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Foliated fields of an extension of a Lie G-foliation having dense leaves on a compact manifold.

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Abstract

Is shown in ([3], [4]) as any extension of a \mathcal{G} -foliation \mathcal{F} having dense leaves on a compact manifold M corresponds a Lie subalgebra \mathcal{H} of \mathcal{G} .

We denote by $\mathcal{F}_{\mathcal{H}}$ the extension corresponding to the subalgebra \mathcal{H} .

In this paper we seek to determine the Lie algebra $\ell(M, \mathcal{F}_{\mathcal{H}})$ of transverse foliated fields of the extension $\mathcal{F}_{\mathcal{H}}$.

This study show us that

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}.$$

Key words and phrases: Lie foliation, Riemannian foliation, foliation having dense leaves, extension of a foliation .

Subject Classifications: 53C05, 53C12, 58A30.

1 Introduction

Is shown in ([3], [4]) as any extension of \mathcal{G} -foliation \mathcal{F} having dense leaves on a compact manifold M corresponds a Lie subalgebra \mathcal{H} of \mathcal{G} .

We denote by $\mathcal{F}_{\mathcal{H}}$ the extension corresponding to the subalgebra \mathcal{H} .

The purpose of this article is to compute the Lie algebra $\ell(M, \mathcal{F}_{\mathcal{H}})$ of $\mathcal{F}_{\mathcal{H}}$ -foliated transverse vectors fields.

The following results were first established:

(i) all extension \mathcal{F}' of a Riemannian foliation \mathcal{F} having dense leaves on a compact manifold is a Riemannian foliation

(ii) If \mathcal{F}'' is an extension of Riemannian foliation \mathcal{F}' on a compact manifold M then $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$.

That said, we show in this paper that all informations about $\ell(M, \mathcal{F}_{\mathcal{H}})$ are contained in the Lie algebra \mathcal{G} of G .

In fact, it is established to an isomorphism of Lie algebra nearly that:

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for all } h \in \mathcal{H}\}.$$

This paper is articulated in three parts:

- The first part is devoted to reminders about extensions of foliation and $\frac{\mathcal{G}}{\mathcal{H}}$ -foliations,
- Extension of Riemannian foliation having dense leaves on a compact manifold,
- The third part is devoted to computation of $\ell(M, \mathcal{F}_{\mathcal{H}})$.

2 Definitions and Reminders

In this section, are formulated in the helpful sense some definitions and theorems which are in ([3], [4], [5]).

Definition 2.1 *An extension of a codimension q foliation (M, \mathcal{F}) is a codimension q' foliation (M, \mathcal{F}') such as $0 < q' < q$ and the leaves of (M, \mathcal{F}') are union of those of (M, \mathcal{F}) (we note $\mathcal{F} \subset \mathcal{F}'$).*

We show that if (M, \mathcal{F}') is a simple extension of a simple foliation (M, \mathcal{F}) and if (M, \mathcal{F}) and (M, \mathcal{F}') are respectively defined by submersions $\pi : M \rightarrow T$ and $\pi' : M \rightarrow T'$, then there exists a submersion $\theta : T \rightarrow T'$ such as $\pi' = \theta \circ \pi$.

We say that the submersion θ is a bond between the foliation (M, \mathcal{F}) and its extension (M, \mathcal{F}') .

It is shown in [3] if the foliation (M, \mathcal{F}) and its extension (M, \mathcal{F}') are respectively defined by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i \in I}$ and $(U_i, f'_i, T', \gamma'_{ij})_{i \in I}$ then we have:

$$f'_i = \theta_i \circ f_i \text{ and } \gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}$$

where θ_s is a bond between the foliation (U_s, \mathcal{F}) and its extension (U_s, \mathcal{F}') .

In what follows \mathcal{G} is the Lie algebra of a Lie group G of dimension q , \mathcal{H} is a Lie subalgebra of \mathcal{G} of codimension q' , (e_1, \dots, e_q) is a base of \mathcal{G} such as $(e_{q'+1}, \dots, e_q)$ may be a base of \mathcal{H} , ω be a 1-form on a manifold M taking values in \mathcal{G} , relatively to (e_1, \dots, e_q) we have $\omega = \sum_{i=1}^q \omega^i \otimes e_i$. What is written $\omega = (\omega^1, \dots, \omega^q)$.

If $d\omega + \frac{1}{2}[\omega, \omega] = 0$ and $\omega^1, \dots, \omega^{q'}$ are linearly independent at any point of M then the differential system $\omega^1 = \dots = \omega^{q'} = 0$ define a foliation $\mathcal{F}_{\mathcal{H}}$ of codimension q' .

Definition 2.2 *The foliation $\mathcal{F}_{\mathcal{H}}$ defined is called $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation defined by the 1-form ω .*

Proposition 2.3 Let $\mathcal{F}_{\mathcal{H}}$ be a $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation defined by a 1-form ω and let $\widetilde{\mathcal{F}}_{\mathcal{H}} = \pi^*\mathcal{F}_{\mathcal{H}}$ be the lifting of the $\mathcal{F}_{\mathcal{H}}$ -foliation on the universal covering $\pi : \widetilde{M} \rightarrow M$ of M .

Then, there is a differentiable map $\mathcal{D} : \widetilde{M} \rightarrow G$ transverse for the foliation $\mathcal{F}_{G,H}$ obtained by the left translations of H that is to say that for every $\tilde{x} \in \widetilde{M}$,

$$T_{\mathcal{D}(\tilde{x})}G = T_{\tilde{x}}\mathcal{D}\left(T_{\tilde{x}}\widetilde{M}\right) + T_{\mathcal{D}(\tilde{x})}\mathcal{F}_{G,H}$$

and a homomorphism $\rho : \pi_1(M) \rightarrow G$ such as:

(i) \mathcal{D} is ρ -equivariant that is to say that

$$\mathcal{D}(\gamma.\tilde{x}) = \rho(\gamma).\mathcal{D}(\tilde{x}),$$

(ii)

$$\pi^*\omega = \mathcal{D}^*\alpha \quad \text{that is to say } \widetilde{\mathcal{F}}_{\mathcal{H}} = \mathcal{D}^*\mathcal{F}_{G,H}$$

where α is the Maurer-Cartan form over G .

It is saying that \mathcal{D} is an involute application on \widetilde{M} of the $\frac{\mathcal{G}}{\mathcal{H}}$ -foliation $\mathcal{F}_{\mathcal{H}}$.

If the 1-form ω is Fedida's 1-form of a Lie-foliation \mathcal{F} , then all involute application of \mathcal{F} is also involute application of $\mathcal{F}_{\mathcal{H}}$.

Before finishing this part of recall, we give the theorem of the biunivocal correspondence between Lie subalgebras of $\mathcal{G} = \text{Lie}(G)$ and the extensions of a Lie G -foliation having dense leaves existing in [4].

Theorem 2.4 Let (M, \mathcal{F}) be a Lie G -foliation having dense leaves on a compact, connected manifold, with Lie algebra \mathcal{G} .

Then:

1- There is a correspondence 1-to-1 between Lie-subalgebra of $\mathcal{G} = \text{Lie}(G)$ (or if you prefer between the conncted Lie subgroup of G) and the extensions of \mathcal{F} .

