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# DEFORMATION SPACES OF DISCRETE GROUPS OF $SU(2,1)$ IN QUATERNIONIC HYPERBOLIC PLANE: A CASE STUDY

ANTONIN GUILLOUX AND INKANG KIM

ABSTRACT. In this note, we study deformations of discrete and Zariski dense subgroups of  $SU(2,1)$  in the isometry group  $Sp(2,1)$  of quaternionic hyperbolic space. Specifically we consider two examples coming from representations of 3-manifold groups (the figure eight knot and Whitehead links complement) and show opposite behaviors: one is not deformable outside  $U(2,1)$ , while the other has a big space of deformations in  $Sp(2,1)$ .

## 1. INTRODUCTION

In 1960's, A. Weil [24] proved a local rigidity of a uniform lattice  $\Gamma \subset G$  inside  $G$ : he showed that  $H^1(\Gamma, \mathfrak{g}) = 0$  for any semisimple Lie group  $G$  not locally isomorphic to  $SL(2, \mathbb{R})$ . This result implies that the canonical inclusion map  $i : \Gamma \hookrightarrow G$  is locally rigid up to conjugacy. In other words, for any local deformation  $\rho_t : \Gamma \rightarrow G$  such that  $\rho_0 = i$ , there exists a continuous family  $g_t \in G$  such that  $\rho_t = g_t \rho_0 g_t^{-1}$ . Weil's idea is further explored by many others but notably by Raghunathan [21] and Matsushima-Murakami [19]. Much later Goldman and Millson [10] considered the embedding of a uniform lattice  $\Gamma$  of  $SU(n, 1)$

$$\Gamma \hookrightarrow SU(n, 1) \hookrightarrow SU(n+1, 1)$$

and proved that there is still a local rigidity inside  $SU(n+1, 1)$  if one ignores a deformation coming from the center. More recently further examples [16, 17, 15, 18] of local rigidity of a complex hyperbolic lattice in quaternionic Kähler manifolds are described in the following situations:

$$\Gamma \hookrightarrow SU(n, 1) \subset Sp(n, 1) \subset SU(2n, 2) \subset SO(4n, 4),$$

$$\Gamma \hookrightarrow SU(n, 1) \subset SU(p, q),$$

$$\Gamma \hookrightarrow SU(n, 1) \subset Sp(n+1, \mathbb{R}),$$

$$\Gamma \hookrightarrow SU(n, 1) \subset SO(2n, 2).$$

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But all these examples deal with the standard inclusion map  $\Gamma \hookrightarrow G'$  to use the Weil's original idea about  $L^2$ -group cohomology. We look in this paper at the more general setting of a representation  $\rho : \Gamma \rightarrow G \subset G'$ . We focus our attention to the case where the representation is discrete and has Zariski-dense image in  $G$ . We seek the possibility of deforming  $\rho$  in  $G'$  without being conjugate to a representation landing in  $G$ .

In general, very little is known on this general problem. We study here deformations of two representations of non-uniform lattices of  $\mathrm{SL}(2, \mathbb{C})$  inside  $\mathrm{Sp}(2, 1)$ . Indeed, let  $M_8$  be the figure eight knot complement and denote by  $\Gamma_8$  its fundamental group, and let  $M_W$  be the Whitehead link complement and  $\Gamma_W$  its fundamental group.

The character variety  $\chi(\Gamma_8, \mathrm{SU}(2, 1))$  is fully understood [7] (see section 3 for the definition of character variety), and it contains 2 (up to some equivalences) boundary unipotent irreducible representations  $\rho_0$  and  $\rho_1$  which are already obtained in [9], see also [8]. We will be mainly interested in the representation  $\rho_0$  whose image is generated by the following matrices in  $\mathrm{SU}(2, 1)$ :

$$\begin{bmatrix} 1 & 1 & \frac{-1-i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{-1-i\sqrt{3}}{2} & -1 & 1 \end{bmatrix}$$

In particular, we see that the image of  $\rho_0$  is included in the Eisenstein-Picard arithmetic lattice of  $\mathrm{SU}(2, 1)$ . It turns out that it is a *thin* subgroup, as it is Zariski-dense. We will show that  $\rho_0$ , as its surrounding lattice, is not deformable outside  $\mathrm{U}(2, 1)$ . Recently some thin subgroups of finite index in  $\Gamma_8$  were constructed inside lattices in  $\mathrm{SL}(4, \mathbb{R})$ , that are indeed deformable (inside  $\mathrm{SL}(4, \mathbb{R})$ ) [2].

