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# WIGNER MEASURES AND EFFECTIVE MASS THEOREMS

VICTOR CHABU, CLOTILDE FERMANIAN-KAMMERER, AND FABRICIO MACIÀ

**ABSTRACT.** We study a semi-classical Schrödinger equation which describes the dynamics of an electron in a crystal in the presence of impurities. It is well-known that under suitable assumptions on the initial data, the wave function can be approximated in the semi-classical limit by the solution of a simpler equation, the effective mass equation. Using Floquet-Bloch decomposition and with a non-degeneracy condition on the critical points of the Bloch bands, as it is classical in this subject, we establish effective mass equations for more general initial data. Then, when the critical points are degenerated (which may occur in dimension strictly larger than one), we prove that a similar analysis can be performed, leading to a new type of effective mass equations which are operator-valued and of Heisenberg form. Our analysis relies on Wigner measure theory and, more precisely, to its applications to the analysis of dispersion effects.

**Keywords:** Bloch modes, semi-classical analysis on manifolds, Wigner measures, two-microlocal measures, effective mass theory.

## 1. INTRODUCTION

**1.1. The dynamics of an electron in a crystal and the effective mass equation.** The dynamics of an electron in a crystal in the presence of impurities is described by a wave function  $\Psi(t, x)$  that solves the Schrödinger equation:

$$(1.1) \quad \begin{cases} i\hbar\partial_t\Psi(t, x) + \frac{\hbar^2}{2m}\Delta_x\Psi(t, x) - Q_{\text{per}}(x)\Psi(t, x) - Q_{\text{ext}}(t, x)\Psi(t, x) = 0, \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

The potential  $Q_{\text{per}}$  is periodic with respect to some lattice in  $\mathbb{R}^d$  and describes the interactions between the electron and the crystal. The external potential  $Q_{\text{ext}}$  takes into account the effects of impurities on the otherwise perfect crystal. Here  $\hbar$  denotes the Planck constant and  $m$  is the mass of the electrons. In many cases of physical interest, the ratio between the mean spacing of the lattice and the characteristic length scale of variation of  $Q_{\text{ext}}$  is very small.

Denote that ratio by  $\varepsilon$ . After performing a suitable change of units and rescaling the external potential and the wave function (see for instance [44]) the Schrödinger equation

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becomes:

$$(1.2) \quad \begin{cases} i\partial_t \psi^\varepsilon(t, x) + \frac{1}{2} \Delta_x \psi^\varepsilon(t, x) - \frac{1}{\varepsilon^2} V_{\text{per}}\left(\frac{x}{\varepsilon}\right) \psi^\varepsilon(t, x) - V_{\text{ext}}(t, x) \psi^\varepsilon(t, x) = 0, \\ \psi^\varepsilon|_{t=0} = \psi_0^\varepsilon. \end{cases}$$

The potential  $V_{\text{per}}$  is periodic with respect to a fixed lattice in  $\mathbb{R}^d$ , which, for the sake of definiteness will be assumed to be  $\mathbb{Z}^d$ .

Effective Mass Theory consists in showing that, under suitable assumptions on the initial data  $\psi_0^\varepsilon$ , the solutions of (1.2) can be approximated for  $\varepsilon$  small by those of a simpler Schrödinger equation, the *effective mass equation*, which is of the form:

$$(1.3) \quad i\partial_t \phi(t, x) + \frac{1}{2} \langle B D_x, D_x \rangle \phi(t, x) - V_{\text{ext}}(t, x) \phi(t, x) = 0,$$

where, as usual,  $D_x = \frac{1}{i} \partial_x$ . Above,  $B$  is a  $d \times d$  matrix called the *effective mass tensor*. It is an experimentally accessible quantity that can be used to study the effect of the impurities on the dynamics of the electrons. Both the question of finding those initial conditions for which the corresponding solutions of (1.2) converge (in a suitable sense) to solutions to the effective mass equation and that of clarifying the dependence of  $B$  on the sequence of initial data have been extensively studied in the literature [11, 44, 3, 29, 9].

Another scaling limit that has been widely studied in this context is the *semiclassical limit*; it deals with the behavior as  $\varepsilon \rightarrow 0$  of the solutions to the semiclassical Schrödinger equation:

$$(1.4) \quad \begin{cases} i\varepsilon \partial_t v^\varepsilon(t, x) + \frac{\varepsilon^2}{2} \Delta_x v^\varepsilon(t, x) - V_{\text{per}}\left(\frac{x}{\varepsilon}\right) v^\varepsilon(t, x) - \varepsilon^2 V_{\text{ext}}(t, x) v^\varepsilon(t, x) = 0, \\ v^\varepsilon|_{t=0} = \psi_0^\varepsilon. \end{cases}$$

whose characteristic wave-lengths are comparable to  $\varepsilon$ . The interested reader can consult, among many others, references [24, 27, 44] for results in that direction, or [30, 10, 2, 14] for extensions of these results that allow to consider the presence external potentials of order one, without the factor  $\varepsilon^2$  in front  $V_{\text{ext}}(x)$ , and of magnetic fields [43, 16].

We again refer to [44] for details on the well-known derivation of (1.4) from (1.1). It is straightforward to check that the solutions to (1.2) and (1.4) are related through:

$$(1.5) \quad \psi^\varepsilon(t, x) = v^\varepsilon\left(\frac{t}{\varepsilon}, x\right).$$

Therefore, at least formally, one should be able to recover the effective mass equation (1.3) by performing the semiclassical limit  $\varepsilon \rightarrow 0$  in (1.4) simultaneously with the limit  $t/\varepsilon \rightarrow +\infty$ .

This type of simultaneous limit, combining high-frequencies ( $\varepsilon \rightarrow 0$ ) and long times ( $t \sim t_\varepsilon \rightarrow +\infty$ ), when the periodic potential is absent is relevant if one wants to understand the behavior of solutions of (1.4) beyond the Ehrenfest time [36, 4, 40]. Note also that, again when  $V_{\text{per}} = 0$ , the change of time scale (1.5) transforms the semiclassical equation (1.4) into the non-semiclassical one (that is, the one corresponding to  $\varepsilon = 1$ ). Therefore, it is possible to derive results on the dynamics of the Schrödinger equation via this scaling limit [37, 8, 7, 5]. The reader can consult the survey articles [38, 6] and the introductory lecture notes [39] for additional details and references on this approach.

