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Romain Dujardin, Charles Favre

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Abstract. We study the asymptotic behavior of the Lyapunov exponent in a meromorphic family of random products of matrices in $\mathrm{SL}(2,\mathbb{C})$, as the parameter converges to a pole. We show that the blow-up of the Lyapunov exponent is governed by a quantity which can be interpreted as the non-Archimedean Lyapunov exponent of the family. We also describe the limit of the corresponding family of stationary measures on $\mathbb{P}^1(\mathbb{C})$.

Contents

Introduction

Let G be a finitely generated group, endowed with a probability measure m, satisfying the following two conditions:

(A1) $\mathrm{Supp}(m)$ generates G;

(A2) $\displaystyle\int \mathrm{length}(g)\,dm(g) < \infty$.

In a few occasions we shall also require the following stronger moment condition:

(A2$^+$) there exists $\delta > 0$ such that $\displaystyle\int (\mathrm{length}(g))^{1+\delta} dm(g) < \infty$.

In (A2) and (A2$^+$), $\mathrm{length}(\cdot)$ denotes the word-length relative to some fixed, unspecified, finite symmetric set of generators of G. It depends of course of the choice of generators but the moment conditions do not.

If $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ is any representation, the random walk on G induced by m gives rise through ρ to a random product of matrices in $\mathrm{SL}(2,\mathbb{C})$. If $\|\cdot\|$ denotes any matrix norm

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on $\mathrm{SL}(2,\mathbb{C})$, then under the moment condition (A2) we can define the *Lyapunov exponent* $\chi = \chi(G, m, \rho)$ by the formula

$$\chi = \lim_{n\to\infty} \frac{1}{n} \int \log \|\rho(g_n \cdots g_1)\| \, dm(g_1) \cdots dm(g_n) = \lim_{n\to\infty} \frac{1}{n} \int \log \|\rho(g)\| \, dm^n(g), \tag{1}$$

where m^n is the image of $m^{\otimes n}$ under the n-fold product map $(g_1, \ldots, g_n) \mapsto g_n \cdots g_1$. Observe that the limit exists because if we choose the matrix norm to be submultiplicative then the sequence of integrals is subadditive. The Lyapunov exponent is the most basic dynamical invariant associated to the random product of matrices, and its properties have been the object of intense research since the seminal work of Furstenberg [Fg] in the 1960's.

It is quite customary that (ρ, m) depends on certain parameters, in which case the dependence of χ as a function of the parameters becomes an interesting problem. One famous instance of this problem, motivated by statistical physics, is the study of discrete Schrödinger operators, which involves random products of matrices of the form

$$\begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix},$$

where v is a real random variable and E (the energy) is a real or complex parameter (see e.g. [BL]).

We are interested in the situation where the representation depends holomorphically on a complex parameter t, that is, we consider a family of representations (ρ_t) such that for every $g \in G$, $t \mapsto \rho_t(g)$ is holomorphic. Then the Lyapunov exponent defines a function $\chi(t) = \chi(G, m, \rho_t)$ on the parameter space. Recall that a representation $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ is said *non-elementary*[1] if there exist two elements $g_1, g_2 \in G$ such that $\rho(g_1)$ and $\rho(g_2)$ are both hyperbolic (i.e. their eigenvalues have modulus $\neq 1$) and have no common eigenvectors. A celebrated result due to Furstenberg [Fg] (see also [FKi]) asserts that if t_0 is such that ρ_{t_0} is non-elementary, then under the assumptions (A1-2), $\chi(t)$ is positive and continuous at t_0. Hölder continuity can also be derived under stronger moment conditions (see [L]). And it was recently proved by Bocker and Viana [BV] that if m is finitely supported then χ is continuous at elementary representations as well.

It is a classical observation that $t \mapsto \chi(t)$ is subharmonic. In [DD1, DD2] Deroin and the first named author have studied the complex analytic properties of the Lyapunov exponent function in relation with the classical bifurcation/stability theory of Kleinian groups (designed by Bers, Maskit, Sullivan, etc.) and established that the harmonicity of χ over some domain is equivalent to the structural stability of the corresponding family of representations.

Our purpose in this paper is to study the asymptotic properties of $\chi(t)$ in non-compact family of representations with "algebraic behavior" at infinity. To be specific, we consider a holomorphic family $(\rho_t)_{t\in\mathbb{D}^*}$ of representations of G into $\mathrm{SL}(2,\mathbb{C})$, parameterized by the punctured unit disk, and such that $t \mapsto \rho_t(g)$ extends meromorphically through the origin for every g. An obvious but crucial observation is that this data is equivalent to that of a single representation with values in $\mathrm{SL}(2,\mathbb{M})$ where $\mathbb{M}$ is the ring of holomorphic functions on the punctured unit disk with meromorphic extension through the origin.

We will show that the behavior of $\chi(t)$ at $t \to 0$ is controlled by a quantity which can be interpreted as a non-Archimedean Lyapunov exponent associated to the family $(\rho_t)_{t\in\mathbb{D}^*}$. To make sense of this statement, observe first that $\mathbb{M}$ may be viewed as a subring of the field of

[1] Another common terminology is "strongly irreducible and proximal".

formal Laurent series[2] $\mathbb{L} := \mathbb{C}(\!(t)\!)$, which is a complete metrized field when endowed with the t-adic norm $|f|_{\mathsf{na}} = \exp(-\operatorname{ord}_{t=0}(f))$. A representation $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ thus canonically yields a representation $\rho_{\mathsf{na}} : G \to \mathrm{SL}(2, \mathbb{L})$ and exactly as in (1) we define χ_{na} to be the Lyapunov exponent of the representation ρ_{na}, where the matrix norm $\|\cdot\|$ is now associated to the t-adic absolute value on $\mathbb{L}$.

We are now in position to state our main result.

Theorem A. *Let (G, m) be a finitely generated group endowed with a measure satisfying (A1) and (A2$^+$), and let $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ be any representation. Then*

$$\frac{1}{\log |t|^{-1}} \chi(t) \longrightarrow \chi_{\mathsf{na}} \text{ as } t \to 0. \tag{2}$$

To illustrate the result in a simple case, consider a random product of Schrödinger matrices of the form

$$\begin{pmatrix} \frac{1}{t}(E - v) & -1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

where v is a bounded random variable, E is fixed, and $t \to 0$. Then it is easy to show in this case that $\chi(t) \sim \log |t|^{-1}$ as $t \to 0$. One reason for this ease is that the pole structure of the n-fold product of such matrices is explicit and easy to describe: such a product will be of the form $\frac{1}{t^n} \begin{pmatrix} O(1) & O(t) \\ O(t) & O(t^2) \end{pmatrix}$. Avron, Craig and Simon [ACS] gave in this situation a refined asymptotics of $\chi(t)$ at the order $o(1)$ (and a conjectural asymptotics at the order $O(t^2)$).

For a general random product of matrices with meromorphic coefficients, the poles can add up or cancel in a rather subtle way, and the non-Archimedean formalism allows to deal efficiently with this algebra. The idea of using non-Archimedean representations to describe the degenerations of $\mathrm{SL}(2, \mathbb{C})$ representations is now classical and was pioneered by Culler and Shalen [CS]. One main input of the present work is the incorporation of this technique into the theory of random matrix products.

Let us explain the strategy of the proof of Theorem A. Since $\mathbb{L}$ is a metrized field it makes sense to talk about hyperbolic elements in $\mathrm{SL}(2, \mathbb{L})$ so we can define natural notions of elementary and non-elementary subgroups. We refer to §2 for a thorough discussion of these concepts.

The proof of Theorem A splits into two quite different parts according to the elementary or non-elementary nature of ρ_{na}. The easier case is when ρ_{na} is elementary: then either the family is holomorphic at the origin (after conjugation by a suitable meromorphic family of Möbius transformations and possibly taking a branched 2-cover of the base) or ρ_t is elementary for all $t \in \mathbb{D}^*$. In the former case, $\chi_{\mathsf{na}} = 0$ and it follows from Furstenberg's theory that $\chi(t) = O(1)$ so we are done. In the latter case, up to meromorphic conjugacy, the image of ρ_t in $\mathrm{SL}(2, \mathbb{C})$ is either upper triangular or lies in an index 2 extension of the diagonal subgroup. The result then follows from a careful application of the law of large numbers. The details of the arguments are explained in §5. Note that this is the only place where we use the stronger integrability condition (A2$^+$).

Let us now assume that ρ_{na} is non-elementary. A first observation is that ρ_t is then non-elementary for small enough t (see Lemma 4.3; by rescaling we may assume that this holds

[2]Beware that in some of the references cited in our bibliography, $\mathbb{L}$ denotes the completion of the algebraic closure of $\mathbb{C}(\!(t)\!)$.

for $|t| < 1$) so we may apply Furstenberg's theory to analyze the Lyapunov exponent $\chi(t)$. The main step of the proof of the continuity of χ in the classical setting is the study of the Markov chain on $\mathbb{P}^1$ induced by the image measure $\rho_* m$ in $\mathrm{SL}(2,\mathbb{C})$. More specifically, the measure $\mu_t = (\rho_t)_* m$ acts by convolution on the set of probability measures on the Riemann sphere by $\nu \mapsto \mu_t * \nu$, where $\mu_t * \nu = \int \gamma_* \nu \, d\mu_t(\gamma)$. A *stationary measure* is by definition a fixed point of this action. A fundamental result is that when ρ_t is non-elementary, there is a *unique* stationary probability measure ν_t. Furthermore, the Lyapunov exponent $\chi(t)$ is positive and can be expressed by an explicit formula involving ν_t:

$$\chi(t) = \int \log \frac{\|\gamma \cdot v\|}{\|v\|} \, d\mu_t(\gamma) \, d\nu_t(v) \ . \tag{4}$$

The continuity of χ at non-elementary representations immediately follows: since ν_t is unique it varies continuously with t, and (4) implies the result.

The first step of the proof consists in extending these results to random products in $\mathrm{SL}(2,\mathbb{L})$. We actually work over an arbitrary complete metrized field k and show in §3 how to generalize the above results to any non-elementary representation $\rho : G \to \mathrm{SL}(2,k)$. Of particular interest to us is the fact that the representation ρ_{na} admits a unique stationary measure ν_{na} which lives on the Berkovich analytification $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ of the projective line and for which a non-Archimedean analog of (4) holds. In particular we obtain the positivity of the non-Archimedean Lyapunov exponent.

Let us point out that this positivity also follows from the recent work of Maher and Tiozzo [MT] on random walks on groups of isometries of non-proper Gromov hyperbolic spaces. Maher and Tiozzo also discuss stationary measures, however, they work in a compactification which is a priori hard to relate to the Berkovich space.

From this point, two different paths lead to the main theorem. Both of them deal with the asymptotic properties of the stationary measure ν_t and can be seen as ways to imitate the Furstenberg argument for the continuity of the Lyapunov exponent.

The first method belongs to complex geometry and is described in §4. It relies on a correspondence between certain finite subsets of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ and *models* of $\mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$. Here by model we mean a complex surface Y endowed with a birational map $\pi : Y \to \mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$ which is a biholomorphism over $\mathbb{D}^* \times \mathbb{P}^1_{\mathbb{C}}$.

For $t \neq 0$ we denote by ν_{Y_t} the pull-back to ν_t on Y, which should be understood as "the measure ν_t viewed on the model Y". We obtain the following result.

Theorem B. *Let (G,m) be a finitely generated group endowed with a measure satisfying (A1) and let $\rho : G \to \mathrm{SL}(2,\mathbb{M})$ be a non-elementary representation.*

Then for every model $Y \to \mathbb{D} \times \mathbb{P}^1$, the canonical family of stationary probability measures ν_{Y_t} converges as $t \to 0$ to a purely atomic measure ν_Y.

Furthermore if ν_{na} denotes the unique stationary probability measure on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$, then $\nu_Y = (\mathrm{res}_Y)_* \nu_{\mathsf{na}}$ is the residual measure of ν_{na} on Y.

Here the residue map res_Y is a canonical anti-continuous map from $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ to the $\mathbb{C}$-scheme $\pi^{-1}(\{0\} \times \mathbb{P}^1_{\mathbb{C}})$. In particular the push-forward of a *non-atomic* measure on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is an *atomic* measure on Y . We refer to §4.3 for a detailed discussion on this map. The asymptotics (2) of the Lyapunov exponent in Theorem A then follows from an analysis of (4) as $t \to 0$ in a carefully chosen family of models.

The second approach to Theorem A relies on the notion of *hybrid (Berkovich) space*. This is a topological space which allows to give a precise meaning to the intuitive idea that ν_t "converges" to ν_{na} as $t \to 0$, and derive the asymptotics (2) directly from this weak convergence.

This space was first constructed by Berkovich in [Ber]. It has been recently realized by Boucksom and Jonsson in [BJ] and by the second named author in [Fav] that it is well-adapted to the description of the limiting behavior of families of measures such as the $(\nu_t)_{t\in\mathbb{D}^*}$. Concretely, $\mathbb{P}^1_{\mathrm{hyb}}$ is a compact topological space endowed with a continuous surjective map[3] $p_{\mathrm{hyb}} : \mathbb{P}^1_{\mathrm{hyb}} \to \overline{\mathbb{D}}_{1/e}$ such that $p_{\mathrm{hyb}}^{-1}(0)$ can be identified with the non-Archimedean analytic space $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ while p_{hyb} is a trivial topological fibration over $\overline{\mathbb{D}}^*_{1/e}$ with $\mathbb{P}^1$ fibers. More precisely there exists a canonical homeomorphism $\psi\colon \overline{\mathbb{D}}^*_{1/e} \times \mathbb{P}^1 \to p_{\mathrm{hyb}}^{-1}(\overline{\mathbb{D}}^*_{1/e})$ such that $p_{\mathrm{hyb}} \circ \psi$ is the projection onto the first factor. Likewise we denote by ψ_{na} the canonical identification $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to p_{\mathrm{hyb}}^{-1}(\{0\})$.

A key point in the construction of $\mathbb{P}^1_{\mathrm{hyb}}$ is that its topology is designed so that for every $g \in G$ the function

$$(t, v) \longmapsto \frac{1}{\log |t|^{-1}} \log \frac{\|\rho_t(g) \cdot v\|}{\|v\|}$$

extends *continuously* to the hybrid space for $t = 0$. Theorem A hence follows immediately from:

Theorem C. *Let (G, m) be a finitely generated group endowed with a measure satisfying (A1) and $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ be a non-elementary representation.*

Then in $\mathbb{P}^1_{\mathrm{hyb}}$, we have that $(\psi_t)_*(\nu_t) \longrightarrow (\psi_{\mathsf{na}})_*\nu_{\mathsf{na}}$ as $t \to 0$ in the weak topology of measures, where ν_t is the unique stationary probability measure under μ_t, and $\psi_t(\cdot) = \psi(t, \cdot)$.

We discuss the construction of the hybrid space and prove Theorem C in §6.

⋄

As for [DD1, DD2], this work was prompted by analogous results in the context of iteration of rational mappings, in accordance with the celebrated *Sullivan dictionary*. Consider a holomorphic family of rational maps $(f_t)_{t\in\mathbb{D}^*}$ of degree $d \geq 2$ that extends meromorphically through 0, and denote by μ_{f_t} their measure of maximal entropy. In this context, the analogue of A is a formula for the blow-up of the Lyapunov exponent of μ_{f_t} which follows from the work of DeMarco [dM1, dM2] (see also [Fav] for generalizations to higher dimension), and Theorem B was proven by DeMarco and Faber [dMF1] . It is particularly interesting to note that proving the convergence of the measures μ_{f_t} relies on pluripotential theory and the interpretation of μ_{f_t} as the Monge-Ampère measure of a suitable metrization on an ample line bundle, whereas in our case it follows from the uniqueness of the stationary measure.

Our work raises several natural open questions.

(1) Is it possible to estimate the error term $\chi(t) - \chi_{\mathsf{na}} \log |t|^{-1}$? The answer is easy when $\chi(t)$ is harmonic in a punctured neighborhood of the origin, in which case one obtains an expansion of the form

$$\chi(t) = \chi_{\mathsf{na}} \log |t|^{-1} + C^{st} + o(1)$$

(this situation happens e.g. in (3)). In the general case, continuity holds under appropriate moment assumptions if ρ_{na} is elementary (see §5.4). However in the

[3] The choice of the value $1/e$ for the radius is convenient, of course any other would do.

nonelementary case our method seems to produce errors of magnitude $\varepsilon \log|t|^{-1}$ (see §4.5) so new ideas have to be developed.

(2) Can our results be extended to higher dimensions? Random matrix products in arbitrary dimension over an Archimedean or a local field are well-understood. Some of our arguments should carry over to the study of the extremal Lyapunov exponents, even if we use the hyperbolic structure of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ at some places. Note however that even the Oseledets theorem does not seem to have received much attention over arbitrary metrized fields.

$\diamond$

The plan of the paper is as follows. In section 1 we recall some basics on Berkovich theory. In section 2 we classify subgroups of $\mathrm{PGL}(2,k)$ for an arbitrary complete metrized field k. In particular we define the notion of non-elementary subgroup and classify elementary ones. Part of this material follows from the classical theory of group acting on trees. In section 3 we develop the non-Archimedean Furstenberg theory. The complex geometric proof to Theorem A and is given in sections 4 (in the non-elementary case, including B) and 5 (for the elementary case). The hybrid formalism and Theorem C are explained in §6.

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1. The Berkovich projective line

In this section, we collect some basic facts on the Berkovich analytification of the projective line over a complete non-trivially metrized field $(k, |\cdot|)$. Observe that we do not assume k to be algebraically closed (since we apply these results to $k = \mathbb{C}((t))$ later on) which leads to a few subtleties. The reader is referred to [Ber, T] for a general discussion on Berkovich spaces, and to [BR, J] for a detailed description of the Berkovich projective line.

1.1. **Analytification of the projective line.** We denote by $\mathbb{P}^1_k$ the projective line over a field k, viewed as an algebraic variety, endowed with its Zariski topology, and by $\mathbb{P}^1(k)$ its set of k-points which is in bijection with $k \cup \{\infty\}$. When $k = \mathbb{C}$, we often simply denote by $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ the Riemann sphere, that is, the complex projective line with its usual structure of compact complex manifold[4].

In the remainder of this section, we suppose that $(k, |\cdot|)$ is a complete metrized *non-Archimedean* field. We also assume that the norm on k is non-trivial, so in particular k is infinite. We denote by $\mathbb{P}^{1,\mathrm{an}}_k$ the Berkovich analytification of $\mathbb{P}^1_k$ which is a compact topological space endowed with a structural sheaf of analytic functions. Only its topological structure will be used in this paper, and we refer the interested reader to [Ber] for the description of the structural sheaf.

The Berkovich space $\mathbb{P}^{1,\mathrm{an}}_k$ is defined as follows. The analytification of the affine line $\mathbb{A}^{1,\mathrm{an}}_k$ is the space of all multiplicative semi-norms on $k[Z]$ whose restriction to k coincides with $|\cdot|$, endowed with the topology of pointwise convergence.

Given a point $x \in \mathbb{A}^{1,\mathrm{an}}_k$ and a polynomial $P \in k[Z]$, the value of the semi-norm defined by x on P is usually denoted by $|P|_x \in \mathbb{R}_+$. It is also customary to denote it by $|P(x)|$, the reason for this notation should be clear from the classification of semi-norms below. The

[4]Note that formally $\mathbb{P}^1$ can be viewed as the analytification $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{C}}$ of the variety $\mathbb{P}^1_{\mathbb{C}}$.

Gauß norm $\sum_i a_i Z^i \mapsto \max|a_i|$ defines a point denoted by x_{g}, and referred to as the Gauß point.

The Berkovich projective line can be defined as a topological space to be the one-point compactification of $\mathbb{A}^{1,\mathrm{an}}_k$ so that we write $\mathbb{P}^{1,\mathrm{an}}_k = \mathbb{A}^{1,\mathrm{an}}_k \cup \{\infty\}$. More formally it is obtained by gluing two copies of $\mathbb{A}^{1,\mathrm{an}}_k$ in a standard way using the transition map $z \mapsto z^{-1}$ on the punctured affine line $(\mathbb{A}^1)^{*,\mathrm{an}}_k$.

A *rigid* point in $\mathbb{P}^{1,\mathrm{an}}_k$ is a point defined by a multiplicative semi-norm having a non-trivial kernel. For any point z lying in a finite extension of k, the semi-norm $|\cdot|_z$ defined by $P \mapsto |P(z)|$ is a rigid point in $\mathbb{A}^{1,\mathrm{an}}_k$. The induced map yields a canonical bijection between closed points of the k-scheme $\mathbb{P}^1_k$ and rigid points in $\mathbb{P}^{1,\mathrm{an}}_k$. In particular $\mathbb{P}^1(k)$ naturally embeds as a set of rigid points in $\mathbb{P}^{1,\mathrm{an}}_k$, and in the following we simply view $\mathbb{P}^1(k)$ as a subset of $\mathbb{P}^{1,\mathrm{an}}_k$.

The Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}_k$ is an $\mathbb{R}$-tree in the sense that it is uniquely pathwise connected, see [J, §2] for precise definitions. In particular for any pair of points $(x, y) \in \mathbb{P}^{1,\mathrm{an}}_k$ there is a well-defined segment $[x, y]$. Recall that the convex hull of a subset F in an $\mathbb{R}$-tree is the smallest connected set $\mathrm{Conv}(F)$ which contains F, that is the union of all segments $[x, y]$ with $x, y \in F$. Any point in $\mathbb{P}^{1,\mathrm{an}}_k$ admit a well defined projection to a closed convex subset.

In this paper, by *measure* on $\mathbb{P}^{1,\mathrm{an}}_k$ we mean a Radon measure, that is a Borel measure which is internally regular, or equivalently a bounded linear functional on the vector space of continuous functions on $\mathbb{P}^{1,\mathrm{an}}_k$ endowed with the sup norm.

Using the tree structure, one can show the following result (see [FJ, Lemma 7.15]).

Lemma 1.1. *The support of any measure in $\mathbb{P}^{1,\mathrm{an}}_k$ is compact and metrizable.*

1.2. **Balls in the projective line and semi-norms.** We still assume that the norm on k is non-Archimedean and non-trivial. Closed and open balls in k (of radius $R \in \mathbb{R}_+$)

$$\overline{B}(z_0, R) = \{z \in k,\, |z - z_0| \le R\} \text{ and } B(z_0, R) = \{z \in k,\, |z - z_0| < R\},$$

are defined as usual. By definition, a ball in $\mathbb{P}^1(k)$ is either a ball in k of the complement of a ball in k.

Any closed (or open) ball $B \subsetneq \mathbb{P}^1(k)$ determines a point $x_B \in \mathbb{P}^{1,\mathrm{an}}_k$. When B or its complement is a singleton $\{z\}$, this point x_B is the rigid point attached to z. When B is a (open or closed) ball of finite radius in k, we let x_B be the point in $\mathbb{A}^{1,\mathrm{an}}_k$ corresponding to the semi-norm $|P(x_B)| := \sup_B |P|$. Otherwise the complement of B is a ball of finite radius in k, and we set $|P(x_B)| := \sup_{k \setminus B} |P|$. Observe that a closed ball x_B is rigid iff its diameter is zero, and that the Gauß point is equal to $x_{\overline{B}(0,1)}$.

Remark 1.2. When $|k^*|$ is dense in $\mathbb{R}_+$, we have $x_{\overline{B}(z_0,R)} = x_{B(z_0,R)}$ for all $z_0 \in k$ and $R \in \mathbb{R}_+$. Otherwise k is discretely valued, $|k^*| = r^{\mathbb{Z}}$ for some $r > 1$, and we have $x_{B(z_0,r^n)} = x_{\overline{B}(z_0,r^{n-1})}$.

1.3. **The spherical metric.** Let $(k, |\cdot|)$ be any non-Archimedean complete metrized field. We can endow $\mathbb{P}^1(k)$ with the spherical metric:

$$d_{\mathrm{sph}}([z_0 : z_1], [w_0 : w_1]) = \frac{|z_0 w_1 - z_1 w_0|}{\max\{|z_0|, |z_1|\} \max\{|w_0|, |w_1|\}} \ . \tag{1.1}$$

and its spherical diameter is equal to 1. For any $z \in \mathbb{P}^1(k)$ and $r \leq 1$ we define closed and open *spherical* balls

$$\overline{B}^{\mathrm{sph}}(z,r) = \{d_{\mathrm{sph}}(\cdot, z) \leq r\} \text{ and } B^{\mathrm{sph}}(z,r) = \{d_{\mathrm{sph}}(\cdot, z) < r\} \ .$$

Observe that for all $r \geq 1$, $\overline{B}^{\mathrm{sph}}(z,r) = \mathbb{P}^1(k)$. A spherical ball is either $\mathbb{P}^1(k)$ or a ball in $\mathbb{P}^1(k)$ in the sense of the previous section. Conversely a ball in $\mathbb{P}^1(k)$ is either a spherical ball or the complement of a spherical ball.