Our knowledge of the character variety  $\chi(\Gamma_W, \mathrm{SU}(2, 1))$  is far less thorough. Boundary unipotent representations are described in [8], whereas a component of this character variety has been described in [13]. We will consider a representation  $\rho_W$  inside this component. Note that the image of  $\rho_W$  is a free product of two copies of  $\mathbb{Z}/3\mathbb{Z}$  and is not contained in an arithmetic lattice. We will prove that  $\rho_W$  has a big space of deformations in  $\mathrm{Sp}(2, 1)$  and is therefore deformable outside  $\mathrm{U}(2, 1)$ .

We will first describe what is known about  $\rho_0$  and  $\rho_W$ , exhibiting structural differences. We then prove that the first one is rigid whereas the second one is deformable. It would be very interesting to understand which properties of these representations lead to the rigidity or deformability.

## 2. TWO OPPOSITE BEHAVIORS

2.1. **Rigidity of  $\rho_0$ .** The fundamental group  $\Gamma_8$  has a presentation [7]:

$$\Gamma_8 = \langle a, b | b^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1} \rangle.$$

We consider the representation  $\rho_0$  defined by the images of the generators:

$$\rho_0(a) = \begin{bmatrix} 1 & 1 & \frac{-1-i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_0(b) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{-1-i\sqrt{3}}{2} & -1 & 1 \end{bmatrix}$$

We prove in this paper that the representation  $\rho_0$  cannot be deformed locally outside  $U(2, 1)$ . The proof is fairly straightforward, though involved computations are tedious. Here are the steps:

- (1) As we will see in section 3.2, at  $[\rho_0]$ , the character variety  $\chi(\Gamma_8, U(2, 1))$  is 3-dimensional.
- (2) We are able to compute the tangent space to  $\chi(\Gamma_8, \text{Sp}(2, 1))$  at  $[\rho_0]$ : it amounts to compute  $H^1(\Gamma_8, \mathfrak{sp}(2, 1)_{\text{ad}(\rho_0)})$ . This homological computation will be explained in section 4. The computed dimension is 3.
- (3) As we will recall in section 3.1, the natural map  $\chi(\Gamma_8, U(2, 1)) \rightarrow \chi(\Gamma_8, \text{Sp}(2, 1))$  is a local diffeomorphism onto its image.

Knowing these three facts, we see:

**Proposition 2.1.** *Every small deformation of  $\rho_0 : \Gamma_8 \rightarrow \text{Sp}(2, 1)$  results in a representation conjugate to a representation  $\Gamma_8 \rightarrow U(2, 1)$ .*

**2.2. Deformability of  $\rho_W$ .** Following [13], the fundamental group  $\Gamma_W$  has a presentation:

$$\Gamma_W = \langle a, b \mid aba^{-3}b^2a^{-1}b^{-1}a^3b^{-2} \rangle.$$

We consider the representation  $\rho_W$  defined by the images of the generators:

$$\rho_W(a) = \begin{bmatrix} 1 & \frac{\sqrt{3}-i\sqrt{5}}{2} & -1 \\ \frac{-\sqrt{3}-i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \rho_W(b) = \begin{bmatrix} 1 & -\frac{\sqrt{3}+i\sqrt{5}}{2} & -1 \\ \frac{\sqrt{3}-i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Unlike the previous example, we prove in this paper that the representation  $\rho_W$  can be deformed locally outside  $U(2, 1)$ . The proof is once again fairly straightforward. Here are the steps:

- (1)  $\rho_W$  factors through a quotient  $\mathbb{Z}_3 * \mathbb{Z}_3$ , and the whole component of the  $SU(2, 1)$ -character variety of  $\Gamma_W$  does (see section 3.3).
- (2) The  $U(2, 1)$ -character variety has dimension 6 at  $\rho_W$  (see section 3.3).
- (3) The  $\text{Sp}(2, 1)$ -character of  $\mathbb{Z}_3 * \mathbb{Z}_3$  at  $\rho_W$  has dimension at least 7 (see section 5).

