , following  $\prec$ .

Let  $f$  be a restriction of  $\text{Rk}$  to the domain  $\{x ; x \prec a\}$  and let  $x \prec a$ . We must show that  $f(x) \simeq \{f(y) ; y \prec x\}$ , in other words, that we have :

$$\text{Rk}(x) \simeq \{\text{Rk}(y) ; y \prec x\}.$$

Let  $\psi$  be any restriction of  $\text{Rk}$  to the domain  $\{y ; y \prec x\}$ . By the induction hypothesis,  $\psi$  is a  $x$ -inductive function for  $\prec$ .

We now show that  $\text{Rk}(x) \simeq \{\text{Rk}(y) ; y \prec x\}$  :

i) If  $u \in \text{Rk}(x)$  then  $u \in \text{O}(\phi, x)$  for some function  $\phi$  which is  $x$ -inductive for  $\prec$ , *provided that there exists such a function*. Now, there exists effectively one, otherwise  $\text{Rk}(x)$  would be void.

Therefore, by definition of  $\text{O}(\phi, x)$ , we have  $u = \phi(y)$  with  $y \prec x$ . But  $\text{Rk}(y) \simeq \phi(y)$ , since  $\phi, \psi$  are both  $x$ -inductive functions for  $\prec$ , and  $\psi(y) = \text{Rk}(y)$  (lemma 9(i)).

Therefore, we have  $u \simeq \text{Rk}(y)$ , with  $y \prec x$ .

ii) Conversely, if  $y \prec x$ , then  $\text{Rk}(y) = \psi(y)$ . Let  $\phi \in \Phi(x)$  ; then  $\phi, \psi$  are  $x$ -inductive for  $\prec$  ; therefore  $\phi(y) \simeq \psi(y)$  (lemma 9(i)).

Now  $\phi(y) \in \text{O}(\phi, x)$ , and therefore  $\phi(y) \in \text{Rk}(x)$  by definition of  $\text{Rk}(x)$ .

It follows that  $\text{Rk}(y) = \psi(y) \in \text{Rk}(x)$ .

Q.E.D.

**Theorem 12.** *We have  $\text{Rk}(x) \simeq \{\text{Rk}(y) ; y \prec x\}$  for every  $x$ .*

Proof by induction on  $x$ , following  $\prec$  ; let  $\psi$  be any restriction of  $\text{Rk}$  to the domain  $\{y ; y \prec x\}$ . By lemma 11,  $\psi$  is a  $x$ -inductive function for  $\prec$ .

Then, we finish the proof, by repeating paragraphs (i) and (ii) of the proof of lemma 11.

Q.E.D.

$\text{Rk}$  is called the *rank function* of the ranked, well founded and transitive relation  $\prec$ .

$\text{Rk}(x)$  is, for every  $x$ , a representative of the ordinal of any  $x$ -inductive function for  $\prec$ .

The values of the rank function  $\text{Rk}$  form an initial segment of  $\text{On}$ , which we shall call *the image of  $\text{Rk}$* . It is therefore, *either an ordinal, or the whole of  $\text{On}$* .

**Proposition 13.** *Let  $\prec_0, \prec_1$  be two ranked transitive well founded relations, and  $f$  a function such that  $\forall x \forall y (x \prec_0 y \rightarrow f(x) \prec_1 f(y))$ .*

*If  $\text{Rk}_0, \text{Rk}_1$  are their rank functions, then we have  $\forall x (\text{Rk}_0(x) \leq \text{Rk}_1(f(x)))$ , and the image of  $\text{Rk}_0$  is an initial segment of the image of  $\text{Rk}_1$ .*

We show immediately  $\forall x (\text{Rk}_0(x) \leq \text{Rk}_1(f(x)))$  by induction following  $\prec_0$ . Hence the result, since the image of a rank function is an initial segment of  $\text{On}$ .

Q.E.D.

## An ultrafilter on $\mathfrak{J}2$

In all of the following, we write  $y < x$  for  $y \in \text{Cl}(x)$  in  $\mathcal{M}$ , where  $\text{Cl}(x)$  denotes the transitive closure of  $x$ . It is a strict well founded ordering (many other such orderings would do the job, for instance the relation  $\text{rank}(y) < \text{rank}(x)$ ).

The binary function symbol  $\langle y < x \rangle$  is therefore defined in  $\mathcal{N}$ , with values in  $\mathfrak{J}2$ .

By theorem 8, the binary relation  $\langle y < x \rangle = 1$  is well founded in  $\mathcal{N}$ .

**Theorem 14.**  $\Vdash$  *There exists an ultrafilter  $\mathcal{D}$  on  $\mathfrak{J}2$ , which is defined as follows :  $\mathcal{D} = \{\alpha \varepsilon \mathfrak{J}2 ; \text{the relation } \langle y < x \rangle \geq \alpha \text{ is well founded}\}$ .*

**Remark.** By lemma 5, the formula  $\langle y < x \rangle \geq \alpha$  may be written  $\langle \alpha y < \alpha x \rangle = \alpha$ .

The formula  $\alpha \varepsilon \mathcal{D}$ , which we shall also write  $\mathcal{D}[\alpha]$ , is therefore :

$$\mathcal{D}[\alpha] \equiv \forall X (\forall x (\forall y (\langle y < x \rangle \geq \alpha \leftrightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X))$$

**Remark.** We have :

$$\mathcal{D}[1] \equiv \forall X (\forall x (\forall y (\langle y < x \rangle = 1 \leftrightarrow y \notin X) \rightarrow x \notin X) \rightarrow \forall x (x \notin X)).$$

$$\mathcal{D}[0] \equiv \forall X ((\emptyset \notin X \rightarrow \emptyset \notin X) \rightarrow \emptyset \notin X).$$

We have immediately :  $\lambda x x \Vdash \neg \mathcal{D}[0] ; \mathbf{Y} \Vdash \mathcal{D}[1] ;$

$\mathbf{I} \Vdash \forall \alpha^{\mathfrak{J}2} \forall \beta^{\mathfrak{J}2} (\alpha \leq \beta \leftrightarrow (\mathcal{D}[\alpha] \rightarrow \mathcal{D}[\beta]))$  (more precisely :  $\|\mathcal{D}[1]\| \subset \|\mathcal{D}[0]\|$ ).