Our goal in this article is to apply the aforementioned viewpoint to obtain new results on the effective mass approximation, by studying the asymptotic behavior of the position densities:

$$|\psi^\varepsilon(t, x)|^2,$$

issued from general sequence of initial data  $(\psi_0^\varepsilon)$ . To do so, we first use the Floquet-Bloch decomposition to show that, under a spectral simplicity assumption on the periodic potential,  $|\psi^\varepsilon|^2$  behaves, as  $\varepsilon \rightarrow 0$ , like a superposition of the form:

$$\sum_{n \in \mathbb{N}} \left| \varphi_n \left( \frac{x}{\varepsilon}, \varepsilon D_x \right) v_n^\varepsilon \left( \frac{t}{\varepsilon}, x \right) \right|^2,$$

where  $\varphi_n$  is the  $n^{\text{th}}$  Bloch wave and  $v_n^\varepsilon$  propagates following an equation of the form:<sup>1</sup>

$$i\varepsilon \partial_t v_n^\varepsilon(t, x) - \lambda_n(\varepsilon D_x) v_n^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) v_n^\varepsilon(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where  $\lambda_n(\varepsilon D_x)$  is the corresponding Bloch energy. See Section 1.2 for definitions and a precise statement of this result. Then we perform a careful analysis of the asymptotic behavior  $|v_n^\varepsilon(t/\varepsilon, \cdot)|^2$  following the approach in [37, 7, 4], whose main consequences are presented in 1.3. Finally, the applications of this analysis to Effective Mass Theory are presented in 1.4. This strategy has two main advantages: first, it allows to sum the infinite number of contributions of each Bloch wave for a very general class of initial data, without making any strong assumption on their structure. Second, it seems to give the first results in the literature on the validity of the Effective Mass Approximation in the presence of degenerate critical points in the Bloch energies.

**1.2. Floquet-Bloch decomposition.** The analysis of Schrödinger operators with periodic potentials has a long history that has its origins, among other sources, in the seminal works by Floquet [21] on ordinary differential equations with periodic coefficients, and by Bloch [12], who developed a spectral theory of periodic Schrödinger operators in the context of solid state physics. Floquet-Bloch theory can be used to study the spectrum of the perturbed periodic Schrödinger operator:

$$-\frac{\varepsilon^2}{2} \Delta_x + V_{\text{per}} \left( \frac{x}{\varepsilon} \right) + \varepsilon^2 V_{\text{ext}}(t, x),$$

---

<sup>1</sup>We recall that if  $\Phi$  is a bounded function  $\Phi(D_x)$  denotes the Fourier multiplier defined by  $\widehat{\Phi(D)f}(\xi) = \Phi(\xi)\widehat{f}(\xi)$  for all  $f \in L^2(\mathbb{R}^d)$ .



Moreover, the Bloch energies  $\varrho_n(\xi)$  are  $2\pi\mathbb{Z}^d$ -periodic<sup>2</sup> whereas the Bloch waves satisfy

$$\varphi_n(y, \xi + 2\pi k) = e^{-i2\pi k \cdot y} \varphi_n(y, \xi), \quad \text{for every } k \in \mathbb{Z}^d.$$

It is proved in [46] that the Bloch energies  $\varrho_n$  are continuous and piecewise analytic functions of  $\xi \in \mathbb{R}^d$ , and that the Bloch waves can be chosen in such a way there exists a subset  $\mathcal{Z}$  of the fundamental domain  $\mathcal{B} := [-\pi, \pi)^d$  ( $\mathcal{B}$  is called the *Brillouin zone*) of zero Lebesgue measure such that each  $\varphi_n$  is analytic in  $\xi \in \mathcal{B} \setminus \mathcal{Z}$ . If the multiplicity of the eigenvalue  $\varrho_n(\xi)$  is constant for  $\xi \in \mathbb{R}^d$ , then  $\varrho_n$  is a globally analytic function of  $\xi$ ; moreover, it is possible to find an orthonormal eigenbasis of the corresponding eigenspace consisting of functions whose dependence on  $\xi$  is also analytic.

Now define:

$$\begin{aligned} U_n^\varepsilon(t, x) &:= \int_{\mathbb{T}^d} \overline{\varphi_n}(y, \varepsilon D_x) U^\varepsilon(t, x, y) dy \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{\varphi_n}(y, \varepsilon \xi) U^\varepsilon(t, w, y) e^{i\xi \cdot (x-w)} \frac{dw d\xi}{(2\pi)^d} dy. \end{aligned}$$

The fact that  $(\varphi_n(\cdot, \xi))_{n \in \mathbb{N}}$  form an orthonormal basis of  $L^2(\mathbb{T}^d)$  implies that

$$U^\varepsilon(t, x, y) = \sum_{n \in \mathbb{N}} \varphi_n(y, \varepsilon D_x) U_n^\varepsilon(t, x).$$

Besides, when the  $n$ -th mode is of constant multiplicity 1, it turns out that for each  $n \in \mathbb{N}$  the functions  $U_n^\varepsilon$  satisfy:

$$(1.8) \quad U_n^\varepsilon(t, \cdot) = u_n^\varepsilon(t, \cdot) + \mathcal{O}(\varepsilon|t|),$$

where  $u_n^\varepsilon$  are the solutions to:

$$(1.9) \quad \begin{cases} i\varepsilon^2 \partial_t u_n^\varepsilon(t, x) = \varrho_n(\varepsilon D_x) u_n^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) u_n^\varepsilon(t, x), \\ u_n^\varepsilon|_{t=0}(x) = \int_{\mathbb{T}^d} \overline{\varphi_n}(y, \varepsilon D_x) \psi_0^\varepsilon(x) dy. \end{cases}$$

Precise statements of these facts, as well as the nature of the norms in which the approximations take place can be found in Section 1.4.

The construction we have described so far is called the *Floquet-Bloch decomposition*. It leads to the following representation formula for the solution of (1.2):

$$(1.10) \quad \psi^\varepsilon(t, x) = \sum_{n \in \mathbb{N}} \varphi_n \left( \frac{x}{\varepsilon}, \varepsilon D_x \right) U_n^\varepsilon(t, x).$$

This formula implies that the solutions of (1.2) can be decomposed as a countable superposition of waves whose dependence on the fast variable is given by a Bloch wave, whereas the profile  $U_n^\varepsilon$  describing the dependence on the slow variable is given by a time-evolution whose dispersion relation involves the corresponding Bloch energy (for suitably chosen  $n$ ).