1.4. **The hyperbolic space (k algebraically closed).** Let $(k, |\cdot|)$ be any *algebraically closed* non-Archimedean complete metrized field, and let us describe the geometry of $\mathbb{P}^{1,\mathrm{an}}_k$ under this assumption. First observe that rigid points are in bijection with balls of zero diameter. Following the Berkovich terminology we say that points corresponding to balls of diameter $\mathrm{diam}(B) \in |k^*|$ (resp. $\mathrm{diam}(B) \notin |k^*|$) are of type 2 (resp. of type 3). Rigid points are said to be of type 1.

More generally, points in $\mathbb{P}^{1,\mathrm{an}}_k$ are in bijection with (equivalence classes of) decreasing sequences of balls, see [BR, Theorem 1.2]. When k is spherically complete, that is, every decreasing intersection of balls is non-empty, then $\mathbb{P}^{1,\mathrm{an}}_k$ consists of type 1, 2 or 3 points. In general, there may exist a fourth type of points, associated with decreasing sequences of balls with empty intersection. All these types of points (whenever non-empty) yield dense subsets of $\mathbb{P}^{1,\mathrm{an}}_k$.

Types of points relate with the tree structure as follows. Type 1 and 4 points are precisely the ones at which the $\mathbb{R}$-tree $\mathbb{P}^{1,\mathrm{an}}_k$ has only one branch, that is, they are endpoints of the tree. Type 2 points are branching points (i.e. $\mathbb{P}^{1,\mathrm{an}}_k$ has at least three branches at these points) and type 3 points are regular points (i.e. $\mathbb{P}^{1,\mathrm{an}}_k$ has exactly two branches at these points).

The hyperbolic space $\mathbb{H}_k$ is by definition the complement of the set of rigid points in $\mathbb{P}^{1,\mathrm{an}}_k$, i.e. $\mathbb{H}_k = \mathbb{P}^{1,\mathrm{an}}_k \setminus \mathbb{P}^1(k)$. It is a proper subtree of $\mathbb{P}^{1,\mathrm{an}}_k \setminus \mathbb{P}^1(k)$ which contains no rigid point and is neither open nor closed.

To describe the structure of $\mathbb{H}_k$, for any $r \in \mathbb{R}_+$ introduce the semi-norm $x_r \in \mathbb{A}^{1,\mathrm{an}}_k$ defined by

$$|P(x_r)| = \max\{|a_n| r^n,\, a_n \neq 0\}$$

where $P(Z) = \sum a_n Z^n$. In particular $x_r = x_{\bar{B}(0,r)}$. Let $\mathbb{H}^\circ_k$ be the orbit under $\mathrm{PGL}(2,k)$ of the ray $\{x_r,\, r \in \mathbb{R}^*_+\}$. Then $\mathbb{H}^\circ_k$ is a dense subtree of $\mathbb{H}_k$, and $\mathbb{H}_k \setminus \mathbb{H}^\circ_k$ coincides with the set of type 4 points in $\mathbb{P}^{1,\mathrm{an}}_k$.

By [BR, Prop. 2.30], one can endow $\mathbb{H}^\circ_k$ with a unique $\mathrm{PGL}(2,k)$-invariant metric such that

$$d_{\mathbb{H}}(x_{r_1}, x_{r_2}) = \log\left|\frac{r_1}{r_2}\right|$$

for any $r_1 \geq r_2 > 0$. Observe that

$$d_{\mathbb{H}}(x_{B_1}, x_{B_2}) = \log\left(\frac{\mathrm{diam}(B_2)}{\mathrm{diam}(B_1)}\right) \ ,$$

for any closed two balls $B_1 \subset B_2 \subset k$ (the diameter is relative to the metric induced by the norm $|\cdot|$).

A proof of the next result can be found in [BR, Prop. 2.29].

Lemma 1.3. *The metric defined above on $\mathbb{H}_k^\circ$ extends to a distance on $\mathbb{H}_k$, which makes $(\mathbb{H}_k, d_{\mathbb{H}})$ a complete metric $\mathbb{R}$-tree upon which $\mathrm{PGL}(2,k)$ acts by isometries.*

Recall that by metric $\mathbb{R}$-tree we mean that for any pair of distinct points x, y there exists a unique isometric embedding $\gamma : [0, d_{\mathbb{H}}(x,y)] \to \mathbb{H}_k$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $d_{\mathbb{H}}(\gamma(t), \gamma(t')) = |t - t'|$.

1.5. **Field extensions.** Let K/k be any complete field extension. The inclusion $k[Z] \subset K[Z]$ yields by restriction a canonical surjective and continuous map $\pi_{K/k} \colon \mathbb{P}^{1,\mathrm{an}}_K \to \mathbb{P}^{1,\mathrm{an}}_k$, and the Galois group $\mathrm{Gal}(K/k)$ acts continuously on $\mathbb{P}^{1,\mathrm{an}}_K$.

Let $\bar{k}^a$ be the completion of an algebraic closure of k. Then $\mathbb{P}^{1,\mathrm{an}}_k$ is homeomorphic to the quotient of $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$ by $\mathrm{Gal}(\bar{k}^a/k)$, see [Ber, Corollary 1.3.6]. The group $\mathrm{Gal}(\bar{k}^a/k)$ preserves the types of points in $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$, so that we may define the type of a point $x \in \mathbb{P}^{1,\mathrm{an}}_k$ as the type of any of its preimage by $\pi_{\bar{k}^a/k}$ in $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$. Note that since the field extension $\bar{k}^a/k$ is not algebraic in general, it may happen that some type 1 points in $\mathbb{P}^{1,\mathrm{an}}_k$ are not rigid (this phenomenon occurs when k is the field of Laurent series over any field).

By [BR, Prop. 2.15], the natural action of $\mathrm{PGL}(2,k)$ on $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$ preserves the types of points so the same holds for its action on $\mathbb{P}^{1,\mathrm{an}}_k$.

Proposition 1.4. *The following assertions are equivalent.*

(1) The point $x \in \mathbb{P}^{1,\mathrm{an}}_k$ belongs to the orbit of the Gauß point under the action of $\mathrm{PGL}(2,k)$ *(in particular it is of type 2).*
(2) There exists $z \in k$ and $r \in |k^*|$ such that $x = x_{\bar{B}(z,r)}$.

Proof. Pick $z \in k$ and assume $r \in |k^*|$ that there exists $y \in k^*$ with $r = |y|$. The image of $\bar{B}(0,1)$ by the affine map $Z \mapsto yZ + z$ is equal to $\bar{B}(z,r)$ hence (2)⇒(1). Conversely any element in $\mathrm{PGL}(2,k)$ can be decomposed as a product of affine maps and the inversion $\Phi(Z) := 1/Z$. Thus we conclude that (1)⇒(2) by observing that $\Phi(x_{\bar{B}(z,r)}) = x_{\bar{B}(z^{-1}, r/|z|^2)}$ if $r \le |z|$, $\Phi(x_{\bar{B}(0,r)}) = x_{\bar{B}(0,r^{-1})}$ and $A(x_{\bar{B}(z,r)}) = x_{\bar{B}(az+b, |a|r)}$ if $A(Z) = aZ + b$. □

Proposition 1.5. *Suppose x, y, z belongs to the orbit of x_{g} under* $\mathrm{PGL}(2,k)$*. Then the projection of z on $[x,y]$ also belongs to the orbit of x_{g} under* $\mathrm{PGL}(2,k)$*.*

Proof. We can normalize the situation so that $x = x_{\mathrm{g}} = x_{\bar{B}(0,1)}$ and $y = x_{\bar{B}(0,r)}$ for some $1 < r \in |k^*|$. Let $z = x_{\bar{B}(a,r')}$. If $\bar{B}(a,r')$ is disjoint from $\bar{B}(0,1)$ or contains it then the projection is the Gauss point and we are done. Otherwise $\bar{B}(a,r') \subset \bar{B}(0,1)$ with $|a| \le 1$ and $r' < 1$. If $\bar{B}(a,r') \subset \bar{B}(0,r)$ the projection equals $\bar{B}(0,r)$ and again we are done. The remaining case is when $|a| > r$, in which case the projection is $\bar{B}(0,|a|)$ and we conclude by Proposition 1.4. □

1.6. **The hyperbolic space (k arbitrary).** The Galois group $\mathrm{Gal}(\bar{k}^a/k)$ acts on $\bar{k}^a$ by isometries, so the diameter of balls is $\mathrm{Gal}(\bar{k}^a/k)$-invariant. As a consequence the action of $\mathrm{Gal}(\bar{k}^a/k)$ on $(\mathbb{H}_{\bar{k}^a}, d_{\mathbb{H}})$ is also isometric. Let $\tilde{\mathbb{H}}_k \subset \mathbb{H}_{\bar{k}^a}$ be the set of fixed points of this action.

Lemma 1.6. *The set of fixed points of the action of* $\mathrm{Gal}(\bar{k}^a/k)$ *on $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$ is $\overline{\mathrm{Conv}(\mathbb{P}^1(k))}$.*

Proof. Denoting by $\mathcal{F}$ this set of fixed points, it is clear that $\mathcal{F}$ contains $\mathbb{P}^1(k)$. Since the Galois action preserves the tree structure we infer that $\mathcal{F} \supset \mathrm{Conv}(\mathbb{P}^1(k))$ and since it is weakly continuous we finally deduce that $\mathcal{F}$ contains $\overline{\mathrm{Conv}(\mathbb{P}^1(k))}$.

Suppose by contradiction that there exists a point $x \in \mathcal{F}$ which does not belong to $\overline{\mathrm{Conv}(\mathbb{P}^1(k))}$. Since type 2 points are dense on any ray in the tree $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$, we may suppose $x = x_{\bar{B}(z,r)}$ for some $z \in \bar{k}^a$ and $r \in |k^*|^{\mathbb{Q}}$. It is enough to show that $\bar{B}(z,r)$ contains a point of k. Indeed in this case we get that $x \in \mathrm{Conv}(\mathbb{P}^1(k))$ which contradicts our assumption.

To show this, first note that algebraic points over k are dense in $\bar{k}^a$, hence we may assume that z is algebraic over k. Let P be its minimal polynomial, and suppose first that its degree d is prime to the characteristic of k. The point x is fixed by $\mathrm{Gal}(\bar{k}^a/k)$ hence so does $\bar{B}(z,r)$, so this ball contains all the roots $z = z_1, \ldots, z_d$ of P (repeated according to their multiplicity if needed). In particular letting $z^* = \frac{1}{d}(z_1 + \cdots + z_d)$ we have that $|z^* - z| \le r$ and $z^* \in k$ and we are done.

When d is not prime to the characteristic, we modify this argument as follows. Fix $a \in k^*$ such that $|a| \cdot |z|^{d+1} \ll |P(x)|$ (recall that $|P(x)| = \sup_B |P|$) and consider the polynomial $\widetilde{P}(X) = aX^{d+1} + P(X)$. By definition of P we have $|\tilde{P}(z)| = |a| \cdot |z|^{d+1}$, and on the other hand $|\tilde{P}(x)| \ge |P(x)|$. This classically implies the existence of a root of $\widetilde{P}$ in $\bar{B}(z,r)$, so $\widetilde{P}$ is a polynomial in $k[X]$ of degree $d+1$ with a root in $\bar{B}(z,r)$ and we can apply the previous argument to $\widetilde{P}$. $\square$

By the previous lemma we have that in $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$,
$\tilde{\mathbb{H}}_k := \overline{\mathrm{Conv}(\mathbb{P}^1(k))} \setminus \mathbb{P}^1(k)$. In particular $(\tilde{\mathbb{H}}_k, d_{\mathbb{H}})$ is a complete metric $\mathbb{R}$-tree by §1.4.

Since $\mathbb{P}^{1,\mathrm{an}}_k$ is homeomorphic to the quotient of $\mathbb{P}^{1,\mathrm{an}}_{\bar{k}^a}$ by $\mathrm{Gal}(\bar{k}^a/k)$, the restriction map $\pi_{\bar{k}^a/k}$ induces a homeomorphism from $\tilde{\mathbb{H}}_k$ onto its image, which we denote by $\mathbb{H}_k$ and call the hyperbolic space over k [5]. We endow it with the metric $d_{\mathbb{H}}$ making $\pi_{\bar{k}^a/k} \colon (\tilde{\mathbb{H}}_k, d_{\mathbb{H}}) \to (\mathbb{H}_k, d_{\mathbb{H}})$ an isometry.

The following proposition summarizes the properties of $\mathbb{H}_k$ obtained so far.

Proposition 1.7. *The hyperbolic space $\mathbb{H}_k$ is the closure of the convex hull of $\mathbb{P}^1(k)$ in $\mathbb{P}^{1,\mathrm{an}}_k$ from which $\mathbb{P}^1(k)$ is removed, that is $\mathbb{H}_k := \overline{\mathrm{Conv}(\mathbb{P}^1(k))} \setminus \mathbb{P}^1(k) \subset \mathbb{P}^{1,\mathrm{an}}_k$. Endowed with the metric $d_{\mathbb{H}}$, it is a complete metric $\mathbb{R}$-tree upon which $\mathrm{PGL}(2,k)$ acts by isometries.*

Let K/k be any complete field extension, then there is a canonical $\mathrm{PGL}(2,k)$-equivariant continuous map $\sigma_{K/k} : \overline{\mathrm{Conv}(\mathbb{P}^1(k))} \to \mathbb{P}^{1,\mathrm{an}}_K$ such that $\pi_{K/k} \circ \sigma_{K/k} = \mathrm{id}$ and which sends the point $x_{\bar{B}(0,r)} \in \mathbb{P}^{1,\mathrm{an}}_k$ to the corresponding point $x_{\bar{B}(0,r)} \in \mathbb{P}^{1,\mathrm{an}}_K$ for all $r \in \mathbb{R}_+$. A detailed discussion of this map can be found in [P]. The map $\sigma_{K/k}$ is injective hence induces a homeomorphism from $\overline{\mathrm{Conv}(\mathbb{P}^1(k))}$ in $\mathbb{P}^{1,\mathrm{an}}_k$ onto its image which is the closure of the convex hull of $\mathbb{P}^1(k))$ in $\mathbb{P}^{1,\mathrm{an}}_K$.

Proposition 1.8. *For any pair (x,y) of points in $\mathbb{H}_k$ lying in the orbit of x_{g} under $\mathrm{PGL}(2,k)$, there exists a quadratic extension K/k and $g \in \mathrm{PGL}(2,K)$ such that $g \cdot x_{\mathrm{g}} \in \mathbb{P}^{1,\mathrm{an}}_K$ is the middle point of the segment $[\sigma_{K/k}(x), \sigma_{K/k}(y)]$.*

Proof. Applying a suitable Möbius transformation, we may assume that $x = x_{\mathrm{g}} = x_{\bar{B}(0,1)}$ and $y = x_{\bar{B}(0,r)}$ with $r \in |k^*|$. Fix $z \in k^*$ such that $|z| = r$, and pick any square root z' of z. The

[5] When k is a p-adic field, $\mathbb{H}_k$ is *not* the Drinfeld upper half-plane which is equal as a set to $\mathbb{P}^{1,\mathrm{an}}_k \setminus \mathbb{P}^1(k)$.

middle point of the segment $[x, y] := \{x_{B(0,t)},\ t \in [1, r]\}$ is the type 2 point $x_{\bar{B}(0,\sqrt{r})}$ which lies in the orbit of $x_{\rm g}$ by $\mathrm{PGL}(2, k(z'))$. The assertion is proved with $K = k(z')$. □

1.7. **Balls and simple domains in $\mathbb{P}^{1,\mathbf{an}}_k$.** For any $a \in k$ and any $r \geq 0$, set

$$\bar{B}^{\rm an}(a, r) = \left\{x \in \mathbb{A}^{1,\rm an}_k,\ |Z - a|_x \leq r\right\} \text{ or } B^{\rm an}(a, r) = \left\{x \in \mathbb{A}^{1,\rm an}_k,\ |Z - a|_x < r\right\} . \tag{1.2}$$

When no confusion can arise, we drop the "an" subscript.

A closed ball in the Berkovich projective line is a set of the form $\bar{B}^{\rm an}(a, r)$ or the complement of a set of the form $B^{\rm an}(a, r)$ in $\mathbb{P}^{1,\rm an}_k$. One defines similarly open balls. Observe that any ball in $\mathbb{P}^{1,\rm an}_k$ not containing ∞ is of the form (1.2). A closed (resp. open) ball is a closed (resp. open) subset of $\mathbb{P}^{1,\rm an}_k$ with one single boundary point. When the ball is $\bar{B}^{\rm an}(a, r)$ or $B^{\rm an}(a, r)$, or their complements, this boundary point is $x_{\bar{B}(a,r)}$.

When k is algebraically closed, a closed (resp. open) ball in $\mathbb{P}^{1,\rm an}_k$ is the convex hull of a closed (resp. open) ball in $\mathbb{P}^1(k)$.

Following the terminology of [BR], by a *simple domain* we mean any open set $U \subset \mathbb{P}^{1,\rm an}_k$ whose boundary is a finite set of type 2 points. Open balls are simple domains, and it follows from [Ber, Thm 4.2.1] that simple domains form a basis for the topology of $\mathbb{P}^{1,\rm an}_k$.

The next result will play an important role in our approach to Theorem A.

Proposition 1.9. *Let ν be any probability measure on $\mathbb{P}^{1,\rm an}_k$ having no atom.*

Then for every $\varepsilon > 0$ there exists a finite set S of type 2 points such that every connected component U of $\mathbb{P}^{1,\rm an}_k \setminus S$ satisfies $\nu(U) < \varepsilon$.

Proof. Pick any $\varepsilon > 0$. Since ν is a Radon measure, any point x is included in a simple domain U_x such that $\nu(U_x) \leq \varepsilon$. By compactness, we may cover the support of ν by finitely many of these domains $U_{x_1}, \ldots, U_{x_n}$. Let S be the union of all boundary points of U_{x_i} for $i = 1, \ldots, n$. Any connected component U of $\mathbb{P}^{1,\rm an}_k \setminus S$ intersects one of the open sets say U_{x_1}. Since $U \cap \partial U_{x_1} = U \cap S = \emptyset$, it follows that $U \subset U_{x_1}$ and $\nu(U) \leq \varepsilon$ as claimed. □

2. Subgroups of PGL(2, k)

In this section $(k, |\cdot|)$ is an arbitrary non-trivially valued field that is complete and non-Archimedean and by k° its valuation ring. The case of most interest to us is $\mathbb{L} := \mathbb{C}(\!(t)\!)$ which is a complete metrized field when endowed with the t-adic norm $|f|_{\mathsf{na}} = \exp(-\operatorname{ord}_{t=0}(f))$.

We consider a subgroup $\Gamma \leq \mathrm{PGL}(2, k) = \mathrm{Aut}(\mathbb{P}^1_k)$ and study the geometric properties of its action on the projective and Berkovich spaces. Much of this material is a reformulation in our context of classical results on groups acting on trees (see e.g. [CM, K, O], and also [YW] for related material).

Over the complex numbers the corresponding results are well-known (see [Bea]).

2.1. **Basics.** As in the complex setting, there is a morphism $\mathrm{SL}(2, k) \to \mathrm{PGL}(2, k)$, defined by associating a Möbius transformation to a 2-by-2 matrix by the usual formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \mapsto \frac{az + b}{cz + d}\right),$$

whose kernel is $\{\pm \mathrm{id}\}$.

Beware that in general this morphism is not surjective. It is so when the field k is algebraically closed, but not for instance in the case $k = \mathbb{C}(\!(t)\!)$. The trouble is that for a general Möbius transformation $\frac{az+b}{cz+d}$, the determinant $ad-bc$ need not be a square in k. Thus, if we denote by ε the generator of the Galois group of the quadratic field extension $\mathbb{C}((t^{1/2}))/\mathbb{C}((t))$ (i.e. $\varepsilon \cdot t^{1/2} = -t^{1/2}$), we have a surjective morphism from the subset of matrices $M \in \mathrm{SL}(2, \mathbb{C}((t^{1/2})))$ for which $\varepsilon \cdot M = \pm M$ onto $\mathrm{PGL}(2, \mathbb{C}((t)))$ whose kernel is again $\{\pm \mathrm{id}\}$. In other words, after a base change we can always lift a meromorphic family of Möbius transformations to a family of matrices in $\mathrm{SL}(2, \mathbb{M})$. The same phenomenon happens for triangular matrices over k and the affine group Aff_k.

Working with matrices is often more convenient for calculations, and when no confusion can arise, we simply identify $\gamma \in \mathrm{SL}(2,k)$ with the corresponding Möbius transformation, denoted by by $z \mapsto \gamma(z)$.

An element of $\mathrm{PGL}(2,k)$ induces an automorphism of $\mathbb{P}^{1,\mathrm{an}}_k$ preserving $\mathbb{P}^1(k)$ and $\mathbb{H}_k$. Recall that it preserves the types of points and acts by isometries on $(\mathbb{H}_k, d_{\mathbb{H}})$.

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,k)$ we denote by $\|A\| = \max(|a|, |b|, |c|, |d|)$, which in the ultrametric case is the matrix norm associated to the sup norm on k^2.

When $\gamma \in \mathrm{PGL}(2,k)$ as explained above there exists a quadratic extension K/k and $A \in \mathrm{SL}(2,K)$ inducing γ on $\mathbb{P}^1(k)$, and we set $\|\gamma\| := \|A\|$. This is well-defined since K carries a unique complete norm whose restriction to k is $|\cdot|$ and A is defined up to multiplication by $\pm\,\mathrm{id}$. Likewise, we define $|\mathrm{tr}(\gamma)| := |\mathrm{tr}(A)|$.

Proposition 2.1. *For all γ and γ' in $\mathrm{SL}(2,k)$, we have that:*

(i) $\|\gamma\| \geq 1$,
(ii) $\|\gamma\gamma'\| \leq \|\gamma\| \cdot \|\gamma'\|$,
(iii) $\|\gamma\| = \left\|\gamma^{-1}\right\|$,
(iv) if furthermore $\gamma \in \mathrm{SL}(2,k^\circ)$, then γ induces an isometry of $(\mathbb{P}^1(k), d_{\mathrm{sph}})$.

The proof is left to the reader (note that *(i)* follows from the ultrametric property of the absolute value).

2.2. Classification of elements in $\mathrm{PGL}(2,k)$.

Proposition 2.2. *Let $\gamma \in \mathrm{PGL}(2,k)$, $\gamma \neq \mathrm{id}$. Then exactly one of the following holds:*

- $|\mathrm{tr}(\gamma)| > 1$*: then γ is diagonalizable over k and has one attracting (resp. repelling) fixed point $x_{\mathrm{att}} \in \mathbb{P}^1(k)$ (resp. $x_{\mathrm{rep}} \in \mathbb{P}^1(k)$). Furthermore for every $x \neq x_{\mathrm{rep}}$ in $\mathbb{P}^{1,\mathrm{an}}_k$, the sequence $\gamma^n \cdot x$ converges to x_{att} when $n \to \infty$.*
- $|\mathrm{tr}(\gamma)| \leq 1$*: then γ admits a fixed point in $\mathbb{H}_k$ and more precisely:*
  - $\mathrm{tr}^2(\gamma) = 4$*: then γ is not diagonalizable, and is conjugate in $\mathrm{PGL}(2,k)$ to $z \mapsto z+1$; thus it fixes a segment $[x,y] \in \mathbb{P}^{1,\mathrm{an}}_k$ where x (resp. y) is a type 2 (resp. type 1) point belonging to the $\mathrm{PGL}(2,k)$ orbit of the Gauß point.*
  - $\mathrm{tr}^2(\gamma) \neq 4$*: then in some at most quadratic extension K/k the matrix γ is diagonalizable; in addition it is conjugate to an element in $\mathrm{PGL}(2,K^\circ)$ and fixes a type 2 point in $\mathbb{H}_k$.*

In accordance with the terminology of group actions on trees, when $|\mathrm{tr}(\gamma)| > 1$ we say that γ is *hyperbolic*, otherwise it is said *elliptic*. When required we can be more precise: if γ is elliptic and $\gamma \neq \mathrm{id}$, we say that γ is *parabolic* when $\mathrm{tr}^2(\gamma) = 4$ and *strictly elliptic* otherwise.

One cannot say much more on the action of γ on $\mathbb{P}^1_k$ in the elliptic case. It depends heavily on the residue field $\tilde{k}$. When the characteristic of $\tilde{k}$ is $p > 0$, then the closure of the subgroup generated by γ is isomorphic to $\mathbb{Z}_p$ and $\gamma^{p^n} \to \mathrm{id}$ when $n \to \infty$.

Following standard terminology, we say that γ has *good reduction* if it fixes the Gauß point (i.e. belongs to $\mathrm{PGL}(2, k^\circ)$), and *potential good reduction* over K/k if it is conjugate in $\mathrm{SL}(2, K)$ to a map having good reduction. With notation as in §1.6 this is equivalent to saying that γ fixes a type 2 point $x \in \mathbb{P}^{1,\mathrm{an}}_k$ such that $\sigma_{K/k}(x)$ lies in the $PGL(2, K)$-orbit of the Gauß point.