Therefore, in order to prove theorem 14, it suffices to show :

$\Vdash \forall \alpha^{\mathfrak{J}2} \forall \beta^{\mathfrak{J}2} (\alpha \wedge \beta = 0 \leftrightarrow (\mathcal{D}[\alpha \vee \beta] \rightarrow \mathcal{D}[\alpha] \vee \mathcal{D}[\beta]))$  ; see theorem 15 ;

$\Vdash \forall \alpha^{\mathfrak{J}2} \forall \beta^{\mathfrak{J}2} (\alpha \wedge \beta = 0 \leftrightarrow (\mathcal{D}[\alpha], \mathcal{D}[\beta] \rightarrow \perp))$  ; or even only :

$\Vdash \forall \alpha^{\mathfrak{J}2} (\mathcal{D}[\alpha], \mathcal{D}[-\alpha] \rightarrow \perp)$  ; see theorem 22.

**Notation.** For  $\alpha \varepsilon \mathfrak{J}2$ , we shall write  $x <_\alpha y$  for  $\langle x < y \rangle \geq \alpha$ .

**Theorem 15.**

i)  $\Vdash \forall \alpha^{\mathfrak{J}2} \forall \beta^{\mathfrak{J}2} (\alpha \wedge \beta = 0 \leftrightarrow (\mathcal{D}[\alpha \vee \beta] \rightarrow \mathcal{D}[\alpha] \vee \mathcal{D}[\beta]))$ .

ii)  $\Vdash \forall \alpha^{\mathfrak{J}2} \forall \beta^{\mathfrak{J}2} (\mathcal{D}[\alpha \vee \beta] \rightarrow \mathcal{D}[\alpha] \vee \mathcal{D}[\beta])$ .

i) Let  $\alpha, \beta \varepsilon \mathfrak{J}2$  be such that  $\alpha \wedge \beta = 0, \neg \mathcal{D}[\alpha], \neg \mathcal{D}[\beta]$ . We have to show  $\neg \mathcal{D}[\alpha \vee \beta]$ .

By hypothesis on  $\alpha$  and  $\beta$ , there exists individuals  $a_0, A$  (resp.  $b_0, B$ ) such that  $a_0 \varepsilon A$  (resp.  $b_0 \varepsilon B$ ) and  $A$  (resp.  $B$ ) has no minimal  $\varepsilon$ -element for  $<_\alpha$  (resp. for  $<_\beta$ ). We set :

$$c_0 = \alpha a_0 \sqcup \beta b_0 \quad \text{and} \quad C = \{\alpha x \sqcup \beta y ; x \varepsilon A, y \varepsilon B\}.$$

Therefore, we have  $c_0 \varepsilon C$  ; it suffices to show that  $C$  has no minimal  $\varepsilon$ -element for  $<_{\alpha \vee \beta}$ .

Let  $c \varepsilon C, c = \alpha a \sqcup \beta b$ , with  $a \varepsilon A, b \varepsilon B$ . By hypothesis on  $A, B$ , there exists  $a' \varepsilon A$  and  $b' \varepsilon B$  such that  $a' <_\alpha a, b' <_\beta b$ . If we set  $c' = \alpha a' \sqcup \beta b'$ , we have  $c' \varepsilon C$ , as needed. We also have :

$\langle c' = a' \rangle \geq \alpha, \langle a' < a \rangle \geq \alpha, \langle c = a \rangle \geq \alpha$  ; it follows that  $\langle c' < c \rangle \geq \alpha$ .

In the same way, we have  $\langle c' < c \rangle \geq \beta$  and therefore, finally,  $\langle c' < c \rangle \geq \alpha \vee \beta$ .

ii) We set  $\beta' = \beta \wedge (-\alpha)$  ; we have  $\alpha \wedge \beta' = 0$  and  $\alpha \vee \beta' = \alpha \vee \beta$ . Therefore, we have :

$\mathcal{D}[\alpha \vee \beta] \rightarrow \mathcal{D}[\alpha] \vee \mathcal{D}[\beta']$ . Now, we have  $\beta' \leq \beta$  and therefore  $\mathcal{D}[\beta'] \rightarrow \mathcal{D}[\beta]$ .

Q.E.D.

**Lemma 16.**

- i)  $I \Vdash \forall x \forall y (\langle x < y \rangle \neq 1 \rightarrow x \notin y)$ .
- ii) If  $\mathcal{M} \models u \in v$ , then  $I \Vdash u \varepsilon \mathfrak{J}v$ .
- iii)  $I \Vdash \forall x \forall y \forall \alpha^{\mathfrak{J}2} (\langle x < y \rangle \geq \alpha \leftrightarrow \alpha x \varepsilon \mathfrak{J}Cl(\{y\}))$ .
- iv)  $\Vdash \forall x \forall y (\langle x < y \rangle = 1 \leftrightarrow x \varepsilon \mathfrak{J}Cl(y))$ .

Let  $a, b$  be two individuals.