<sup>2</sup>This follows from the fact that for every  $k \in \mathbb{Z}^d$ , the operator  $P(\xi + 2\pi k)$  is unitarily equivalent to  $P(\xi)$  since  $P(\xi + 2\pi k) = e^{-i2\pi k \cdot y} P(\xi) e^{i2\pi k \cdot y}$ .

**1.3. Quantifying the lack of dispersion.** As the previous discussion shows, one of the main steps in understanding the limiting behavior as  $\varepsilon \rightarrow 0$  of the position densities of solutions to the Schrödinger equation (1.2) relies on a careful analysis of the solutions of equations of the form:

$$(1.11) \quad \begin{cases} i\varepsilon^2 \partial_t u^\varepsilon(t, x) = \lambda(\varepsilon D_x) u^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) u^\varepsilon(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon. \end{cases}$$

This equation ceases to be dispersive as soon as  $\lambda(\xi)$  has critical points  $\xi \neq 0$ , and this is always the case if  $\lambda$  is a Bloch energy. Heuristically, one can think that one of the consequences of a dispersive time evolution is a regularization of the high-frequency effects (that is associated to frequencies  $\varepsilon\xi = c \neq 0$ ) caused by the sequence of initial data. These heuristics have been made precise in many cases; a presentation of our results from this point of view can be found in [15]. The reader can also find there a detailed account on the literature on the subject.

Here we show that, in the presence of critical points of  $\lambda$ , some of the high-frequency effects exhibited by the sequence of initial data persist after applying the time evolution (1.11). We provide a quantitative picture of this persistence by giving a complete description of the asymptotic behavior of the densities  $|u^\varepsilon(t, x)|^2$  associated to a bounded sequence  $(u^\varepsilon)$  of solutions to (1.11). We are going to give an explicit procedure to compute all weak- $\star$  accumulation points of the sequence of positive measures  $(|u^\varepsilon(t, x)|^2)$  in terms of quantities that can be obtained from the sequence of initial data  $(u_0^\varepsilon)$ . These results are of independent interest; we have thus chosen to present them in a more general framework than what it is necessary in our applications to effective mass theory.

In order to obtain a non trivial result we must make sure that the characteristic length-scale of the oscillations carried by the sequence of initial data is of the order of  $\varepsilon$ . The following assumption is sufficient for our purposes:

**H0** The sequence  $(u_0^\varepsilon)$  is uniformly bounded in  $L^2(\mathbb{R}^d)$  and  $\varepsilon$ -oscillating, in the sense that its energy is concentrated on frequencies smaller or equal than  $1/\varepsilon$ :

$$(1.12) \quad \limsup_{\varepsilon \rightarrow 0^+} \int_{|\xi| > R/\varepsilon} |\widehat{u_0^\varepsilon}(\xi)|^2 d\xi \xrightarrow{R \rightarrow +\infty} 0.$$

We shall assume that  $\lambda$  is smooth and grows at most polynomially, and that its set of critical points is a submanifold of  $\mathbb{R}^d$ . More precisely, we impose the following hypotheses on  $\lambda$  and  $V$ :

**H1**  $V_{\text{ext}} \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$  is bounded together with its derivatives and  $\lambda \in C^\infty(\mathbb{R}^d)$ , together with its derivatives, grows at most polynomially; *i.e.* there exists  $N > 0$  such that, for every  $\alpha \in \mathbb{N}_+^d$ , one has:

$$\sup_{\xi \in \mathbb{R}^d} |\partial_\xi^\alpha \lambda(\xi)| (1 + |\xi|^N)^{-1} < \infty.$$

**H2** The set

$$\Lambda := \{\xi \in \mathbb{R}^d : \nabla \lambda(\xi) = 0\}$$

is a connected, closed embedded submanifold of  $\mathbb{R}^d$  of codimension  $0 < p \leq d$  and the Hessian  $\nabla^2 \lambda$  is of maximal rank over  $\Lambda$ .

The hypothesis **H2** implies the existence of tubular coordinates in a neighborhood of  $\Lambda$ . A stronger version of **H2** is to suppose that all critical points of  $\lambda$  are non-degenerate (that is,  $\nabla^2 \lambda(\xi)$  is a non-degenerate quadratic form for every  $\xi \in \Lambda$ ), which implies that  $p = d$  and  $\Lambda$  is a discrete set<sup>3</sup> in  $\mathbb{R}^d$ . We first state the main result of this section under this stronger hypothesis.

**Theorem 1.1.** *Suppose that the sequence of initial data  $(u_0^\varepsilon)$  verifies **H0**, denote by  $(u^\varepsilon)$  the corresponding sequence of solutions to (1.11). Suppose in addition that **H1** is satisfied and all critical points of  $\lambda$  are non-degenerate.*

*Then there exists a subsequence  $(u_0^{\varepsilon_k})$  such that for every  $a < b$  and every  $\phi \in C_c(\mathbb{R}^d)$  the following holds:*

$$(1.13) \quad \lim_{k \rightarrow \infty} \int_a^b \int_{\mathbb{R}^d} \phi(x) |u^{\varepsilon_k}(t, x)|^2 dx dt = \sum_{\xi \in \Lambda} \int_a^b \int_{\mathbb{R}^d} \phi(x) |u_\xi(t, x)|^2 dx dt,$$

where  $u_\xi$  solves the following Schrödinger equation:

$$(1.14) \quad i \partial_t u_\xi(t, x) = \nabla^2 \lambda(\xi) D_x \cdot D_x u_\xi(t, x) + V_{\text{ext}}(t, x) u_\xi(t, x),$$

with initial data:

$$u_\xi|_{t=0} \text{ is the weak limit in } L^2(\mathbb{R}^d) \text{ of the sequence } \left( e^{-\frac{i}{\varepsilon_k} \xi \cdot x} u_0^{\varepsilon_k} \right).$$

If  $\Lambda = \emptyset$  then the right-hand side of (1.13) is equal to zero.

Note that  $u_\xi$  may be identically equal to zero even if the sequence  $(u_0^\varepsilon)$  oscillates in the direction  $\xi$ . For instance, if the sequence of initial data is a coherent state:

$$u_0^\varepsilon(x) = \frac{1}{\varepsilon^{d/4}} \rho \left( \frac{x - x_0}{\sqrt{\varepsilon}} \right) e^{\frac{i}{\varepsilon} \xi_0 \cdot x},$$

centered at a point  $(x_0, \xi_0)$  in phase space with  $\rho \in C_0(\mathbb{R}^d)$ , then  $u_\xi|_{t=0} = 0$  for every  $\xi \in \mathbb{R}^d$ . Theorem (1.1) allows us to conclude that the corresponding solutions  $(u^\varepsilon)$  converge to zero in  $L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d)$ .