2- An extension of \mathcal{F} is a $\frac{\mathcal{G}}{\mathcal{H}}$ transverse riemannian foliation with trivial normal fiber bundle, defined by 1-forme valued in $\frac{\mathcal{G}}{\mathcal{H}}$.

3- An extension of \mathcal{F} is transversally homogeneous (resp. of Lie) if and only if the Lie subgroup of G corresponding to it is a closed subgroup (resp. a normal subgroup) of G .

Remark 2.5 Let $\mathcal{F}_{\mathcal{H}}$ be an extension of a Lie G -foliation with dense leaves on a compact manifold corresponding to the Lie-subalgebra \mathcal{H} of $\mathcal{G} = \text{Lie}(G)$, let $\pi : \widetilde{M} \rightarrow M$ be the universal covering of M , let $\mathcal{D} : \widetilde{M} \rightarrow G$ be the involute application of \mathcal{F} associated to a 1-form of Fedida ω , let α be the Maurer-Cartan form on G , let $\widetilde{\mathcal{F}}_{\mathcal{H}}$ be the lift of $\mathcal{F}_{\mathcal{H}}$ on \widetilde{M} , let U be a distinguished open set in \widetilde{M} for both \mathcal{F} and $\mathcal{F}_{\mathcal{H}}$, let f' be a submersion defining $\mathcal{F}_{\mathcal{H}}$ on U and let \overline{U} be a local quotient manifold of $\widetilde{\mathcal{F}}_{\mathcal{H}}$ on U .

i) There is an open set \widetilde{U} of \widetilde{M} , an open set \overline{U} of G , a riemannian submersion $f : U \rightarrow \overline{U}$ and a riemannian submersion $\theta : \overline{U} \rightarrow \overline{U}$ such that following diagram

commutes:

$$\begin{array}{ccccc} \tilde{U} & \xrightarrow{\mathcal{D}} & \bar{U} & & \\ \pi \downarrow & \nearrow f & \downarrow \theta & & \\ U & \xrightarrow{f'} & \bar{\bar{U}} & & \end{array} .$$

We have $f : U \rightarrow \bar{U}$ defining \mathcal{F} on U and θ is a bond between (U, \mathcal{F}) and $(U, \mathcal{F}_{\mathcal{H}})$.

ii) The Darboux's differential of f is ω that is to say that

$$\omega_x(X_x) = (L_{f(x)*})^{-1} \circ f_{*x}(X_x)$$

for every $X_x \in T_x U$.

iii) There is

$$f_{*x}(T_x \mathcal{F}_{\mathcal{H}}) = L_{f(x)*}(\mathcal{H}) = T_x \mathcal{F}_{\theta}$$

for every $x \in U$ that is to say that

$$f(\mathcal{F}_{\mathcal{H}}) = \mathcal{F}_{\bar{U}, H} = \mathcal{F}_{\theta}$$

where $\mathcal{F}_{\bar{U}, H}$ is the restriction to \bar{U} of $\mathcal{F}_{G, H}$.

iv) For $X_x \in T_x M$,

$$X_x \in T_x \mathcal{F}_{\mathcal{H}} \iff \omega_x(X_x) \in \mathcal{H}.$$

v) \mathcal{F} being with dense leaves, we note that for every vector fields X and Y , \mathcal{F} -transverse foliated leads

$$\omega([X, Y]) = [\omega(X), \omega(Y)].$$

vi) Let $\mathcal{X}(\mathcal{F}_{\mathcal{H}})$ (resp. $\mathcal{X}(\mathcal{F})$) be the Lie algebra of tangent vectors fields of $\mathcal{F}_{\mathcal{H}}$ (resp. \mathcal{F}), let $\mathcal{A}^1(M)$ be the space of differentiable fonctions on M and let $\tilde{\mathcal{H}}$ be the Lie algebra of vectors fields \mathcal{F} -foliated obtained from \mathcal{H} .

We have $([\mathfrak{g}], [4])$

$$\mathcal{X}(\mathcal{F}_{\mathcal{H}}) = \mathcal{X}(\mathcal{F}) \oplus (\mathcal{A}^1(M) \otimes \tilde{\mathcal{H}}).$$

We end this part with the two following propositions found in [9] and which will be useful in the following.

Proposition 2.6 [9] Let $\mathcal{K}(M, \lambda)$ be a Lie algebra of Killing vectors fields on a Riemannian connected manifold (M, λ) , let $i_x = \{X \in \mathcal{K}(M, \lambda) / X_x = 0\}$ be the isotropy of $\mathcal{K}(M, \lambda)$ at the point $x \in (M, \lambda)$, let \mathcal{O}_x be the orbit at the point x of $\mathcal{K}(M, \lambda)$ and let $\mathcal{O}_{(x_0, x)}$ be the orbit at the point $x \in (M, \lambda)$ of the isotropy i_{x_0} where $x_0 \in (M, \lambda)$.

If for $x \in (M, \lambda)$, all orbits \mathcal{O}_x have the same dimension then $\mathcal{O}_{(x_0, x)} = \{x\}$ for every $(x_0, x) \in (M, \lambda)^2$

This proposition implies that if X and Y are two Killing vectors fields on a related open set U and $a \in U$ then

$$X_a = Y_a \iff X = Y.$$

Proposition 2.7 ([7], [9]) *Let \mathcal{F} be a codimension q Riemannian foliation on a compact connected manifold M , let $\overline{F^{\mathfrak{d}}}$ be the closure of a leaf $F^{\mathfrak{d}}$ of lifted foliation $\mathcal{F}^{\mathfrak{d}}$ of \mathcal{F} on the orthonormal transverse frame bundle $M^{\mathfrak{d}}$, let $\phi: M^{\mathfrak{d}} \rightarrow M$ be the projection which to a frame at x associates x .*

Then:

- i) $\phi(F^{\mathfrak{d}})$ is a leaf of \mathcal{F} and $\phi(\overline{F^{\mathfrak{d}}}) = \overline{\phi(F^{\mathfrak{d}})}$.
- ii) the map $\phi: \overline{F^{\mathfrak{d}}} \rightarrow \phi(\overline{F^{\mathfrak{d}}})$ is a locally trivial fibration .
- iii) $\phi: \overline{F^{\mathfrak{d}}} \rightarrow M$ is a covering.

3 Extension of Riemannian foliation having dense leaves on a compact manifold

In the following we designate by \tilde{k} the transverse vector field associated with $k \in \mathcal{G}$ where $\mathcal{G} = \text{Lie}(G)$.