- i) Let  $\xi \Vdash \langle a < b \rangle \neq 1$ ,  $\pi \in \|a \notin b\|$ ; then  $(a, \pi) \in b$ , therefore  $\langle a < b \rangle = 1$  and  $\xi \Vdash \perp$ ; therefore  $\xi \star \pi \in \perp$ .
- ii) Indeed, we have  $\|u \notin \mathfrak{J}v\| = \{\pi \in \Pi; (u, \pi) \in v \times \Pi\} = \Pi$ .
- iii) Let  $\alpha \in \{0, 1\}$  and  $a, b \in \mathcal{M}$  such that  $\langle a < b \rangle \geq \alpha$ .  
If  $\alpha = 0$ , we must show  $I \Vdash \emptyset \varepsilon \mathfrak{J}Cl(\{y\})$  which follows from (ii).  
If  $\alpha = 1$ , then  $\langle a < b \rangle = 1$ , that is  $a \in Cl(b)$ , therefore  $a \in Cl(\{b\})$ .  
From (ii), it follows that  $I \Vdash a \varepsilon \mathfrak{J}Cl(\{b\})$ .
- iv) Indeed, if  $a, b$  are individuals of  $\mathcal{M}$ , we have trivially :  
 $\|\langle a < b \rangle \neq 1\| = \|a \notin \mathfrak{J}Cl(b)\|$ .

Q.E.D.

**Lemma 17.** *The well founded relation  $\langle x < y \rangle = 1$  is ranked, and its rank function  $R$  has for image the whole of  $On$ .*

Lemma 16(iv) shows that this relation is ranked.

Let  $\rho$  be an ordinal and  $r$  an individual  $\simeq \rho$ . We show, by induction on  $\rho$ , that  $R(r) \geq \rho$ . Indeed, for every  $\rho' \in \rho$ , there exists  $r' \varepsilon r$  such that  $r' \simeq \rho'$ . We have  $R(r') \geq \rho'$  by induction hypothesis, and  $\langle r' < r \rangle = 1$  from lemma 16(i). Therefore, we have  $\rho' \in R(r)$  by definition of  $R$ , and finally  $R(r) \geq \rho$ . This shows that the image of  $R$  is not bounded in  $On$ . Since it is an initial segment, it is the whole of  $On$ .

Q.E.D.

**Theorem 18.** *Let  $F(x, y)$  be a formula of  $ZF_\varepsilon$ , with parameters. Then, we have :*

$$I \Vdash \forall x \forall y (\forall \varpi^{\mathfrak{J}III} F(x, f(x, \varpi)) \rightarrow F(x, y))$$

for some function symbol  $f$ , defined dans  $\mathcal{M}$ , with domain  $\mathcal{M} \times \Pi$ .

Since the ground model  $\mathcal{M}$  satisfies  $V = L$  (or only the *choice principle*), we can define, in  $\mathcal{M}$ , a function symbol  $f$  such that :

$$\forall x \forall y (\forall \varpi \in \Pi) (\varpi \in \|F(x, y)\| \rightarrow \varpi \in \|F(x, f(x, \varpi))\|).$$

Let  $a, b$  be individuals,  $\xi \Vdash \forall \varpi^{\mathfrak{J}II} F(a, f(a, \varpi))$  and  $\pi \in \|F(a, b)\|$ .

Thus, we have  $\pi \in \|F(a, f(a, \pi))\|$ , and therefore  $\xi \star \pi \in \perp$ .

Q.E.D.

**Definitions.** Let  $a$  be any individual of  $\mathcal{N}$  and  $\kappa$  an ordinal (therefore,  $\kappa$  is not an individual of  $\mathcal{N}$ , but an equivalence class for  $\simeq$ ).

A *function* or *application* from  $\kappa$  into  $a$  is, by definition, a binary relation  $R(\rho, x)$  such that :  $\forall x \forall x' (\forall \rho, \rho' \in \kappa) (R(\rho, x), R(\rho', x'), \rho \simeq \rho' \rightarrow x = x')$  ;  $(\forall \rho \in \kappa) (\exists x \varepsilon a) R(\rho, x)$ .

It is an *injection* if we have  $\forall x (\forall \rho, \rho' \in \kappa) (R(\rho, x), R(\rho', x) \rightarrow \rho \simeq \rho')$ .

A *surjection* from  $a$  onto  $\kappa$  is a function  $f$  of domain  $a$  such that :

$$(\forall \rho \in \kappa) (\exists x \varepsilon a) f(x) \simeq \rho.$$

**Theorem 19.**

For any individual  $a$ , there exists an ordinal  $\kappa$ , such that there is no surjection from  $a$  onto  $\kappa$ .

Let  $f$  be a surjection from  $a$  onto an ordinal  $\rho$ . We define a strict ordering relation  $\prec_f$  by setting  $x \prec_f y \Leftrightarrow x \in a \wedge y \in a \wedge f(x) < f(y)$ . It is clear that this relation is well founded, that  $f$  is an  $a$ -inductive function, and that  $\mathbf{O}(f, a) \simeq \rho$ .

We may consider this relation as a subset of  $a \times a$ .

By means of the axioms of union, power set and collection given above (theorems 1–4), we define an ordinal  $\kappa_0$ , which is the union of the  $\mathbf{O}(f, a)$  for all the functions  $f$  which are  $a$ -inductive for some well founded strict ordering relation on  $a$ .

In fact, we consider the set :

$$\mathcal{B}(a) = \{X \in \overline{\mathcal{P}}(a \times a) ; X \text{ is a well founded strict ordering relation on } a\}.$$

Then, we set  $\kappa_0 = \bigcup \{\mathbf{O}(f, a) ; X \in \mathcal{B}(a), f \in \Phi(X, a)\}$ .

In this definition, we use the function symbol  $\Phi$ , defined after lemma 10, which associates with each well founded strict ordering relation  $X$  on  $a$ , a *non void* set of  $a$ -inductive functions for this relation.