Theorem 1.1 can be interpreted as a description of the obstructions to the validity of smoothing-type estimates for the solutions to equation (1.11) in the presence of critical points of the symbol of the Fourier multiplier. We refer again the reader to [15] for additional details concerning this issue.

We will obtain Theorem 1.1 as a particular case of a much more general result; it is however more complicated to state and requires some geometric preliminaries. Reference [15] gives in a self-contained presentation the (simpler) proof of Theorem 1.1.

<sup>3</sup>If moreover one has that  $\lambda$  is  $\mathbb{Z}^d$ -periodic, which is the situation when  $\lambda$  is a Bloch energy, this set is finite modulo  $\mathbb{Z}^d$ .



First, it will be convenient to identify  $\lambda$  to a function defined on  $(\mathbb{R}^d)^*$  rather than  $\mathbb{R}^d$  (via the standard identification given by the scalar product). Define the cotangent bundle of  $\Lambda$  as the union of all cotangent spaces to  $\Lambda$

$$T^*\Lambda := \{(x, \xi) \in \mathbb{R}^d \times \Lambda : x \in T_\xi^*\Lambda\},$$

each fiber  $T_\xi^*\Lambda$  is the dual space of the tangent space  $T_\xi\Lambda$ . Note that this is well-defined, since  $T^*\Lambda \subset (\mathbb{R}^d)^{**} = \mathbb{R}^d$ . We also define the normal bundle of  $\Lambda$  which is the union of those linear subspaces of  $\mathbb{R}^d$  that are normal to  $\Lambda$ :

$$N\Lambda := \{(z, \xi) \in \mathbb{R}^d \times \Lambda : z \in N_\xi\Lambda\},$$

where  $N_\xi\Lambda$  consists of those  $x \in (\mathbb{R}^d)^{**} = \mathbb{R}^d$  that annihilate  $T_\xi\Lambda$ . Every point  $x \in \mathbb{R}^d$  can be uniquely written as  $x = v + z$ , where  $v \in T_\xi^*\Lambda$  and  $z \in N_\xi\Lambda$ . Given a function  $\phi \in L^\infty(\mathbb{R}^d)$  we write  $m_\phi(v, \xi)$ , where  $v \in T_\xi^*\Lambda$ , to denote the operator acting on  $L^2(N_\xi\Lambda)$  by multiplication by  $\phi(v + \cdot)$ . Note that assumption **H2** implies that the Hessian of  $\lambda$  defines an operator  $\nabla^2\lambda(\xi)D_z \cdot D_z$  acting on  $N_\xi\Lambda$  for any  $\xi \in \Lambda$ .

The next result is the analogue of Theorem 1.1 in this context. The sum over critical points is replaced by an integral with respect to a measure over  $T^*\Lambda$ , and the Schrödinger equation (1.22) becomes a Heisenberg equation for a time-dependent family  $M$  of trace-class operators acting on a certain  $L^2$ -space. More precisely, the operators  $M$  depend on  $t \in \mathbb{R}$  and on  $\xi \in \Lambda$ ,  $v \in T_\xi^*\Lambda$ ; for every choice of these parameters,  $M_t(v, \xi)$  is a trace-class operator acting on  $L^2$  functions of the vector space  $N_\xi\Lambda$ . Note that  $M_t$  can also be viewed as a section of a vector bundle over  $T^*\Lambda$ , namely:  $\bigsqcup_{(v, \xi) \in T^*\Lambda} \mathcal{L}_+^1(L^2(N_\xi\Lambda))$ . In the following, given a complete, locally compact, metric space  $X$  we denote by  $\mathcal{M}_+(X)$  the set of positive Radon measures on  $X$ .

**Theorem 1.2.** *Let  $(u_0^\varepsilon)$  be a sequence of initial data satisfying **H0**, and denote by  $(u^\varepsilon)$  the corresponding sequence of solutions to (1.11). If **H1** and **H2** hold, then there exist a subsequence  $(u_0^{\varepsilon_k})$ , a positive measure  $\nu \in \mathcal{M}_+(T^*\Lambda)$  and a measurable family of self-adjoint, positive, trace-class operators*

$$M_0 : T_\xi^*\Lambda \ni (v, \xi) \longmapsto M_0(z, \xi) \in \mathcal{L}_+^1(L^2(N_\xi\Lambda)), \quad \text{Tr}_{L^2(N_\xi\Lambda)} M_0(v, \xi) = 1,$$

such that for every  $a < b$  and every  $\phi \in \mathcal{C}_c(\mathbb{R}^d)$  one has:

$$(1.15) \quad \lim_{k \rightarrow \infty} \int_a^b \int_{\mathbb{R}^d} \phi(x) |u^{\varepsilon_k}(t, x)|^2 dx dt = \int_a^b \int_{T^*\Lambda} \text{Tr}_{L^2(N_\xi\Lambda)} [m_\phi(v, \xi) M_t(v, \xi)] \nu(dv, d\xi) dt,$$

where  $t \mapsto M_t(v, \xi) \in \mathcal{C}(\mathbb{R}; \mathcal{L}_+^1(L^2(N_\xi\Lambda)))$  solves the following Heisenberg equation:

$$(1.16) \quad \begin{cases} i\partial_t M_t(v, \xi) = \left[ \frac{1}{2} \nabla^2\lambda(\xi) D_z \cdot D_z + m_{V_{\text{ext}}(t, \cdot)}(v, \xi), M_t(v, \xi) \right], \\ M|_{t=0} = M_0. \end{cases}$$

*Remark 1.3.* When  $\Lambda$  consists of a set of isolated critical points, Theorems 1.1 and 1.2 are completely equivalent. Note that in this case,  $T^*\Lambda = \{0\} \times \Lambda$  and the measure  $\nu$  (which in this case is a measure depending on  $\xi \in \mathbb{R}^d$  only) is simply

$$\nu = \sum_{\xi \in \Lambda} \alpha_\xi \delta_\xi,$$

where  $\alpha_\xi = \|u_\xi|_{t=0}\|_{L^2(\mathbb{R}^d)}^2$ . In addition,  $N_\xi \Lambda = \mathbb{R}^d$  and the operator  $M_t(\xi)$  (which again does not depend on  $z$ ) is the orthogonal projection onto  $u_\xi(t, \cdot)$  in  $L^2(\mathbb{R}^d)$  (recall that  $u_\xi$  solves the Schrödinger equation (1.22)). These orthogonal projections satisfy the Heisenberg equation (1.16).