We hence see that the  $\text{Sp}(2, 1)$ -character variety of  $\Gamma_W$  has dimension, at  $\rho_W$ , at least 1 more than the dimension of the  $U(2, 1)$ -character variety. It yields:

**Proposition 2.2.** *There are small deformations of  $\rho_W : \Gamma_W \rightarrow \text{Sp}(2, 1)$  which are not conjugate to any representation  $\Gamma_W \rightarrow U(2, 1)$ .*

## 3. CHARACTER VARIETIES

The  $G$ -character variety of  $\pi_1(M)$ , denoted  $\chi(\pi_1(M), G)$ , is the geometric invariant theory quotient of  $\text{Hom}(\pi_1(M), G)$  by inner automorphisms of  $G$ . Often, some components of the character variety are realized as the space of  $(G, X)$ -structures on a given manifold  $M$ . Thurston studied the Dehn surgery space of a hyperbolic knot complement in the early 70s using the idea of gluing tetrahedra in hyperbolic 3-space. In his case, the variety appears as defined by his gluing equations [23]. Thurston's approach is generalized to several different directions corresponding to different geometric structures such as spherical CR structure and real projective structure associated with Lie groups  $\text{SU}(2, 1)$  and  $\text{SL}(3, \mathbb{R})$  respectively. The latter one is known as a Hitchin component consisting of convex real projective structures on a closed surface [14].

**3.1. General facts and definitions.** For a given reductive algebraic group  $G \subset \text{GL}(m, k)$  defined over  $k$ , and a finitely generated group  $\Gamma$  with  $n$ -generators, the **representation variety** is  $R(\Gamma, G) = \text{Hom}(\Gamma, G) \subset G^n$ , defined by the zero set of polynomials in  $k[x_1, \dots, x_{nm^2}]$ . In this paper,  $k = \mathbb{R}$  or  $\mathbb{C}$ . A representation  $\rho : \Gamma \rightarrow G$  is **Zariski dense** if the Zariski closure of the image is  $G$ . The group  $G$  acts on  $R(\Gamma, G)$  by conjugation, and it is well-known that the orbit of  $\rho$  under conjugation is closed if  $\rho$  is Zariski dense.

Since the orbit under the conjugation is not closed in general, the quotient space of  $R(\Gamma, G)$  under conjugation is not in general a Hausdorff space. To avoid this phenomenon, one takes the GIT quotient  $\chi(\Gamma, G) = R(\Gamma, G)//G$  to get again an algebraic set, called the **character variety**.

In this paper, all the representations we are considering are not contained in  $P \times Z(G)$  where  $P$  is a parabolic subgroup and  $Z(G)$  is the center of  $G$ . In this case, the quotient of  $R(\Gamma, G)$  by the conjugation action of  $G$  is nice around  $\rho$  [11, Section 1.3], and we can assure that the Zariski tangent space of the character variety at  $[\rho]$  can be computed by the first group cohomology of  $\Gamma$  with coefficient in  $\mathfrak{g}_{Ad\rho}$ .

We will need the following later.