Then, there exists no surjection from  $a$  onto  $\kappa_0 + 1$ .

Q.E.D.

We denote by  $\Delta$  the first ordinal of  $\mathcal{N}$  such that there is no surjection from  $\mathbb{N}$  onto  $\Delta$  : for every function  $\phi$ , there exists  $\delta \in \Delta$  such that  $\forall x^{\mathbb{N}}(\phi(x) \neq \delta)$ .

For each  $\alpha \in \mathbb{N}$ , we denote by  $\mathcal{N}_\alpha$  the class defined by the formula  $x = \alpha x$ .

**Lemma 20.** Let  $\alpha_0, \alpha_1 \in \mathbb{N}$ ,  $\alpha_0 \wedge \alpha_1 = 0$  and  $R_0$  (resp.  $R_1$ ) be a functional relation of domain  $\mathcal{N}_{\alpha_0}$  (resp.  $\mathcal{N}_{\alpha_1}$ ) with values in  $\text{On}$ . Then, either  $R_0$ , or  $R_1$ , is not surjective onto  $\Delta$ .

Proof by contradiction : we suppose that  $R_0$  and  $R_1$  are both surjective onto  $\Delta$ .

We apply theorem 18 to the formula  $F(x_0, x_1) \equiv \neg(R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1))$ , and we get :

$$\forall x_0 (\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \rightarrow \exists \varpi^{\mathbb{N}} (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(x_0, \varpi))))$$

where  $f$  is a suitable function symbol (therefore *defined in  $\mathcal{M}$* ).

Replacing  $x_0$  with  $\alpha_0 x_0$ , we obtain :

$$\forall x_0 (\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \rightarrow \exists \varpi^{\mathbb{N}} (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\alpha_0 x_0, \varpi)))).$$

But, by lemma 5(i), we have  $\alpha_1 f(\alpha_0 x, \varpi) = \alpha_1 f(\alpha_1 \alpha_0 x, \varpi) = \alpha_1 f(\emptyset, \varpi)$ . It follows that :

$$\forall x_0 (\exists x_1 (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1)) \rightarrow \exists \varpi^{\mathbb{N}} (R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\emptyset, \varpi)))).$$

By hypothesis, we have  $(\forall \rho \in \Delta) \exists x_0 \exists x_1 (\rho \simeq R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 x_1))$ . It follows that :

$(\forall \rho \in \Delta) \exists x_0 \exists \varpi^{\mathbb{N}} (\rho \simeq R_0(\alpha_0 x_0) \simeq R_1(\alpha_1 f(\emptyset, \varpi)))$  ; therefore, we have :

$(\forall \rho \in \Delta) \exists \varpi^{\mathbb{N}} (\rho \simeq R_1(\alpha_1 f(\emptyset, \varpi)))$ .

Therefore, the function  $\varpi \mapsto R_1(\alpha_1 f(\emptyset, \varpi))$  is a surjection from  $\mathbb{N}$  onto  $\Delta$ . But this is a contradiction with the definition de  $\Delta$ .

**Remark.** We should write  $f(\alpha_0, \alpha_1, x_0, \varpi)$  instead of  $f(x_0, \varpi)$ , since the function symbol  $f$  depends on the four variables  $\alpha_0, \alpha_1, x_0, \varpi$ . In fact, it depends also on the parameters which appear in  $R_0, R_1$ . The proof does not change.

Q.E.D.

**Corollary 21.** *Let  $\alpha_0, \alpha_1 \in \mathfrak{I}2$ ,  $\alpha_0 \wedge \alpha_1 = 0$ , and  $\prec_0, \prec_1$  be two well founded ranked strict ordering relations with respective domains  $\mathcal{N}_{\alpha_0}, \mathcal{N}_{\alpha_1}$ . Let  $Rk_0, Rk_1$  be their rank functions. Then, either the image of  $Rk_0$ , or that of  $Rk_1$  is an ordinal  $< \Delta$ .*

In order to be able to define the rank functions  $Rk_0, Rk_1$ , we consider the relations  $\prec'_0, \prec'_1$ , with domain the whole of  $\mathcal{N}$ , defined by  $x \prec'_i y \equiv (x = \alpha_i x) \wedge (y = \alpha_i y) \wedge (x \prec_i y)$  for  $i = 0, 1$ .

These strict ordering relations are well founded and ranked.

Their rank functions  $Rk'_0, Rk'_1$  take the value 0 outside  $\mathcal{N}_{\alpha_0}, \mathcal{N}_{\alpha_1}$  respectively : indeed, all the individuals outside  $\mathcal{N}_{\alpha_i}$  are minimal for  $\prec'_i$ .

By lemma 20, one of them,  $Rk'_0$  for instance, is not surjective onto  $\Delta$ .

Since the image of any rank function is an initial segment of  $\text{On}$ , the image of  $Rk_0$  is an ordinal  $< \Delta$ .

Q.E.D.

**Theorem 22.**

i)  $\Vdash \forall \alpha_0 \mathfrak{I}2 \forall \alpha_1 \mathfrak{I}2 (\alpha_0 \wedge \alpha_1 = 0 \leftrightarrow (\mathcal{D}[\alpha_0], \mathcal{D}[\alpha_1] \rightarrow \perp))$ .

ii)  $\Vdash \forall \alpha_0 \mathfrak{I}2 \forall \alpha_1 \mathfrak{I}2 (\mathcal{D}[\alpha_0], \mathcal{D}[\alpha_1] \rightarrow \mathcal{D}[\alpha_0 \wedge \alpha_1])$ .

i) In  $\mathcal{N}$ , let  $\alpha_0, \alpha_1 \in \mathfrak{I}2$  be such that  $\alpha_0 \wedge \alpha_1 = 0$  and the relations  $\langle x < y \rangle \geq \alpha_0, \langle x < y \rangle \geq \alpha_1$  be well founded. Therefore, we have  $\alpha_0, \alpha_1 \neq 0, 1$ .