*Remark 1.4.* As it will be clear from the proof of this theorem, the measure  $\nu$  and the family of operators  $M_0(v, \xi)$ , depending on  $v \in T_\xi^* \Lambda$ , only depend on the subsequence of initial data  $(u_0^{\varepsilon_k})$ . At this point, it might be useful to stress out that in this regime the limiting objects  $M, \nu$  cannot be computed in terms of the Wigner/semiclassical measure of the sequence of initial data, as it is the case when dealing with the semiclassical limit (see Section 2 for the definitions of these objects). We will see later that they are defined as two microlocal Wigner measures of  $(u_0^{\varepsilon_k})$  in the sense of [18, 19, 37]. In [15], we have explicitly constructed sequences of initial data having the same semiclassical measure but such that their time dependent measures differ. This type of behavior was first remarked in this context in the case of the Schrödinger equation on the torus, see [36, 37].

As soon as  $\Lambda$  has strictly positive dimension (*i.e.* it is not a union of isolated critical points), the measure  $\nu$  may be singular with respect to the  $z$  variable. See [15] for specific examples exhibiting this type of behavior. See also that reference for examples proving the necessity of hypothesis **H2**; it is shown there that different types of behavior can happen whenever the Hessian of  $\lambda$  is not of full rank on  $\Lambda$ .

**1.4. Back to effective mass equations.** The Floquet-Bloch decomposition and Theorems 1.1 and 1.2 have interesting consequences in Effective Mass Theory. Denote by  $I \subset \mathbb{N}$  the set of indices  $n$  such that the multiplicity of the Bloch energy  $\varrho_n(\xi)$  as an eigenvalue of  $P(\xi)$  is constant for every  $\xi \in \mathbb{R}^d$  (note that it is possible that this set is not finite). Recall that this implies that  $\varrho_n$  is real analytic [46]. Chose a strictly increasing subsequence of  $(\varrho_n)_{n \in I}$  taking the same values as the original sequence. From now on, we are going to denote this new sequence of distinct Bloch eigenvalues by:

$$\lambda_1(\xi) < \lambda_2(\xi) < \dots < \lambda_n(\xi) \rightarrow \infty.$$

For every  $j \in \mathbb{N}$  and  $w \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$  we define:

$$(1.17) \quad P_{\varphi_j}^\varepsilon w(x, y) := \varphi_j(y, \varepsilon D_x) \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \varepsilon D_x) w(z, x) dz,$$

and, for every distinct Bloch eigenvalue  $\lambda_n$ , we construct the corresponding Bloch spectral projector:

$$(1.18) \quad \Pi_{\lambda_n}^\varepsilon := \sum_{P(\xi)\varphi_j(\xi, \cdot) = \lambda_n(\xi)\varphi_j(\xi, \cdot)} P_{\varphi_j}^\varepsilon.$$

The action of these operators extend to functions  $v \in L^2(\mathbb{R}^d)$  by identifying  $v$  to  $v \otimes 1 \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ .

**H0'** The sequence  $(\psi_0^\varepsilon)$  is strongly  $\varepsilon$ -oscillating:

$$\forall s \geq 0, \exists C_s > 0 : \quad \|\langle \varepsilon D_x \rangle^s \psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_s, \quad \text{uniformly in } \varepsilon.$$

and its energy is concentrated on well-separated Bloch eigenvalues:

$$(1.19) \quad \psi_0^\varepsilon(x) = \sum_{n \in I} \Pi_{\lambda_n}^\varepsilon(\psi_0^\varepsilon \otimes 1) \left(x, \frac{x}{\varepsilon}\right) + r^\varepsilon(x),$$

with  $\|r^\varepsilon\|_{L^2(\mathbb{R}^d)} \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ .

In what follows, we shall denote by  $\Lambda_n \subset \mathbb{R}^d$  the set of critical points of the Bloch eigenvalue  $\lambda_n(\xi)$  for  $j \in I$ .

*Remark 1.5.* The fact that the sum presented in equation (1.19) is infinite induces a difficulty consisting in showing that the series converges. We shall see in Section 4 (Lemma 4.4) that the strong  $\varepsilon$ -oscillation property of **H0'** is crucial in establishing this convergence.

**Theorem 1.6.** *Assume **H0'**, and that **H2** holds for any distinct Bloch eigenvalue  $\lambda_n(\xi)$  for  $n \in I$ . Then, there exist a subsequence  $(\varepsilon_k)_{k \in \mathbb{N}}$ , positive measures  $\nu_n \in \mathcal{M}_+(T^*\Lambda_n)$ ,  $n \in I$ , and a measurable family of self-adjoint, positive, trace-class operators*

$$M_{0,n} : T_\xi^*\Lambda_n \ni (v, \xi) \longmapsto M_{0,n}(v, \xi) \in \mathcal{L}_+^1(L^2(N_\xi\Lambda_n)), \quad \text{Tr}_{L^2(N_\xi\Lambda_n)} M_{0,n}(v, \xi) = 1,$$

*such that for every  $a < b$  and every  $\phi \in \mathcal{C}_c(\mathbb{R}^d)$  one has:*

$$(1.20) \quad \lim_{k \rightarrow \infty} \int_a^b \int_{\mathbb{R}^d} \phi(x) |\psi^{\varepsilon_k}(t, x)|^2 dx dt = \sum_{j \in I} \int_a^b \int_{T^*\Lambda_n} \text{Tr}_{L^2(N_\xi\Lambda_n)} [m_\phi(v, \xi) M_n(t, v, \xi)] \nu_j(dv, d\xi) dt,$$

*where  $M_n(\cdot, v, \xi) \in \mathcal{C}(\mathbb{R}; \mathcal{L}_+^1(L^2(N_\xi\Lambda_n)))$  solves the following Heisenberg equation:*

$$(1.21) \quad \begin{cases} i\partial_t M_n(t, v, \xi) + \left[ \frac{1}{2} \nabla^2 \lambda_n(\xi) D_y \cdot D_y + m_{V_{\text{ext}}(t, \cdot)}(v, \xi), M_n(t, v, \xi) \right] = 0, \\ M_n|_{t=0} = M_{0,n}. \end{cases}$$

Let us now precise Theorem 1.6 when one of the Bloch eigenvalues  $\lambda_n(\xi)$  have a finite set of critical points.