Proposition 3.1 *Let (M, λ) be a Riemannian manifold, let $\tilde{\mathcal{H}}$ be a Lie algebra of Killing vector fields on the Riemannian manifold (M, λ) and let \mathcal{H}_M^\perp be a set of all vector fields orthogonal to any vector fields of \mathcal{H} .*

Then for every $\tilde{h} \in \tilde{\mathcal{H}}$ and every $\tilde{u} \in \mathcal{H}_M^\perp$ we have

$$[\tilde{h}, \tilde{u}] \in \mathcal{H}_M^\perp.$$

Proof. Let $\tilde{k} \in \tilde{\mathcal{H}}$ be a vector field, Let $(\tilde{h}, \tilde{h}') \in \tilde{\mathcal{H}} \times \tilde{\mathcal{H}}$ and let $\tilde{u} \in \mathcal{H}_M^\perp$ we have

$$\begin{aligned} 0 &= \tilde{h}'\lambda(\tilde{u}, \tilde{h}) \\ &= \lambda([\tilde{h}', \tilde{u}], \tilde{h}) + \lambda(\tilde{u}, [\tilde{h}', \tilde{h}]) \text{ because } \tilde{h}' \text{ is a Killing vector field for } \lambda \\ &= \lambda([\tilde{h}', \tilde{u}], \tilde{h}) + 0 \\ &= \lambda([\tilde{h}', \tilde{u}], \tilde{h}). \end{aligned}$$

The equality $\lambda([\tilde{h}', \tilde{u}], \tilde{h}) = 0$ for all $(\tilde{h}, \tilde{u}) \in \tilde{\mathcal{H}} \times \mathcal{H}_M^\perp$ implies that $[\tilde{h}', \tilde{u}] \in \mathcal{H}_M^\perp$ for all $(\tilde{h}', \tilde{u}) \in \tilde{\mathcal{H}} \times \mathcal{H}_M^\perp$. ■

Proposition 3.2 [7] *Let (M, \mathcal{F}) be a transversaly oriental Riemannian foliation on a manifold compact connected manifolde .*

The adhesions of the leaves of (M, \mathcal{F}) correspond to a differential system \mathcal{P} on M such as $\frac{\mathcal{P}}{\mathcal{F}}$ can be generated locally by transversal Killing vectors fields Z_1, \dots, Z_r for (M, \mathcal{F}) having the following properties:

- i) Z_1, \dots, Z_r switch with transverse foliated fields.*
- ii) Z_1, \dots, Z_r freely generate a Lie algebra isomorphic to structural Lie algebra \mathcal{G} of (M, \mathcal{F}) .*

The previous proposition implies that for any global foliated transverse vector field \overline{X} and $a \in M$, there existe an open set \mathcal{F} -distinguished U having \overline{U} for \mathcal{F} -local quotient containing \overline{a} such as $\overline{X}_{/\overline{U}}$ is a Killing vector fields where \overline{a} is the project of a following \mathcal{F} . Indeed consider the previous \mathcal{F} -distinguished set U , let \overline{U} be the \mathcal{F} -quotient local of U on where the Killing \mathcal{F} -transverse vectors fields Z_1, \dots, Z_r switching with globaltransverse foliated fields are defined where $r = \text{codim} \mathcal{F}$ (cf . proposition [7]). Let $Z_k \in \mathcal{G}$ and let \mathcal{F}_{Z_k} (resp. \mathcal{F}_X) be foliations on \overline{U} defined by the vector field Z_k (resp. X) , we assume that the open set \overline{U} is both distinguished for \mathcal{F}_{Z_k} and \mathcal{F}_X .

In all that follows it will be assumed that U will be chosen so that \overline{U} is a connected set . Let $\overline{\overline{U}}$ be the \mathcal{F}_{Z_k} -local quotient of \overline{U} and let $\overline{a} \in U$. For all vector $\overline{X}_{\overline{a}}$ at a point \overline{a} there existe a \mathcal{F} -transverse Killing vectors field Z_r such as

$$Z_r(\overline{a}) = \overline{X}_{\overline{a}}.$$

We have

$$(Z_r - \overline{X})(\overline{a}) = 0$$

As

$$[Z_r - \overline{X}, Z_j] = 0 \text{ for all } j \in \{1, \dots, r\}$$

and the transverse vectors fields Z_1, \dots, Z_r form a parallelism of \overline{U} then

$$\Omega = \{\overline{x} \in \overline{U} / Y(\overline{x}) = 0\}$$

where

$$Y = Z_r - \overline{X}$$

is an open set [7] closed not empty. Therefore the fact \overline{U} is a connected set implies that $\Omega = \overline{U}$.

Thus, its established that $\overline{X}_{/\overline{U}}$ is a Killinvector fields.

Corollary 3.3 *Let \mathcal{H} be a Lie subalgebra of the Lie algebra \mathcal{G} of a Lie foliation \mathcal{F} having dense leaves*

on a compact manifold and $u \in \mathcal{H}^\perp$.

then the following assertions are equivalent:

- i) $[u, h] = 0$ for every $h \in \mathcal{H}$.*
- ii) $[u, h] \in \mathcal{H}$ for every $h \in \mathcal{H}$*

Proof. According the propositions (3 · 1) and (3 · 2), $[u, h] \in \mathcal{H}^\perp$. Therefore, the fact that

$$[u, h] \in \mathcal{H} \iff [u, h] = 0.$$

■

In what follows we consider the proposal 3·2.

We designate by U an open set on which the *transverses Killing vector* Z_1, \dots, Z_r are defined.

We will say that the adhesions of the leaves of (M, \mathcal{F}) are the orbits $[1, 1]$ of the transverse beam $\mathcal{C}_U(\mathcal{F}, \mathcal{G})$ of \mathcal{F} having \mathcal{G} for fiber type .

In the following we will show that any extension of a Riemannian foliation having dense leaves have its leaves which are orbits of a transverse beam.

Theorem 3.4 *Let \mathcal{F} be a Riemannian foliation having dense leaves on a Riemannian compact manifold (M, λ) . Let M^\natural be the fiber of the transverse orthonormal frame bundle of \mathcal{F} , let \mathcal{F}^\natural be the lifted foliation of \mathcal{F} on M^\natural , let F^\natural be a leaf of \mathcal{F}^\natural , let $\overline{F^\natural}$ be the closure of F^\natural , let $\phi : M^\natural \rightarrow M$ be the projection to a frame at x associated x , let \mathcal{G} be the Lie structural algebral of \mathcal{F} and let $\tilde{\mathcal{F}}$ be an extension of \mathcal{F} .*

Then :

(i) *The leaves of foliation $\tilde{\mathcal{F}}^\natural = \phi^* \left(\tilde{\mathcal{F}} \right) = \phi^{-1} \left(\tilde{\mathcal{F}} \right)$ are the orbits $([1, 1])$ of a transverse beam $\mathcal{C}_U \left(\tilde{\mathcal{F}}, \mathcal{H} \right)$ of $\tilde{\mathcal{F}}$ having $\mathcal{H} \subset \mathcal{G}$ for fiber type.*

(ii) *$\tilde{\mathcal{F}}$ is a riemannian foliation.*

Proof. (i) Let $\tilde{\lambda} = \phi^*(\lambda)$ the metric pulls back of λ and let $\mathcal{V}_x(\mathcal{F}) = (T_x \mathcal{F})^\perp$ be the fiber at x of orthonormal fiber $\mathcal{V}(\mathcal{F})$. $\tilde{\lambda}$ is a metrique on $\overline{F^\natural}$ because $\phi : \overline{F^\natural} \rightarrow M$ is a covering [9].