**Lemma 3.1.** *Let  $\nu_1 : \Gamma \rightarrow \text{U}(2, 1)$  be a Zariski dense representation which is conjugate to  $\nu_2 : \Gamma \rightarrow \text{U}(2, 1)$  in  $\text{Sp}(2, 1)$ . Then  $\nu_1$  is conjugate in  $\text{SU}(2, 1)$  to either  $\nu_2$  or  $\overline{\nu_2}$ .*

**Proof:** Suppose  $Q\nu_1Q^{-1} = \nu_2$  for  $Q \in \text{Sp}(2, 1)$ . Since  $\nu_1$  is Zariski dense,  $Q$  stabilizes  $H_{\mathbb{C}}^2$  inside  $H_{\mathbb{H}}^2$ . If it is holomorphic,  $Q \in \text{SU}(2, 1)$ . Suppose it is anti-holomorphic. Any anti-holomorphic element in  $H_{\mathbb{C}}^2$  can be written as  $\iota$  followed by an element in  $U(2, 1)$  where  $\iota$  is a reflection along  $H_{\mathbb{R}}^2$ . By absorbing the element in  $U(2, 1)$  we may assume that  $Q$  restricted to  $H_{\mathbb{C}}^2$  is  $\iota$ . Now,  $\iota$  can be realized as a complex conjugate  $(z, w) \rightarrow (\bar{z}, \bar{w})$  in unit ball model.

Then  $Q$  is realized by a diagonal matrix with entries  $(j, j, j)$ . Hence we get  $Q\nu_1Q^{-1} = \overline{\nu_1}$  and  $\nu_1 = \overline{\nu_2}$ . ■

**3.2. Description of the  $U(2, 1)$ -character variety for  $\Gamma_8$ .** Let  $\Gamma_8$  denote the fundamental group of the figure eight knot complement in  $\mathbb{S}^3$ . Falbel-Guilloux-Koseleff-Rouillier-Thistlethwaite [7] studied the character variety of  $\Gamma_8$  in  $\mathrm{PGL}(3, \mathbb{C})$  and  $\mathrm{PU}(2, 1)$ . They describe a Zariski open set, through a variant of the character variety: the deformation variety. They show that there exist three irreducible components of the deformation variety. Each one of these components is smooth of complex dimension two and contains a real-dimension 2 subvariety of representations landing in  $\mathrm{PU}(2, 1)$  [7, Section 5.3].

**Proposition 3.2.** *The component of the  $U(2, 1)$ -character variety  $\chi(\Gamma_8, U(2, 1))$  through  $\rho_0$  has dimension 3.*

*Proof.* First of all,  $\rho_0$  belongs to one of the components described in [7]: it corresponds to the point  $(u, v) = (-\sqrt{3}i, 2)$  from [7, Section 5.3]. Hence, we know that the component of the  $SU(2, 1)$ -character variety  $\chi(\Gamma_8, SU(2, 1))$  through  $\rho_0$  has real dimension 2.

Then  $\Gamma_8$  is the fundamental group of a knot complement, so its abelianization is  $\mathbb{Z}$ . Hence the character variety from  $\Gamma_8$  to the center  $U(1)$  of  $U(2, 1)$  is of real dimension 1.

Now any representation  $\Gamma_8 \rightarrow U(2, 1)$  can be locally decomposed as product of a representation in its center and a representation in  $SU(2, 1)$ . We get that the component of the  $U(2, 1)$ -character variety  $\chi(\Gamma_8, U(2, 1))$  through  $\rho_0$  has dimension 3. □

**3.3. A known component of the  $U(2, 1)$ -character variety for  $\Gamma_W$ .** Guilloux-Will studied the character variety  $\chi(\Gamma_W, \mathrm{SL}(3, \mathbb{C}))$ . They showed that the representations studied by Schwartz, Deraux, Falbel, Acosta, Parker, Will [22, 5, 6, 1, 20] all belong to a common algebraic component  $X_0$  consisting of representations that factor through the group  $\pi' = \mathbb{Z}_3 * \mathbb{Z}_3$ . Here  $X_0$  is the character variety of  $\pi'$  consisting of representations whose images are generated by two regular order 3 elements in  $\mathrm{SL}(3, \mathbb{C})$ .  $X_0$  is of complex dimension 4, and the subset of representations in  $SU(2, 1)$  is of real dimension 4.