Proof. The diagonalizability of γ depends on the roots of its characteristic polynomial. If $\mathrm{char}(k) \neq 2$, this can be read off the discriminant $\mathrm{tr}^2(\gamma) - 4$.

Suppose that $|\mathrm{tr}(\gamma)| > 1$.

Then γ is diagonalizable in a quadratic extension K of k and its eigenvalues have respective norms norm larger and smaller than 1. This implies that there exists a global attracting point $\mathbb{P}^1(K)$ which in particular attracts all elements of k. Hence by completeness this fixed point belongs to $\mathbb{P}^1(k)$. Applying the same reasoning to the inverse, we conclude that γ is diagonalizable over k.

Suppose now that $\mathrm{char}(k) \neq 2$ and $\mathrm{tr}^2(\gamma) = 4$. Then (up to sign) 1 is an eigenvalue of multiplicity 2, hence since γ is not the identity it is conjugate in $\mathrm{PGL}(2, k)$ to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma(z)$ is conjugate to a translation. If $\mathrm{char}(k) = 2$ and $\mathrm{tr}(\gamma) = 0$, then the characteristic polynomial is $X^2 + 1 = (X - 1)^2$ and the same discussion applies.

Suppose finally that $|\mathrm{tr}(\gamma)| \leq 1$ and $\mathrm{tr}(\gamma) \neq 2$. Then γ is diagonalizable over an extension K of k of degree at most 2, and both eigenvalues belong to K°. If $K = k$, then the existence of the announced fixed point is clear. Otherwise consider the geodesic in $\mathbb{P}^{1,\mathrm{an}}_K$ joining the two fixed points. The Galois group K/k acts on this geodesic and permutes these two points. Thus it admits a fixed point which is of type 2 and lies in $\mathbb{H}_k$. □

Here is a noteworthy consequence of this classification.

Corollary 2.3. *For $\gamma \in \mathrm{PGL}(2, k)$, $\|\gamma^n\| \to \infty$ as $n \to \infty$ if and only if γ is hyperbolic.*

The next result is an analogue of the Cartan KAK decomposition in the non-Archimedean setting. It will play an important role in the following.

Proposition 2.4. *Any element $\gamma \in \mathrm{SL}(2, k)$ can be decomposed as a product $\gamma = m \cdot a \cdot n$ with $m, n \in \mathrm{SL}(2, k^\circ)$ and $a = \mathrm{diag}(\lambda, \lambda^{-1})$ with $\lambda \in k$, $|\lambda| \geq 1$. Furthermore $\|\gamma\| = \|a\| = |\lambda|$.*

Proof. Let x_{g} be the Gauß point. Pick an element $m \in \mathrm{SL}(2, k^\circ)$ such that $m^{-1}\gamma \cdot x_{\mathrm{g}}$ belongs to the segment $[x_{\mathrm{g}}, \infty]$. Likewise, choose $n \in \mathrm{SL}(2, k^\circ)$ such that $n^{-1}\gamma^{-1} \cdot x_{\mathrm{g}}$ belongs to the segment $[x_{\mathrm{g}}, 0]$. Then $\gamma' = m^{-1}\gamma n$ either fixes x_{g} or maps x_{g} into (x_{g}, ∞) and its inverse into $(x_{\mathrm{g}}, 0)$.

In the former case γ' belongs to $\mathrm{SL}(2, k^\circ)$ hence γ too and we can choose $a = \mathrm{id}$, $m = \gamma$, $n = \mathrm{id}$.

In the latter case, γ' is hyperbolic with two fixed points $|c^+| > 1$ and $|c^-| < 1$. We claim that in this case we can conjugate it by an element in $\mathrm{SL}(2, k^\circ)$ so that it becomes diagonal. Indeed we first conjugate by $\left(\begin{smallmatrix} 1 & -c^- \\ 0 & 1 \end{smallmatrix}\right)$ (i.e. by the translation $z \mapsto z - c^-$), which belongs to $\mathrm{SL}(2, k^\circ)$, to send c^- to 0. This maps c^+ to $\tilde{c}^+ = c^+ - c^-$ which has the same norm. Then we use the element $\left(\begin{smallmatrix} 1 & 0 \\ -1/\tilde{c}^+ & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2, k^\circ)$ to send $\tilde{c}^+$ to ∞, and we are done.

To prove the identity on $\|\gamma\|$ simply observe that $\|\gamma\| = \|man\| \leq \|a\|$ and $\|a\| = \left\|m^{-1}\gamma n^{-1}\right\| \leq \|\gamma\|$. □

2.3. **Norms of elements in** $\mathrm{SL}(2,k)$**.**

Lemma 2.5. *For $\gamma \in \mathrm{PGL}(2,k)$, one has the identity $d_{\mathbb{H}_k}(x_{\mathrm{g}}, \gamma \cdot x_{\mathrm{g}}) = \log\|\gamma\|$.*

Proof. Any element in $\mathrm{PGL}(2,k^\circ)$ has norm 1 and also fixes the Gauß point so the formula is clear in this case. In the general case we use the KAK decomposition and write $\gamma = man$. Then $d_{\mathbb{H}_k}(x_{\mathrm{g}}, \gamma \cdot x_{\mathrm{g}}) = d_{\mathbb{H}_k}(x_{\mathrm{g}}, a \cdot x_{\mathrm{g}}) = \|a\| = \|\gamma\|$. □

A similar argument shows:

Lemma 2.6. *For $\gamma \in \mathrm{PGL}(2,k)$ and $x, y \in \mathbb{P}^1(k)$, then*

$$\|\gamma\|^{-2} d_{\mathrm{sph}}(x,y) \le d_{\mathrm{sph}}(\gamma x, \gamma y) \le \|\gamma\|^2 d_{\mathrm{sph}}(x,y)$$

and this bound is optimal.

The following geometric consequence of Proposition 2.4 will be very useful.

Proposition 2.7. *For every $\gamma \in \mathrm{PGL}(2,k)$, there exist two closed balls $B_{\mathrm{att}}(\gamma)$ and $B_{\mathrm{rep}}(\gamma)$ in $\mathbb{P}^{1,\mathrm{an}}_k$ of spherical radius $\|\gamma\|^{-1}$ and such that $\gamma(\mathbb{P}^{1,\mathrm{an}}_k \setminus B_{\mathrm{rep}}(\gamma)) \subset B_{\mathrm{att}}(\gamma)$.*

Proof. This is a straightforward consequence of Proposition 2.4. Indeed, with notation as in Proposition 2.4, the result is obvious for a, with $B_{\mathrm{rep}}(a) = \overline{B}^{an}(0, |\lambda|^{-1})$ and $B_{\mathrm{att}}(a) = \overline{B}^{an}(\infty, |\lambda|^{-1}) = \mathbb{P}^{1,\mathrm{an}}_k \setminus B(0, |\lambda|)$.

In the general case, writing $\gamma = man$, it is enough to put $B_{\mathrm{rep}}(\gamma) = n^{-1}(\overline{B}^{an}(0, |\lambda|^{-1}))$ and $B_{\mathrm{att}}(\gamma) = m(\overline{B}^{an}(\infty, |\lambda|^{-1}))$. □

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,k)$ and $v \in \mathbb{P}^1(k)$, let

$$\sigma(\gamma, v) = \log \frac{\|\gamma V\|}{\|V\|} = \log\left(\max(|ax+by|, |cx+dy|)\right), \tag{2.1}$$

where $V \in k^2$ is any representative of v, and $(x,y) \in k^2$ is a representative of v with $\max(|x|, |y|) = 1$. The second equality shows that $\sigma(\gamma, \cdot)$ extends continuously to $\mathbb{P}^{1,\mathrm{an}}_k$, and we have the cocycle relation

$$\sigma(\gamma_1\gamma_2, v) = \sigma(\gamma_1, \gamma_2 \cdot v) + \sigma(\gamma_2, v), \tag{2.2}$$

for all $\gamma_1, \gamma_2 \in \mathrm{SL}(2,k)$ and for any $v \in \mathbb{P}^{1,\mathrm{an}}_k$.

Lemma 2.8. *For any $\gamma \in \mathrm{SL}(2,k)$ we have that*

$$-\log\|\gamma\| \le \sigma(\gamma, v) \le \log\|\gamma\|,$$

and if furthermore $v \notin B_{\mathrm{rep}}(\gamma)$, then $\sigma(\gamma, v) = \log\|\gamma\|$.

Proof. The first assertion is obvious from (2.1). For the second one, observe first that $\sigma(\gamma, v) = 0$ for all v when $\gamma \in \mathrm{SL}(2,k^\circ)$. Then, writing $\gamma = man$ with as in Proposition 2.4 we are reduced to the case of $a = \mathrm{diag}(\lambda, \lambda^{-1})$ and the result follows easily. □

Lemma 2.9. *For any* hyperbolic *element $\gamma \in \mathrm{SL}(2,k)$ the balls $B_{\mathrm{att}}(\gamma)$ and $B_{\mathrm{rep}}(\gamma)$ are disjoint and we have*

$$\|\gamma\| = \frac{|\mathrm{tr}(\gamma)|}{\min\{\delta, 1\}} \tag{2.3}$$

where $\delta = d_{\mathrm{sph}}(B_{\mathrm{att}}(\gamma), B_{\mathrm{rep}}(\gamma)) = \sup_{x \in B_{\mathrm{att}}(\gamma)} \inf_{y \in B_{\mathrm{rep}}(\gamma)} d_{\mathrm{sph}}(x,y)$.

Proof. If γ is hyperbolic the disjointness of the two balls $B_{\mathrm{att}}(\gamma)$ and $B_{\mathrm{rep}}(\gamma)$ follows from their construction and the ultrametric property implies that $\delta = d_{\mathrm{sph}}(x, y)$ for any pair $(x, y) \in B_{\mathrm{att}}(\gamma) \times B_{\mathrm{rep}}(\gamma)$.

We follow the reasoning of [DD1, Lem. 2.1]. Conjugate γ by some element in $\mathrm{SL}(2, k^\circ)$ to send the attracting fixed point to ∞. This does not affect neither $\|\gamma\|$ nor $\mathrm{tr}(\gamma)$, and after this conjugacy we have

$$\gamma = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

so that as a Möbius transformation $\gamma(z) = a^2 z + ab$ for some $|a| > 1$. The repelling fixed point is $ab/(1 - a^2)$, and its distance to ∞ is equal to $\delta = \min\{1, |(1 - a^2)/ab|\} = \min\{1, |a|/|b|\}$. We then have

$$\|\gamma\| = \max\{|a|, |b|\} = \frac{|a|}{\min\{1, |a|/|b|\}} = \frac{|\operatorname{tr}(\gamma)|}{\min\{1, \delta\}} ,$$

as was to be shown. □

Let us point out a kind of converse to the previous lemma.

Lemma 2.10. *Let $\gamma \in \mathrm{PGL}(2, k)$ be such that there exist two disjoint balls B_{a} and B_{r} of radius < 1 such that $\gamma(B^c_{\mathrm{r}}) \subset B_{\mathrm{a}}$. Then γ is hyperbolic with attracting and repelling fixed points respectively contained in B_{a} and B_{r}.*

Proof. Since the complement of a ball is a ball, the existence of an attracting (resp. repelling) type 1 fixed point in B_{a} (resp. B_{r}) follows from [BR, Thm. 10.69]. The result follows. □

Lemma 2.11. *For any pair of distinct points $z_1, z_2 \in \mathbb{P}^1(k)$, there exists a constant $C = C(z_1, z_2) > 0$ such that*

$$|\max\{\sigma(\gamma, z_1), \sigma(\gamma, z_2)\} - \log \|\gamma\|| \le C$$

for all $\gamma \in \mathrm{SL}(2, k)$.

Proof. Since $\sigma(\gamma, z) \le \log \|\gamma\|$, we only need to prove the lower bound $\max\{\sigma(\gamma, z_1), \sigma(\gamma, z_2)\} \ge \log \|\gamma\| - C$. Pick $g \in \mathrm{SL}(2, k)$ sending z_1 to 0 and z_2 to ∞. We have

$$\begin{aligned} \max\{\sigma(\gamma, z_1), \sigma(\gamma, z_2)\} &= \max\left\{\sigma(\gamma g^{-1}, 0) + \sigma(g, z_1), \sigma(\gamma g^{-1}, \infty) + \sigma(g, z_2)\right\} \\ &\ge \log \|\gamma g^{-1}\| - \log \|g\| \ge \log \|\gamma\| - 2 \log \|g\| \ . \end{aligned}$$

This concludes the proof. □

2.4. **Elementary and non-elementary subgroups.** A subgroup $\Gamma \le \mathrm{SL}(2, k)$ (resp. $\Gamma \le \mathrm{PGL}(2, k)$) is said *reducible* if its action on $\mathbb{P}^1(k)$ fixes a point, and *irreducible* otherwise. It is *strongly irreducible* if it does not admit a finite orbit in $\mathbb{P}^1(k)$.

We say that Γ has *good reduction* if it takes values in $\mathrm{SL}(2, k^\circ)$, or equivalently, fixes the Gauß point. It has *potential good reduction* if there exists a finite field extension K/k such that Γ is conjugate in $\mathrm{SL}(2, K)$ to a subgroup of $\mathrm{SL}(2, K^\circ)$. Finally, Γ is *proximal* if it contains at least one hyperbolic element.

Proposition 2.12. *A finitely generated subgroup Γ of $\mathrm{PGL}(2, k)$ is either proximal or has potential good reduction. If moreover k is discretely valued, then Γ is conjugate to a subgroup of $\mathrm{SL}(2, K^\circ)$ in some quadratic extension K/k.*

Observe that the groups of translations is not proximal but has not potential good reduction when the norm on k is non-trivial, so that the assumption that Γ is finitely generated is necessary in the previous statement.

This proposition is essentially a formulation in our language of the well-known fact that a group acting on a tree with only elliptic elements has a global fixed point. We sketch the proof for convenience.

The key is the following lemma (see [K, Lemma 10.4] or [O, Lemme 40]).

Lemma 2.13. *Any finitely generated semi-group of* $\mathrm{SL}(2,k)$ *which does not contain any hyperbolic element fixes a type 2 point lying in* $\mathbb{H}_k$.

Proof. Let S be a finitely generated semi-group which does not contain any hyperbolic element. We shall prove by induction on the number of generators the existence of a type 2 point in $\mathbb{H}_k$ fixed by S.

When S is generated by a single element thisis a direct consequence of Proposition 2.2. If S is generated by two elements g and h, pick $x, y \in \mathbb{H}_k$ two type 2 points fixed by g and h respectively. Let x' be the unique point satisfying $[x, g(y)] \cap [x, y] = [x, x']$. By Proposition 1.5 this is a type 2 point. Similarly define y' to be the unique type 2 point satisfying $[y, h(x))] \cap [y, x] = [y, y']$. If the segment $[x', y']$ is degenerate, the segment $[y', h(y')]$ is a fundamental domain for the action of gh which is therefore hyperbolic. Otherwise gh fixes pointwise $[x', y']$. This proves the result when S is generated by two elements.

Now suppose S is generated by $g_1, \ldots, g_l$ with $l \geq 3$, and that the result is known for semi-groups generated by $l-1$ elements. For $i = 1, 2, 3$, let S_i be generated by $\{g_1, \ldots, g_l\} \setminus \{g_i\}$. By the induction hypothesis S_i admits a type 2 fixed point $x_i \in \mathbb{H}_k$. Then the projection of x_3 on $[x_1, x_2]$ is a type 2 point in $\mathbb{H}_k$ fixed by S and we are done. □

Proof of Proposition 2.12. It follows from Lemma 2.13 that if Γ is not proximal then it fixes a type 2 point $x_\star$ in $\mathbb{H}_k$. Using the notation of §1.6 this means that $\sigma_{\bar{k}^a/k}(x_\star)$ lies in the $\mathrm{PGL}(2, \bar{k}^a)$-orbit of the Gauß point. Since algebraic points over k are dense in $\bar{k}^a$, the ball corresponding to $\sigma_{\bar{k}^a/k}(x_\star)$ contains a point of k^a so we get that $\sigma_{\bar{k}^a/k}(x_\star)$ lies in fact in the $\mathrm{PGL}(2, k^a)$-orbit of the Gauß point. In other words, we can find a finite field extension K/k and conjugate Γ by a matrix in $\mathrm{PGL}(2, K)$ so that it fixes the Gauß point.

Assume now that k is discretely valued so that $\mathbb{H}_k$ is a simplicial tree. The point $x_\star$ is either in the $\mathrm{PGL}(2, k)$-orbit of the Gauß point or it belongs to a unique segment $[x_0, x_1]$ of $\mathbb{H}_k$ whose extremities lie in the $\mathrm{PGL}(2, k)$-orbit of the Gauß point. Any element fixing $x_\star$ either fixes pointwise $[x_0, x_1]$ or acts upon it as an involution switching the two extremities. It follows that the middle point of $[x_0, x_1]$ is fixed by Γ. We conclude using Proposition 1.8. □

There is a simple classification of subgroups that are not strongly irreducible, analogous to the Archimedean case.

Proposition 2.14. *Let* $\Gamma \leq \mathrm{PGL}(2, k)$ *be a finitely generated subgroup that is not strongly irreducible. Then one of the following situations occurs:*

(1) Γ *has potential good reduction;*
(2) Γ *is conjugate to a subgroup of the affine group* $\{z \mapsto az + b,\, a \in k^\times,\, b \in k\}$*;*
(3) Γ *is conjugate to a subgroup of* $\{z \mapsto \lambda z^{\pm 1},\, \lambda \in k\}$.

Proof. By assumption there exists a finite Γ-orbit $x_1, \ldots, x_n$ on $\mathbb{P}^1(k)$. If $n = 1$ then Γ is conjugate to a subgroup of the affine group. If $n = 2$, we may assume that $x_1 = 0$ and $x_2 = \infty$ and it follows that any element Γ is conjugate to λ/z or λz for some $\lambda \in k^*$.

Assume now that Γ leaves invariant a set of $n \geq 3$ distinct points $E = \{x_1, \ldots, x_n\}$ in $\mathbb{P}^1(k)$. The first observation is that for every $\gamma \in \Gamma$, some iterate γ^m fixes E pointwise, therefore $\gamma^m = \mathrm{id}$. All elements of Γ are thus elliptic and the previous proposition shows that Γ has potential good reduction. □

Remark 2.15. If $\mathrm{char}(k) = 0$, then by the Selberg lemma (see e.g. [A]) the existence of a finite orbit of cardinality $n \geq 3$ implies that Γ is finite.

Proposition 2.16. *Let $\Gamma \leq \mathrm{PGL}(2, k)$ be a finitely generated subgroup. If Γ is proximal and strongly irreducible then it contains two hyperbolic elements with disjoint sets of fixed points.*

Proof. Since Γ is proximal, it contains a hyperbolic element g. Denote by $x_{\mathrm{att/rep}}$ its fixed points. We claim that there exists an element $h \in \Gamma$ such that $h(\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}) \cap \{x_{\mathrm{att}}, x_{\mathrm{rep}}\} = \emptyset$. Indeed since Γ is strongly irreducible, $\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$ is not a Γ-orbit so that there exists $h \in \Gamma$ satisfying $h(x_{\mathrm{att}}) \notin \{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$. There are 3 possibilities:

- $h(x_{\mathrm{rep}}) \notin \{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$;
- $h(x_{\mathrm{rep}}) = x_{\mathrm{att}}$: then either hgh or h^2 sends $\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$ to a disjoint pair;
- $h(x_{\mathrm{rep}}) = x_{\mathrm{rep}}$: then there exists $j \in \Gamma$ such that $j(x_{\mathrm{rep}}) \notin \{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$ and $hg^n j^{-1}$ is convenient for large n (use Proposition 2.7).

In any case there exists k such that $k(\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}) \cap \{x_{\mathrm{att}}, x_{\mathrm{rep}}\} = \emptyset$, thus $k^{-1}\gamma k$ is a hyperbolic element whose fixed points are disjoint from $\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$. □

Proposition 2.16 motivates the following definition.

Definition 2.17. A finitely generated subgroup Γ of $\mathrm{PGL}(2, k)$ is *non-elementary* if it is proximal and strongly irreducible.

A finitely generated subgroup Γ of $\mathrm{SL}(2, k)$ if its image in $\mathrm{PGL}(2, k)$ is non-elementary.

Propositions 2.12 and 2.14 imply the following characterization of non-elementary subgroups. The details are left to the reader.

Proposition 2.18. *Let $\Gamma \leq \mathrm{PGL}(2, k)$ be a finitely generated subgroup. The following assertions are equivalent:*

(1) Γ is non-elementary;
(2) Γ does not admit a finite orbit on $\mathbb{P}^{1,\mathrm{an}}_k$;
(3) for every $z \in \mathbb{P}^{1,\mathrm{an}}_k$, $\#\Gamma \cdot z \geq 3$.

Let us note for further reference the following variation on Proposition 2.16.

Lemma 2.19. *Let Γ be a non-elementary finitely generated subgroup of $\mathrm{PGL}(2, k)$. Then for every set S of generators of Γ, the semi-group generated by S contains two hyperbolic elements with distinct attracting fixed points.*

Proof. First note that since Γ is finitely generated, there is a finite subset $S' \subset S$ such that $\langle S' \rangle$ contains a finite set of generators of Γ, hence $\langle S' \rangle = \Gamma$, so, replacing S by S' we may assume that S is finite. Denote by G_0 the semi group generated by S. Since $\langle S \rangle = \Gamma$ the elements of S do not admit a common fixed point, hence by Lemma 2.13 there exists a hyperbolic element $g \in G_0$. Thus, letting $\rho = \|g\| > 1$ we infer that for $n \geq 0$, g^n maps $B(x_{\mathrm{rep}}(g), \rho^n)^c$ into $B(x_{\mathrm{att}}(g), \rho^n)$ (all the balls here are in $\mathbb{P}^{1,an}_k$). Since S has no fixed point there exists $h \in S$ such that $h(x_{\mathrm{att}}(g)) \neq x_{\mathrm{att}}(g)$. Then for every $n \geq 1$, hg^n belongs to G_0 and maps $B(x_{\mathrm{rep}}(g), \rho^n)^c$ into $B(h(x_{\mathrm{att}}(g)), C\rho^n)$ for some $C = C(h)$. If $h(x_{\mathrm{att}}(g)) \neq x_{\mathrm{rep}}(g)$, from Lemma 2.10 we infer that for large n hg^n is hyperbolic, and its attracting fixed point is close to $h(x_{\mathrm{att}}(g))$, hence distinct from $x_{\mathrm{att}}(g)$.

If $h(x_{\mathrm{att}}(g)) = x_{\mathrm{rep}}(g)$, then $h^{-1}(x_{\mathrm{rep}}(g)) = x_{\mathrm{att}}(g)$, and we consider hg^nh instead of hg^n. Indeed for some C we have that

$$B(x_{\mathrm{att}}(g), C^{-1}\rho^n)^c \xrightarrow{h} B(x_{\mathrm{rep}}(g), \rho^n)^c \xrightarrow{g^n} B(x_{\mathrm{att}}(g), \rho^n) \xrightarrow{h} B(x_{\mathrm{rep}}(g), C\rho^n),$$

so again we see that hg^nh is hyperbolic for large n, and its attracting fixed point is distinct from $x_{\mathrm{att}}(g)$. □

2.5. **The limit set.**

Theorem 2.20. *Let Γ be a finitely generated and non-elementary subgroup of* $\mathrm{PGL}(2,k)$*. Then the following sets coincide:*

– *the closure in $\mathbb{P}^{1,\mathrm{an}}_k$ of the set of fixed points of all hyperbolic elements of Γ;*
– *the smallest non-empty Γ-invariant closed subset of $\mathbb{P}^{1,\mathrm{an}}_k$;*
– *for any given $x \in \mathbb{P}^{1,\mathrm{an}}_k$, the set of points y such that there exists a sequence $(g_n) \in \Gamma^{\mathbb{N}}$ such that $\|g_n\| \to \infty$ and $g_n \cdot x \to y$.*

This set is compact, metrizable, and included in $\mathrm{Conv}(\mathbb{P}^1(k)) = \mathbb{H}_k \cup \mathbb{P}^1(k)$*. It is by definition the limit set* $\mathrm{Lim}(\Gamma)$ *of* Γ*.*

Proof. Denote by Λ_0 the set of fixed points of all hyperbolic elements, and by Λ the smallest closed Γ-invariant subset of $\mathbb{P}^{1,\mathrm{an}}_k$. If x is fixed by some hyperbolic element g, then for every $h \in \Gamma$, $h(x)$ is fixed by hgh^{-1}. We infer that $\overline{\Lambda}_0$ is a closed Γ-invariant set thus $\Lambda \subset \overline{\Lambda}_0$.