Note that the restriction $\mathcal{F}_{\overline{F^\natural}}^\natural$ of $\tilde{\mathcal{F}}^\natural$ on $\overline{F^\natural}$ is a Lie \mathcal{G} foliation having dense leaves [7], We also note that the foliaton $\tilde{\mathcal{F}}^\natural = \phi^* \left(\tilde{\mathcal{F}} \right)$ is by construction an extension of the Lie foliation having dense leaves $\mathcal{F}_{\overline{F^\natural}}^\natural$ because $\mathcal{F} \subset \tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^\natural = \phi^* \left(\tilde{\mathcal{F}} \right)$ and \mathcal{F}^\natural (resp. $\tilde{\mathcal{F}}^\natural$) are the lifted foliation of \mathcal{F} (resp. $\tilde{\mathcal{F}}$) on M^\natural . According to the theoreme 2 · 4, $\tilde{\mathcal{F}}^\natural$ is a Riemannian foliation having dense leaves on the Riemannian compact manifold (M, λ) then there existe r local Killing vecteurs fields $Z_1^\natural, \dots, Z_r^\natural$ linearly independent such as according to the propositions ([7], [9])

$$\left(T_{x^\natural} \tilde{\mathcal{F}}^\natural \right)^\perp = \text{ev}_{x^\natural}(\mathcal{G}) = \langle Z_1^\natural, \dots, Z_r^\natural \rangle$$

where $ev_x(X) = X_x$. And according the proposition [9] $\phi : M^{\natural} \rightarrow M$ is a covering, Then

$$\begin{aligned}
(T_x \tilde{\mathcal{F}})^{\perp} &= \phi_* \left((T_{x^{\natural}} \tilde{\mathcal{F}}^{\natural})^{\perp} \right) \text{ because relatively to } \lambda \text{ and } \tilde{\lambda}, \phi \text{ is a local isometry and } \phi(x^{\natural}) = x \\
&= \langle \phi_* \left(Z_{1x^{\natural}}^{\natural} \right), \left(\phi_* \left(Z_{2x^{\natural}}^{\natural} \right) \right), \dots, \phi_* \left(Z_{rx^{\natural}}^{\natural} \right) \rangle \\
&\simeq ev_{x^{\natural}}(\mathcal{G}) \\
&\simeq \mathcal{G}
\end{aligned}$$

From the foregoing it is argued that the leaves of $\tilde{\mathcal{F}}$ are the orbits ([1]) of transverse beam $\mathcal{C}_U(\tilde{\mathcal{F}}, \mathcal{G})$ of $\tilde{\mathcal{F}}$ having $\bar{\mathcal{G}} = (\phi)_*(\mathcal{G}) \simeq \mathcal{G}$ for fiber type.

(ii) what precedes shows that $\tilde{\mathcal{F}}$ is a riemannian foliation. In deed let's show now that foliation $\tilde{\mathcal{F}}$ is Riemannian. For it, as the problem is local, notice that:
an extension $\tilde{\mathcal{F}}$ of a Riemannian foliation \mathcal{F} is a Riemannian foliation if and only if there existe an open set U such (U, \mathcal{F}) is a Riemannian foliation.

As $\phi : M^{\natural} \rightarrow M$ is a covering, then there existe an open set U^{\natural} such as:

- (i) $\phi : U^{\natural} \rightarrow U$ is a difeomorphism
- (ii) $(U^{\natural}, \tilde{\mathcal{F}}^{\natural})$ and $(U, \tilde{\mathcal{F}})$ are simple.

Note that (U, \mathcal{F}) and $(U^{\natural}, \tilde{\mathcal{F}}^{\natural})$ are difeomorphic then (U, \mathcal{F}) is Riemannian if and only if $(U^{\natural}, \tilde{\mathcal{F}}^{\natural})$ is Riemannian. Therefore the fact that $\tilde{\mathcal{F}}^{\natural}$ is a Riemannian foliation implies that $(U, \tilde{\mathcal{F}})$ is a Riemannian foliation. Thus all extension $\tilde{\mathcal{F}}$ of a Riemannian foliation \mathcal{F} having dense leaves is a Riemannian foliation. ■

4 Computation of the Lie algebra of the set of foliated transverse vectors fields of an extension of a Lie G foliation having dense leaves on a compact manifold

Proposition 4.1 *Let \mathcal{G} be the Lie structural algebra of a Lie G foliation having dense leaves, let \mathcal{H} be a Lie subalgebra of $\mathcal{G} = Lie(G)$, and let \mathcal{H}^{\perp} be the orthocomplementary of \mathcal{H} in \mathcal{G} relavely to λ .*

Then

$$\{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$$

is a Lie subagebra of \mathcal{G} .

Proof. We consider two vectors u and v of $\mathcal{B} = \{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$. Note Above all that

$$[t, z] = 0 \text{ for all } t \in \mathcal{B} \text{ and } z \in \mathcal{H}$$

On other hand for $k \in \mathcal{H}$ we have

$$0 = [[u, v], k] + [[v, k], u] + [[k, u], v] = [[u, v], k] + [0, u] + [0, v] = [[u, v], k] \quad (\star\star)$$

moreover, according to Molino there exists an open neighborhood V_e of the neutral element e of G such as for all $b \in \mathcal{G}$ the restriction \tilde{b}_{V_e} of the transverse vector fields associated at $b \in \mathcal{G}$ is a Killing vector. From where, we have

$$\begin{aligned} 0 &= \tilde{u}_{V_e} \cdot \lambda \left(\tilde{v}_{V_e}, \tilde{k}_{V_e} \right) \text{ because } \lambda \left(\tilde{v}_{V_e}, \tilde{k}_{V_e} \right) = \lambda(v, k) \\ &= \lambda \left([\tilde{u}_{V_e}, \tilde{v}_{V_e}], \tilde{k}_{V_e} \right) + \lambda \left(\tilde{v}_{V_e}, [\tilde{u}_{V_e}, \tilde{k}_{V_e}] \right) \\ &= \lambda \left([\tilde{u}_{V_e}, \tilde{v}_{V_e}], \tilde{k}_{V_e} \right) + \tilde{k}_{V_e}(\tilde{v}_{V_e}, 0) \\ &= \lambda \left([\tilde{u}_{V_e}, \tilde{v}_{V_e}], \tilde{k}_{V_e} \right) \\ &= \lambda \left(\widetilde{[u, v]}_{V_e}, \tilde{k}_{V_e} \right) \\ &= \lambda([u, v], k). \quad (\star\star\star) \quad \text{Thus } [u, v] \in \mathcal{H}^\perp. \end{aligned}$$

The equality $(\star\star\star)$ and $(\star\star)$ implies that

$$\{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$$

is a Lie subalgebra of \mathcal{H} ■

Let \mathcal{G} be the Lie algebra of a Lie group G and let h_T be a left-invariant metric on G .

Let $\mathcal{F}_{\mathcal{H}}$ the extension of Lie G -foliation having dense leaves \mathcal{F} on a compact, connected

manifold corresponding to subalgebra \mathcal{H} of $\mathcal{G} = \text{Lie}(G)$, \mathcal{H}^g The Lie algebra of left-invariant vector fields on G obtained from vectors of Lie subalgebra \mathcal{H} of \mathcal{G} and $\mathcal{F}_{G,H}$ the foliation on G whose leaves are left classes of H .