**Theorem 1.7.** *Assume **H0'**, and that **H2** holds for any Bloch eigenvalue  $\lambda_n(\xi)$  for  $n \in I$  and assume that for some  $n_0 \in I$ , the set  $\Lambda_{n_0}$  of critical points of  $\lambda_{n_0}$  is finite. Then the measure  $\nu_{n_0}$  and the operator  $M_{n_0}$  of Theorem 1.6 above satisfy:*

(1) The operator  $M_{n_0}(t, \xi)$  is the orthogonal projection on  $\psi_\xi$  which solves the effective mass equation:

$$(1.22) \quad i\partial_t \psi_\xi(t, x) = \nabla^2 \lambda_{n_0}(\xi) D_x \cdot D_x \psi_\xi(t, x) + V_{\text{ext}}(t, x) \psi_\xi(t, x),$$

with initial data:

$$\psi_\xi|_{t=0} \text{ is the weak limit in } L^2(\mathbb{R}^d) \text{ of the sequence } \left( e^{-\frac{i}{\varepsilon k} \xi \cdot x} \psi_{0, n_0}^{\varepsilon k} \right) \text{ where}$$

$$\psi_{0, n_0}^\varepsilon = \Pi_{\lambda_{n_0}}^\varepsilon (\psi_0^\varepsilon \otimes 1) \left( x, \frac{x}{\varepsilon} \right).$$

(2) The measure  $\nu_{n_0}$  is given by

$$\nu_{n_0} = \sum_{\xi \in \Lambda_{n_0}} \alpha_\xi \delta_\xi, \quad \alpha_\xi = \|\psi_\xi|_{t=0}\|_{L^2(\mathbb{R}^d)}.$$

Besides, we have the following characterization of the initial data of the effective mass equation.

**Proposition 1.8.** *Let  $\xi \in \mathbb{R}^d$  and let us assume that for every  $k \in \mathbb{Z}^d$  the functions*

$$x \mapsto \psi_0^\varepsilon(x) e^{-i \frac{\xi - 2\pi k}{\varepsilon} \cdot x}$$

*converge weakly to some limit  $\psi_0^k$ . Then, with the notations of Theorem 1.7,*

$$\psi_\xi|_{t=0}(x) = \sum_{P(\xi)\varphi_j(\xi, \cdot) = \lambda_{n_0}(\xi)\varphi_j(\xi, \cdot)} c^j(\xi) \sum_{k \in \mathbb{Z}^d} \overline{c^j}(\xi - 2\pi k) \psi_0^k(x).$$

Some results that are related to Theorem 1.7 can be found in the work by Allaire and Piatnitski [3] (see also [2] for similar results in a related problem); in that work homogenization and two-scale convergence techniques are used to obtain a precise description of the solution profile for a particular class of initial data admitting only one term in the decomposition (1.19). In [9], Barletti and Ben Abdallah obtained a result similar to Theorem 1.7 by following the approach initiated by Kohn and Luttinger in [35] consisting in introducing a (non-canonical) basis of modified Bloch functions. The starting point in our approach is conceptually closer to that in [44], in the sense that we analyse the structure of Wigner measures associated to sequences of solutions. The main novelty here is the use of two-microlocal Wigner measures, that give a more explicit geometric description of the mechanism that underlies the Effective Mass Approximation, showing that it is a result of the dispersive effects associated to high-frequency solutions to the semiclassical Bloch band equations. Moreover, we are able to deal with the presence of non-isolated critical points on the Bloch energies (which may appear in dimension larger than one) and to treat the infinite sum of (1.19).

**1.5. Structure of the article.** Section 2 is devoted to preliminaries about pseudodifferential operators and semi-classical measures. Then, in Sections 3, we introduce the two-microlocal tools that we use for the analysis of the dispersion properties of solutions to equations of the form (1.11) and for proving Theorems 1.1 and 1.2. Finally, in Section 4 we come back to the effective mass equations and prove Theorems 1.6

and 1.7. Some Appendices are devoted to basic results about pseudodifferential calculus and trace-class operator-valued measures, and to the proof of technical lemma.

## 2. PSEUDODIFFERENTIAL OPERATORS AND SEMICLASSICAL MEASURES – PRELIMINARIES

In this section we recall some basic facts on Wigner distributions and semiclassical measures, which are the tools we are going to use to prove Theorem 1.2, and derive preliminary results about Wigner measures associated with families of solutions of equations of the form (1.11).

**2.1. Wigner transform and Wigner measures.** Wigner distributions provide a useful way for computing weak- $\star$  accumulation points of a sequence of densities ( $|u^\varepsilon|^2$ ) constructed from a  $L^2$ -bounded sequence ( $u^\varepsilon$ ) of solutions of a semiclassical (pseudo) differential equation. They provide a joint physical/Fourier space description of the energy distribution of functions in  $\mathbb{R}^d$ . The Wigner distribution of a function  $f \in L^2(\mathbb{R}^d)$  is defined as:

$$W_f^\varepsilon(x, \xi) := \int_{\mathbb{R}^d} f\left(x - \frac{\varepsilon v}{2}\right) \overline{f\left(x + \frac{\varepsilon v}{2}\right)} e^{i\xi \cdot v} \frac{dv}{(2\pi)^d},$$

and has several interesting properties (see, for instance, [22]).

- $W_f^\varepsilon \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ .
- Projecting  $W_f^\varepsilon$  on  $x$  or  $\xi$  gives the position or momentum densities of  $f$  respectively :

$$\int_{\mathbb{R}^d} W_f^\varepsilon(x, \xi) d\xi = |f(x)|^2, \quad \int_{\mathbb{R}^d} W_f^\varepsilon(x, \xi) dx = \frac{1}{(2\pi\varepsilon)^d} \left| \widehat{f}\left(\frac{\xi}{\varepsilon}\right) \right|^2.$$

Note that despite this,  $W_f^\varepsilon$  is not positive in general.