Moreover, the representation  $\rho_W$  belongs to this component  $X_0$  [13, Section 3.4]. Using that the abelianization of  $\Gamma_W$  is  $\mathbb{Z}^2$ , we get as before:

**Proposition 3.3.** *The component of the  $U(2, 1)$ -character variety  $\chi(\Gamma_W, U(2, 1))$  through  $\rho_0$  has dimension 6.*

## 4. FOX CALCULUS AND HOMOLOGICAL COMPUTATIONS

**4.1. General presentation.** In this section, we briefly introduce a Fox calculus which is necessary for the calculation of the first group cohomology

and the Zariski tangent space of  $\text{Hom}(\pi, G)$ . For a detailed exposition, refer to [11] Section 3. Such computations have already been used, e.g. in [3]. Let  $F_n$  be a free group on  $n$ -generators  $x_1, \dots, x_n$  and  $\mathbb{Z}F_n$  the integral group ring. The augmentation homomorphism is a ring homomorphism

$$\epsilon : \mathbb{Z}F_n \rightarrow \mathbb{Z}$$

which maps an element  $\sum_{\sigma \in F_n} m_\sigma \sigma$  to the coefficient sum  $\sum_{\sigma \in F_n} m_\sigma$ . A derivation is a  $\mathbb{Z}$ -linear map  $D : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$  satisfying

$$D(m_1 m_2) = D(m_1)\epsilon(m_2) + m_1 D(m_2).$$

Then the set of derivations  $\text{Der}(F_n)$  is freely generated as a right  $\mathbb{Z}F_n$ -module by  $n$  elements  $\partial_i = \frac{\partial}{\partial x_i}$  which satisfy  $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$ . This derivation satisfies a useful rule of differential calculus, a mean value theorem,

$$u - \epsilon(u) = \sum (\partial_i u)(x_i - 1)$$

for any  $u \in \mathbb{Z}F_n$ .

Let  $\phi : F_n \rightarrow \text{GL}(V)$  be a linear representation, which extends to a ring homomorphism  $\mathbb{Z}F_n \rightarrow \text{End}(V)$ . Then a cocycle  $u : \mathbb{Z}F_n \rightarrow V$  which satisfies the cocycle identity  $u(ab) = u(a)\epsilon(b) + \phi(a)u(b)$ , can be written using the mean value theorem as

$$u(w) = \sum_{i=1}^n \phi(\partial_i w)u(x_i).$$

Using this Fox calculus, we can describe the Zariski tangent space to  $\text{Hom}(\pi, G) \subset G^n$  for a group  $\pi = F_n/\mathcal{R}$  where  $\mathcal{R}$  is a normal subgroup of  $F_n$  consisting of relations and  $G$  is a Lie group whose Lie algebra is denoted by  $\mathfrak{g}$ . Since an element in  $\text{Hom}(\pi, G)$  corresponds to an element  $\phi \in \text{Hom}(F_n, G)$  satisfying  $\phi(R) = 1$  for all  $R \in \mathcal{R}$ , the Zariski tangent space to  $\text{Hom}(\pi, G)$  at  $\phi \in \text{Hom}(\pi, G)$  is the space of cocycles

$$Z^1(\pi, \mathfrak{g}_{\text{Ad}\phi}) = \{(u_1, \dots, u_n) \in \mathfrak{g}^n \mid \sum_{i=1}^n \text{Ad}\phi(\partial_i R)u_i = 0, \text{ for all } R \in \mathcal{R}\}$$

by associating  $(\mu(x_1), \dots, \mu(x_n))$  to each 1-cocycle  $\mu$ .

Moreover, in order to have the Zariski tangent space to the character variety, you have to mod out by the coboundaries  $B^1(\pi, \mathfrak{g}_{\text{Ad}\phi})$ . In this setting, a coboundary is an element  $(u_1, \dots, u_n) \in \mathfrak{g}^n$  such that there exist some  $u \in \mathfrak{g}$  with:

$$\forall 1 \leq i \leq n, \quad u_i = \text{Ad}\phi(x_i)u - u.$$

**4.2. Effective computations for  $\Gamma_8$ .** The material presented above can be tackled in a very concrete and effective manner. Let us describe the involved computations for the representation  $\rho_0 : \Gamma_8 \rightarrow \text{Sp}(2, 1)$ . The actual computations are basic linear algebra, but with matrices a bit too big to be fully displayed here. A Sage Notebook [12] is available showing the computations done by a computer algebra system.