Conversely, pick any $x \in \Lambda$. By Proposition 2.16 there exists a hyperbolic element $g \in \Gamma$ whose fixed point set $\{x_{\mathrm{att}}, x_{\mathrm{rep}}\}$ is disjoint from x. Since $g^{\pm n}(x) \to x_{\mathrm{att/rep}}$ it follows that $\{x_{\mathrm{att}}, x_{\mathrm{rep}}\} \subset \Lambda$ so Λ admits at least three points. Therefore, for an arbitrary hyperbolic element $g' \in \Gamma$ with fixed point set $\{x'_{\mathrm{att}}, x'_{\mathrm{rep}}\}$, there exists $y \in \Lambda \setminus \{x'_{\mathrm{att}}, x'_{\mathrm{rep}}\}$. Then $(g')^{\pm n}(y) \to x'_{\mathrm{att/rep}}$ as $n \to \infty$, from which we infer that $\Lambda_0 \subset \Lambda$. We conclude that $\Lambda = \overline{\Lambda}_0$.

Fix now any point $x \in \mathbb{P}^{1,\mathrm{an}}_k$ and denote by Λ_1 the set of all y for which there exists a sequence (g_n) with $\|g_n\| \to \infty$ and $g_n \cdot x \to y$. Observe that Λ_1 is Γ-invariant and non-empty since Γ contains a hyperbolic element. We claim that it is also closed. Indeed by a theorem of Poineau [P, Théorème 5.3], for any y' in the closure of Λ_1 there exists a sequence $y_n \in \Lambda_1$ such that $y_n \to y'$. For each n, pick a sequence with $\|g_{m,n}\| \geq m + n$ such that $g_{m,n} \cdot x \to y_n$. The set $\{g_{m,n} \cdot x\}$ contains $\{y_n\}$ in its closure hence y' too. Again by Poineau's theorem, there exists a subsequence $g_{m_j,n_j} \cdot x \to y$. This shows that $\Lambda_1 \supset \Lambda$.

Now suppose $\|g_n\| \to \infty$ and $g_n \cdot x \to y \in \Lambda_1$. We want to show that y belongs to Λ. If x belongs to Λ then the closure of $\Gamma \cdot x$ is contained in Λ and the result follows. So suppose that x does not belong to Λ.

Recall from Proposition 2.7 that we can associate to every $g \in \mathrm{PGL}(2,k)$ two closed balls $B_{\mathrm{att}}(g)$ and $B_{\mathrm{rep}}(g)$ in $\mathbb{P}^{1,\mathrm{an}}_k$ such that $g(\mathbb{P}^{1,\mathrm{an}}_k \setminus B_{\mathrm{rep}}(g)) \subset B_{\mathrm{att}}(g)$. We claim that for large enough n, $B_{\mathrm{att}}(g_n)$ intersects Λ. Indeed pick any 2 distinct points in Λ. Then since $\|g_n\| \to \infty$, for large n one of these points does not belong to $B_{\mathrm{rep}}(g_n)$, hence its image under g_n belongs to $B_{\mathrm{att}}(g_n)$, and also to Λ by invariance, so we get that $B_{\mathrm{att}}(g_n) \cap \Lambda \neq \emptyset$. Similarly, $B_{\mathrm{rep}}(g_n) \cap \Lambda \neq \emptyset$.

In particular we see that for large n, $x \notin B_{\mathrm{rep}}(g_n)$. Indeed otherwise since the diameter of $B_{\mathrm{rep}}(g_n)$ tends to zero we would infer that x belongs to Λ, which is not the case. Thus we conclude that $g_n \cdot x \in B_{\rm att}(g_n)$ for large n, so every neighborhood of y intersects Λ and it follows that $\Lambda_1 \subset \Lambda$.

The limit set is a closed subset of $\mathbb{P}^{1,\rm an}_k$ which is compact, hence it is also compact. Since Γ is countable hence Λ_0 is countable too. It follows that Λ is included in the closure of the convex hull of a countable set. Such a set is always metrizable (see e.g. the proof of [FJ, Lemma 7.15], or [BR, Lemma 5.7]). Finally Λ_0 is a subset of $\mathbb{P}^1(k)$, hence $\mathrm{Lim}(\Gamma)$ is included in its closure which is contained in $\mathbb{H}_k \cup \mathbb{P}^1(k)$. □

3. Random products of matrices in $\mathrm{SL}(2,k)$

In this section we work on an arbitrary complete non-trivially valued field $(k, |\cdot|)$ –shortly to be assumed non-Archimedean. We keep the notation of the previous section. We consider a measure μ with at most countable support in $\mathrm{SL}(2,k)$, and make the following assumptions:

(B1) $\Gamma = \langle \mathrm{Supp}(\mu)\rangle$ is non-elementary.

(B2) μ has finite first moment $\displaystyle\int \log\|\gamma\|\, d\mu(\gamma) < \infty$.

The measure μ acts by convolution on the set of probability measures on $\mathbb{P}^{1,\rm an}_k$ by $\nu \mapsto \mu * \nu$. The measures invariant under this action are called *stationary*. We use the probabilistic notation $(\Omega, \mathsf{P}) = (\mathrm{SL}(2,k)^{\mathbb{N}^*}, \mu^{\mathbb{N}^*})$, and for $\omega = (\gamma_n)_{n\geq 1} \in \Omega$ we respectively let $r_n(\omega) = \gamma_1 \cdots \gamma_n$ and $\ell_n(\omega) = \gamma_n \cdots \gamma_1$ the $n^{\rm th}$ step of the right and left random walk on Γ with transition probabilities given by μ. We denote by μ^n the $n^{\rm th}$ convolution power of μ, that is the image of $\mu^{\otimes n}$ under the map $(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$. It is also the distribution of $r_n(\omega)$ and $\ell_n(\omega)$.

The *Lyapunov exponent* of μ is defined to be the following non-negative real number:

$$\chi(\mu) := \lim_{n\to\infty} \frac{1}{n} \int \log\|\gamma\|\, d\mu^n(\gamma) = \lim_{n\to\infty} \frac{1}{n} \int \log\|\gamma_1\cdots\gamma_n\|\, d\mu(\gamma_1)\cdots d\mu(\gamma_n)\ . \tag{3.1}$$

It follows from Kingman's sub-multiplicative ergodic theorem that $\chi(\mu) = \lim_{n\to\infty} \frac{1}{n}\log\|\ell_n(\omega)\|$ for P-a.e. ω.

The main result of this section is the following theorem. Recall the notation $\sigma(\gamma, v) = \log\|\gamma V\|$, where $V = (V_1, V_2)$ is a lift of norm 1 of v and $\|V\| = \max(|V_1|, |V_2|)$.

Theorem 3.1. *Let μ be a probability measure with countable support in $\mathrm{SL}(2,k)$, satisfying (B1). Then there a unique stationary probability measure ν on $\mathbb{P}^{1,\rm an}_k$, that is stationary under the action of μ. This measure has no atoms, it is supported on the limit set of Γ, and when k is non-Archimedean it gives full mass to $\mathbb{P}^1(k)$ so that $\nu(\mathbb{H}_k) = 0$.*

If furthermore (B2) holds, then the Lyapunov exponent $\chi(\mu)$ of the associated random product of matrices is positive and satisfies the following formula

$$\chi(\mu) = \int \sigma(\gamma, v)\, d\mu(\gamma)\, d\nu(v). \tag{3.2}$$

Finally for P-a.e. ω and for ν-a.e. v, we have:

$$\chi(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \sigma(\ell_n(\omega), v). \tag{3.3}$$

Remark 3.2. If Γ is generated by $\mathrm{Supp}(\mu)$ as a semi-group then $\mathrm{Supp}(\nu) = \mathrm{Lim}(\Gamma)$. Indeed $\mathrm{Supp}(\nu)$ is contained in $\mathrm{Lim}(\Gamma)$, closed, and $\mathrm{Supp}(\mu)$-invariant, hence Γ-invariant. In the general case, however, the inclusion $\mathrm{Supp}(\nu) \subset \mathrm{Lim}(\Gamma)$ can be strict.

There are many statements of this kind in the literature, and when k is archimedean this statement is precisely Furstenberg's theorem on random products of matrices [Fg] (see [BL, Chap. II] for a simple exposition). In the non-Archimedean setting, it was observed by several authors (see in particular [G]) that Furstenberg's theory can be adapted without much harm to local fields. The novelty here is that k is arbitrary, in particular may not be locally compact. This leads us to resort to Berkovich theory, and also prevents us from taking cluster limits of sequences of elements in $\mathrm{SL}(2,k)$, which is commonplace in the classical presentation of the topic.

On the other hand it was recently proved by Maher and Tiozzo [MT] that non-elementary random walks on non-necessarily proper Gromov hyperbolic spaces have positive drift, from which the first conclusion of the theorem follows.

Nevertheless we provide a complete proof of the theorem for at least two reasons: first our algebraic setting allows us to provide a relatively short and self-contained proof, and also the conclusion on the stationary measure does not straightforwardly follow from [MT] since Maher and Tiozzo work in a horofunction compactification that is not directly related to the Berkovich projective line.

As in the classical case, the uniqueness of the stationary measure and the positivity of the Lyapunov exponent follow from a contraction statement, which asserts that if ν is any stationary measure, then for P-a.e. ω, $\|r_n(\omega)\| \to \infty$ and $(r_n(\omega))_*\nu$ converges to a Dirac mass at a point $e(\omega)$ which does not depend on ν (see below Lemma 3.5). To prove this result, we adapt the arguments of Guivarc'h and Raugi [GR].

In the remaining of this section we assume that the norm on k is *non-Archimedean*.

3.1. **Uniqueness of the stationary measure.** As a first step towards Theorem 3.1 in this section we prove the following result.

Theorem 3.3. *Let μ be a probability measure with countable support in $\mathrm{SL}(2,k)$, satisfying (B1), and suppose ν is a μ-stationary measure having no atom and such that $\nu(\mathbb{P}^1(k)) = 1$.*

1. *There exists a measurable map $e : \Omega \to \mathbb{P}^1(k)$ such for a.e. ω, $r_n(\omega)_*\nu \to \delta_{e(\omega)}$. Moreover for a.e. ω we have $\|r_n(\omega)\| \to \infty$, and $d_{sph}(e(\omega), B_{\mathrm{att}}(r_n(\omega))) \leq 2\|r_n(\omega)\|^{-1}$.*
2. *The identity $\nu = \int \delta_{e(\omega)}\, d\mathsf{P}(\omega)$ holds, and ν is the unique μ-stationary measure which has no atom and gives full mass to $\mathbb{P}^1(k)$.*

We start with the following classical lemma.

Lemma 3.4. *Under the assumptions of Theorem 3.3, for P-a.e. ω there exists a probability measure ν_ω such that for every γ belonging to the semi-group generated by $\mathrm{Supp}(\mu)$, the sequence $r_n(\omega)_*g_*\nu$ converges weakly to ν_ω. This measure a.s. puts full mass on $\mathbb{P}^1(k)$ and $\nu = \int \nu_\omega\, d\mathsf{P}(\omega)$.*

Proof. The support of ν is a compact metrizable space by Lemma 1.1, so that we may apply [BL, Lemma 2.1 p.19]. We obtain for P-a.e. ω the existence of a probability measure ν_ω such that for μ^∞-a.e. γ, $r_n(\omega)_*\gamma_*\nu$ converges weakly to ν_ω, where $\mu^\infty = \sum_{n=0}^\infty 2^{-n-1}\mu^n$. Since Γ is countable the measure μ is purely atomic and its support is precisely the semi-group generated by $\mathrm{Supp}(\mu)$. Thus for every γ in this semi-group, we obtain $r_n(\omega)_*\gamma_*\nu \rightharpoonup \nu_\omega$.

The stationarity property implies that for every n, $\nu = \int r_n(\omega)_*\nu\, d\mathsf{P}(\omega)$, and the dominated convergence theorem implies $\nu = \int \nu_\omega\, d\mathsf{P}(\omega)$. In particular $\nu_\omega(\mathbb{P}^1(k)) = 1$ almost surely. □

Next we prove the divergence of the norms of generic random products.

Lemma 3.5. *For* P-*a.e.* ω *we have that* $\|r_n(\omega)\| \to \infty$.

Proof. Let $\omega \in G^{\mathbb{N}}$ be a sequence satisfying the conclusion of Lemma 3.4, and let us show that $\|r_n(\omega)\| \to \infty$. We proceed by contradiction, so assume there exists a subsequence (n_j) such that $\|r_{n_j}(\omega))\| \leq C$ for some $C \geq 1$. Since Γ is non elementary, by Lemma 2.19 the semi-group generated by $\mathrm{Supp}(\mu)$ contains two hyperbolic elements γ_1, γ_2 with distinct attracting fixed points, that we fix from now on. Denoting by $x_{\mathrm{att}}(\gamma_i)$ the respective attractive fixed points, we fix r small enough so that $d(x_{\mathrm{att}}(\gamma_1), x_{\mathrm{att}}(\gamma_2)) > 2r(C^2 + 1)$.

The measure ν_ω charges $\mathbb{P}^1(k)$, so that one can find a point $z \in \mathbb{P}^1(k)$ such that $m := \nu_\omega(B(z, r)) > 0$. Since the measure ν has no atom, for $i = 1, 2$ $\nu(B(x_{\mathrm{rep}}(\gamma_i), \rho))$ tends to 0 when $\rho \to 0$, so we can fix N (large) such that

$$((\gamma_1^N)_*\nu)(B(x_{\mathrm{att}}(\gamma_1), r)) \geq 1 - \frac{m}{4} \text{ and } ((\gamma_2^N)_*\nu)(B(x_{\mathrm{att}}(\gamma_2), r)) \geq 1 - \frac{m}{4}.$$

Since on the other hand $B(z, r)$ is open, from the choice of ω, there exists j such that

$$\left(r_{n_j}(\omega)_*(\gamma_1^N)_*\nu\right)(B(z, r)) \geq \frac{m}{2} \text{ and } \left(r_{n_j}(\omega)_*(\gamma_2^N)_*\right)(B(z, r)) \geq \frac{m}{2}.$$

Applying Lemma 2.6 it follows that for $i = 1, 2$,

$$(\gamma_i^N)_*\nu(B(r_{n_j}(\omega)^{-1}z, rC^2)) \geq m/2,$$

hence

$$B(r_{n_j}(\omega)^{-1}z, rC^2) \cap B(x_{\mathrm{att}}(\gamma_i), r) \neq \emptyset.$$

which implies that $d(x_{\mathrm{att}}(\gamma_1), x_{\mathrm{att}}(\gamma_2)) \leq 2rC^2 + 2r$, a contradiction. □

The proof of Theorem 3.3 will be complete if we prove that

Lemma 3.6. *For* P-*a.e.* ω, *the measure* ν_ω *is a Dirac mass at a point* $e(\omega) \in \mathbb{P}^1(k)$ *which does not depend on* ν, *and satisfies* $d_{\mathrm{sph}}(e(\omega), B_{\mathrm{att}}(r_n(\omega))) \leq 2\|r_n(\omega)\|^{-1}$.

The proof of this lemma relies on the following elementary fact which asserts that the measure of small balls is uniformly small.

Lemma 3.7. *Let* ν *be an atomless Borel probability measure on a complete metric space* (X, d). *There exists a function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\eta(r) \to 0$ *as* $r \to 0$ *such that for every* $x \in X$ *we have that* $\nu(B(x, r)) \leq \eta(r)$.

Proof. Assume by way of contradiction that there exists $\eta > 0$, a sequence $(x_n) \in X^{\mathbb{N}}$ and a sequence of radii $r_n \to 0$ such that $\nu(B(x_n, r_n)) \geq \eta$. Extracting a subsequence we may assume that $\sum r_n$ converges. We will show that (x_n) admits a Cauchy subsequence (x_{n_j}), thus converging to some x. Since for any $r > 0$, we have that $B(x_{n_j}, r_{n_j}) \subset B(x, r)$ for large j, it follows that $\nu(B(x, r)) \geq \eta$ for every r, contradicting the assumption on ν.

To show that (x_n) admits a Cauchy subsequence, we define the set

$$A = \{n \in \mathbb{N},\ \forall p > n,\ B(x_p, r_p) \cap B(x_n, r_n) = \emptyset\}.$$

Write A as an increasing sequence $A = \{a_1, a_2, \ldots\}$. By construction, for every $k > l$, $B(x_{a_k}, r_{a_k}) \cap B(x_{a_l}, r_{a_l}) = \emptyset$. Since each of these ball has mass at least η, there are at most $1/\eta$ of them, hence A is finite.

Now if $n_1 > \max A$, by assumption there exists $n_2 > n_1$ such that $B(x_{n_2}, r_{n_2}) \cap B(x_{n_1}, r_{n_1}) \neq \emptyset$. Repeating this process we construct a subsequence $(n_j)_{j \geq 1}$. Since the series $\sum r_n$ converges, the sequence (x_{n_j}) is Cauchy, and the lemma follows. □

Proof of Lemma 3.6. Recall the definition of $B_{\rm att}(\gamma)$ and $B_{\rm rep}(\gamma)$ from Proposition 2.7: these are two balls in $\mathbb{P}^{1,\rm an}_k$ of spherical diameter $\|\gamma\|^{-1}$ such that $\gamma(\mathbb{P}^{1,\rm an}_k \setminus B_{\rm rep}(\gamma)) \subset B_{\rm att}(\gamma)$. Observe that they necessarily intersect $\mathbb{P}^1(k)$.

From Lemmas 3.4 and 3.5, for almost every ω we have that $(r_n(\omega))_*\nu \rightharpoonup \nu_\omega$ and $\|r_n(\omega)\| \to \infty$. With η as in Lemma 3.7, we have that $\nu(B_{\rm rep}(r_n(\omega))) \leq \eta(\|r_n(\omega)\|^{-1})$, hence

$$\nu(B_{\rm att}(r_n(\omega))) \geq 1 - \eta(\|r_n(\omega)\|^{-1}) \geq \frac{3}{4} \tag{3.4}$$

for n large enough. For each n choose any $x_n \in B_{\rm att}(r_n(\omega)) \cap \mathbb{P}^1(k)$. Then (3.4) implies $d_{\rm sph}(x_n, x_m) \leq 2\|r_n(\omega)\|^{-1}$ for all $m \geq n$ hence x_n forms a Cauchy sequence. Observe that the limit of this sequence belongs to $\mathbb{P}^1(k)$, is at distance at most $2\|r_n(\omega)\|^{-1}$ from $B_{\rm att}(r_n(\omega))$, and does depends neither on the choice of the sequence (x_n), nor on the stationary measure. We may thus denote it by $e(\omega)$.

Finally, for each $r > 0$, we have that $B_{\rm att}(r_n(\omega)) \subset B(x_\infty, r)$ for large n since $\|r_n(\omega)\|^{-1} \to 0$. We conclude that $\nu(B(e(\omega), r)) \geq \liminf_n B_{\rm att}(r_n(\omega)) = 1$, so that $\nu_\omega = \delta_{e(\omega)}$. □

3.2. **Proof of Theorem 3.1.** Recall that μ is a measure with countable support on $\mathrm{SL}(2,k)$ satisfying condition (B1). We first show the existence and uniqueness of a μ-stationary measure, and then prove that this measure has no atom and puts full mass on $\mathbb{P}^1(k)$. Then assuming (B2) we establish the analog of Furstenberg's formula that expresses the Lyapunov exponent $\chi(\mu)$ in terms of the stationary measure. The positivity of $\chi(\mu)$ and the identities (3.2) and (3.3) follow from this formula.

Recall first from Theorem 2.20 that the limit set of Γ is a compact metrizable space. For any $x \in \mathrm{Lim}(\Gamma)$, one can thus extract a converging subsequence from $\frac{1}{n}\sum_{i=0}^{n-1} \mu^i * \delta_x$, and the limit measure ν is μ-stationary. The uniqueness of the stationary measure then follows from Theorem 3.3 together with the next lemma whose proof will be given at the end of this section.

Lemma 3.8. *Let μ be a probability measure with countable support in $\mathrm{SL}(2,k)$, satisfying (B1). If ν is any μ-stationary probability measure on $\mathbb{P}^{1,\rm an}_k$, then ν has no atom and gives full mass to $\mathbb{P}^1(k)$.*

By Lemma 3.4 the support of ν is contained in the limit set. It remains to prove (3.2), (3.3) and the positivity of the Lyapunov exponent. From now on we assume that the moment condition (B2) holds, and proceed in several steps.

Step 1. We first claim that for $(P \times \nu)$ a.e. (ω, v), one has $\sigma(\ell_n(\omega), v) \to \infty$.

To see this we introduce the reversed random walk on Γ, that is the random walk associated to $\check{\mu}$, the image of μ under the involution $\gamma \mapsto \gamma^{-1}$. It satisfies the assumptions (B1-2) so from what precedes we know that it admits a unique stationary measure $\check{\nu}$ on $\mathbb{P}^{1,\rm an}_k$ having no atom and putting full mass on $\mathbb{P}^1(k)$. Define an involution $G^{\mathbb{N}} \to G^{\mathbb{N}}$ by $\omega = (\gamma_n)_{n\geq 1} \mapsto \check{\omega} = (\gamma_n^{-1})_{n\geq 1}$, so that $\ell_n(\omega) = r_n(\check{\omega})^{-1}$, and $B_{\rm rep}(\ell_n(\omega)) = B_{\rm att}(r_n(\check{\omega}))$.

By Theorem 3.3, we have $\|l_n(\omega)\| \to \infty$ for a.e. ω, and there exists a measurable map $\check{e} : \Omega \to \mathbb{P}^1(k)$ such that $\check{\nu} = \int \delta_{\check{e}(\omega)} d\check{\mu}(\omega)$, and $d_{\rm sph}(\check{e}(\omega), B_{\rm att}(l_n(\omega))) \leq 2\|l_n(\omega)\|^{-1}$.

If $v \in \mathbb{P}^1(k)$ is different from $\check{e}(\omega)$, then for n large enough, $v \notin B_{\rm rep}(\ell_n(\omega))$ hence $\sigma(\ell_n(\omega), v) = \|\ell_n(\omega)\|$ by Lemma 2.8. Since ν gives no mass to points, for P-a.e. ω and for ν-a.e. z, we have $\sigma(\ell_n(\omega), v) \to \infty$, thus by Fubini's theorem, $\sigma(\ell_n(\omega), v) \to \infty$ holds $(P \times \nu)$ a.s., as claimed.

Denote by $\theta : \Omega \to \Omega$ the shift map $\omega = (\gamma_n)_{n\geq 1} \mapsto \theta(\omega) = (\gamma_{n+1})_{n\geq 1}$, and observe that the skew product map $\Theta(\omega, v) = (\theta(\omega), l_1(\omega) \cdot v)$ preserves the measure $m = \mathsf{P} \times \nu$.

Step 2. We next show that $\mathsf{P} \times \nu$ is ergodic under Θ.

From (Kifer's version of) the Kakutani random ergodic theorem (see [Fn, Thm. 3.1]), it is sufficient to prove that for any Borel set E such that $\nu(E\Delta\gamma E) = 0$ for μ-a.e. γ, then we have $\nu(E) = 0$ or 1.

Pick such a set E. First note that by stationarity and by the countability of $\mathrm{Supp}(\mu)$, for μ-a.e. $\gamma \in \Gamma$, we have $\gamma_*\nu \ll \nu$. In particular, we get $\gamma_*\nu(E\Delta\gamma E) = 0$ so that

$$\mu * (\nu|_E) = \int \gamma_*\nu \, \mathbf{1}_{\gamma(E)} \, d\mu(\gamma) = \int \gamma_*\nu \, \mathbf{1}_E \, d\mu(\gamma) = \nu|_E \ .$$

By the uniqueness of the μ-stationary measure, we conclude that $\nu(E) = 0$ or 1, as required.

Step 3. Proof of (3.2) and (3.3).

The Birkhoff ergodic theorem applied to Θ and the function $\sigma(l_1(\omega), v)$, together with the cocycle relation (2.2) now yield for (P, ν)-a.e. (ω, v)

$$\lim_{n\to\infty} \frac{1}{n}\sigma(l_n(\omega), v) = \int \sigma(l_1(\omega), v) \, d\mathsf{P}(\omega) \, d\nu(v) = \int \sigma(\gamma, v) d\mu(\gamma) d\nu(v). \tag{3.5}$$

By Fubini's theorem, for P-a.e. ω we have that for ν-a.e. v, the limit in (3.5) exists. Since ν gives no mass to points, this holds for at least two distinct points so by Lemma 2.11 we get that for such ω

$$\lim_{n\to\infty} \frac{1}{n} \log \|l_n(\omega)\| = \lim_{n\to\infty} \frac{1}{n}\sigma(l_n(\omega), v) \ . \tag{3.6}$$

Thus, combining (3.5) and (3.6) we get (3.2) and (3.3).

Step 4. Positivity of the Lyapunov exponent.

To that end we rely on the following general lemma, a proof of which can be found in [BL, Lemma 2.3, p. 22]. For simplicity we write $F^+ = \max\{0, F\}$.