Therefore, the relations  $x \prec_i y \equiv (x = \alpha_i x) \wedge (y = \alpha_i y) \wedge (\langle x < y \rangle = \alpha_i)$  for  $i = 0, 1$ , are well founded strict orderings.

From lemma 16(iii), it follows that these relations are *ranked*.

Now, by lemma 5, we have :  $\Vdash \forall x \forall y \forall \alpha \mathfrak{I}2 (\langle x < y \rangle = 1 \rightarrow \langle \alpha x < \alpha y \rangle = \alpha)$ .

But, by lemma 17, the rank function of the well founded relation  $\langle x < y \rangle = 1$  has for image the whole of  $\text{On}$ . Therefore, by proposition 13, the same is true for the rank functions of the well founded strict order relations  $x \prec_0 y$  and  $x \prec_1 y$ .

But this contradicts corollary 21.

ii) We have  $\alpha_0 \leq (\alpha_0 \wedge \alpha_1) \vee (-\alpha_1)$ . Therefore, by  $\mathcal{D}[\alpha_0]$  and theorem 15, we have  $\mathcal{D}[\alpha_0 \wedge \alpha_1]$  or  $\mathcal{D}[-\alpha_1]$ . But  $\mathcal{D}[-\alpha_1]$  is impossible, by  $\mathcal{D}[\alpha_1]$  and (i).

Q.E.D.

**Corollary 23.**  $\mathcal{D}[\alpha]$  is equivalent with each one of the following propositions :

i) *There exists a well founded ranked strict ordering relation  $\prec$  with domain  $\mathcal{N}_\alpha$ , the rank function of which has an image  $\geq \Delta$ .*

ii) *There exists a function with domain  $\mathcal{N}_\alpha$  which is surjective onto  $\Delta$ .*

$\mathcal{D}[\alpha] \Rightarrow$  (i) :

By definition of  $\mathcal{D}[\alpha]$ , the binary relation  $(x = \alpha x) \wedge (y = \alpha y) \wedge (\langle x < y \rangle = \alpha)$  is well founded. By lemma 16(iii), this relation is ranked. We have seen, in the proof of theorem 22, that the image of its rank function is the whole of  $\text{On}$ .

(i)  $\Rightarrow$  (ii) : obvious.

(ii)  $\Rightarrow \mathcal{D}[\alpha]$  :

Since  $\mathcal{D}$  is an ultrafilter, it suffices to show  $\neg \mathcal{D}[-\alpha]$ . But, (ii) and  $\mathcal{D}[-\alpha]$  contradict lemma 20.

Q.E.D.

**Theorem 24.**

If  $\mathfrak{J}2$  is non trivial, there exists no set, which is totally ordered by  $\varepsilon$ , the ordinal of which is  $\geq \Delta$ .

Let  $\alpha \varepsilon \mathfrak{J}2, \alpha \neq 0, 1$  and  $X$  be a set which is totally ordered by  $\varepsilon$ , and equipotent with  $\Delta$ .

Then, we show that the application  $x \mapsto \alpha x$  is an injection from  $X$  into  $\mathcal{N}_\alpha$  :

Indeed, by lemma 16(i), we have  $x \varepsilon y \rightarrow \langle x < y \rangle = 1$  and, by lemma 5, we have :

$\langle x < y \rangle = 1 \rightarrow \langle \alpha x < \alpha y \rangle = \alpha$ . Therefore, if  $x, y \varepsilon X$  and  $x \neq y$ , we have, for instance  $x \varepsilon y$ , therefore  $\langle \alpha x < \alpha y \rangle = \alpha$  and therefore  $\alpha x \neq \alpha y$  since  $\alpha \neq 0$ .

Thus, there exists a function with domain  $\mathcal{N}_\alpha$  which is surjective onto  $\Delta$ . The same reasoning, applied to  $-\alpha$  gives the same result for  $-\alpha$ . But this contradicts lemma 20.

Q.E.D.

**Remark.** Theorem 24 shows that it is impossible to define Von Neumann ordinals in  $\mathcal{N}$ , with  $\varepsilon$  instead of  $\in$ , unless  $\mathfrak{J}2$  is trivial, i.e. the realizability model is, in fact, a forcing model.

## The model $\mathcal{M}_\mathcal{D}$

For each formula  $F[x_1, \dots, x_n]$  of ZF, we have defined, in the ground model  $\mathcal{M}$ , an  $n$ -ary function symbol with values in  $\{0, 1\}$ , denoted by  $\langle F[x_1, \dots, x_n] \rangle$ , by setting, for any individuals  $a_1, \dots, a_n$  of  $\mathcal{M}$  :  $\langle F[a_1, \dots, a_n] \rangle = 1 \Leftrightarrow \mathcal{M} \models F[a_1, \dots, a_n]$ .

In  $\mathcal{N}$ , the function symbol  $\langle F[x_1, \dots, x_n] \rangle$  takes its values in the Boolean algebra  $\mathfrak{J}2$ .

We define, in  $\mathcal{N}$ , two binary relations  $\in_\mathcal{D}$  and  $=_\mathcal{D}$ , by setting :

$$(x \in_\mathcal{D} y) \equiv \mathcal{D}[\langle x \in y \rangle] ; (x =_\mathcal{D} y) \equiv \mathcal{D}[\langle x = y \rangle].$$

The class  $\mathcal{N}$ , equipped with these relations, will be denoted  $\mathcal{M}_\mathcal{D}$ .