The above theorem allows us to say that the leaves of the foliation $\mathcal{F}_{G,H}$ (resp. $\mathcal{F}_{\mathcal{H}}$) are orbits (resp. transversally to \mathcal{F}) of a Lie algebra of Killing fields. And this Lie algebra is of course \mathcal{H}^g .

In the following if there is a metric h on a manifold M , we will identify for every

foliation \mathcal{F}' on M , the normal bundle $\mathcal{V}(\mathcal{F}')$ of \mathcal{F}' and the orthogonal bundle $(T\mathcal{F}')^\perp$ of $T\mathcal{F}'$.

The purpose of this section is to calculate the algebra $\ell(M, \mathcal{F}_{\mathcal{H}})$ of vector fields $\mathcal{F}_{\mathcal{H}}$ -transverse foliate where $\mathcal{F}_{\mathcal{H}}$ is an extension of Lie G -foliation having dense

leaves \mathcal{F} on compact, connected manifold corresponding to the subalgebra \mathcal{H} of $\mathcal{G} = \text{Lie}(G)$.

Before stating the next proposal, we recall some remarks on the transversally parallelizable foliations on a compact manifold.

Let \mathcal{F}' a transversally parallelizable foliation on a compact manifold M , \mathcal{F}'_b the basic foliation of \mathcal{F}' and $\pi : M \rightarrow W$ the basic fibration of \mathcal{F}' .

It is known [7] that π defined \mathcal{F}'_b and we have the following exact sequence:

$$0 \rightarrow \varepsilon_w \rightarrow \ell(M, \mathcal{F}') \rightarrow \mathcal{X}(W) \rightarrow 0$$

where ε_w is the Lie algebra of vector fields \mathcal{F}' -transverse foliate which projection by π is zero and $\mathcal{X}(W)$ is the Lie algebra of vector fields of the basic manifold W . As \mathcal{F}'_b is a simple foliation defined by π , we have $\mathcal{X}(W) = \ell(M, \mathcal{F}'_b)$. whence the previous exact sequence becomes:

$$0 \rightarrow \varepsilon_w \rightarrow \ell(M, \mathcal{F}') \rightarrow \ell(M, \mathcal{F}'_b) \rightarrow 0.$$

hence $\ell(M, \mathcal{F}'_b) \subset \ell(M, \mathcal{F}')$.

This inducted us asking about the nature of relations existing between $\ell(M, \mathcal{F}')$ and $\ell(M, \mathcal{F}'')$ for any extension \mathcal{F}'' of riemannian foliation \mathcal{F}' .

Proposition 4.2 *Let \mathcal{F}'' an extension of a riemannian foliation \mathcal{F}' on a manifold M .*

Then $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$.

Proof. Let X be a vector field \mathcal{F}'' -transverse foliate, h a \mathcal{F}' -bundle-like metric of \mathcal{F}' -transverse metric h_T and U a distinguished open set of M both for \mathcal{F}' and \mathcal{F}'' . There are three submersions $f' : U \rightarrow \overline{U}$, $f'' : U \rightarrow \overline{\overline{U}}$ and $\theta : \overline{U} \rightarrow \overline{\overline{U}}$ such that θ be the bond between (U, \mathcal{F}') and (U, \mathcal{F}'') and the restriction foliations (U, \mathcal{F}') and (U, \mathcal{F}'') are defined respectively by f' and f'' .

The following diagram commutes:

$$\begin{array}{ccc} & & \overline{U} \\ & \nearrow f' & \downarrow \theta \\ U & \xrightarrow{f''} & \overline{\overline{U}} \end{array}.$$

The submersion f' is riemannian submersion which defines the riemannian foliation (U, \mathcal{F}') .

Now considering $P_{(x, \mathcal{F}')}^U$ and $P_{(x, \mathcal{F}'')}^U$ plaques respectively of \mathcal{F}' and \mathcal{F}'' in U via $x \in U$.

Let $z \in P_{(x, \mathcal{F}')}^U$ and \mathcal{F}_θ the foliation in \overline{U} defined by the submersion θ . Let show that

$$f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) = T_{f'(x)} \mathcal{F}_\theta$$

We have $f'' = \theta \circ f'$. whence $f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) \subset \ker \theta_{f'(x)}$.

The vector spaces $f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right)$ and $(T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'')$ have the same dimension because $f'_{*z} : (T_z \mathcal{F}')^\perp \rightarrow T_{f'(x)} \overline{U}$ is an isometry.

Therefore

$$\begin{aligned} \dim (T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') &= \dim (T_z \mathcal{F}')^\perp + \dim T_z \mathcal{F}'' - \dim < (T_z \mathcal{F}')^\perp \cup T_z \mathcal{F}'' > \\ &= \dim \mathcal{F}'' - \dim \mathcal{F}' \\ &= \text{codim } \mathcal{F}' - \text{codim } \mathcal{F}'' \\ &= \dim \ker \theta_{f'(x)}. \end{aligned}$$

so $f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) \subset \ker \theta_{f'(x)}$ ensures that

$$f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) = \ker \theta_{f'(x)} = T_{f'(x)} \mathcal{F}_\theta.$$

That said, X being \mathcal{F}'' -transverse foliate, we have $f''_*(X)$ is a vector field on \overline{U} .

It is noted \overline{X} . More X is valued in $(T \mathcal{F}'')^\perp$ because we identify the normal bundle $\mathcal{V}(\mathcal{F}'')$ of \mathcal{F}'' and the orthogonal bundle $(T \mathcal{F}'')^\perp$ of $T \mathcal{F}''$. Since

$$(T \mathcal{F}')^\perp = (T \mathcal{F}')^\perp \cap (T \mathcal{F}'') \oplus (T \mathcal{F}'')^\perp$$

and $f'_{*z} : (T_z \mathcal{F}')^\perp \rightarrow T_{f'(x)} \overline{U}$ is an isometry then for any $z \in P_{(x, \mathcal{F}')}^U$

and for any $Y_z \in (T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'')$ we have

$$0 = h_T(Y_z, X_z) = \overline{h}_T(f'_{*z}(Y_z), f'_{*z}(X_z))$$

where \overline{h}_T is the metric on \overline{U} obtained by projection of h_T on \overline{U} .

$\overline{h}_T(f'_{*z}(Y_z), f'_{*z}(X_z)) = 0$ leads up that

$f'_{*z}(X_z) \in \left(f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) \right)^\perp$ for any $z \in P_{(x, \mathcal{F}')}^U$. Therefore

$$f'_{*z} \left((T_z \mathcal{F}')^\perp \cap (T_z \mathcal{F}'') \right) = \ker \theta_{f'(x)} = T_{f'(x)} \mathcal{F}_\theta$$

for every $z \in P_{(x, \mathcal{F}')}^U$, so

$$f'_{*z}(X_z) \in (T_{f'(x)} \mathcal{F}_\theta)^\perp$$

for every $z \in P_{(x, \mathcal{F}')}^U$.