First of all, we use Fox calculus on our presentation of  $\Gamma_8$ :

$$\Gamma_8 = \langle a, b | b^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1} \rangle.$$

Let us denote by  $R$  the relation  $b^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1}$ . A straightforward computation gives:

- $\partial_a R = b^{-1} - b^{-1}aba^{-1} + b^{-1}aba^{-1}b - b^{-1}aba^{-1}bab^{-1}a^{-1} - b^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1}$ .
- $\partial_b R = -b^{-1} + b^{-1}a + b^{-1}aba^{-1} - b^{-1}aba^{-1}bab^{-1} + b^{-1}aba^{-1}bab^{-1}a^{-1}$ .

Let us note that in Sagemath, the Fox calculus is implemented and the result of this computation is given by the so-called Alexander matrix.

From this, the whole computation of the Zariski tangent space follows. This computation can be seen in the notebook, and the steps are:

- Choose a basis for  $\mathfrak{sp}(2, 1)$ : its cardinality is 21. Pairs of vectors in this basis give a basis of the cochains  $C^1$ : as presented above, a cochain is seen as an element of  $\mathfrak{sp}(2, 1)^2$ .
- Compute both  $21 \times 21$  matrices representing in this basis the adjoint action  $\text{Ad}(x)$  and  $\text{Ad}(y)$  of the generators,  $x = \rho_0(a)$ ,  $y = \rho_0(b)$ .
- Compute  $B^1$  as the image of the  $42 \times 21$  matrix  $\begin{pmatrix} \text{Ad}(x) - \text{id} \\ \text{Ad}(y) - \text{id} \end{pmatrix}$  in the chosen basis.
- Using the different terms  $\partial_a R$ ,  $\partial_b R$  appearing in the definition of  $Z^1$  as in the previous lemma, applying  $\text{Ad}(\rho_0)$  to these expressions, we get the  $21 \times 42$  matrix whose kernel is  $Z^1$ :

$$(\text{Ad } \rho_0(\partial_a R), \text{Ad } \rho_0(\partial_b R)) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

- Compute the dimension of  $Z^1/B^1$ . Note that  $\rho_0$  has entries in a number field: the computation can be done exactly and the computed dimension has a true meaning.

As a result of this computation, we get:

**Proposition 4.1.** *The component of the character variety  $\chi(\Gamma_8, \text{Sp}(2, 1))$  through  $\rho_0$  has dimension 3.*

Proof of Proposition 2.1: The Lie algebra  $\mathfrak{sp}(2, 1)$  decomposes as  $\mathfrak{u}(2, 1) \oplus S^2\mathbb{C}^3$  under  $\rho_0$  as a real representation, see [16]. Hence

$$H^1(\Gamma_8, \mathfrak{sp}(2, 1)) = H^1(\Gamma_8, \mathfrak{u}(2, 1)) + H^1(\Gamma_8, S^2\mathbb{C}^3).$$

By above Propositions 3.2 and 4.1,  $H^1(\Gamma_8, S^2\mathbb{C}^3) = 0$ , which implies that all the small deformations of  $\rho_0$  in  $\text{Sp}(2, 1)$  are conjugate to the ones in  $\text{U}(2, 1)$ .

## 5. ORDER 3 ELEMENTS AND THE DEFORMATION OF $\rho_W$

We compute in this section a lower bound on the dimension of a component of the character variety  $\chi(\Gamma_W, \text{Sp}(2, 1))$ :

**Proposition 5.1.** *The dimension around  $[\rho_W]$  of the character variety  $\chi(\Gamma_W, \text{Sp}(2, 1))$  is at least 7.*



*Proof.* As we saw in Section 2.2, the image of  $\rho_W$  is isomorphic to  $\mathbb{Z}_3 \star \mathbb{Z}_3$ , with  $\rho_W(a)$  and  $\rho_W(b)$  being two order 3 generators.