- For every  $a \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  one has:

$$(2.1) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} a(x, \xi) W_f^\varepsilon(x, \xi) dx d\xi = (\text{op}_\varepsilon(a)f, f)_{L^2(\mathbb{R}^d)},$$

where  $\text{op}_\varepsilon(a)$  is the semiclassical pseudodifferential operator of symbol  $a$  obtained through the Weyl quantization rule:

$$\text{op}_\varepsilon(a)f(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} a\left(\frac{x+y}{2}, \varepsilon\xi\right) e^{i\xi \cdot (x-y)} f(y) dy \frac{d\xi}{(2\pi)^d}.$$

If  $(f^\varepsilon)$  is a bounded sequence in  $L^2(\mathbb{R}^d)$  then  $(W_{f^\varepsilon}^\varepsilon)$  is a bounded sequence of tempered distributions in  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ . This is proved using identity (2.1) combined with the fact that the operators  $\text{op}_\varepsilon(a)$  are uniformly bounded by a suitable semi-norm in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , see (A.1). Appendix A contains additional facts on the theory of pseudodifferential operators, as well as references to the literature.

In addition, every accumulation point of  $(W_{f^\varepsilon}^\varepsilon)$  in  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$  is a positive distribution and therefore, by Schwartz's theorem, a positive measure on  $\mathbb{R}^d \times \mathbb{R}^d$ . These

measures are called *semiclassical* or *Wigner measures*. See references [24, 34, 26, 27] for different proofs of the results we have presented so far.

Now, if  $\mu \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$  is an accumulation point of  $(W_{f^\varepsilon}^\varepsilon)$  along some subsequence  $(\varepsilon_k)$  and  $(|f^{\varepsilon_k}|^2)$  converges weakly- $\star$  towards a measure  $\nu \in \mathcal{M}_+(\mathbb{R}^d)$  then one has:

$$(2.2) \quad \int_{\mathbb{R}^d} \mu(\cdot, d\xi) \leq \nu.$$

Equality holds if and only if  $(f^\varepsilon)$  is  $\varepsilon$ -oscillating:

$$(2.3) \quad \limsup_{\varepsilon \rightarrow 0^+} \int_{|\xi| > R/\varepsilon} |\widehat{f^\varepsilon}(\xi)|^2 d\xi \xrightarrow{R \rightarrow +\infty} 0,$$

see [24, 26, 27]. The hypothesis **H0** that we made on the initial data for equation (1.11), is this  $\varepsilon$ -oscillating property. Note also that (2.2) implies that  $\mu$  is always a finite measure of total mass bounded by  $\sup_\varepsilon \|f^\varepsilon\|_{L^2(\mathbb{R}^d)}^2$ .

**2.2. Space-time Wigner measures.** We need to define semiclassical measures for sequences  $(f^\varepsilon)$  which are solutions to a certain evolution equation; these sequences are bounded in  $\mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$  and therefore are only locally integrable with respect to the first variable (the time variable). We introduce a localized-in-time Wigner distribution acting on test functions  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  through identity (2.1):

$$\left\langle \tilde{W}_{\phi f^\varepsilon}^\varepsilon, a \right\rangle := (\text{op}_\varepsilon(a^\varepsilon) \phi f^\varepsilon, \phi f^\varepsilon)_{L^2(\mathbb{R} \times \mathbb{R}^d)},$$

where  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_t)$  and  $a^\varepsilon$  stands for:

$$a^\varepsilon(t, \tau, x, \xi) := a(t, \varepsilon\tau, x, \xi).$$

This particular scaling in the variable  $\tau$  corresponds to the scaling of the dispersion relation in the equations that we are considering (namely, equations (1.2) and (1.11)). The following result holds in our context:

**Proposition 2.1.** *Let  $(f^\varepsilon)$  be bounded in  $L_{\text{loc}}^2(\mathbb{R}; L^2(\mathbb{R}^d))$ . Then  $(\tilde{W}_{\phi f^\varepsilon}^\varepsilon)$  is bounded in  $\mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$ , uniformly in  $\varepsilon > 0$  and  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_t)$ . It is possible to extract weakly converging subsequences  $(\tilde{W}_{\phi f^{\varepsilon_k}}^{\varepsilon_k})$  such that, for every  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  and every  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  one has:*

$$\lim_{k \rightarrow \infty} \left\langle \tilde{W}_{\phi f^{\varepsilon_k}}^{\varepsilon_k}, a \right\rangle = \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} a(t, \tau, x, \xi) |\phi(t)|^2 \tilde{\mu}(dt, d\tau, dx, d\xi).$$

The accumulation points  $\tilde{\mu}$  are always positive Radon measures on  $\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d}$ . If, in addition,  $(|f^{\varepsilon_k}|^2)$  converges weak- $\star$  in the sense of measure to some measure  $\tilde{\nu}$  then:

$$\int_{\mathbb{R}_\tau \times \mathbb{R}_\xi} \tilde{\mu}(\cdot, d\tau, \cdot, d\xi) \leq \tilde{\nu}.$$

Equality takes place if and only if for every  $\phi \in C_0^\infty(\mathbb{R}_t)$  the sequence  $(\phi f^\varepsilon)$  satisfies:

$$(2.4) \quad \limsup_{\varepsilon \rightarrow 0^+} \int_{|\varepsilon^2 \tau| + |\varepsilon \xi| > R} |\widehat{\phi f^\varepsilon}(\tau, \xi)|^2 d\tau d\xi \xrightarrow{R \rightarrow \infty} 0.$$

The proof of this result is by now classical; it can be obtained by following the lines of the proofs of the analogous results in [25, 24, 27].

**2.3. Wigner measure and family of solutions of dispersive equations.** We will now consider Wigner distributions associated to solutions of the evolution equation:

$$(2.5) \quad \begin{cases} i\varepsilon^2 \partial_t u^\varepsilon(t, x) = \lambda(\varepsilon D_x) u^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) u^\varepsilon(t, x) + \varepsilon^3 g^\varepsilon(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u^\varepsilon|_{t=0} = u_0^\varepsilon, \end{cases}$$

where  $V_{\text{ext}}$  and  $\lambda$  satisfy hypothesis **H1** and  $\|g^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)}$  is locally uniformly bounded with respect to  $t$

When the sequence  $(u_0^\varepsilon)$  of initial data is uniformly bounded in  $L^2(\mathbb{R}^d)$ , so is the corresponding sequence  $(u^\varepsilon(t)) := (u^\varepsilon(t, \cdot))$  of solutions to (2.5) for every  $t \in \mathbb{R}$ . Therefore the sequence of Wigner distributions  $(W_{u^\varepsilon(t)}^\varepsilon)$  is bounded in  $\mathcal{C}(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d))$ . Nevertheless, its time derivatives are unbounded and, in general, one cannot hope to find a subsequence that converges pointwise (or even almost everywhere) in  $t$  (see Proposition 2.6 below). This difficulty can be overcome if one considers Wigner distributions in  $(t, x)$ . Our first result states that the corresponding measures  $\tilde{\mu}$  are localized on the dispersion relation (or characteristic set) of the principal part of equation (2.5).