For each formula  $F[\vec{x}, y]$  of ZF, with  $n + 1$  free variables  $x_1, \dots, x_n, y$ , we can define, by means of the *choice principle* in  $\mathcal{M}$ , an  $n$ -ary function symbol  $f_F$ , such that :

$$\mathcal{M} \models \forall \vec{x} (F[\vec{x}, f_F(\vec{x})] \rightarrow \forall y F[\vec{x}, y]) ;$$

$f_F$  is called the *Skolem function* of the formula  $F[\vec{x}, y]$ .

**Lemma 25.**

- i)  $I \Vdash \forall \vec{x} \forall y (\langle \forall y F[\vec{x}, y] \rangle \leq \langle F[\vec{x}, y] \rangle)$
- ii)  $I \Vdash \forall \vec{x} \forall y (\langle \forall y F[\vec{x}, y] \rangle = \langle F[\vec{x}, f_F(\vec{x})] \rangle)$ .

Trivial.

Q.E.D.

For each formula  $F[\vec{x}]$  of ZF, we define, by recurrence on  $F$ , a formula of  $ZF_\varepsilon$ , which has the same free variables, and that we denote  $\mathcal{M}_\mathcal{D} \models F[\vec{x}]$ .

- $F$  is atomic :  
 $(\mathcal{M}_\mathcal{D} \models x_1 \in x_2)$  is  $x_1 \in_\mathcal{D} x_2$  ;  $(\mathcal{M}_\mathcal{D} \models x_1 = x_2)$  is  $x_1 =_\mathcal{D} x_2$  ;  $(\mathcal{M}_\mathcal{D} \models \perp)$  is  $\perp$ .
- $F \equiv F_0 \rightarrow F_1$  : then  $(\mathcal{M}_\mathcal{D} \models F)$  is the formula  $(\mathcal{M}_\mathcal{D} \models F_0) \rightarrow (\mathcal{M}_\mathcal{D} \models F_1)$ .
- $F[\vec{x}] \equiv \forall y G[\vec{x}, y]$  : then  $(\mathcal{M}_\mathcal{D} \models F[\vec{x}])$  is the formula  $\forall y (\mathcal{M}_\mathcal{D} \models G[\vec{x}, y])$ .

**Lemma 26.** For each formula  $F[\vec{x}]$  of ZF, we have  $\Vdash \forall \vec{x} ((\mathcal{M}_\mathcal{D} \models F[\vec{x}]) \leftrightarrow \mathcal{D}[\langle F[\vec{x}] \rangle])$ .

Proof by recurrence on the length of  $F$ .

If  $F$  is atomic, we have  $\mathbb{1} \Vdash \forall \vec{x} \left( (\mathcal{M}_{\mathcal{D}} \models F[\vec{x}]) \rightarrow \mathcal{D}\langle F[\vec{x}] \rangle \right)$

and  $\mathbb{1} \Vdash \forall \vec{x} \left( \mathcal{D}\langle F[\vec{x}] \rangle \rightarrow (\mathcal{M}_{\mathcal{D}} \models F[\vec{x}]) \right)$  since  $(\mathcal{M}_{\mathcal{D}} \models F[\vec{x}])$  is identical with  $\mathcal{D}\langle F[\vec{x}] \rangle$ .

If  $F \equiv F_0 \rightarrow F_1$ , the formula  $(\mathcal{M}_{\mathcal{D}} \models F) \leftrightarrow \mathcal{D}\langle F \rangle$  is :

$$((\mathcal{M}_{\mathcal{D}} \models F_0) \rightarrow (\mathcal{M}_{\mathcal{D}} \models F_1)) \leftrightarrow \mathcal{D}\langle F_0 \rightarrow F_1 \rangle.$$

Since  $\mathcal{D}$  is an ultrafilter, this formula is equivalent with :

$$((\mathcal{M}_{\mathcal{D}} \models F_0) \rightarrow (\mathcal{M}_{\mathcal{D}} \models F_1)) \leftrightarrow (\mathcal{D}\langle F_0 \rangle \rightarrow \mathcal{D}\langle F_1 \rangle),$$

which is a logical consequence of :  
 $(\mathcal{M}_{\mathcal{D}} \models F_0) \leftrightarrow \mathcal{D}\langle F_0 \rangle$  and  $(\mathcal{M}_{\mathcal{D}} \models F_1) \leftrightarrow \mathcal{D}\langle F_1 \rangle$ .

Hence the result, by the recurrence hypothesis.

If  $F[\vec{x}] \equiv \forall y G[\vec{x}, y]$ , let  $f_G(\vec{x})$  be the Skolem function of  $G$ .

Then, we have  $(\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y]) \equiv \forall y (\mathcal{M}_{\mathcal{D}} \models G[\vec{x}, y])$ , and therefore :

$$\mathbb{1} \Vdash (\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y]) \rightarrow (\mathcal{M}_{\mathcal{D}} \models G[\vec{x}, f_G(\vec{x})]).$$

Therefore, by the recurrence hypothesis, we have :

$$\Vdash (\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y]) \rightarrow \mathcal{D}\langle G[\vec{x}, f_G(\vec{x})] \rangle.$$

Applying lemma 25(ii), we obtain  $\Vdash (\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y]) \rightarrow \mathcal{D}\langle \forall y G[\vec{x}, y] \rangle$ .

Conversely, by lemma 25(i), we have  $\Vdash \forall y (\mathcal{D}\langle \forall y G[\vec{x}, y] \rangle \rightarrow \mathcal{D}\langle G[\vec{x}, y] \rangle)$ .