We have $P_{(x, \mathcal{F}')}^U \subset P_{(x, \mathcal{F}'')}^U$. Whence for any $z, z' \in P_{(x, \mathcal{F}')}^U$

$$f''_{*z}(X_z) = f''_{*z'}(X_{z'}) = \overline{X}_{f''(x)}.$$

$f'' = \theta \circ f'$ leads up that

$$\overline{\overline{X}}_{f''(x)} = f''_{*z}(X_z) = \theta_{*f'(x)}(f'_{*z}(X_z))$$

and

$$\overline{\overline{X}}_{f''(x)} = f''_{*z'}(X_{z'}) = \theta_{*f'(x)}(f'_{*z'}(X_{z'}))$$

hence

$$\theta_{*f'(x)}(f'_{*z}(X_z)) = \theta_{*f'(x)}(f'_{*z'}(X_{z'})).$$

The fact that $\theta_{*f'(x)} : (T_{f'(x)}\mathcal{F}_\theta)^\perp \rightarrow T_{f''(x)}\overline{\overline{U}}$ be an isomorphism of vector spaces implies by using

$$(f'_{*z}(X_z), f'_{*z'}(X_{z'})) \in (T_{f'(x)}\mathcal{F}_\theta)^\perp \times (T_{f'(x)}\mathcal{F}_\theta)^\perp \text{ that}$$

$$f'_{*z}(X_z) = f'_{*z'}(X_{z'})$$

for any $z, z' \in P_{(x, \mathcal{F}')}^U$.

It follows from that precedes the vector field \mathcal{F}'' -transverse foliate X is projected according the plaques of \mathcal{F}' on the manifolds \mathcal{F}' -local quotients of distinguished open sets for \mathcal{F}' and \mathcal{F}'' . This shows that X is a vector field \mathcal{F}' -foliate.

Since $(T\mathcal{F}'')^\perp \subset (T\mathcal{F}')^\perp$ and X is tangent to $(T\mathcal{F}'')^\perp$ then it shows that X is tangent to $(T\mathcal{F}')^\perp$, hence X is a vector field \mathcal{F}' -foliate tangent to $(T\mathcal{F}')^\perp$. That shows that X is a vector field \mathcal{F}' -transverse foliate.

Properly $\ell(M, \mathcal{F}'') \subset \ell(M, \mathcal{F}')$. ■

Remark 4.3 We put $\overline{X} = f'_*(X_U)$ where X_U is a restriction to U of the field \mathcal{F}'' -transverse

foliate X . Note that \overline{X} is tangent to $(T\mathcal{F}_\theta)^\perp$ and is projected by θ on $\overline{\overline{X}}$.

It results that \overline{X} is \mathcal{F}_θ -transverse foliate.

Defitively, thanks to the commutative diagram

$$\begin{array}{ccc} & & \overline{U} \\ & \nearrow f' & \downarrow \theta \\ U & \xrightarrow{f''} & \overline{\overline{U}} \end{array},$$

that f'_* is a linear isomorphism between $\ell(U, \mathcal{F}'')$ and $\ell(\overline{U}, \mathcal{F}_\theta)$.

Using the above proposal and the theorems 2.4 and 3.1 we establish the following theorem which is the main result of this paper.

Theorem 4.4 Let \mathcal{H} a Lie subalgebra of the Lie algebra \mathcal{G} of a Lie-foliation having dense leaves \mathcal{F} on a compact manifold, ω a 1-form of Fedida defining

\mathcal{F} and $\mathcal{F}_\mathcal{H}$ the extension of \mathcal{F} corresponding to \mathcal{H} .

Then:

i)

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{\langle h \rangle})$$

where $\mathcal{F}_{\langle h \rangle}$ is the extension of \mathcal{F} corresponding to the Lie subalgebra $\langle h \rangle$ of \mathcal{G} generated by h .

ii)

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \mathcal{H}^{\perp}$$

where \mathcal{H}^{\perp} is the ortho-complementary of \mathcal{H} in \mathcal{G} by the transverse metric associated to \mathcal{F} .

iii) For $h \in \mathcal{G}$,

$$\omega(\ell(M, \mathcal{F}_{\langle h \rangle})) = \{u \in \langle h \rangle^{\perp} / [u, h] = 0\}.$$

iv)

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \{u \in \mathcal{H}^{\perp} / [u, h] = 0 \text{ for any } h \in \mathcal{H}\}.$$

Proof. i) For every $h \in \mathcal{H}$, $\mathcal{F}_{\langle h \rangle} \subset \mathcal{F}_{\mathcal{H}}$ and $\mathcal{F}_{\langle h \rangle}$ is a riemannian foliation. Whence $\ell(M, \mathcal{F}_{\mathcal{H}}) \subset \ell(M, \mathcal{F}_{\langle h \rangle})$ for any $h \in \mathcal{H}$. This implies that $\ell(M, \mathcal{F}_{\mathcal{H}}) \subset \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{\langle h \rangle})$.

Let $Y \in \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{\langle h \rangle})$, Z a vector field tangent to $\mathcal{F}_{\mathcal{H}}$.

There is $X_{\mathcal{F}} \in \mathcal{X}(\mathcal{F})$, $X_{\mathcal{H}} \in \widetilde{\mathcal{H}}$ and $s \in \mathcal{A}^1(M)$ such that

$$Z = X_{\mathcal{F}} + s.X_{\mathcal{H}}.$$

We have

$$\begin{aligned} [Y, Z] &= [Y, X_{\mathcal{F}}] + [Y, s.X_{\mathcal{H}}] \\ &= [Y, X_{\mathcal{F}}] + Y(s).X_{\mathcal{H}} + s.[Y, X_{\mathcal{H}}] \end{aligned}$$

However Y is foliate for $\mathcal{F}_{\langle h \rangle}$ so Y is foliate for \mathcal{F} because $\mathcal{F} \subset \mathcal{F}_{\langle h \rangle}$ and \mathcal{F} is riemannian. This shows that $[Y, X_{\mathcal{F}}]$ is tangent to $\mathcal{F} \subset \mathcal{F}_{\mathcal{H}}$.

More $\omega(X_{\mathcal{H}}) \in \mathcal{H}$. whence $[Y, X_{\mathcal{H}}]$ is tangent to $\mathcal{F}_{\langle \omega(X_{\mathcal{H}}) \rangle} \subset \mathcal{F}_{\mathcal{H}}$.

Given $Y(s).X_{\mathcal{H}}$ tangent to $\mathcal{F}_{\mathcal{H}}$, It results from that precedes that $[Y, Z]$ is tangent to $\mathcal{F}_{\mathcal{H}}$. This means that Y is $\mathcal{F}_{\mathcal{H}}$ -transverse foliate.

Hence $\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{\langle h \rangle}) \subset \ell(M, \mathcal{F}_{\mathcal{H}})$. Properly $\ell(M, \mathcal{F}_{\mathcal{H}}) = \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{\langle h \rangle})$.

ii) Noted that for every $X \in \ell(M, \mathcal{F})$ and for every $x \in M$ we have $\omega_x(X_x) = \omega(X)$ because the Lie-foliation \mathcal{F} is with dense leaves.

Let U a local open of trivialization of universal covering \widetilde{M} of M distinguished both for \mathcal{F} and $\mathcal{F}_{\mathcal{H}}$.