Lemma 3.9. *Let Θ be a measurable map on a probability space (X, m). If $F : X \to \mathbb{R}$ is a measurable function such that $\int F^+ dm < \infty$ and $\lim_{n\to\infty} \sum_{i=0}^{n-1} F \circ \Theta^i = +\infty$ a.s., then $F \in L^1(X, m)$ and $\int F \, dm > 0$.*

We apply the previous lemma to the function $F : (\omega, v) \mapsto \sigma(\ell_1(\omega), v)$ on $X = \Omega \times \mathbb{P}^{1,\mathrm{an}}_k$, and to the skew product map $\Theta(\omega, v) = (\theta(\omega), l_1(\omega) \cdot v)$. By Step 1 for $\mathsf{P} \times \nu$-a.e (ω, v) we have that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} F \circ \Theta^i(\omega, v) = \lim_{n\to\infty} \sigma(l_n(\omega), v) = +\infty.$$

Therefore Lemma 3.9 yields

$$\int \sigma(\gamma, v) \, d\mu(\gamma) d\nu(v) > 0 \ , \tag{3.7}$$

and the proof of Theorem 3.1 is complete.

Proof of Lemma 3.8. Assume by contradiction that ν charges a point in $\mathbb{P}^{1,\mathrm{an}}_k$. Then for every $\alpha > 0$, the set $\{x, \nu(\{x\}) > \alpha\}$ is finite, thus there exists an atom of maximal mass α_0. It follows that the set $\{x, \nu(\{x\}) = \alpha_0\}$ is finite and invariant under every element of $\mathrm{Supp}(\mu)$, hence Γ-invariant, which is impossible since Γ is non-elementary.

For second assertion, we start by proving that that $\nu(\mathbb{H}_k) = 0$. Observe first that any closed ball for the hyperbolic metric is also closed in $\mathbb{P}^{1,\mathrm{an}}_k$ hence is a Borel set. Denote by $\mathbb{B}(x, R) = \{d_{\mathbb{H}}(\cdot, x) < R\}$ (resp. $\overline{\mathbb{B}}(x, R) = \{d_{\mathbb{H}}(\cdot, x) \leq R\}$) the open (resp. closed) ball of radius R relative to the hyperbolic metric. It is thus enough to prove that for every $x \in \mathbb{H}_k$ and every $R > 0$, $\nu(\overline{\mathbb{B}}(x, R)) = 0$.

By Lemma 2.19, the semi-group generated by $\mathrm{Supp}(\mu)$ contains a hyperbolic element h. Replacing μ by some μ^k if necessary (this does not affect the stationarity of ν) we may assume $h \in \mathrm{Supp}(\mu)$. Put $s = \sup_{x\in\mathbb{H}_k} \nu(\mathbb{B}(x, R))$ and suppose by way contradiction that $s > 0$.

Since the series $\sum_{n\geq 0} \nu\left(\overline{\mathbb{B}}(x_{\mathrm{g}}, (n + 1)R) \setminus \mathbb{B}(x_{\mathrm{g}}, R)\right)$ converges, there exists B such that for $d_{\mathbb{H}}(x, x_{\mathrm{g}}) > B$, $\nu(\mathbb{B}(x, R)) \leq s/2$. Since h is hyperbolic and is an isometry for $d_{\mathbb{H}}$, we have $d_{\mathbb{H}}(h^m(x_{\mathrm{g}}), x_{\mathrm{g}}) \to \infty$, hence there exists an integer m such that $d_{\mathbb{H}}(x, x_{\mathrm{g}}) \leq B$ implies $d_{\mathbb{H}}(h^m(x), x_{\mathrm{g}}) \geq 2B$ hence $\nu(\mathbb{B}(h^m(x), R)) \leq s/2$.

Put $\varepsilon = \mu^m(\{h^m\})$ and pick $x \in \mathbb{H}_k$ such that $\nu(\mathbb{B}(x, R)) \geq s(1 - \varepsilon/3)$. In particular $d_{\mathbb{H}_k}(x, x_{\mathrm{g}}) \leq B$. The invariance relation $\mu^m * \nu = \nu$ yields

$$\begin{aligned} s\left(1 - \frac{\varepsilon}{3}\right) \leq \nu(\mathbb{B}(x, R)) &= \sum_{\gamma\in\Gamma} \mu^m(\{\gamma\})\nu(\gamma(\mathbb{B}(x, R))) \\ &\leq \sum_{\gamma\in\Gamma\setminus h^m} \mu^m(\{\gamma\})\nu(\mathbb{B}(\gamma x, R)) + \mu^m(\{h^m\})\nu(\mathbb{B}(h^m(x), R)) \\ &\leq (1 - \mu^m(\{h^m\}))s + \mu^m(\{h^m\})\frac{s}{2} = s\left(1 - \frac{\varepsilon}{2}\right). \end{aligned}$$

From this contradiction we conclude that $\nu(\mathbb{H}_k) = 0$.

To prove that ν gives full mass to $\mathbb{P}^1(k)$, consider the projection π on the closed subtree $\overline{\mathrm{Conv}(\mathbb{P}^1(k))} = \mathbb{H}_k \cup \mathbb{P}^1(k)$. Since $\mathbb{H}_k \cup \mathbb{P}^1(k)$ is Γ-invariant, for any $x \in \mathbb{P}^{1,\mathrm{an}}_k$ and any $\gamma \in \Gamma$ we have that $\pi(\gamma(x)) = \gamma(\pi(x))$. It follows that $\pi_*\nu$ is stationary. By the first part of the proof, $\pi_*\nu$ gives full mass to $\mathbb{P}^1(k)$ (which is a Borel set). Hence ν gives full mass to $\pi^{-1}(\mathbb{P}^1(k)) = \mathbb{P}^1(k)$. This completes the proof. □

3.3. **Distribution of attracting and repelling fixed points.** In the course of the proof of our main result we shall need the following interpretation of the stationary measures as the distribution of fixed points of hyperbolic elements.

Recall that the dual measure $\check{\mu}$ is defined as the image of μ under the involution $\gamma \mapsto \gamma^{-1}$, and that the measure μ satisfies (B1) (resp. (B2)) iff $\check{\mu}$ does. We also set $\check{\omega} = (\gamma_n^{-1})$ when $\omega = (\gamma_n)$ so that $l_n(\omega) = r_n(\check{\omega})^{-1}$. Observe that $\|l_n(\omega)\| = \|r_n(\check{\omega})^{-1}\| = \|r_n(\check{\omega})\|$ hence $\chi(\check{\mu}) = \chi(\mu)$.

Theorem 3.10. *Let μ be a probability measure with countable support in* $\mathrm{SL}(2, k)$*, satisfying (B1). Then*

$$\mathsf{P}\left(\{\omega,\ r_n(\omega) \text{ is hyperbolic}\}\right) \underset{n\to\infty}{\longrightarrow} 1. \tag{3.8}$$

In addition, the asymptotic distribution for the weak-$*$ topology on $\mathbb{P}^{1,\mathrm{an}}_k$ of the attracting (resp. repelling) fixed point of $r_n(\omega)$ is given by the unique μ-stationary (resp. $\check{\mu}$-stationary) probability measure ν (resp. $\check{\nu}$).

If furthermore μ satisfies (B2), then for every $\varepsilon > 0$

$$\mathsf{P}\left(\left\{\left|\frac{1}{n}\log|\mathrm{tr}(r_n(\omega))| - \chi(\mu)\right| < \varepsilon\right\}\right) \underset{n\to\infty}{\longrightarrow} 1. \tag{3.9}$$

Remark 3.11. The meaning of the statement on the distribution of periodic points is the following. For each n let Ω_n be the set of $\omega \in \Omega$ such that $r_n(\omega)$ is hyperbolic. One can then define the measurable map $\mathrm{att}^n, \mathrm{rep}^n : \Omega_n \to \mathbb{P}^1(k)$ by sending ω to the attracting and repelling fixed points of $r_n(\omega)$. The theorem asserts that $\mathsf{P}(\Omega_n) \to 1$ when $n \to \infty$, and $\mathrm{att}^n_* \mathsf{P} \to \nu$, and $\mathrm{rep}^n_* \mathsf{P} \to \check{\nu}$ as $n \to \infty$.

Remark 3.12. Under stronger moment assumptions on μ, (3.8) can be turned into an almost sure limit. For instance it is shown in [MT, Thm. 1.4] that if μ has bounded support the probability in (3.8) is exponentially close to 1, thus by Borel-Cantelli, $\frac{1}{n} \log |\mathrm{tr}(r_n(\omega))|$ converges a.s. to $\chi(\mu)$.

Proof. Consider the probability measure $\nu \times \check{\nu}$ on the product space $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. This measure is the image of P under the measurable map $\omega \mapsto (e(\omega), e(\check{\omega}))$.

By Fubini, and since the measures ν and $\check{\nu}$ do not have any atom, we have $\nu \times \check{\nu}(\Delta) = 0$ where Δ denotes the diagonal in $\mathbb{P}^1(k) \times \mathbb{P}^1(k)$. It follows that

$$(\nu \times \check{\nu}) \left\{ (x, y) \in \mathbb{P}^1(k)^2,\ d_{\mathrm{sph}}(x, y) \leq \delta \right\} \underset{\delta \to 0}{\longrightarrow} 0 \ . \tag{3.10}$$

By Theorem 3.3 $\|r_n(\omega)\| \to \infty$ a.s. and the asymptotic distribution of the $B_{\mathrm{att}}(r_n(\omega))$ is given by ν. Similarly the asymptotic distribution of the $B_{\mathrm{rep}}(r_n(\omega))$ is given by $\check{\nu}$, so that for each $\delta > 0$, we infer that

$$\limsup_{n \to \infty} \mathsf{P} \left\{ d_{\mathrm{sph}}(B_{\mathrm{att}}(r_n(\omega)), B_{\mathrm{rep}}(r_n(\omega))) \leq \delta \right\} \leq (\nu \times \check{\nu}) \left\{ (x, y) \in \mathbb{P}^1(k)^2,\ d_{\mathrm{sph}}(x, y) \leq \delta \right\} \ ,$$

which can be made as small as we wish.

Fix any real number $\varepsilon > 0$, and choose $\delta > 0$ such that the left hand side in (3.10) is at most $\varepsilon/2$. Then there exists $N = N(\varepsilon)$ and a set Ω_ε with $\mathsf{P}(\Omega_\varepsilon) \geq 1 - \varepsilon/2$, such that if $\Omega \in \Omega_\varepsilon$ and for $n \geq N(\omega)$, we have

$$d_{\mathrm{sph}}(B_{\mathrm{att}}(r_n(\omega)), B_{\mathrm{rep}}(r_n(\omega))) > \delta \ .$$

In addition $\|r_n(\omega)\| \to \infty$ a.s, so increasing N and discarding a set of probability $\varepsilon/2$ we may further assume that

$$\|r_n(\omega)\| \geq 2\delta^{-1} \text{ on } \Omega_\varepsilon \ .$$

Now if we pick $\omega \in \Omega_\varepsilon$ and $n \geq N$. The two closed balls $B_{\mathrm{att}}(r_n(\omega)), B_{\mathrm{rep}}(r_n(\omega))$ have diameter at most $\delta/2$ hence are disjoint. Since $r_n(\omega)$ maps $\mathbb{P}^{1,\mathrm{an}}_k \setminus B_{\mathrm{rep}}(r_n(\omega))$ into $B_{\mathrm{att}}(r_n(\omega))$, we conclude by Lemma 2.10.

If furthermore (B2) holds then from Lemma 2.9, we conclude that

$$|\log |\operatorname{tr}(r_n(\omega)| - \log \|r_n(\omega)\|| \leq -\log \min\{1, \delta\} \ ,$$

and (3.9) follows since $\lim_{n \to \infty} \frac{1}{n} \log \|r_n(\omega)\| = \chi(\check{\mu}) = \chi(\mu)$. □

4. Degenerations: non elementary representations

In this section we fix a finitely generated group G, endowed with some probability measure m. Recall the notation $\mathbb{L} = \mathbb{C}(\!(t)\!)$ and that $\mathbb{M}$ is the ring of holomorphic functions in $\mathbb{D}$ with meromorphic extension at the origin. We fix a representation $\rho : G \to \mathrm{SL}(2, \mathbb{M})$, that is a family of representations $\rho_t : G \to \mathrm{SL}(2, \mathbb{C})$ for $t \in \mathbb{D}^*$ such that for any $g \in G$ $t \mapsto \rho_t(g)$ is holomorphic on $\mathbb{D}^*$ and extends meromorphically through the origin. We suppose that

(A1) $\mathrm{Supp}(m)$ generates G;

(A2) m has finite first moment $\displaystyle\int \mathrm{length}(g)dm(g) < \infty$.

We also assume that if ρ_{na} denotes the induced representation $\rho_{\mathsf{na}} : G \to \mathrm{SL}(2, \mathbb{L})$, then $\rho_{\mathsf{na}}(G)$ is non-elementary. Our aim is to prove Theorem B and to infer the non-elementary case of Theorem A.

4.1. **Basic remarks on meromorphic families of representations.** Since we have to deal with both Archimedean and non-Archimedean objects, we slightly change notation: $\|\cdot\|$ denotes the operator norm in $\mathrm{SL}(2, \mathbb{C})$ associated to any norm on $\mathbb{C}^2$, say $\|(x, y)\| = \max(|x|, |y|)$, and $\|\cdot\|_{\mathsf{na}}$ denotes the non-Archimedean norm on $\mathrm{SL}(2, \mathbb{L})$ associated to the t-adic norm in $\mathbb{M}$ given by $|f|_{\mathsf{na}} = \exp(-\operatorname{ord}_{t=0}(f))$. More generally we use the subscript na to label non-Archimedean objects.

For any $t \in \mathbb{D}^*$, we set $\Gamma_t = \rho_t(G)$ and let μ_t be the push-forward of μ under ρ_t. Analogously, denoting by $\rho_{\mathsf{na}} : G \to \mathrm{SL}(2, \mathbb{L})$ the non-Archimedean representation naturally associated to ρ, we let $\Gamma_{\mathsf{na}} = \rho_{\mathsf{na}}(G)$ and $\mu_{\mathsf{na}} = (\rho_{\mathsf{na}})_*\mu$.

Observe that for every $g \in G$, $\log \|\rho(g)\|_{\mathsf{na}} \leq C \,\mathrm{length}(g)$ for some uniform constant $C > 0$, and likewise for $\log \|\rho_t(g)\|$. In particular we have

Lemma 4.1. *The condition (A2) implies the moment condition (B2) for the measures μ_t and μ_{na}.*

We will need some uniformity on the control of $\|\rho_t(g)\|$.

Lemma 4.2. *For every homomorphism $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ there exists $C = C(\rho) > 0$ such that for every $g \in G$ one can write $\rho_t(g) = t^{-\log\|\rho(g)\|_{\mathsf{na}}} \cdot \widetilde{\gamma}(t)$, with $\widetilde{\gamma} \in \mathrm{SL}(2, \mathcal{O}(\mathbb{D}))$, and*

$$\left|\log \|\widetilde{\gamma}\|_{L^\infty(\overline{\mathbb{D}}(0,1/2))}\right| \leq C \,\mathrm{length}(g). \tag{4.1}$$

Proof. let (s_i) be a finite symmetric set of generators of G and write g as a reduced word in G, $g = s_{i_1} \cdots s_{i_n}$, $n = \mathrm{length(g)}$. Let $\sigma_i = \rho(s_i)$ and write $\sigma_i = t^{-\alpha_i}\widetilde{\sigma}_i$ with $\alpha_i = \log \|\sigma_i\|_{\mathsf{na}}$ and $\widetilde{\sigma}_i$ holomorphic and non vanishing at 0, so that

$$\rho_t(g) = t^{-\sum_{j=1}^n \alpha_{i_j}} \ \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_n} \ .$$

Set $A := \max \alpha_i$, and write $\rho_t(g) = t^{-\log\|\rho(g)\|_{\mathsf{na}}} \cdot \widetilde{\gamma}(t)$ with $\widetilde{\gamma} \in \mathrm{SL}(2, \mathcal{O}(\mathbb{D}))$. Then we get that $\widetilde{\gamma}(t) = t^{-\alpha} \cdot \widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_n}(t)$ with $0 \leq \alpha = \sum_{j=1}^n \alpha_{i_j} - \log \|\rho(g)\|_{\mathsf{na}} \leq An$. By using the maximum principle we can estimate

$$\begin{aligned} \sup_{|t|\leq 1/2} |\widetilde{\gamma}(t)| \leq \sup_{|t|=1/2} |\widetilde{\gamma}(t)| &\leq (1/2)^{-\alpha} \prod_{j=1}^n \left(\sup_{|t|=1/2} \left|\widetilde{\sigma}_{i_j}(t)\right| \right) \\ &\leq 2^{An} \left(\max_i \|\widetilde{\sigma}_i\|_{L^\infty(D(0,1/2))} \right)^n \leq D^n \end{aligned}$$

for some $D \geq 1$. Likewise we have that

$$\sup_{|t|\leq 1/2} |\widetilde{\gamma}(t)| \geq \sup_{|t|=1/2} |\widetilde{\gamma}(t)| \geq (1/2)^{-\alpha} \sup_{|t|=1/2} |\widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_n}(t)| = (1/2)^{-\alpha} |\widetilde{\sigma}_{i_1} \cdots \widetilde{\sigma}_{i_n}(0)| \geq E^n$$

for some $E > 0$ and we are done. □

Let us also note the following basic but crucial observation.

Lemma 4.3. *If Γ_{na} is non-elementary, then for small enough $t \in \mathbb{D}^*$, $\Gamma_t \leq \mathrm{SL}(2,\mathbb{C})$ is non-elementary.*

Proof. By definition Γ contains two hyperbolic elements γ_1 and γ_2 with disjoint fixed points on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Since for $i = 1, 2$ $|\mathrm{tr}(\gamma_i)|_{\mathsf{na}} > 1$, it follows that $|\mathrm{tr}(\gamma_{i,t})| \to \infty$ as $t \to 0$. Thus $\gamma_{1,t}$ and $\gamma_{2,t}$ are loxodromic for small t, and they have well-defined attracting and repelling fixed points $\mathrm{att}(\gamma_{i,t})$, $\mathrm{rep}(\gamma_{i,t})$. Saying that γ_1 and γ_2 have disjoint fixed points sets on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ implies that the formal expansions of the curves $t \mapsto \mathrm{att}(\gamma_{i,t})$ and $t \mapsto \mathrm{rep}(\gamma_{i,t})$ are all distinct, thus the corresponding points in $\mathbb{P}^1$ must be disjoint in some punctured neighborhood of the origin. □

4.2. **Models and $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$.** In this paragraph we review the notion of model. Set $X = \mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$. A (bimeromorphic) model of X is a surface Y together with a bimeromorphic holomorphic map $\pi_Y : Y \to X$ that is biholomorphic above $X \setminus (\{0\} \times \mathbb{P}^1)$. The fiber $\pi_Y^{-1}(\{t\} \times \mathbb{P}^1)$ will be denoted by Y_t. We will only consider the case where Y is smooth, in which case π_Y is simply a composition of point blow-ups above the central fiber and Y_0 is a divisor with simple normal crossings.

We say that a model Y' dominates a model Y if the birational map $\pi_{Y'} : Y' \to X$ factors through Y. The set of models is then directed in the sense that given two models Y and Y', there exists a third one Y'' dominating both.

We now explain the basic correspondence between models and finite subsets of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Recall as a set, $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is the one point compactification of the space of multiplicative semi-norms on $\mathbb{L}[z]$ whose restriction to $\mathbb{L}$ is the t-adic norm. Let Y be any model, and pick any irreducible component E of Y_0: we will define a type 2 point $\zeta_E \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$.

Observe that any element $f \in \mathbb{C}(t)[z]$ defines a rational function on $\mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$ so that we can define

$$|f|_{\zeta_E} = |f(\zeta_E)| = \exp\left(-\frac{1}{b_E}\,\mathrm{ord}_E(f \circ \pi_Y)\right)$$

where $b_E = \mathrm{ord}_E(t \circ \pi_Y)$. Dividing by b_E guarantees that $\zeta_E|_{\mathbb{L}} = |\cdot|$ and also that the definition is model-independent in the sense that if Y' dominates Y and E' is the strict transform of E then $\zeta_E = \zeta_{E'}$. Since the field $\mathbb{C}(t)$ is dense in $\mathbb{L}$, it follows that ζ_E extends uniquely to a semi-norm on $\mathbb{L}[z]$ hence defines a point in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. This point is of type 2 and is defined over the field extension $\mathbb{C}(\!(t^{1/b_E})\!)$. Conversely, any type 2 point of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is equal to ζ_E for some irreducible component E of the central fiber of some model Y over X, see [Fan, Lemma 7.16].

We denote by $S(Y)$ the set of all type 2 points $\zeta_E \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ where E ranges over the set of irreducible components of Y_0. It is an elementary fact that $S(Y') \supset S(Y)$ if and only if Y' dominates Y.

Proposition 4.4. *(see* [dMF2, §4.1]*) Let S be any finite set of type 2 points in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Then there exists a (smooth) model Y such that $S \subset S(Y)$.*

Remark 4.5. In fact there is an isomorphism of partially ordered sets between finite sets of type 2 points endowed with the inclusion and proper bimeromorphic maps $\pi : Y \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{D}$ with Y a normal complex analytic variety, see [Fan, Theorem 7.18].

The previous proposition together with Proposition 1.9 yield the following corollary.

Corollary 4.6. *Let ν be any probability measure on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ which is non atomic and gives zero mass to $\mathbb{H}_{\mathbb{L}}$. Then for every $\varepsilon > 0$ there exists a model Y such that every connected component U of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus S(Y)$ satisfies $\nu(U) < \varepsilon$.*

Remark 4.7. Suppose that ν puts full mass on the set $\mathbb{P}^{1,\mathrm{an}}(\mathbb{L})$ of points of type 1 defined over $\mathbb{L}$. Then one can actually choose $\pi : Y \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{D}$ to be a composition of blow-ups at *free* points, i.e. lying in the regular locus of the central fiber. In other words, one can choose the central divisor $\pi^*(\mathbb{P}^1_{\mathbb{C}} \times \{0\})$ to be *reduced*.

4.3. **Reduction map and action of** $\mathrm{SL}(2,\mathbb{M})$**.** Fix any model Y, and let ζ be any type 2 point in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Then we can find a model Y' dominating Y and an irreducible component E of Y'_0 such that $\zeta = \zeta_E$. If the natural map $\pi_{Y',Y} : Y' \to Y$ contracts E to a point, we set $\mathrm{red}_Y(\zeta_E) = \pi_{Y',Y}(E)$. Otherwise we let $\mathrm{red}_Y(\zeta_E)$ be the generic point of the curve $\pi_{Y',Y}(E)$ which is a (non-closed) point in the $\mathbb{C}$-scheme Y_0. The mapping red_Y is called the *reduction map*. It extends canonically to an anti-continuous map $\mathrm{red}_Y : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to Y_0$ (i.e. the preimage of a closed set in the Zariski topology is open for the Berkovich topology), see e.g. [T, §5.2.4], or [dMF2, §4.2].

It can be shown that it η_E is the generic point of an irreducible component E of Y_0, then $\mathrm{red}_Y^{-1}(\eta_E) = \{\zeta_E\}$. If $p \in Y_0$ is a closed point then $\mathrm{red}_Y^{-1}(p)$ is a connected component of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus S(Y)$ whose boundary consists of the points ζ_E where E ranges over all irreducible components of Y_0 containing p. In particular the boundary of $\mathrm{red}_Y^{-1}(p)$ consists of one or two points.

The reduction map behaves well under proper modifications.

Lemma 4.8. *If $\pi : Y' \to Y$ is a birational morphism, then $\mathrm{red}_Y = \pi \circ \mathrm{red}_{Y'}$.*

Proof. If ζ is any type 2 point, there exists a model Y'' dominating Y' such that $\zeta = \zeta_E$ for some component E of Y''_0. Then it follows immediately from the definitions that $\mathrm{red}_Y(\zeta) = \pi \circ \mathrm{red}_{Y'}(\zeta)$. This identity then extends to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ because red_Y and $\mathrm{red}_{Y'}$ admit unique anti-continuous extensions to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ and π is continuous for the Zariski topology. $\square$

Let us now pick $\gamma \in \mathrm{SL}(2,\mathbb{M})$, and denote by γ_{na} its natural image in $\mathrm{SL}(2,\mathbb{L})$. Observe that γ induces a biholomorphism from $\mathbb{D}^* \times \mathbb{P}^1_{\mathbb{C}}$ to itself commuting with the first projection, and extending meromorphically to $\mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$. More generally given any two models Y, Y' this biholomorphism extends to a bimeromorphic map $\gamma_{Y,Y'} : Y \dashrightarrow Y'$. Its properties can be described from γ_{na} and the reduction map as follows.