Therefore, applying the recurrence hypothesis, we obtain :

$$\Vdash \mathcal{D}\langle \forall y G[\vec{x}, y] \rangle \rightarrow \forall y (\mathcal{M}_{\mathcal{D}} \models G[\vec{x}, y]),$$

and thus, by definition of  $(\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y])$  :

$$\Vdash \mathcal{D}\langle \forall y G[\vec{x}, y] \rangle \rightarrow (\mathcal{M}_{\mathcal{D}} \models \forall y G[\vec{x}, y]).$$

Q.E.D.

**Theorem 27.**  $\mathcal{M}_{\mathcal{D}}$  is an elementary extension of the ground model  $\mathcal{M}$ .

Let  $F[\vec{a}]$  be a closed formula of ZF, with parameters  $a_1, \dots, a_n$  in  $\mathcal{M}$ .

If  $\mathcal{M} \models F[\vec{a}]$ , we have  $\langle F[\vec{a}] \rangle = 1$  (by definition), and therefore, of course,  $\Vdash \mathcal{D}\langle F[\vec{a}] \rangle$ .

Therefore, by lemma 26, we have  $\Vdash (\mathcal{M}_{\mathcal{D}} \models F[\vec{a}])$ .

If  $\mathcal{M} \not\models F[\vec{a}]$ , then  $\mathcal{M} \models \neg F[\vec{a}]$ ; therefore, we have  $\Vdash (\mathcal{M}_{\mathcal{D}} \models \neg F[\vec{a}])$ .

Q.E.D.

**Remark.** Theorem 27 is, in fact, true for any ultrafilter on  $\mathfrak{I}2$ , with the same proof.

**Theorem 28.** Let  $\sqsubset$  be a well founded binary relation, defined in the ground model  $\mathcal{M}$ . Then the relation  $\mathcal{D}\langle x \sqsubset y \rangle$  is well founded in the realizability model  $\mathcal{N}$ .

**Remark.** Theorem 28 is an improvement on theorem 8.

**Notations.** We shall write  $x \sqsubset_{\mathcal{D}} y$  for  $\langle x \sqsubset y \rangle \in \mathcal{D}$ .

Recall that  $x < y$  means  $x \in \text{Cl}(y)$ ; and that  $x <_{\alpha} y$  means  $\langle x < y \rangle \geq \alpha$ , for  $\alpha \in \mathfrak{I}2$ .

We define, in the model  $\mathcal{M}$ , a binary relation  $\sqsubset$  on the class  $\{0, 1\} \times \mathcal{M}$ , by setting, for any  $\alpha, \alpha' \in \{0, 1\}$  and  $a, a'$  in  $\mathcal{M}$  :

$$(\alpha', a') \sqsubset (\alpha, a) \Leftrightarrow (\alpha' < \alpha) \vee (\alpha = \alpha' = 0 \wedge a' < a) \vee (\alpha = \alpha' = 1 \wedge a' \sqsubset a).$$

The relation  $\sqsubset$  is the *ordered direct sum* of the relations  $\sqsubset, <$ .

It is easily shown that it is *well founded* in  $\mathcal{M}$ .

The binary function symbol associated with this relation, of domain  $\{0, 1\} \times \mathcal{M}$  and values in  $\{0, 1\}$ , is given by :

$$\langle (\alpha', a') \sqsubset (\alpha, a) \rangle = (\neg \alpha' \wedge \alpha) \vee (\neg \alpha' \wedge \alpha \wedge \langle a' < a \rangle) \vee (\alpha' \wedge \alpha \wedge \langle a' \sqsubset a \rangle).$$

This definition gives, in  $\mathcal{N}$ , a binary function symbol with arguments in  $\mathbb{J}2 \times \mathcal{N}$ , and values in  $\mathbb{J}2$ .

By theorem 8, the binary relation  $\langle (\alpha', a') \sqsubset (\alpha, a) \rangle = 1$  is well founded in  $\mathcal{N}$ .

*Proof of theorem 28.*

Proof by contradiction : we assume that the binary relation  $\sqsubset_{\mathcal{D}}$  is not well founded.

Thus, there exists  $a_0, A_0$  such that  $a_0 \in A_0$  and  $A_0$  has no minimal  $\varepsilon$ -element for  $\sqsubset_{\mathcal{D}}$ .

We define, in  $\mathcal{N}$ , the class  $\mathcal{X}$  of ordered pairs  $(\alpha, x)$ , such that :

*There exists  $X$  such that  $x \in X$  and  $X$  has no minimal  $\varepsilon$ -element, neither for  $\sqsubset_{\mathcal{D}}$  nor for  $<_{-\alpha}$ .*

Therefore, the formula  $\mathcal{X}(\alpha, x)$  is :

$$\alpha \varepsilon \mathbb{J}2 \wedge \exists X \left\{ x \in X, (\forall u \varepsilon X) \{ (\exists v \varepsilon X) (v \sqsubset_{\mathcal{D}} u), (\exists w \varepsilon X) (w <_{-\alpha} u) \} \right\}.$$

If  $(\alpha, x)$  is in  $\mathcal{X}$ , then we have  $\mathcal{D}(\alpha)$  : indeed, the set  $X$  is non void and has no minimal  $\varepsilon$ -element for  $<_{-\alpha}$ . Therefore, we have  $\neg \mathcal{D}(-\alpha)$ , and thus  $\mathcal{D}(\alpha)$ , since  $\mathcal{D}$  is an ultrafilter.

We obtain the desired contradiction by showing that the class  $\mathcal{X}$  is non void and has no minimal element for the binary relation  $\langle (\alpha', x') \sqsubset (\alpha, x) \rangle = 1$ .

The ordered pair  $(1, a_0)$  is in  $\mathcal{X}$  : indeed, we have  $x <_0 x$  for every  $x$ , and therefore  $A_0$  has no minimal  $\varepsilon$ -element for  $<_0$ .