There is (cf.remark 2.5) a riemannian submersion f defining \mathcal{F} on U and checking

$$\omega_x(X_x) = (L_{f(x)*})^{-1} \circ f_{*x}(X_x)$$

for every $X_x \in T_x U$.

That said, we have

$$\begin{aligned}\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) &= \omega_x(\ell(M, \mathcal{F}_{\mathcal{H}})) \\ &= (L_{f(x)*})^{-1} \circ f_{*x}(\ell(M, \mathcal{F}_{\mathcal{H}})).\end{aligned}$$

But according to the remark 4.2

$$f_{*x}(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset (Ker \theta_{f(x)})^{\perp} = (L_{f(x)*}(\mathcal{H}))^{\perp}$$

then $\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset (L_{f(x)*})^{-1} (L_{f(x)*}(\mathcal{H}))^{\perp}$.

we also have $(L_{f(x)*})^{-1} (L_{f(x)*}(\mathcal{H}))^{\perp} = \mathcal{H}^{\perp}$ because $L_{f(x)*}$ is an isometry. It results from the foregoing that $\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \mathcal{H}^{\perp}$.

iii) Let $v \in \omega(\ell(M, \mathcal{F}_{<h>}))$ and \tilde{h} field \mathcal{F} -transverse foliate associated to h .

Since $v \in \omega(\ell(M, \mathcal{F}_{<h>}))$ then there is $\tilde{v} \in \ell(M, \mathcal{F}_{<h>})$ such that $\omega(\tilde{v}) = v$.

We have $[\tilde{v}, \tilde{h}]$ tangent to $\mathcal{F}_{<h>}$ as \tilde{h} is tangent to $\mathcal{F}_{<h>}$ and $\tilde{v} \in \ell(M, \mathcal{F}_{<h>})$.

Accordingly

$$\omega([\tilde{v}, \tilde{h}]) \in \omega(T\mathcal{F}_{<h>}) = \langle h \rangle$$

where $\langle h \rangle$ is the Lie subalgebra generated by h .

More, the Lie-foliation \mathcal{F} being with dense leaves, we have

$$\omega([\tilde{v}, \tilde{h}]) = [\omega(\tilde{v}), \omega(\tilde{h})] = [v, h].$$

It results from foregoing that $[v, h] \in \langle h \rangle$. This implies that $[v, h] = 0$ as $[v, h]$ is orthogonal to vectors v and h .

hence the fact that $\omega(\ell(M, \mathcal{F}_{<h>})) \subset \langle h \rangle^{\perp}$ ensures that $v \in \{u \in \langle h \rangle^{\perp} / [u, h] = 0\}$.

Let $b \in \{u \in \langle h \rangle^{\perp} / [u, h] = 0\}$, \tilde{b} the field \mathcal{F} -transverse foliate associated to b and X a vector field tangent to $\mathcal{F}_{<h>}$.

There is $X_{\mathcal{F}} \in \mathcal{X}(\mathcal{F})$ and $s \in \mathcal{A}^1(M)$ such that

$$X = X_{\mathcal{F}} + s.\tilde{h}.$$

whence

$$\begin{aligned}[\tilde{b}, X] &= [\tilde{b}, X_{\mathcal{F}}] + [\tilde{b}, s.\tilde{h}] \\ &= [\tilde{b}, X_{\mathcal{F}}] + \tilde{b}(s).\tilde{h} + s. [\tilde{b}, \tilde{h}].\end{aligned}$$

Hence

$$\begin{aligned}\omega([\tilde{b}, X]) &= \omega([\tilde{b}, X_{\mathcal{F}}]) + \omega(\tilde{b}(s).\tilde{h}) + \omega(s. [\tilde{b}, \tilde{h}]) \\ &= \tilde{b}(s).\omega(\tilde{h}) + s.\omega([\tilde{b}, \tilde{h}]) \\ &= \tilde{b}(s).h.\end{aligned}$$

$\omega \left(\left[\tilde{b}, X \right] \right) = \tilde{b}(s) \cdot h$ shows that $\left[\tilde{b}, X \right]$ is tangent to $\mathcal{F}_{<h>}$

(cf. remark 2.5). Accordingly \tilde{b} is foliate for $\mathcal{F}_{<h>}$.

Let U a local open of trivialization of universal covering \widetilde{M} of M distinguished both for \mathcal{F} and $\mathcal{F}_{<h>}$, h_T the bundle-like metric \mathcal{F} -transverse on U and \bar{h}_T the left-invariant metric on \bar{U} associated to h_T .

There is (cf. remark 2.5) a riemannian submersion f defining \mathcal{F} on U and verifying

$$\omega_x(X_x) = (L_{f(x)*})^{-1} \circ f_{*x}(X_x)$$

For every $X_x \in T_x U$.

For every $x \in U$ we have:

$$\begin{aligned} h_T(\tilde{b}_x, \tilde{h}_x) &= \bar{h}_T(f_{*x}(\tilde{b}_x), f_{*x}(\tilde{h}_x)) \\ &= \bar{h}_T((L_{f(x)*})^{-1} \circ f_{*x}(\tilde{b}_x), (L_{f(x)*})^{-1} \circ f_{*x}(\tilde{h}_x)) \\ &= \bar{h}_T(\omega_x(\tilde{b}_x), \omega_x(\tilde{h}_x)) \\ &= \bar{h}_T(b, h) \\ &= 0 \end{aligned}$$

hence the vector field \tilde{b} foliate for $\mathcal{F}_{<h>}$ is $\mathcal{F}_{<h>}$ -transverse foliate .

i.e. $\tilde{b} \in \ell(M, \mathcal{F}_{<h>})$. This implies that $b \in \omega(\ell(M, \mathcal{F}_{<h>}))$.

It results from the above that

$$\omega(\ell(M, \mathcal{F}_{<h>})) = \{u \in <h>^\perp / [u, h] = 0\}.$$

iv) It is easy to see that

$$\omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right) \subset \bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>})).$$

Now, let us prove that

$$\bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>})) \subset \omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right).$$

Let $u \in \bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>}))$.

For every $h \in \mathcal{H}$ there is $\tilde{u}_h \in \ell(M, \mathcal{F}_{<h>})$ such that $\omega(\tilde{u}_h) = u$.

Since $\mathcal{F}_{<h>}$ is an extension of foliation \mathcal{F} riemannian then $\tilde{u}_h \in \ell(M, \mathcal{F})$.

The foliation \mathcal{F} is a Lie-foliation with dense leaves, whence ω is an linear isomorphism between $\ell(M, \mathcal{F})$ and \mathcal{G} . Hence $\omega(\tilde{u}_h) = u$ for any

$h \in \mathcal{H}$ leads up that for any $h_0 \in \mathcal{H}$ and any $h_1 \in \mathcal{H}$ we have

$\tilde{u}_{h_0} = \tilde{u}_{h_1}$. We can state $\tilde{u}_h = \tilde{u}$ for all $h \in \mathcal{H}$.