Proposition 4.9. *Let Y, Y' be two models over X, and pick $\gamma \in \mathrm{SL}(2,\mathbb{M})$.*

(1) The induced bimeromorphic map $\gamma_{Y,Y'} : Y \dashrightarrow Y'$ has an indeterminacy point at $p \in Y_0$ iff there exists a type 2 point $\zeta \in \mathrm{red}_Y^{-1}(p)$ such that $\gamma_{\mathsf{na}}(\zeta) \in S(Y')$.

(2) Suppose $\gamma_{Y,Y'}$ is holomorphic at $p \in Y_0$. Then for any $\zeta \in \mathrm{red}_Y^{-1}(p)$, we have $\gamma_{Y,Y'}(p) = \mathrm{red}'_Y(\gamma_{\mathsf{na}}(\zeta))$.

Proof. The proposition follows easily from the basic properties of the reduction map together with the following lemma. $\square$

Lemma 4.10. *Let $\zeta \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ be a type 2 point. Fix a model Y_1 dominating Y and a component E of $(Y_1)_0$ such that $\zeta = \zeta_E$. Let $\zeta' = \gamma_{\mathsf{na}}(\zeta)$, and fix a model Y'_1 dominating Y' and a component E' of $(Y'_1)_0$ such that $\zeta' = \zeta_{E'}$. Then $\gamma_{Y_1,Y'_1}(\eta_E) = \eta_{E'}$, where η_E (resp. $\eta_{E'}$) is the generic point of E (resp. E').*

Conversely, given any two models Y_1*,* Y_1' *and respective irreducible components* $E \subset (Y_1)_0$ *and* $E' \subset (Y_1')_0$*, if* $\gamma_{Y_1,Y_1'}(\eta_E) = \eta_{E'}$ *then* $\zeta_{E'} = \gamma_{\mathsf{na}}(\zeta_E)$*.*

Proof. The pull-back by γ of any rational function on Y_1' vanishing at $\eta_{E'}$ necessarily vanishes at η_E since $\zeta_{E'} = \gamma_{\mathsf{na}}(\zeta_E)$. Since for any point p not lying on E' there exists a rational function on Y_1' which is non-zero at p, and zero on E', the first claim follows.

For the second claim, pick f a rational function on Y_1', and choose points $p \in E$, $p' \in E'$ such that $\gamma_{Y_1,Y_1'}$ is regular at p, $p' = \gamma_{Y_1,Y_1'}(p)$, and E (resp. E') is regular at p (resp. at p'). Then we may choose coordinate (x, y) at p and (x', y') at p' such that $E = \{x = 0\}$, $E' = \{x' = 0\}$ and $\gamma_{Y_1,Y_1'}(x, y) = (x^k, \star)$. It follows that

$$-\log|f(\gamma_{\mathsf{na}}(\zeta_E))| = \frac{1}{b_E}\operatorname{ord}_E(f \circ \gamma_{Y_1,Y_1'}) = \frac{k}{b_E}\operatorname{ord}_{E'}(f)$$

which implies $b_{E'} = b_E/k$ and $\zeta_{E'} = \gamma_{\mathsf{na}}(\zeta_E)$. □

Using the reduction map, one can see that for large $\|\gamma\|_{\mathsf{na}}$, the meromorphic map γ_Y acts on the central fiber Y_0 like a (brutal) North-South transformation.

Proposition 4.11. *Let* Y *be any model.*

There exists a constant $C = C(Y)$ *depending only on* Y *such that for any* $\gamma \in \mathrm{SL}(2, \mathbb{M})$ *such that* $\|\gamma\|_{\mathsf{na}} \geq C$*, there exist two points* $\operatorname{att}(\gamma_Y)$ *and* $\operatorname{rep}(\gamma_Y)$ *in* Y_0 *(not necessarily distinct) such that the induced bimeromorphism* $\gamma_Y : Y \dashrightarrow Y$ *is holomorphic on* $Y_0 \setminus \operatorname{rep}(\gamma_Y)$ *and* $\gamma_Y(Y_0 \setminus \operatorname{rep}(\gamma_Y)) = \operatorname{att}(\gamma_Y)$*.*

Proof. Choose $C > \max_{S(Y)} \exp(d_{\mathbb{H}}(\zeta, x_{\mathrm{g}}))$, and pick γ of norm $\geq C$.

By Proposition 2.7 there exist two disjoint closed Berkovich disks $B_{\mathrm{att}}(\gamma)$ and $B_{\mathrm{rep}}(\gamma)$ of spherical diameter $\|\gamma\|_{\mathsf{na}}^{-1}$ such that $\gamma(\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus B_{\mathrm{rep}}(\gamma)) \subset B_{\mathrm{att}}(\gamma)$.

Observe that $B_{\mathrm{rep}}(\gamma)$ cannot contain any point $\zeta \in S(Y)$ since otherwise we would get that $d_{\mathbb{H}}(\zeta, x_{\mathrm{g}}) \geq \log\|\gamma\|_{\mathsf{na}}^{-1}$, contradicting the choice of C. Thus, $B_{\mathrm{rep}}(\gamma) \cap S(Y) = \emptyset$, and since $B_{\mathrm{rep}}(\gamma)$ is connected, it is contained in a connected component of $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus S(Y)$. It follows that $B_{\mathrm{rep}}(\gamma) \subset \operatorname{red}_Y^{-1}(\operatorname{rep}(\gamma_Y))$. Similarly we have $B_{\mathrm{att}}(\gamma) \cap S(Y) = \emptyset$, and $B_{\mathrm{att}}(\gamma) \subset \operatorname{red}_Y^{-1}(\operatorname{att}(\gamma_Y))$.

Now pick any point $p \in Y_0$ different from $\operatorname{rep}(\gamma_Y)$. Then $\operatorname{red}_Y^{-1}(p)$ is disjoint from $B_{\mathrm{rep}}(\gamma)$ so it is mapped into $B_{\mathrm{att}}(\gamma)$ by γ_{na}. The first item of Proposition 4.9 then asserts that γ_Y is holomorphic at p and the second one that $\gamma_Y(p) = \operatorname{red}_Y(B_{\mathrm{att}}(\gamma)) = \operatorname{att}(\gamma_Y)$. □

Remark 4.12. The proof shows that $\operatorname{rep}(\gamma_Y) = \operatorname{red}_Y(\zeta)$ for any $\zeta \in B_{\mathrm{rep}}(\gamma)$. In particular if γ is hyperbolic and ζ_{rep} is its repelling fixed point, then $\operatorname{rep}(\gamma_Y) = \operatorname{red}_Y(\zeta_{\mathrm{rep}})$. Also if $S(Y)$ contains the Gauß point then taking $C > \exp(\operatorname{diam}_{\mathbb{H}}(S(Y)))$ is enough.

Let ν be any Radon measure on the Berkovich projective line $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$. Then by definition the *residual measure* $(\operatorname{red}_Y)_*\nu$ is the atomic measure on Y_0 satisfying

$$(\operatorname{red}_Y)_*\nu(\{p\}) = \nu(\operatorname{red}_Y^{-1}(p)) ,$$

for any closed point $p \in Y_0$. Note that if ν gives no mass to $S(Y)$ $(\operatorname{red}_Y)_*\nu$ is the push forward of ν in the usual sense. Since the union of the open sets $\operatorname{red}_Y^{-1}(p)$ as p ranges through closed points of Y_0 is equal to the complement of $S(Y)$ in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$, it follows that the total mass of $(\operatorname{red}_Y)_*\nu$ is equal to the mass of the restriction of ν to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus S(Y)$.

Lemma 4.13. *If ν_n converges weakly to ν and $\nu(S(Y)) = 0$, then $(\mathrm{red}_Y)_*\nu_n$ converges in mass to $(\mathrm{red}_Y)_*\nu$.*

Proof. Pick any closed point $p \in Y_0$. Recall that Radon measures are regular. Since $\nu_n \to \nu$ and $\mathrm{red}_Y^{-1}(p)$ is open we deduce that

$$\liminf_{n\to\infty} (\mathrm{red}_Y)_*\nu_n\{p\} \geq (\mathrm{red}_Y)_*\nu\{p\} \ .$$

On the other hand, since the boundary of $\mathrm{red}_Y^{-1}\{p\}$ is included in $S(Y)$, the assumption $\nu(S(Y)) = 0$ implies that

$$\limsup_{n\to\infty} (\mathrm{red}_Y)_*\nu_n(\{p\}) \leq \nu\left(\overline{\mathrm{red}_Y^{-1}\{p\}}\right) = (\mathrm{red}_Y)_*\nu(\{p\}) \ .$$

Therefore we conclude that for every $p \in Y_0$, $(\mathrm{red}_Y)_*\nu_n(\{p\}) \to (\mathrm{red}_Y)_*\nu(\{p\})$. Since all these measures are atomic and of uniformly bounded mass, the result follows. □

4.4. **Proof of Theorem B.** Let (G, m) satisfying (A1), and a representation $\rho : G \to \mathrm{SL}(2, \mathbb{M})$, and suppose that the induced representation $\rho_{\mathsf{na}} : G \to \mathrm{SL}(2, \mathbb{L})$ is non-elementary. Lemma 4.3 implies that $\rho_t : G \to \mathrm{SL}(2, \mathbb{C})$ is also non-elementary for small t so by Theorem 3.1 (in the complex case) it makes sense to talk about the unique probability measure ν_t on $\mathbb{P}^1(\mathbb{C})$ that is stationary under $\mu_t = (\rho_t)_*m$. We also denote by ν_{na} the unique probability measure on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ that is stationary under $\mu_{\mathsf{na}} = (\rho_{\mathsf{na}})_*m$.

Fix a model Y. Since $\pi_Y : Y \to \mathbb{D} \times \mathbb{P}^1_{\mathbb{C}}$ is a biholomorphism outside the central fiber, we can view ν_t as a probability measure on Y_t, which we denote by ν_{Y_t}. Our aim is to prove that ν_{Y_t} converges as $t \to 0$ to an atomic measure on the central fiber given by $(\mathrm{res}_Y)_*\nu_{\mathsf{na}}$.

Fix $\varepsilon > 0$, and choose a model[6] Y' such that the conclusion of Corollary 4.6 holds for the unique stationary probability measure $\check{\nu}_{\mathsf{na}}$ associated to the reversed random walk. We can assume that Y' dominates Y, and write $\nu_{Y'_t}$ for the stationary measure on Y'_t.

Consider the Markov operator $P_t = \int (\gamma_t)_* d\mu_t(\gamma)$ acting on the space of probability measures on Y'_t, whose n-fold iterate is $P_t^n = \int (\gamma_t)_* d\mu_t^n(\gamma)$. Observe that for every $n \geq 1$ we have that $P_t^n \nu_{Y'_t} = \nu_{Y'_t}$. To analyze this identity as t tends to 0, we extract a sequence $t_j \to 0$ such that $\nu_{Y_{t_j}}$ and $\nu_{Y'_{t_j}}$ converge to respective probability measures ν_{Y_0} on Y_0, and $\nu_{Y'_0}$ on Y'_0. Note that by construction the push-forward of $\nu_{Y'_0}$ under the canonical projection map $Y' \to Y$ is ν_{Y_0}. We will show that $\nu_{Y_0} = (\mathrm{res}_Y)_*\nu_{\mathsf{na}}$.

Let $C = C(Y')$ be the constant given by Proposition 4.11, and define $A = A(C) = \{\gamma \in \mathrm{SL}(2, \mathbb{L}),\ \|\gamma\| \geq C\}$. We define two measurable maps on A_n with values in Y'_0, namely $\mathrm{att}_{Y'}(\gamma) := \mathrm{att}(\gamma_{Y'})$ and $\mathrm{rep}_{Y'}(\gamma) := \mathrm{rep}(\gamma_{Y'})$. The image $\mu^n_{\mathsf{na}}|_A$ under $\mathrm{rep}_{Y'}$ (resp. $\mathrm{att}_{Y'}(\gamma)$) is by definition the distribution of repelling (resp. attracting) points on Y' for the random walk at time n.

Lemma 4.14. *The sequence of atomic measures $(\mathrm{rep}_{Y'})_*(\mu^n_{\mathsf{na}}|_A)$ (resp. $(\mathrm{att}_{Y'})_*(\mu^n_{\mathsf{na}}|_A)$) converges in mass to $(\mathrm{res}_{Y'})_*\check{\nu}_{\mathsf{na}}$ (resp. to $(\mathrm{res}_{Y'})_*\nu_{\mathsf{na}}$).*

Taking this lemma for granted for the moment, let us complete the proof of the theorem. Observe first that for any $\gamma \in A$, from the description of the action of γ in Proposition 4.11, we see that any cluster value of $(\gamma_{t_j})_*\nu_{Y'_{t_j}}$ is of the form

$$\left(1 - \nu_{Y'_0}(\{\mathrm{rep}(\gamma_{Y'})\})\right)\delta_{\mathrm{att}(\gamma_{Y'})} + \mathrm{error} = \delta_{\mathrm{att}(\gamma_{Y'})} + \mathrm{error}$$

[6] Observe that Remark 4.7 applies here so that we can further assume the central fiber to be reduced.

where the error on the right hand side is a signed measure of total mass $\leq 2\nu_{Y'_0}(\{\mathrm{rep}(\gamma_{Y'})\})$. Using the identity $\nu_{Y'_t} = P^n_t \nu_{Y'_t}$ and letting $t = t_j \to 0$ we infer that

$$\nu_{Y'_0} = \int_A \delta_{\mathrm{att}(\gamma_{Y'})} d\mu^n_{\mathsf{na}}(\gamma) + \text{error} \tag{4.2}$$

where the mass of the error is

$$\begin{aligned}\mathbf{M}(\text{error}) &\leq 2\int_A \nu_{Y'_0}(\{\mathrm{rep}(\gamma_{Y'})\})\, d\mu^n_{\mathsf{na}}(\gamma) + |1-\mu^n_{\mathsf{na}}(A)|\\ &= 2\int_{x\in Y'_0} \nu_{Y'_0}(\{x\}) d\left((\mathrm{rep}_{Y'})_*(\mu^n_{\mathsf{na}}|_A)\right) + |1-\mu^n_{\mathsf{na}}(A)|\ . \end{aligned} \tag{4.3}$$

By Lemma 4.14, $(\mathrm{rep}_{Y'})_*(\mu^n_{\mathsf{na}}|_A)$ converges in mass towards $(\mathrm{res}_{Y'})_*\check{\nu}_{\mathsf{na}}$. Since every atom of the latter measure has mass $\leq \varepsilon$ by construction, we get that every atom of $(\mathrm{rep}_{Y'})_*(\mu^n|_A)$ has mass $\leq 2\varepsilon$ for n large enough. It follows that the integral in (4.3) is bounded by $\leq 2\varepsilon$. Since in addition $\mu^n(A) \to 1$ by Lemma 3.5, we conclude that for n large enough

$$\nu_{Y'_0} - \int_A \delta_{\mathrm{att}(\gamma_{Y'})} d\mu^n_{\mathsf{na}}(\gamma) \tag{4.4}$$

has total mass $\leq 5\varepsilon$.

To conclude, we observe that applying Lemma 4.14 again, the sequence of measures

$$\int_A \delta_{\mathrm{att}(\gamma_{Y'})} d\mu^n_{\mathsf{na}}(\gamma) = \int_{x\in Y'_0} \delta_x d\left((\mathrm{att}_{Y'})_*(\mu^n_{\mathsf{na}}|_A)\right)$$

converges in mass to $(\mathrm{res}_{Y'})_*\nu$. We thus infer that $\nu_{Y'_0} - (\mathrm{res}_{Y'})_*\nu_{\mathsf{na}}$ has total mass $\leq 5\varepsilon$. Pushing down this information to Y we get the same bound for $\nu_{Y_0} - (\mathrm{res}_Y)_*\nu_{\mathsf{na}}$ on Y. Since Y does not depend on ε and ε can be made arbitrarily small, we conclude that $\nu_{Y_0} = (\mathrm{res}_Y)_*\nu_{\mathsf{na}}$, as required. □

Proof of Lemma 4.14. Let $(\Omega, \mathsf{P}) = (\mathrm{SL}(2,\mathbb{L})^{\mathbb{N}^*}, \mu_{\mathsf{na}}^{\mathbb{N}^*})$, and consider the set

$$\Omega_n = \{\omega \in \Omega,\ r_n(\omega) \in \mathrm{SL}(2,\mathbb{L}) \text{ is hyperbolic}\}\ .$$

If $\mathrm{rep}^n_{\mathsf{na}} : \Omega_n \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ denotes the measurable map sending ω to the repelling fixed point of $r_n(\omega)$, then by Theorem 3.10 $\mathsf{P}(\Omega_n) \to 1$ and $(\mathrm{rep}^n_{\mathsf{na}})_*\mathsf{P} \to \check{\nu}_{\mathsf{na}}$. Pushing forward this convergence by the residue map $\mathrm{red}_{Y'}$ and applying Lemma 4.13, we thus get that $(\mathrm{red}_{Y'})_*(\mathrm{rep}^n_{\mathsf{na}})_*\mathsf{P}$ converges in mass to $(\mathrm{red}_{Y'})_*\check{\nu}_{\mathsf{na}}$.

Now define $\Omega'_n = \Omega_n \cap r_n^{-1}(A)$, which also satisfies $\mathsf{P}(\Omega'_n) \to 1$ by Lemma 3.5. It follows from Remark 4.12 that for $\omega \in \Omega'_n$, $\mathrm{rep}_{Y'}(r_n(\omega)) = \mathrm{red}_{Y'}(\mathrm{rep}^n_{\mathsf{na}}(\omega))$, in other words, the repelling fixed point of $r_n(\omega)$ is mapped under $\mathrm{red}_{Y'}$ to $\mathrm{rep}_{Y'}(r_n(\omega))$. We thus obtain that

$$(\mathrm{red}_{Y'})_*(\mathrm{rep}^n_{\mathsf{na}})_*\mathsf{P} = (\mathrm{rep}_{Y'})_*(\mu^n_{\mathsf{na}}|_A) + \text{error}$$

where the mass of the error tends to 0 as $n \to \infty$. This completes the proof. □

4.5. **Proof of Theorem A in the non-elementary case.** As in the previous section we work with a representation $\rho : G \to \mathrm{SL}(2,\mathbb{M})$ such that the induced representation $\rho_{\mathsf{na}} : G \to \mathrm{SL}(2,\mathbb{L})$ is non-elementary, and further assume that (A1) and (A2) hold. For $t \in \mathbb{D}^*$ we set $\mu_t = (\rho_t)_* m$ and we also put $\mu_{\mathsf{na}} = (\rho_{\mathsf{na}})_* m$. Recall that for $t \neq 0$ the Lyapunov exponent

$$\chi(t) := \lim_{n\to\infty} \frac{1}{n} \int \log \|\gamma\|\, d\mu^n_t(\gamma)$$

is a well defined positive number, and that the non-Archimedean Lyapunov exponent

$$\chi_{\mathsf{na}} := \lim_{n\to\infty} \frac{1}{n} \int \log \|\gamma\|_{\mathsf{na}} \, d\mu_{\mathsf{na}}^n(\gamma)$$

is also well-defined and positive by Theorem 3.1.

We have to show that

$$\frac{1}{\log |t|^{-1}} \chi(t) \longrightarrow \chi_{\mathsf{na}} \text{ as } t \to 0. \tag{4.5}$$

If $(K, |\cdot|)$ is any metrized field, and $z = [z_1 : z_2] \in \mathbb{P}^1(K)$, recall the notation

$$\sigma(\gamma, z) = \log \frac{\|\gamma Z\|}{\|Z\|} \ ,$$

where $Z = (z_1, z_2) \in K^2 \setminus \{0\}$ and $\|Z\| = \max(|z_1|, |z_2|)$. To establish (4.5), we first need to relate the classical and non-Archimedean expansion rates for a single group element. We start with the following consequence of Lemma 4.2.

Lemma 4.15. *There exists a constant $C > 0$, such that for any model Y, any $g \in G$, and any point $y_t \in Y_t$ with $|t| \leq 1/2$, we have*

$$\left| \frac{1}{\log |t|^{-1}} \sigma(\gamma_t, y) \right| \leq \log \|\gamma\|_{\mathsf{na}} + \frac{C \operatorname{length}(g)}{\log |t|^{-1}} \ , \tag{4.6}$$

where $\gamma = \rho(g) \in \mathrm{SL}(2, \mathbb{M})$.

Proof. As before for $t \neq 0$ we can naturally identify the fibers Y_t and $\{t\} \times \mathbb{P}^1$, and we write $y_t = [z_{1t} : z_{2t}]$, with $Z_t = (z_{1t}, z_{2t})$ and $\max\{|z_{1t}|, |z_{2t}|\} = 1$. The upper bound

$$\sigma(\gamma_t, y_t) = \log \|\gamma_t Z_t\| \leq \log \|\gamma_t\| \leq (\log |t|^{-1}) \log \|\gamma\|_{\mathsf{na}} + C \operatorname{length}(g)$$

follows directly from the definitions and Lemma 4.2. To get the lower bound it is enough to write $\|\gamma_t Z_t\| \geq \left\|\gamma_t^{-1}\right\|^{-1} \|Z_t\|$ and remind that $\|\gamma_t\| = \left\|\gamma_t^{-1}\right\|$. □

The main step of the proof is the following proposition.

Proposition 4.16. *For every model Y, there exists a constant $c(Y) > 0$ satisfying the following property. For every $\gamma \in \mathrm{SL}(2, \mathbb{M})$, there exists a point $\alpha(\gamma) \in Y$ such that if $t_j \to 0$ and $(y_{t_j}) \in Y_{t_j}$ is any sequence not accumulating $\alpha(\gamma)$, then*

$$\liminf_{j\to\infty} \frac{1}{\log |t_j|^{-1}} \sigma(\gamma_{t_j}, y_{t_j}) \geq \log \|\gamma\|_{\mathsf{na}} - c(Y). \tag{4.7}$$

This says that the (positive) upper bound that was obtained in (4.6) is almost achieved everywhere on Y when $t \to 0$, except at one point, up to an error that is uniform in γ (compare Lemma 2.8).

Let us postpone the proof of the proposition to the end of the section, and first complete the proof of (4.5).

Fix $\varepsilon > 0$. Apply Corollary 4.6 to get a model Y in which all the atoms of the residual measure $\nu_{Y_0} = (\mathrm{red}_Y)_* \nu_{\mathsf{na}}$ are smaller than ε. By Theorem B, we have that $\nu_{Y_t} \to \nu_{Y_0}$ as $t \to 0$ where ν_{Y_t} is the pull-back of ν_t to Y.