Now let  $(\alpha, a)$  be in  $\mathcal{X}$  ; we search for  $(\alpha', a')$  in  $\mathcal{X}$  such that  $\langle (\alpha', a') \sqsubset (\alpha, a) \rangle = 1$ .

By hypothesis on  $(\alpha, a)$ , there exists  $A$  such that  $a \in A$  and  $A$  has no minimal  $\varepsilon$ -element, neither for  $\sqsubset_{\mathcal{D}}$  nor for  $<_{-\alpha}$ . Thus, there exists  $a^0, a^1 \in A$  such that we have  $\mathcal{D}(a^0 \sqsubset a)$  and  $a^1 <_{-\alpha} a$ .

We set  $\alpha' = (\alpha \wedge \langle a^0 \sqsubset a \rangle)$  and therefore, we have  $\mathcal{D}(\alpha')$ . We set  $\beta = -\alpha' \wedge \alpha$  ; therefore  $\alpha', -\alpha, \beta$  form a partition of 1 in the Boolean algebra  $\mathbb{J}2$ .

We have  $\neg \mathcal{D}(\beta)$  ; therefore, by definition of  $\mathcal{D}$ , the relation  $<_{\beta}$  is not well founded. Thus, there exists  $b, B$  such that  $b \in B$  and  $B$  has no minimal  $\varepsilon$ -element for  $<_{\beta}$ . Then, we set :

$$a' = \alpha' a^0 \sqcup (-\alpha) a^1 \sqcup \beta b \quad \text{and} \quad A' = \{ \alpha' x \sqcup (-\alpha) y \sqcup \beta z ; x, y \in A, z \in B \}.$$

Therefore, we have  $a' \in A'$ , as needed ; moreover :

$$-\alpha' \wedge -\alpha \wedge \langle a' < a \rangle = -\alpha, \text{ since } -\alpha' \geq -\alpha \text{ and } \langle a' < a \rangle \geq -\alpha \wedge \langle a^1 < a \rangle = -\alpha ;$$

$$\alpha' \wedge \alpha \wedge \langle a' \sqsubset a \rangle = \alpha' \wedge \langle a' \sqsubset a \rangle = \alpha' \wedge \langle a^0 \sqsubset a \rangle = \alpha'.$$

By definition of  $\langle (\alpha', a') \sqsubset (\alpha, a) \rangle$ , it follows that  $\langle (\alpha', a') \sqsubset (\alpha, a) \rangle = \beta \vee -\alpha \vee \alpha' = 1$ .

It remains to show that  $A'$  has no minimal  $\varepsilon$ -element for  $\sqsubset_{\mathcal{D}}$  and for  $<_{-\alpha'}$ .

Therefore, let  $u \in A'$ , thus  $u = \alpha' x \sqcup (-\alpha) y \sqcup \beta z$  with  $x, y \in A$  and  $z \in B$ .

By hypothesis on  $A, B$ , there exists  $x', y' \in A, x' \sqsubset_{\mathcal{D}} x, y' <_{-\alpha} y$  and  $z' \in B, z' <_{\beta} z$ .

Then, if we set  $u' = \alpha' x' \sqcup (-\alpha) y' \sqcup \beta z'$ , we have  $u' \in A'$ .

Moreover, we have  $\langle u' \sqsubset u \rangle \geq \alpha' \wedge \langle x' \sqsubset x \rangle$ , and therefore  $\mathcal{D}(u' \sqsubset u)$ , that is  $u' \sqsubset_{\mathcal{D}} u$ .

Finally,  $\langle u' < u \rangle \geq (-\alpha \wedge \langle y' < y \rangle) \vee (\beta \wedge \langle z' < z \rangle) = -\alpha \vee \beta = -\alpha'$  ; therefore, we have  $u' <_{-\alpha'} u$ .

Q.E.D.

**Theorem 29.**  $\mathcal{M}_{\mathcal{D}}$  is well founded, and therefore has the same ordinals as  $\mathcal{N}'_{\varepsilon}$ .

We apply theorem 28 to the binary relation  $\in$  which is well founded in  $\mathcal{M}$ . We deduce that the relation  $\mathcal{D}(x \in y)$ , that is  $x \in_{\mathcal{D}} y$ , is well founded in  $\mathcal{N}$ .

Q.E.D.

The relation  $\in_{\mathcal{D}}$  is well founded *and extensional*, which means that we have, in  $\mathcal{N}$  :

$$\forall x \forall y (\forall z (z \in_{\mathcal{D}} x \leftrightarrow z \in_{\mathcal{D}} y) \rightarrow \forall z (x \in_{\mathcal{D}} z \rightarrow y \in_{\mathcal{D}} z)).$$

It follows that we can define a *collapsing*, by means of a function symbol  $\Phi$ , which is an *isomorphism* of  $(\mathcal{M}_{\mathcal{D}}, \in_{\mathcal{D}})$  on a transitive class in the model  $\mathcal{N}_{\in}$  of ZF, which contains the ordinals. This means that we have :

$$\forall x \forall y (y \in_{\mathcal{D}} x \rightarrow \Phi(y) \in \Phi(x)) ; \forall x (\forall z \in \Phi(x)) (\exists y \in_{\mathcal{D}} x) z \simeq \Phi(y).$$

The definition of  $\Phi$  is analogous with that of the *rank function* already defined for a *transitive* well founded relation. The details will be given in a later version of this paper. It follows that :

**Theorem 30.** *The realizability model  $\mathcal{N}_{\in}$  contains a transitive class, which contains the ordinals and is an elementary extension of the ground model  $\mathcal{M}$ .*

**Corollary 31.** *The class  $L^{\mathcal{M}}$  of constructible sets in  $\mathcal{M}$  is an elementary submodel of  $L^{\mathcal{N}}$ .*

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