Moreover, as recalled in section 3.3, the whole component of the  $SU(2, 1)$ -character variety containing  $[\rho_W]$  is made from representations  $[\rho]$  with  $\rho(a)$  and  $\rho(b)$  being two order 3 elements of  $SU(2, 1)$ .

Let  $\mathcal{E} = \{(A, B) \in \mathrm{Sp}(2, 1)^2 \mid A^3 = B^3 = 1\}$ . Then we have an inclusion  $\mathcal{E}/\mathrm{Sp}(2, 1) \rightarrow \chi(\Gamma_W, \mathrm{Sp}(2, 1))$ .

As a matrix of  $SU(2, 1)$ , the eigenvalues of  $\rho_W(\alpha)$  are  $1, \omega, \omega^2$ , where  $\omega^3 = 1$  in  $\mathbb{C}$ . So, inside  $\mathrm{Sp}(2, 1)$ ,  $\rho_W(\alpha)$  is conjugate [4] to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

By deforming the pair  $(\rho_W(\alpha), \rho_W(\beta))$  to a pair of order 3 matrices in  $\mathrm{Sp}(2, 1)$  and up to conjugation, we may assume that the first one always equals  $A$ . Its centralizer [4, Section 5.1] in  $\mathrm{Sp}(2, 1)$  is the subgroup of block-diagonal matrices:

$$Z = \left\{ \begin{pmatrix} x & \\ & X \end{pmatrix} \in \mathrm{Sp}(2, 1) \mid \text{where } x \in \mathbb{H}, X \in U(2) \right\}.$$

Note that the dimension of  $Z$  is 7.

The second matrix  $B$  of the pair is another order 3 matrix, conjugate to  $A$ . So we are indeed looking at the set of pairs  $(A, gAg^{-1})$  up to conjugation. In other terms, let

$$\mathcal{E}' = \{(A, gAg^{-1}) \text{ for } g \in \mathrm{Sp}(2, 1)\}.$$

Then locally around  $[\rho_W]$  we have  $\mathcal{E}/\mathrm{Sp}(2, 1) = \mathcal{E}'/Z$ .

Eventually, we see that for any  $g \in \mathrm{Sp}(2, 1)$  and  $h \in \mathrm{Sp}(2, 1)$ , the two pairs  $(A, gAg^{-1})$  and  $(A, hAh^{-1})$  are conjugate if and only if there exist  $z_1$  and  $z_2$  in  $Z$  such that  $h = z_1gz_2$ .

Hence the dimension of  $\mathcal{E}'/Z$  is at least  $\dim(\mathrm{Sp}(2, 1)) - 2\dim(Z) = 7$ . This implies that the dimension around  $[\rho_W]$  of  $\chi(\Gamma_W, \mathrm{Sp}(2, 1))$  is at least 7.  $\square$

Indeed, the dimension of  $\chi(\Gamma_W, \mathrm{Sp}(2, 1))$  around any point in the component  $\mathcal{C}$  containing  $[\rho_W]$  of  $\chi(\Gamma_W, U(2, 1))$  is at least 7 as we can see as follows. Note that any point in  $\mathcal{C}$  can be written as a pair  $(\alpha C, \beta B)$  with  $\alpha, \beta \in U(1)$  and  $C^3 = B^3 = I$  in  $SU(2, 1)$ . This point is conjugate to  $(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}h_0Ah_0^{-1})$  for some  $g_0, h_0 \in \mathrm{Sp}(2, 1)$ . Now by varying  $h \in \mathrm{Sp}(2, 1)$ , as in the proof above, there are at least 7-dimensional space of  $\{(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}hAh^{-1}) \mid h \in \mathrm{Sp}(2, 1)\}$  near  $(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}h_0Ah_0^{-1})$  in  $\chi(\Gamma_W, \mathrm{Sp}(2, 1))$ .

Note that the proposition 2.2 is now proven: the space of deformations of  $\rho_W$  in  $\mathrm{Sp}(2, 1)$  has bigger dimension than the space of deformations in  $U(2, 1)$  showing that some deformations are not conjugate to  $U(2, 1)$ .

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