We first work with a fixed $g \in G$, and as usual we write $\gamma = \rho(g) \in \mathrm{SL}(2, \mathbb{M})$. By Lemma 4.15, we have that

$$\limsup_{t \to 0} \frac{1}{\log |t|^{-1}} \int \sigma(\gamma_t, y) d\nu_t(y) \leq \log \|\gamma\|_{\mathsf{na}} . \tag{4.8}$$

To obtain a lower bound, we fix a small neighborhood U of $\alpha(\gamma)$ in Y such that $\nu_t(U) \leq 2\varepsilon$ for any t. This is possible because $\nu_t \to \nu_{Y_0}$ and $\nu_{Y_0}(\alpha(\gamma)) \leq \varepsilon$. Then Proposition 4.16 shows that if $\eta \ll 1$ is fixed, then for every small enough t, for $y \in Y_t \setminus U$ we get

$$\frac{1}{\log |t|^{-1}} \sigma(\gamma_t, y) \geq \log \|\gamma\|_{\mathsf{na}} - c(Y) - \eta,$$

whereas for $y \in Y_t \cap U$, Lemma 4.15 implies that

$$\frac{1}{\log |t|^{-1}} \sigma(\gamma_t, y) \geq - \log \|\gamma\|_{\mathsf{na}} - \eta.$$

Combining these two estimates we infer that

$$\frac{1}{\log |t|^{-1}} \int_{Y_t} \sigma(\gamma_t, y) d\nu_t(y) \geq (1 - 2\varepsilon)(\log \|\gamma\|_{\mathsf{na}} - c(Y) - \eta) - 2\varepsilon(\log \|\gamma\|_{\mathsf{na}} + \eta),$$

therefore since η is arbitrary,

$$\liminf_{t \to 0} \frac{1}{\log |t|^{-1}} \int_{Y_t} \sigma(\gamma_t, y) d\nu_t(y) \geq \log \|\gamma\|_{\mathsf{na}} - (1 - 2\varepsilon)c(Y) - 2\varepsilon \log \|\gamma\|_{\mathsf{na}} .$$

Using this inequality and (4.8) we finally obtain

$$\limsup_{t \to 0} \left| \frac{1}{\log |t|^{-1}} \int_{Y_t} \sigma(\gamma_t, y) d\nu_t(y) - \log \|\gamma\|_{\mathsf{na}} \right| \leq 2c(Y) + 4\varepsilon \log \|\gamma\|_{\mathsf{na}} . \tag{4.9}$$

To conclude the argument we integrate this estimate with respect to g. Fix an integer n so large that

$$\frac{2c(Y)}{n} < \varepsilon \text{ and } \left| \frac{1}{n} \int \log \|\gamma\|_{\mathsf{na}} \, d\mu^n_{\mathsf{na}}(\gamma) - \chi_{\mathsf{na}} \right| < \varepsilon.$$

We observe that the Furstenberg formula for the Lyapunov exponent iterated n times and read in the model Y expresses as

$$\chi(t) = \frac{1}{n} \int_{Y_t \times \Gamma} \sigma(\gamma_t, y) d\nu_t(y) d\mu^n_t(\gamma),$$

so we can write

$$\begin{aligned}
\left| \frac{1}{\log |t|^{-1}} \chi(t) - \chi_{\mathsf{na}} \right| &= \left| \frac{1}{n} \int_{Y_t \times \Gamma_t} \frac{\sigma(\gamma_t, y)}{\log |t|^{-1}} d\nu_t(y) d\mu^n_t(\gamma) - \chi_{\mathsf{na}} \right| \\
&\leq \left| \frac{1}{n} \int_{Y_t \times \Gamma_t} \frac{\sigma(\gamma_t, y)}{\log |t|^{-1}} d\nu_t(y) d\mu^n_t(\gamma) - \frac{1}{n} \int \log \|\gamma\|_{\mathsf{na}} \, d\mu^n_{\mathsf{na}}(\gamma) \right| + \varepsilon \\
&=: \Delta(t) + \varepsilon \ .
\end{aligned}$$

By the moment assumption (A2), there exists a finite subset $G' \subset G$ such that

$$\int_{G \setminus G'} \mathrm{length}(g) dm^n(g) \leq \varepsilon \text{ and } \frac{1}{n} \int_{\Gamma_{\mathsf{na}} \setminus \Gamma'_{\mathsf{na}}} \log \|\gamma\|_{\mathsf{na}} \, d\mu^n_{\mathsf{na}}(\gamma) \leq \varepsilon \tag{4.10}$$

where $\Gamma'_{\mathsf{na}} := \rho_{\mathsf{na}}(G')$. To bound the quantity $\Delta(t)$ we split the integrals according to the decomposition $G = G' \cup G \setminus G'$. If $\Delta'(t)$ denotes the contribution coming from G', using (4.10) and Lemma 4.15, we get

$$\begin{aligned}\Delta(t) &\leq \Delta'(t) + 2\int_{\Gamma_{\mathsf{na}}\setminus\Gamma'_{\mathsf{na}}} \frac{1}{n}\log\|\gamma\|_{\mathsf{na}}\, d\mu^n_{\mathsf{na}}(\gamma) + \frac{C}{\log|t|^{-1}}\int_{G\setminus G'} \mathrm{length}(g) dm^n(g)\\ &\leq \Delta'(t) + (C+2)\varepsilon\ ,\end{aligned}$$

From (4.9) we have that

$$\begin{aligned}\limsup_{t\to 0}\Delta'(t) &= \limsup_{t\to 0}\left|\frac{1}{n}\int_{Y_t\times\Gamma'_t}\frac{\sigma(\gamma_t,y)}{\log|t|^{-1}}d\nu_t(y)d\mu^n_t(\gamma) - \int_{\Gamma'_{\mathsf{na}}}\frac{1}{n}\log\|\gamma\|_{\mathsf{na}}\, d\mu^n_{\mathsf{na}}(\gamma)\right|\\ &\leq \frac{1}{n}\int_{G'}\limsup_{t\to 0}\left|\int_{Y_t}\frac{\sigma(\rho_t(g),y)}{\log|t|^{-1}}d\nu_t(y) - \log\|\rho_{\mathsf{na}}(g)\|\right| dm^n(g)\\ &\leq \frac{2c(Y)}{n} + \frac{4\varepsilon}{n}\int_{\Gamma'}\log\|\gamma\|_{\mathsf{na}}\, d\mu^n_{\mathsf{na}}(\gamma) \leq \varepsilon + 4\varepsilon(\chi_{\mathsf{na}}+\varepsilon)\ ,\end{aligned}$$

where in the second line we use $\mu_t = (\rho_t)_* m$ and $\mu_{\mathsf{na}} = (\rho_{\mathsf{na}})_* m$. Finally we conclude that

$$\limsup_{t\to 0}\left|\frac{1}{\log|t|^{-1}}\chi(t) - \chi_{\mathsf{na}}\right| \leq \varepsilon + (C+2)\varepsilon + \varepsilon + 4\varepsilon(\chi_{\mathsf{na}}+\varepsilon)$$

Since this estimate makes no reference to the model Y and ε is arbitrary, the theorem follows.

Proof of Proposition 4.16. We first need a version of Proposition 2.4 for $\mathrm{SL}(2,\mathbb{M})$. Indeed since $\mathbb{M}$ is neither a field, nor complete this proposition cannot be applied directly. Let us explain how to adapt the argument to this concrete setting. We first introduce some notation: for $r > 0$ denote by $\mathcal{O}_r$ (resp. $\mathbb{M}_r$) the ring of holomorphic functions in $D(0,r)$ (resp. of holomorphic functions in $D(0,r)\setminus 0$ admitting a meromorphic extension at the origin).

Lemma 4.17. *For every $\gamma \in \mathrm{SL}(2,\mathbb{M})$ there exist $r > 0$, $m, n \in \mathrm{SL}(2,\mathcal{O}_r)$ and $a \in \mathrm{SL}(2,\mathbb{M}_r)$ diagonal such that $\gamma = m\cdot a\cdot n$ and $\|a\|_{\mathsf{na}} = \|\gamma\|_{\mathsf{na}}$.*

Proof. Observe first that a meromorphic family of matrices (γ_t) in $\mathrm{SL}(2,\mathbb{M})$ extends holomorphically at the origin if and only if for any triple $\{a,b,c\}$ of distinct points in $\mathbb{P}^1(\mathbb{C})$, then as $t\to 0$, there exists distinct a', b', c' such that $\gamma_t(a)\to a'$, $\gamma_t(b)\to b'$, and $\gamma_t(c)\to c'$.

Let now $\gamma\in\mathrm{SL}(2,\mathbb{M})$ and assume that $\gamma\notin\mathrm{SL}(2,\mathcal{O})$. Then on $X = \mathbb{D}\times\mathbb{P}^1_{\mathbb{C}}$, γ contracts $X_0\setminus\{\mathrm{rep}(\gamma_X)\}$ to $\{\mathrm{att}(\gamma_X)\}$. Pick $m\in\mathrm{SL}(2,\mathcal{O})$ such that $m^{-1}(\mathrm{att}(\gamma_X)) = \infty$ and $n\in\mathrm{SL}(2,\mathcal{O})$ such that $n(\mathrm{rep}(\gamma_X)) = \infty$. Then $\gamma' = m^{-1}\gamma n^{-1}$ maps $X_0\setminus\{0\}$ to ∞, which implies that for small t, γ'_t is loxodromic with an attracting fixed point $\mathrm{att}(\gamma_t)$ close to ∞ and a repelling fixed point $\mathrm{rep}(\gamma_t)$ close to 0. Thus there exists $r>0$ and $h\in\mathrm{SL}(2,\mathcal{O}_r)$ such that $h_t(\mathrm{att}(\gamma_t)) = \infty$, $h_t(\mathrm{rep}(\gamma_t)) = 0$ and $h_t(1) = 1$. Then $a = h\gamma' h^{-1}$ fixes 0 and ∞, so it is diagonal.

By the first observation, $t\mapsto h_t$ extends holomorphically at the origin, that is $h\in\mathrm{SL}(2,\mathcal{O}_r)$. So the desired decomposition is $\gamma = (h^{-1}m)a(nh)$. The equality $\|a\|_{\mathsf{na}} = \|\gamma\|_{\mathsf{na}}$ follows easily. ☐

We are now ready to prove Proposition 4.16. We start by working on X. Pick a sequence of points (x_{t_j}) converging to the central fiber, and consider the quantities $\sigma(\gamma_{t_j}, x_{t_j}) = \left\|\gamma_{t_j} X_{t_j}\right\|$. Extract so that (x_{t_j}) converges and drop the index j for notational simplicity. If $m\in\mathrm{SL}(2,\mathcal{O})$

then for every $Z \in \mathcal{O}^2$, $\|m_t Z_t\| \asymp \|Z_t\|$ so by the previous lemma we can assume that γ is diagonal, $\gamma_t = \mathrm{diag}(\lambda_t, \lambda_t^{-1})$. If (x_t) does not converge to $[0:1]$, then $\|\gamma_t X_t\| \asymp \|\gamma_t\|$ so the desired estimate holds, therefore the interesting case is when (x_t) converges to $[0:1]$. In this case a lift of norm 1 of x_t will be of the form $X_t = (\xi_t, \eta_t)$ with $|\eta_t| = 1$, so $\sigma(\gamma_t, x_t) = \max(|\lambda_t \xi_t|, |\lambda_t|^{-1})$. From this formula we infer that if for some $l > 0$, $|\xi_t| \geq |t|^l$ when $t \to 0$ then

$$\liminf_{t\to 0} \frac{1}{\log |t|^{-1}} \sigma(\gamma_t, x_t) \geq |\lambda|_{\mathsf{na}} - l = \|\gamma\|_{\mathsf{na}} - l.$$

We rely on the following elementary geometric fact.

Lemma 4.18. *Let $\pi : M \to \mathbb{D}^2$ be a composition of N blow-ups above the vertical fiber $\{0\}\times\mathbb{D}$ in the unit bidisk, and denote $M_0 = \pi^{-1}(\{0\} \times \mathbb{D})$. Then if $\ell > N$, the open set*

$$\pi^{-1}\left(\left\{(t, x) \in \mathbb{D}^* \times \mathbb{D},\ |x| < |t|^{\ell}\right\}\right)$$

clusters at a unique point of M_0.

Proof. The open set $\left\{(t, x) \in \mathbb{D}^* \times \mathbb{D},\ |x| < |t|^{\ell}\right\}$ is the union of the curves $\left\{x = ct^{\ell}\right\}$ in $\mathbb{D}^* \times \mathbb{D}$, where c ranges over $|c| < 1$. These curves get separated after exactly ℓ blow-ups. □

In order to conclude the proof, pick a model $\pi : Y \to X$ and a sequence (y_{t_j}) as in the statement of the proposition. Extract so that (y_{t_j}) converges. Let N be the number of blow-ups required to obtain Y. We put $x_{t_j} = \pi(y_{t_j})$ and do the analysis of the first part of the proof. Then, Lemma 4.18 applied to $l = N+1$ provides a point $\alpha = \alpha(\gamma)$ in the central fiber Y_0 such that if (y_{t_j}) does not converge to α, then

$$\liminf_{j\to\infty} \frac{1}{\log |t_j|^{-1}} \sigma(\gamma_{t_j}, y_{t_j}) \geq \|\gamma\|_{\mathsf{na}} - (N+1).$$

The result follows. □

5. Degenerations: elementary representations

In this section we complete the proof of Theorem A by addressing the case of elementary representations. Let as before G be a finitely generated group endowed with some probability measure m satisfying

(A2$^+$) there exists $\delta > 0$ such that $\int (\mathrm{length}(g))^{1+\delta} dm(g) < \infty$,

and let $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ be any meromorphic family of representations.

With notation as in §4.4, Viewing $\mathbb{M}$ as a subring of $\mathbb{L}$ we denote by ρ_{na} the corresponding non-Archimedean representation $G \to \mathrm{SL}(2, \mathbb{L})$ and $\mu_{\mathsf{na}} = (\rho_{\mathsf{na}})_* m$, which satisfies the moment condition

(B2$^+$) there exists $\delta > 0$ such that $\int \log \|\gamma\|^{1+\delta} d\mu(\gamma_{\mathsf{na}}) < \infty$

in $\mathrm{SL}(2, \mathbb{L})$. In particular the non-Archimedean Lyapunov exponent $\chi_{\mathsf{na}} = \chi(\mu_{\mathsf{na}})$ is well-defined. Likewise for $t \in \mathbb{D}^*$ we let $\mu_t = (\rho_t)_* \mu$ and $\chi(t) = \chi(\rho_t(G), \mu_t)$.

Theorem 5.1. *Let (G, m) be a finitely generated group endowed with a probability measure satisfying (A2$^+$), and let $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ be such that $\rho_{\mathsf{na}}(G) \subset \mathrm{SL}(2, \mathbb{L})$ is elementary. Then*

$$\chi(t) = \left(\log |t|^{-1}\right) \chi_{\mathsf{na}} + O(1) \ \text{as}\ t \to 0. \tag{5.1}$$

If in addition μ is symmetric, then $\chi_{\mathsf{na}} = 0$.

Under mild assumptions, the error term can be understood more precisely, see §5.4

Put $\Gamma_{\mathsf{na}} = \rho_{\mathsf{na}}(G)$. According to the discussion in §2.4, if $\Gamma_{\mathsf{na}} \leq \mathrm{SL}(2, \mathbb{L})$ is elementary then it is either non-proximal or non-strongly irreducible, so there are three possibilities:

(1) Γ_{na} has potential good reduction;
(2) Γ_{na} is conjugate to a subgroup of the affine group $\{z \mapsto az + b, a \in \mathbb{L}^\times, b \in \mathbb{L}\}$;
(3) Γ_{na} is conjugate to a subgroup of the group of transformations fixing $\{0, \infty\}$, that is, $\left\{z \mapsto \lambda z^{\pm 1}, \lambda \in \mathbb{L}^\times\right\}$.

Note that if we are not in case (1), then the projection of Γ_{na} in $\mathrm{PGL}(2, \mathbb{L})$ is not purely elliptic, and it follows from the analysis of §2.4 that the conjugacy in (2) and (3) lies in $\mathrm{SL}(2, \mathbb{L})$ (i.e. no field extension is required).

In the remaining part of this section we split the proof of Theorem 5.1 according to these three cases.

5.1. **Potential good reduction.** In case (1), Γ_{na} is conjugate in $\mathrm{SL}(2, \mathbb{C}(\!(t^{1/2})\!))$ to a representation fixing the Gauß point. Lifting to a branched 2-cover (which amounts to making the change of variables $t = u^2$), we can assume that the conjugacy lies in $\mathrm{SL}(2, \mathbb{L})$, that is there exists $\alpha \in \mathrm{SL}(2, \mathbb{L})$ such that for every $\gamma \in \Gamma_{\mathsf{na}}$, $\left\|\alpha^{-1}\gamma\alpha\right\| \leq 1$.

Observe first that $\mathbb{M}$ is dense in $\mathbb{L}$ so that there exists a sequence $\alpha_n \in \mathrm{SL}(2, \mathbb{M})$ such that $\|\alpha - \alpha_n\| \to 0$. From the continuity of the matrix product, and the ultrametric property, for any sufficiently large integer n we get $\left\|\alpha_n^{-1}\rho_{\mathsf{na}}(s)\alpha_n\right\| \leq 1$ for all s in a fixed finite set of generators of G. In particular, we have $\left\|\alpha_n^{-1}\rho_{\mathsf{na}}(g)\alpha_n\right\| \leq 1$ for all $g \in G$ so that we may suppose that our original conjugacy α belongs to $\mathrm{SL}(2, \mathbb{M})$.

Since the Lyapunov exponent is insensitive to conjugacy, by replacing ρ by $\alpha^{-1}\rho(\cdot)\alpha$ we can assume that ρ extends holomorphically at the origin. For every $t \neq 0$, by sub-additivity we have the bound $0 \leq \chi(t) \leq \int \log \|\rho_t(g)\|\, dm(g)$. Therefore applying Lemma 4.2 and the moment condition we infer that $\chi(t) = O(1)$ as $t \to 0$. On the other hand, since Γ_{na} has good reduction, χ_{na} vanishes, and we are done.

5.2. **Affine representations.** Let $(k, |\cdot|)$ be any complete valued field and consider any subgroup Γ of $\mathrm{SL}(2, k)$ endowed with a measure μ, such that the projection of Γ in $\mathrm{PGL}(2, k)$ lies in the affine group $\mathrm{Aff}(k)$. An element $\gamma \in \Gamma$ can be written in matrix form as

$$\gamma = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} , \tag{5.2}$$

corresponding to the Möbius transformation $\gamma(z) = az + b$, with $a = \alpha^2$ and $b = \beta\alpha$. Thus its norm is

$$\|\gamma\| = \max\left(|\alpha|, |\beta|, \left|\alpha^{-1}\right|\right) = \max\left(|a|^{1/2}, |a|^{-1/2}, \left|ba^{-1/2}\right|\right) . \tag{5.3}$$

Proposition 5.2. *Let $(k, |\cdot|)$ be a complete valued field and let μ be a measure with countable support in $\mathrm{SL}(2, k)$, contained in the affine group, and satisfying (B2$^+$). Then with notation as above we have*

$$\chi(\mu) = \left|\int \log|\alpha(\gamma)|\, d\mu(\gamma)\right| = \frac{1}{2}\left|\int \log|a(\gamma)|\, d\mu(\gamma)\right| \tag{5.4}$$

In particular if μ is symmetric, $\chi(\mu) = 0$.

Proof. For any $\omega = (\gamma_n) \in \Omega$, we write $\gamma_n(z) = a_n(\omega)z + b_n(\omega)$ so that

$$\ell_n(\omega) = \gamma_n \cdots \gamma_1(z) = A_n(\omega)z + B_n(\omega) = a_n \cdots a_1 z + \sum_{j=1}^{n-1} a_n \cdots a_{j+1} b_j + b_n \ .$$

By the law of large numbers (or equivalently the Birkhoff ergodic theorem) we have that

$$\frac{1}{n} \log |A_n| \to \lambda := \int \log |a| d\mu \quad \text{a.s.} \tag{5.5}$$

Fix $\varepsilon > 0$. For a.e. ω, $e^{n(\lambda-\varepsilon)} \leq |A_n(\omega)| \leq e^{n(\lambda+\varepsilon)}$ for large n. The moment condition (B2$^+$) and Chebyshev's inequality yield $\mu\{|b| > e^{\varepsilon j}\} \leq C j^{-1-\delta}$, so that by the Borel-Cantelli lemma we get that $b_n(\omega) \leq e^{\varepsilon n}$ a.s. for large n.

At this point we split the proof into two cases according to the sign of $\lambda = \int \log |a| d\mu$. Write

$$\gamma_n \circ \cdots \circ \gamma_1(z) = a_1 \cdots a_n \left(z + \sum_{j=1}^{n} \frac{b_j}{a_1 \cdots a_j} \right) = A_n \left(z + \sum_{j=1}^{n} b_j A_j^{-1} \right).$$

If $\lambda > 0$ we infer from (5.5) that a.s. the partial sums of the series $\sum_{j \geq 1} b_j A_j^{-1}$ are bounded, from which it follows that $|B_n| = O(|A_n|)$. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \log |A_n| = \frac{\lambda}{2} \ \text{ and } \ \limsup_{n\to\infty} \left(\frac{1}{2n} \log |B_n| - \frac{1}{2n} \log |A_n| \right) \leq \frac{\lambda}{2} \ .$$

By (5.3), we have

$$\frac{1}{n} \log \|\ell_n(\omega)\| = \max \left\{ \frac{1}{2n} \log |A_n|, -\frac{1}{2n} \log |A_n|, \frac{1}{n} \log |B_n| - \frac{1}{2n} \log |A_n| \right\} \ , \tag{5.6}$$

so we conclude that $\chi(\mu) = \lambda/2$.

On the other hand, if $\lambda \leq 0$, then since almost surely for large j, $|b_j| \leq e^{\varepsilon j}$ and $e^{(\lambda-\varepsilon)j} \leq |A_j| \leq e^{(\lambda+\varepsilon)j}$ we deduce that

$$\left| \sum_{j=1}^{n} b_j A_j^{-1} \right| = O\left(e^{(-\lambda+2\varepsilon)n}\right) \ \text{ hence } \ |B_n| = O\left(e^{3\varepsilon n}\right).$$

Thus from (5.6) we get that for large n,

$$-\frac{\lambda}{2} - \varepsilon \leq \frac{1}{n} \log \|\ell_n(\omega)\| \leq -\frac{\lambda}{2} + 4\varepsilon \ ,$$

and $\chi(\mu) = -\lambda/2$, as required. □

Proof of Theorem 5.1 in the affine case. Under the assumptions of the theorem, assume that the projection of Γ_{na} into $\mathrm{PGL}(2, \mathbb{L})$ lies in the affine group. For any $g \in G$, we use the same notation as above, writing $\alpha(\rho(g)) \in \mathbb{M}$ for the upper diagonal term of $\rho(g)$, and $\beta(\rho(g)) \in \mathbb{M}$ for its upper right term. We get corresponding coefficients $\alpha(\rho_t(g)) \in \mathbb{C}^*$ (for fixed $t \neq 0$) and $\alpha(\rho_{\mathsf{na}}(g)) \in \mathbb{L}^*$.

By Lemma 4.2 we have that $\alpha(\rho_t(g)) = t^{-\log|\alpha(\rho_{\mathsf{na}}(g))|} \widetilde{\alpha}(\rho_t(g))$, where $t \mapsto \widetilde{\alpha}(\rho_t(g))$ is holomorphic in $\mathbb{D}$, and $\left| \log \|\widetilde{\alpha}\|_{L^\infty(\overline{\mathbb{D}}(0,1/2))} \right| \leq C(\rho) \,\mathrm{length}(g)$. Hence

$$\int \log |\alpha(\rho_t(g))| \, dm(g) = (\log |t|^{-1}) \int \log |\alpha(\rho_{\mathsf{na}}(g))| \ dm + \mathcal{E}(t) \tag{5.7}$$

where

$$|\mathcal{E}(t)| = \left| \int \log |\widetilde{\alpha}(\rho_t(g))| \; dm \right| \leq C \int \mathrm{length}(g) dm < +\infty \; . \tag{5.8}$$

Therefore applying the formula of Proposition 5.2 to $k = \mathbb{C}$ and $k = \mathbb{L}$ we infer the desired estimate (5.1) in the affine case. □

5.3. **Representations fixing** $\{0, \infty\}$**.** Back to the general setting, consider now a subgroup $\Gamma \leq \mathrm{SL}(2, k)$ whose projection in $\mathrm{PGL}(2, k)$ fixes $\{0, \infty\}$. Then every matrix in Γ is of the form

$$\text{either } \gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ or } \gamma = \begin{pmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix} ,$$

and $\|\gamma\| = \max\{|\alpha|, |\alpha|^{-1}\}$. As Möbius transformations, we have $\gamma(z) = az$ or $-a/z$ with $a = \alpha^2$, and $\|\gamma\| = \max\{|a|^{1/2}, |a|^{-1/2}\}$.

Proposition 5.3. *Let $(k, |\cdot|)$ be a complete valued field and let μ be a measure with countable support in* $\mathrm{SL}(2, k)$*, satisfying the moment condition (B2). Suppose that any element in the support of μ leaves the pair $\{0, \infty\}$ invariant and at least one element permutes 0 and ∞.*

Then $\chi(\mu) = 0$.

The last case of Theorem 5.1 immediately follows, since in this case we have that $\chi(t) \equiv 0 = \chi_{\mathsf{na}}$.

Proof. It is more convenient here to use probabilistic language. We denote by $\mathsf{E}(\cdot)$ the expectation of a random variable.

In terms of Möbius transformations, we are considering a random composition of maps of the form $\gamma_j(z) = \lambda_j z^{\varepsilon_j}$ where $\lambda_j \in k^\times$ and $\varepsilon_j \in \{\pm 1\}$ are iid random variables.

Write $\ell_n(\omega) = (\gamma_n \circ \cdots \circ \gamma_1) = \Lambda_n z^{\mathcal{E}_n}$, and let $X_n = \log |\Lambda_n|$ and $x_n = \log |\lambda_n|$. A simple computation shows that $\mathcal{E}_n = \prod_{i=1}^n \varepsilon_i$ and

$$X_n = x_n + \varepsilon_n x_{n-1} + \varepsilon_n \varepsilon_{n-1} x_{n-2} + \ldots + \varepsilon_n \cdots \varepsilon_1 x_1 \tag{5.9}$$

Note that (x_n) is a sequence of iid real random variables with $\mathsf{E}(|x_1|) < \infty$. Kingman's theorem implies that the sequence (X_n/n) converges a.s. We have to show that its limit is 0.

Let $(n_l)_{l \geq 0}$ be the increasing sequence of random times where $\varepsilon_{n_l} = -1$, that is, (n_l) is defined by $n_0 = 0$ and $n_{l+1} = \min\{j > n_l, \; \varepsilon_j = -1\}$. Since the ε_n are iid and μ does not give full mass to the affine group, $(n_{l+1} - n_l)_{l \geq 0}$ is a sequence of iid random variables with a geometric distribution of non-zero parameter $p > 0$ (which is the probability that γ is not affine).