It is easy to see that $\tilde{u} \in \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})$. This implies that

$$u \in \omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right).$$

It results from the above that

$$\bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>})) \subset \omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right).$$

Properly we have

$$\bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>})) = \omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right).$$

By

$$\ell(M, \mathcal{F}_{\mathcal{H}}) = \bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})$$

and

$$\omega(\ell(M, \mathcal{F}_{<h>})) = \{u \in <h>^\perp / [u, h] = 0\}$$

we obtain

$$\begin{aligned} \omega(\ell(M, \mathcal{F}_{\mathcal{H}})) &= \omega\left(\bigcap_{h \in \mathcal{H}} \ell(M, \mathcal{F}_{<h>})\right) \\ &= \bigcap_{h \in \mathcal{H}} \omega(\ell(M, \mathcal{F}_{<h>})) \\ &= \bigcap_{h \in \mathcal{H}} \{u \in <h>^\perp / [u, h] = 0\} \\ &= \{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}. \end{aligned}$$

■

We note using *vi*) of remark 2.5 that ω realizes a Lie algebra isomorphism between $\ell(M, \mathcal{F}_{\mathcal{H}})$ and $\{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$.

We also note that we can show that $\{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for all } h \in \mathcal{H}\}$ is a Lie subalgebra without using ω . As an illustration, let's observe two vectors u and v of $\{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$. We have for $k \in \mathcal{H}$,

$$\begin{aligned} 0 &= [[u, v], k] + [[v, k], u] + [[k, u], v] \\ &= [[u, v], k] + [0, u] + [0, v] \\ &= [[u, v], k] \end{aligned}$$

Moreover if \bar{h}_T is a left-invariant metric on the Lie group G we have

$$\begin{aligned} 0 &= \tilde{u} \cdot \bar{h}_T(\tilde{v}, \tilde{k}) \\ &= \bar{h}_T([\tilde{u}, \tilde{v}], \tilde{k}) + \bar{h}_T(\tilde{v}, [\tilde{u}, \tilde{k}]) \\ &= \bar{h}_T([\tilde{u}, \tilde{v}], k^g) + \bar{h}_T(\tilde{v}, 0) \\ &= \bar{h}_T([\tilde{u}, \tilde{v}], \tilde{k}) \\ &= \bar{h}_T([u, v], k) \end{aligned}$$

because the foliated transverse fields \tilde{u} , \tilde{v} and \tilde{k} associated respectively to u ,

v and k are Killing vectors fields.

$[[u, v], k] = 0$ and $\tilde{h}_T([u, v], k) = 0$ for every $k \in \mathcal{H}$ ensures that $[u, v]$ is a vector of $\{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$.

Noted at the end that if $\tilde{\mathcal{H}}$ is the Lie algebra of vector fields \mathcal{F} -transverse foliate obtained from \mathcal{H} then

$$\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$$

that shows $X \in \ell(M, \mathcal{F}_{\mathcal{H}})$, $Y \in \tilde{\mathcal{H}}$ and the vector field \mathcal{F} -transverse foliate $[X, Y]$ is zero. Indeed

$$\omega([X, Y]) = [\omega(X), \omega(Y)] = 0$$

because $\omega(X) \in \{u \in \mathcal{H}^\perp / [u, h] = 0 \text{ for every } h \in \mathcal{H}\}$ and $\omega(Y) \in \mathcal{H}$. The respective projection \bar{X} and \bar{Y} of X and Y on a local quotient manifold \bar{U} for \mathcal{F} are left-invariant fields. whence for every $g \in \bar{U}$ we have

$$[\bar{X}, \bar{Y}]_g = L_{g*}([\omega(X), \omega(Y)]) = 0.$$

Hence $[\bar{X}, \bar{Y}] = 0$. This shows that for every $x \in M$

$$(\varphi_t^Y)_{*x}(X_x) = X_{\varphi_t^Y(x)}$$

where φ_t^Y is the flow of Y .

Corollary 4.5 *Let \mathcal{G} the Lie algebra of a Lie-foliation having dense leaves \mathcal{F} on a compact manifold.*

We have:

- i) If \mathcal{H} is an ideal of \mathcal{G} then the ortho-complémentary \mathcal{H}^\perp of \mathcal{H} in \mathcal{G} is also an ideal,*
- ii) If \mathcal{F} has an extension \mathcal{F}' of Lie then \mathcal{F} has also an other extension \mathcal{F}'' of Lie such that $(T\mathcal{F})^\perp \cap (T\mathcal{F}')$ be orthogonal to $(T\mathcal{F})^\perp \cap (T\mathcal{F}'')$.*

Proof. Let ω be a 1-form of f  dida defining the Lie-foliation having dense leaves \mathcal{F} .

i) Supposed that \mathcal{H} is an ideal of \mathcal{G} .

From theorem 2.4, the extension $\mathcal{F}_{\mathcal{H}}$ of \mathcal{F} corresponding to \mathcal{H} is a Lie-foliation. Whence ([3], [4])

$$\dim \omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \dim(\ell(M, \mathcal{F}_{\mathcal{H}})) = \dim\left(\frac{\mathcal{G}}{\mathcal{H}}\right) = \dim(\mathcal{H}^\perp).$$

But $\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) \subset \mathcal{H}^\perp$ then $\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \mathcal{H}^\perp$. This shows that \mathcal{H}^\perp is a Lie subalgebra of \mathcal{G} .

Since $\omega(\ell(M, \mathcal{F}_{\mathcal{H}})) = \mathcal{H}^{\perp}$ then from the theorem 4.3 we have $[h, u] = 0$ for every $h \in \mathcal{H}$ and all $u \in \mathcal{H}^{\perp}$.
Let $v \in \mathcal{G}$,
there is $v_{\mathcal{H}} \in \mathcal{H}$ and $v_{\mathcal{H}^{\perp}} \in \mathcal{H}^{\perp}$ such that $v = v_{\mathcal{H}} + v_{\mathcal{H}^{\perp}}$.
Whence for every $u \in \mathcal{H}$,

$$[v, u] = [v_{\mathcal{H}}, u] + [v_{\mathcal{H}^{\perp}}, u] = [v_{\mathcal{H}}, u].$$

It results of the fact that \mathcal{H}^{\perp} be a Lie subalgebra that $[v, u] \in \mathcal{H}^{\perp}$ for every $u \in \mathcal{H}$ and all $v \in \mathcal{G}$. i.e. \mathcal{H}^{\perp} is an ideal of \mathcal{G} .
ii) Supposed that \mathcal{F} has an extension \mathcal{F}' of Lie.
From the theorem 2.4 there is an ideal \mathcal{H} of \mathcal{G} corresponding to \mathcal{F}' .
It results from i) that \mathcal{H}^{\perp} is also an ideal of \mathcal{G} .
Let \mathcal{F}'' be the extension of \mathcal{F} corresponding to \mathcal{H}^{\perp} .
we have

$$\omega\left((T\mathcal{F})^{\perp} \cap (T\mathcal{F}')\right) = \mathcal{H} \text{ and } \omega\left((T\mathcal{F})^{\perp} \cap (T\mathcal{F}'')\right) = \mathcal{H}^{\perp}.$$

According to remark 2.5 it is taken ω which can be locally the darbox's differential of riemannian submersion. This implies that $(T\mathcal{F})^{\perp} \cap (T\mathcal{F}')$ is orthogonal to $(T\mathcal{F})^{\perp} \cap (T\mathcal{F}'')$. ■

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