For $q \geq 1$ put $Y_q = \sum_{j=n_{q-1}}^{n_q - 1} x_j$ with the convention that $x_0 = 0$. Observe that (Y_q) forms a sequence of iid random variables with finite first moment, and such that

$$\mathsf{E}(Y_1) = \mathsf{E}(x_1) + \sum_{j=1}^{\infty} j(1-p)^j p \mathsf{E}(x_1) = \frac{1}{p} \mathsf{E}(x_1) \; .$$

It follows from (5.9) that for every $l \geq 0$, $X_{n_l - 1} = \sum_{j=1}^{l} (-1)^{l-j} Y_j$.

Finally, let $(Z_l)_{l \geq 1} = (Y_{2l-1} - Y_{2l})_{l \geq 1}$ which is a sequence of iid random variables with $\mathsf{E}(|Z_1|) < \infty$ and $\mathsf{E}(Z_1) = 0$. Up to sign we have that

$$X_{n_{2l} - 1} = \pm \sum_{j=1}^{l} Z_j,$$

thus from the strong law of large numbers we infer that $\frac{1}{l}X_{n_{2l}-1} \to 0$ a.s. as $l \to \infty$, hence since $l \leq n_l$ the same holds for $\frac{1}{n_{2l}-1}X_{n_{2l}-1}$. The proof is complete. □

5.4. **Continuity of the error term.** If m is finitely supported, the proof of Theorem 5.1 actually yields a finer estimate in (5.1) of the form

$$\chi(t) = \left(\log |t|^{-1}\right)\chi_{\mathsf{na}} + C + o(1) \text{ as } t \to 0.$$

Indeed:

- if Γ_{na} has potential good reduction, the proof reduces the situation to that of a holomorphic family of representations, in which case the result follows from the Furstenberg theory when ρ_0 is non elementary in $\mathrm{SL}(2,\mathbb{C})$ and from Bocker-Viana [BV] when ρ_0 is elementary (the finiteness assumption on m is used here);
- if Γ_{na} is affine we have to show that the error $\mathcal{E}(t)$ in (5.7) admits a limit when $t \to 0$, which by virtue of (5.8) and Lemma 4.2 follows from the dominated convergence theorem;
- finally in the case of representations fixing $\{0, \infty\}$ there is nothing to prove because $\chi(t) \equiv 0$.

6. Degenerations: the hybrid approach

We propose an alternative approach to the analysis of the blow-up of the Lyapunov exponent, which is based on the hybrid space constructed by Berkovich and used by Boucksom and Jonsson in [BJ] and by the first author in [Fav]. The introduction of this space allows us to make sense of the convergence of measures $\nu_t \to \nu_{\mathsf{na}}$ and leads to a proof of Theorem C.

6.1. **The hybrid space.** We start by briefly recalling the definition the hybrid space, referring to [BJ, Fav] for more details.

Let $\mathcal{A}$ be the subring of $\mathbb{L}$ consisting of those series f such that $\|f\|_{\mathrm{hyb}} < +\infty$, where

$$\|f\|_{\mathrm{hyb}} := \sum_{n=-\infty}^{+\infty} |a_n|_{\mathrm{hyb}}\, e^{-n} \ , \text{ and } \begin{cases} |a|_{\mathrm{hyb}} = \max\{|a|, 1\} \text{ if } a \in \mathbb{C}^* \\ |0|_{\mathrm{hyb}} = 0 \end{cases} .$$

Observe that for any $f \in \mathcal{A}$, the sum has only finitely many negative terms and the series defining f converges in $\overline{\mathbb{D}}^*_{1/e}$. Endowed with the hybrid norm $\|\cdot\|_{\mathrm{hyb}}$, $\mathcal{A}$ is a Banach ring, and its Berkovich spectrum $\mathcal{D} := M_{\mathrm{ber}}(\mathcal{A})$ is defined as usual to be the space of multiplicative semi-norms $|\cdot|$ on $\mathcal{A}$ such that $|\cdot| \leq \|\cdot\|_{\mathrm{hyb}}$, endowed with the topology of pointwise convergence.

It turns out that $\mathcal{D}$ is naturally a closed disk. To see this, introduce the map τ from the closed disk of radius $1/e$ to $\mathcal{D}$ by the formula:

$$\begin{cases} |f(\tau(0))| = e^{-\operatorname{ord}_{t=0}(f)}; \\ |f(\tau(t))| = |f(t)|^{\frac{-1}{\log|t|}} \text{ if } 0 < |t| \leq 1/e. \end{cases} \tag{6.1}$$

for any $f \in \mathcal{A}$. One can show that this map is a homeomorphism, see e.g. [Fav, Prop. 1.1]. A note on terminology: an element $x \in \mathcal{D}$ is a non-negative real valued function on $\mathcal{A}$, nevertheless as already said it is customary to write $f \mapsto |f(x)| = |f|_x \in \mathbb{R}_+$ for the evaluation map.

The hybrid affine line $\mathbb{A}^1_{\mathrm{hyb}} := M_{\mathrm{ber}}(\mathcal{A}[Z])$ is by definition the set of multiplicative semi-norms $|\cdot|$ on $\mathcal{A}[Z]$ such that $|\cdot| \leq \|\cdot\|_{\mathrm{hyb}}$ on $\mathcal{A}$. We endow it with the topology of the pointwise convergence which makes it locally compact. The restriction to $\mathscr{A}$ of any semi-norm x is a point $p_{\rm hyb}(x) \in \mathscr{D}$, and the projection $p_{\rm hyb} : \mathbb{A}^1_{\rm hyb} \to \mathscr{D}$ is a continuous surjective map. Given $x \in \mathbb{A}^1_{\rm hyb}$, according to the value (zero or non-zero) of $\tau^{-1} \circ p_{\rm hyb}(x)$, the semi-norm x will carry non-Archimedean or Archimedean information.

It follows from the Gelfand-Mazur theorem (see e.g. the proof of [Fav, Prop. 1.1]) that $M_{\rm ber}(\mathbb{C}[Z]) \simeq \mathbb{C}$, so if $t \neq 0$, the fiber $p^{-1}_{\rm hyb}(\tau(t))$ is homeomorphic to $\mathbb{C}$. Furthermore, there exists a unique homeomorphism $\tilde{\psi} : \overline{\mathbb{D}}^*_{1/e} \times \mathbb{C} \to p^{-1}_{\rm hyb}(\tau(\overline{\mathbb{D}}^*_{1/e}))$ satisfying

$$|g(\tilde{\psi}(t,z))| = |g(t,z)|^{\frac{-1}{\log|t|}} \tag{6.2}$$

for any $(t,z) \in \overline{\mathbb{D}}^*_{1/e} \times \mathbb{C}$ and any $g \in \mathscr{A}[Z]$. By construction, we have $p_{\rm hyb} \circ \tilde{\psi} = \tau \circ \pi$, where $\pi : \overline{\mathbb{D}}^*_{1/e} \times \mathbb{C} \to \overline{\mathbb{D}}^*_{1/e}$ is the first projection. On the other hand, any semi-norm on $\mathbb{L}[Z]$ can be restricted to $\mathscr{A}[Z]$ which yields a canonical map $\tilde{\psi}_{\sf na} : \mathbb{A}^{1,\rm an}_{\mathbb{L}} \to p^{-1}_{\rm hyb}(\tau(0))$. This map is a homeomorphism since the completion of $\mathscr{A}$ with respect to the t-adic norm is the field $\mathbb{L}$.

The hybrid space $(\mathbb{A}^1)^*_{\rm hyb}$ associated to the punctured affine line $(\mathbb{A}^1)^*$ is the Berkovich spectrum of $\mathscr{A}[Z, Z^{-1}]$ and can be identified with an open subset of $\mathbb{A}^1_{\rm hyb}$ whose complement is the set $g \mapsto |g(0)|$ with $|\cdot| \in \mathscr{D}$. The latter set is the closure in $\mathbb{A}^1_{\rm hyb}$ of $\psi\left(\overline{\mathbb{D}}^*_{1/e} \times \{0\}\right)$.

The hybrid projective line is constructed as the union of two copies of $\mathbb{A}^1_{\rm hyb}$ patched in the usual way. Specifically, the natural inclusions $\mathscr{A}[z_1] \to \mathscr{A}[Z, Z^{-1}]$ and $\mathscr{A}[z_2] \to \mathscr{A}[Z, Z^{-1}]$ sending z_1 to Z, and z_2 to Z^{-1} yield two open embeddings $\imath_1, \imath_2 \colon (\mathbb{A}^1)^*_{\rm hyb} \to \mathbb{A}^1_{\rm hyb}$, and $\mathbb{P}^1_{\rm hyb}$ is defined to be the union of $U_1 := M_{\rm ber}(\mathscr{A}[z_1])$ and $U_2 := M_{\rm ber}(\mathscr{A}[z_2])$ glued together using the identification $\imath_1(x) = \imath_2(x)$ for any $x \in (\mathbb{A}^1)^*_{\rm hyb}$.

The inclusion $U_1 \subset \mathbb{P}^1_{\rm hyb}$ yields an open and dense embedding of $\mathbb{A}^1_{\rm hyb}$ into $\mathbb{P}^1_{\rm hyb}$, and the following proposition holds.

Proposition 6.1 ([Fav])**.** *The hybrid space $\mathbb{P}^1_{\rm hyb}$ is compact, and there exists a homeomorphism $\psi : \overline{\mathbb{D}}^*_{1/e} \times \mathbb{P}^1_{\mathbb{C}} \to p^{-1}_{\rm hyb}(\tau(\overline{\mathbb{D}}^*_{1/e}))$ whose restriction to $\overline{\mathbb{D}}^*_{1/e} \times \mathbb{C}$ is equal to $\tilde{\psi}$. Likewise, there is a canonical homeomorphism $\psi_{\sf na} : \mathbb{P}^{1,\rm an}_{\mathbb{L}} \to p^{-1}_{\rm hyb}(\tau(0))$ whose restriction to $\mathbb{A}^{1,\rm an}_{\mathbb{L}}$ is equal to $\tilde{\psi}_{\sf na}$.*

Remark 6.2. In other words, there exists a topology on the disjoint union $\left(\overline{\mathbb{D}}^*_{1/e} \times \mathbb{P}^1_{\mathbb{C}}\right) \bigsqcup \mathbb{P}^{1,\rm an}_{\mathbb{L}}$ such that the map defined by ψ on $\overline{\mathbb{D}}^*_{1/e} \times \mathbb{P}^1$ and $\psi_{\sf na}$ on $\mathbb{P}^{1,\rm an}_{\mathbb{L}}$ is a homeomorphism onto $\mathbb{P}^1_{\rm hyb}$.

The group $\mathrm{SL}(2,\mathbb{M})$ is contained in $\mathrm{SL}(2,\mathbb{L})$ so it admits a natural action on $\mathbb{P}^{1,\rm an}_{\mathbb{L}}$ preserving the analytic structure on this space. It also acts by biholomorphisms on $\mathbb{D}^* \times \mathbb{P}^1_{\mathbb{C}}$ commuting with the second projection. The next proposition shows that these two actions fit together nicely in the hybrid space. To ease notation we write $\psi_t(z) = \psi(z,t)$.

Proposition 6.3. *The group $\mathrm{SL}(2,\mathbb{M})$ admits a unique action by homeomorphisms on the hybrid space $\mathbb{P}^1_{\rm hyb}$ which is compatible with its natural action on $\mathbb{P}^{1,\rm an}_{\mathbb{L}}$, and such that*

$$\psi_t(\gamma_t \cdot z) = \gamma_t \cdot \psi_t(z) \tag{6.3}$$

for any $\gamma \in \mathrm{SL}(2,\mathbb{M})$, and any $(t,z) \in \overline{\mathbb{D}}^*_{1/e} \times \mathbb{P}^1_{\mathbb{C}}$. In particular, for all $\gamma \in \mathrm{SL}(2,\mathbb{M})$ and $x \in \mathbb{P}^1_{\rm hyb}$, we have $p_{\rm hyb}(\gamma \cdot x) = p_{\rm hyb}(x)$.

Proof. We define an action of $\mathrm{SL}(2,\mathbb{M})$ on $\mathbb{P}^1_{\mathrm{hyb}}$ by setting $\gamma\cdot x=\gamma_{\mathsf{na}}\cdot x$ when $x\in\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$, and such that (6.3) holds true. It is only necessary to check that this action is continuous which will follow from the very definition of the hybrid space.

Recall that $\mathbb{P}^1_{\mathrm{hyb}}$ is the union of two copies U_1 and U_2 of $\mathbb{A}^1_{\mathrm{hyb}}$. We pick $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{M})$ and look at the action of γ in the first chart in $U_1=M_{\mathrm{ber}}(\mathscr{A}[z])$. Observe that for any $f\in\mathscr{A}[z]$, $f(\frac{az+b}{cz+d})$ is the quotient of some $\tilde{f}\in\mathscr{A}[z]$ by a polynomial of the form $(cz+d)^N\in\mathbb{M}[z]$ for some integer N. It follows that

$$|f(\gamma\cdot x)|=\left|f\left(\frac{az+b}{cz+d}\right)(x)\right|$$

depends continuously on x on the open set $U:=\{x\in U_1,\ |(cz+d)|_x\neq 0\}$, so that γ defines a continuous map from U to U_1.

If now $f\in\mathscr{A}[z^{-1}]$, then $f(\frac{az+b}{cz+d})$ is the quotient of an element $g\in\mathscr{A}[z]$ by a polynomial of the form $(az+b)^N\in\mathbb{M}[z]$ for some N, and we conclude similarly that γ defines a continuous map from $U':=\{x\in U_1,\ |(az+b)|_x\neq 0\}$ to U_2.

Since $ad-bc=1$, the two open sets U and U' cover U_1, which completes the proof. □

Recall from (2.1) the definition of the cocycle σ.

Proposition 6.4. *For any $\gamma\in\mathrm{SL}(2,\mathbb{M})$, the function defined by*

$$\sigma_{\mathrm{hyb}}(\gamma,x):=\begin{cases}\sigma(\gamma_{\mathsf{na}},\psi_{\mathsf{na}}^{-1}(x)) & \text{if } x\in\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}\\ \frac{\sigma(\gamma_t,z)}{\log|t|^{-1}} & \text{if } p_{\mathrm{hyb}}(x)\neq 0 \text{ and } \psi^{-1}(x)=(z,t)\end{cases}$$

is continuous on $\mathbb{P}^1_{\mathrm{hyb}}$.

Proof. It is enough to check that the restriction of $\sigma_{\mathrm{hyb}}(\gamma,\cdot)$ to one of the two defining charts of the hybrid projective line is continuous, say on $U=M_{\mathrm{ber}}(\mathscr{A}[Z])$. Since $\sigma_{\mathrm{hyb}}(g,\cdot)$ is continuous on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ and on $\psi(\overline{\mathbb{D}}^*_{1/e}\times\mathbb{P}^1_{\mathbb{C}})$ separately , we only need to check that $\sigma_{\mathrm{hyb}}(g,x_i)\to\sigma_{\mathrm{hyb}}(g,x)$ for any net of points $x_i=\psi(t_i,z_i)$, $(t_i,z_i)\in\mathbb{D}^*_{1/e}\times\mathbb{C}$ indexed by some inductive set I and such that $x_i\to x\in\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ along I. Note that this implies $t_i\to 0$, and $|g(x_i)|\to|g(x)|$ for any $g\in\mathscr{A}[Z]$, i.e.

$$|g_{t_i}(z_i)|^{\frac{-1}{\log|t_i|}}\to|g(x)|\ .$$

By switching the chart we are working in and extracting a subfamily if necessary, we may also suppose that $|Z(x)|\leq 1$ and $|z_i|\leq 1$ for all $I\in I$. Let a,b,c and $d\in\mathbb{M}$ be the coefficients of g. By definition, we have

$$\sigma_{\mathrm{hyb}}(g,x_i)=\frac{\log\max\{|a(t_i)z_i+b(t_i)|,|c(t_i)z_i+d(t_i)|\}}{\log|t_i|^{-1}}\ .$$

On the other hand for any $h\in\mathscr{A}[Z]$ we have $\lim_i|h(x_i)|=|h(x)|$. Since $\mathbb{M}\subset\mathscr{A}$, (6.2) yields

$$\lim_i|(aZ+b)(x_i)|=\lim_i|a(t_i)(z_i)+b(t_i)|^{\frac{-1}{\log|t_i|}}=|(aZ+b)(x)|$$

which implies $\sigma_{\mathrm{hyb}}(g,x_i)\to\sigma_{\mathrm{hyb}}(g,x)$ as required. □

6.2. **Convergence of measures in the hybrid space: proof of Theorem C.** Consider a representation $\rho : G \to \mathrm{SL}(2, \mathbb{M})$ such that ρ_{na} is non-elementary, and m a measure on G satisfying (A1).

By Theorem 3.1, we may consider the unique stationary measure ν_{na} associated to ρ_{na}: this is a probability measure supported in $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} = p_{\mathrm{hyb}}^{-1}(\tau(0)) \subset \mathbb{P}^1_{\mathrm{hyb}}$. By Lemma 4.3, for any small enough $t \neq 0$ the representation ρ_t is non-elementary, and we denote by ν_t the image in $\{t\} \times \mathbb{P}^1_{\mathbb{C}}$ of the unique stationary measure associated to ρ_t under the natural inclusion $\mathbb{P}^1_{\mathbb{C}} \subset \{t\} \times \mathbb{P}^1_{\mathbb{C}}$. We shall see that any limit point of ν_t is a stationary measure on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ hence equal to ν_{na} so that $\nu_t \to \nu_{\mathsf{na}}$ (here for simplicity we drop the mention to the embeddings ψ and ψ_{na}). Since the hybrid space is not metrizable, some care needs to be taken when arguing in this way, and we thus proceed as follows.

Consider the set of probability measures $M = \{\psi_*(\nu_t),\, 0 < |t| \leq 1/e\}$ in $\mathbb{P}^1_{\mathrm{hyb}}$. Let $\overline{M}$ be the closure of M in the space of probability measures, for the weak-$\star$ topology associated to the hybrid topology. Since $\mathbb{P}^1_{\mathrm{hyb}}$ is compact, so does $\overline{M}$.

Let us prove that $\overline{M} \setminus M = \{(\psi_{\mathsf{na}})_* \nu_{\mathsf{na}}\}$.

We claim that $\overline{M} \setminus M \neq \emptyset$. Inded for any $\delta \in (0, 1/e)$, define $M_\delta = \{\psi_*(\nu_t),\, 0 < |t| \leq \delta\}$. This forms an increasing family of subsets of M. Observe that any measure in M_δ has its support included in $p_{\mathrm{hyb}}^{-1}(\tau(\overline{\mathbb{D}}^*_\delta))$. The intersection $\bigcap_{\delta > 0} \overline{M_\delta}$ is non-empty as an intersection of compact sets, and it is included in $\overline{M} \setminus M$ since any measure in this intersection has its support included in $p_{\mathrm{hyb}}^{-1}(\tau(0))$. This proves our claim. This also proves that any measure in $\overline{M} \setminus M$ is supported on $p_{\mathrm{hyb}}^{-1}(\tau(0)) = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$.

Pick now any measure $\nu \in \overline{M} \setminus M$. Let $\varphi_{\mathsf{na}} : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{R}$ be an arbitrary continuous function. Since $\mathbb{P}^1_{\mathrm{hyb}}$ is compact, it is a normal topological space, so the Tietze-Urysohn extension lemma applies. We can thus find a continuous function $\phi : \mathbb{P}^1_{\mathrm{hyb}} \to \mathbb{R}$ whose restriction to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is equal to $\varphi_{\mathsf{na}} \circ \psi_{\mathsf{na}}^{-1}$. Let us introduce the convolution operator acting on continuous functions of $\mathbb{P}^1_{\mathrm{hyb}}$ by setting

$$m * \phi(x) = \int_G \phi(\rho(g)^{-1} \cdot x) dm(g) \ .$$

Observe that for any $0 < |t| \leq 1/e$, we have

$$\begin{aligned}
\int_{\mathbb{P}^1(\mathbb{C})} (m * \phi) d(\psi_*(\nu_t)) &= \int_{\mathbb{P}^1} \int_G \phi(\rho(g)^{-1} \cdot x) d(\psi_*(\nu_t))\; dm(g) \\
&= \int_{\mathbb{P}^1} \int_G \phi(\rho(g)^{-1} \cdot \psi(z,t)) d\nu_t\; dm(g) \\
&\overset{(6.3)}{=} \int_{\mathbb{P}^1} \int_G (\phi \circ \psi_t)(\rho_t(g)^{-1} \cdot z) d\nu_t \\
&= \int_{\mathbb{P}^1} (\phi \circ \psi_t) d((\rho_t)_* m) * \nu_t) = \int_{\mathbb{P}^1} \phi\, d(\psi_*(\nu_t))
\end{aligned}$$

so by definition of the weak-$\star$ topology we get that $\int (m * \phi) d\nu = \int \phi d\nu$. This implies that $(\rho_{\mathsf{na}*}(m)) * (\psi^*_{\mathsf{na}} \nu) = \psi^*_{\mathsf{na}} \nu$, hence $\nu = (\psi_{\mathsf{na}})_* \nu_{\mathsf{na}}$ since ρ_{na} admits a unique stationary measure.

Finally let us show that $\psi_*(\nu_t) \to (\psi_{\mathsf{na}})_* \nu_{\mathsf{na}}$. We argue by contradiction, and pick $\varepsilon > 0$, a continuous function ϕ on $\mathbb{P}^1_{\mathrm{hyb}}$, a sequence $t_n \to 0$ such that $\int \phi\, d(\psi_*(\nu_{t_n})) \geq \int \phi\, d((\psi_{\mathsf{na}})_* \nu_{\mathsf{na}}) +$ ε. Since ν_{na} belongs to the accumulation set $\bigcap_m \overline{\bigcup_{n \geq m} \{\psi_*(\nu_{t_n})\}}$ of the sequence $\psi_*(\nu_{t_n})$, it follows that the open set $\{\nu, \int \phi\, d\nu < \int \phi\, d((\psi_{\mathsf{na}})_*\nu_{\mathsf{na}}) + \varepsilon\}$ contains infinitely many measures of the form $\psi_*(\nu_{t_n})$, which is contradictory, thereby finishing the proof. $\square$

6.3. **The hybrid approach to Theorem A for non-elementary representations.** Let $\varepsilon > 0$ be any positive small real number. By condition (A2) there exists a finite subset G' of G such that $\int_{G\setminus G'} \mathrm{length}(g) dm(g) \leq \varepsilon$. By Lemma 4.15 we have

$$\left| \int_{G\setminus G'} \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\sigma(\rho_t(g), v)}{\log |t|^{-1}} dm(g) d\nu_t(v) \right| \overset{(4.6)}{\leq} \int_{G\setminus G'} \left(\log \|\rho(g)\|_{\mathsf{na}} + \frac{C}{\log |t|^{-1}} \mathrm{length}(g) \right) dm(g) \leq 2C\varepsilon \ ,$$

for small enough t. Likewise from Lemma 2.8 we infer that

$$\left| \int_{G\setminus G'} \int_{\mathbb{P}^1_{\mathbb{L}}} \sigma(\rho_{\mathsf{na}}(g), v) dm(g) \right| \leq \left| \int_{G\setminus G'} \log \|\rho(g)\|_{\mathsf{na}} \right| \leq C\varepsilon \ .$$

So using the above and Furstenberg's formula for the Lyapunov exponent we get

$$\begin{aligned} \left| \frac{\chi(t)}{\log |t|^{-1}} - \chi_{\mathsf{na}} \right| &= \left| \int_G \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\sigma(\rho_t(g), v)}{\log |t|^{-1}} dm(g) d\nu_t(v) - \int_G \int_{\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}} \sigma(\rho_{\mathsf{na}}(g), v) dm(g) d\nu_{\mathsf{na}}(v) \right| \\ &\leq 2C\varepsilon + \left| \int_{G'} \left(\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\sigma(\rho_t(g), v)}{\log |t|^{-1}} d\nu_t(v) - \int_{\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}} \sigma(\rho_{\mathsf{na}}(g), v) d\nu_{\mathsf{na}}(v) \right) dm(g) \right| . \end{aligned}$$

Viewed in the hybrid space, the difference of integrals in the last line rewrites as

$$\int_{\mathbb{P}^1_{\mathrm{hyb}}} \frac{\sigma_{\mathrm{hyb}}(\rho(g), x)}{\log |t|^{-1}} d\left((\psi_t)_* \nu_t\right)(x) - \int_{\mathbb{P}^1_{\mathrm{hyb}}} \sigma_{\mathrm{hyb}}(\rho(g), x) d\left((\psi_{\mathsf{na}})_* \nu_{\mathsf{na}}\right)(x) \,,$$

so using the finiteness of G', Proposition 6.4 and Theorem C we deduce that

$$\limsup_{t\to 0} \left| \log |t|^{-1} \chi(t) - \chi_{\mathsf{na}} \right| \leq 2C\varepsilon,$$

and we conclude by letting $\varepsilon \to 0$. $\square$

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