**Proposition 2.2.** *Let  $(u^\varepsilon)$  be a sequence of solutions to (2.5) issued from an  $L^2(\mathbb{R}^d)$ -bounded sequence of initial data  $(u_0^\varepsilon)$ , and suppose  $\tilde{\mu} \in \mathcal{M}_+(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  is obtained from Proposition 2.1 along some subsequence. Then, for every  $a \in C_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$ :*

$$\int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} a(t, \tau, x, \xi) \tilde{\mu}(dt, d\tau, dx, d\xi) = \int_{\mathbb{R} \times \mathbb{R}^{2d}} a(t, \lambda(\xi), x, \xi) \mu(dt, dx, d\xi),$$

for some positive measure  $\mu \in \mathcal{M}_+(\mathbb{R}_t \times \mathbb{R}_{x,\xi}^{2d})$ .

The reader can consult [24] or [36] for a proof of this result. If, in addition, the sequence of initial data is  $\varepsilon$ -oscillating then, by Proposition 2.1, it is possible to describe the projection of  $\mu$  on the time variable  $t$ .

**Proposition 2.3.** *Let  $(u^\varepsilon)$  be a sequence of solutions to (2.5) issued from an  $L^2(\mathbb{R}^d)$ -bounded and  $\varepsilon$ -oscillating sequence of initial data  $(u_0^\varepsilon)$ . Then there exist a subsequence  $(\varepsilon_k)$  tending to zero as  $k \rightarrow \infty$  and a  $t$ -measurable family  $\mu_t \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$  of finite measures, with total mass essentially uniformly bounded in  $t \in \mathbb{R}$ , such that, for every  $\Xi \in L^1(\mathbb{R})$  and  $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ :*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} \Xi(t) a(x, \xi) W_{u^{\varepsilon_k}(t)}^{\varepsilon_k}(x, \xi) dx d\xi dt = \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} \Xi(t) a(x, \xi) \mu_t(dx, d\xi) dt.$$

Moreover, for every  $\Xi \in L^1(\mathbb{R})$  and  $\phi \in \mathcal{C}_0(\mathbb{R}^d)$ :

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Xi(t) \phi(x) |u^{\varepsilon^k}(t, x)|^2 dx dt = \int_{\mathbb{R}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \Xi(t) \phi(x) \mu_t(dx, d\xi) dt.$$

*Remark 2.4.* If the limits in Propositions 2.2 and 2.3 take place along the same subsequence, then the corresponding limit measures are related by:

$$\mu(t, x, \xi) = \mu_t(x, \xi) dt.$$

This result is proved in [36], Theorem 1; see also Appendix B in [40]. Later, we shall also use the following observation.

**Lemma 2.5.** *Suppose H1 holds. Let  $(u^\varepsilon)$  be a sequence of solutions to (2.5) issued from an  $L^2(\mathbb{R}^d)$ -bounded and  $\varepsilon$ -oscillating sequence of initial data  $(u_0^\varepsilon)$ , then (2.4) also holds.*

*Proof.* Let us first prove that  $(u^\varepsilon(t))$  is  $\varepsilon$  oscillating for all  $t \in \mathbb{R}$ . Indeed, consider  $\chi \in \mathcal{C}^\infty(\mathbb{R}_\tau)$  with  $\chi(\tau) = 1$  for  $|\tau| \leq 1$  and  $\chi(\tau) = 0$  for  $|\tau| > 2$ , and set

$$u_R^\varepsilon(t, x) = \left( 1 - \chi \left( \frac{|\varepsilon D_x|}{R} \right) \right) u^\varepsilon(t, x).$$

Then, the family  $(u_R^\varepsilon)$  satisfies

$$i\varepsilon^2 \partial_t u_R^\varepsilon(t, x) = \lambda(\varepsilon D_x) u_R^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) u_R^\varepsilon(t, x) - \varepsilon^2 \left[ \chi \left( \frac{|\varepsilon D_x|}{R} \right), V_{\text{ext}}(t, x) \right] u^\varepsilon(t, x).$$

Symbolic calculus gives

$$\left[ \chi \left( \frac{|\varepsilon D_x|}{R} \right), V_{\text{ext}}(t, x) \right] u^\varepsilon(t, \cdot) = O \left( \frac{\varepsilon}{R} \right) \quad \text{in } L^2(\mathbb{R}^d).$$

Therefore

$$(2.6) \quad \|u_R^\varepsilon(t)\|_{L^2(\mathbb{R}^d)} = \|u_R^\varepsilon(0)\|_{L^2(\mathbb{R}^d)} + O \left( \frac{\varepsilon|t|}{R} \right),$$

and the  $\varepsilon$ -oscillation of  $(u_0^\varepsilon)$  implies the  $\varepsilon$ -oscillation of  $(u^\varepsilon(t))$ .

Let us now prove that we have (2.4). Let  $B \in \mathcal{C}^\infty(\mathbb{R})$  such that  $1 - \chi(\tau) = \tau B(\tau)$ , using equation (2.5), we obtain

$$(1 - \chi) (\varepsilon^2 D_t + \lambda(\varepsilon D_x)) (\phi u^\varepsilon) = \varepsilon^2 B (\varepsilon^2 D_t + \lambda(\varepsilon D_x)) (-V_{\text{ext}} \phi u^\varepsilon) + \varepsilon^2 \tilde{g}^\varepsilon,$$

where

$$\tilde{g}^\varepsilon := B (\varepsilon^2 D_t + \lambda(\varepsilon D_x)) (u^\varepsilon D_t \phi - \varepsilon g^\varepsilon \phi).$$

Clearly,  $(\tilde{g}^\varepsilon)$  is uniformly bounded in  $L^2(\mathbb{R}_t \times \mathbb{R}^d)$ . As a consequence, for  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_t)$ , we have

$$(2.7) \quad \phi u^\varepsilon = \chi (\varepsilon^2 D_t + \lambda(\varepsilon D_x)) (\phi u^\varepsilon) + \varepsilon^2 r^\varepsilon,$$



where the remainder  $r^\varepsilon$  satisfies that  $(\text{op}_\varepsilon(a)r^\varepsilon)$  is uniformly bounded in  $L^2(\mathbb{R}_t \times \mathbb{R}_x^d)$  for any  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,x,\tau,\xi}^{2d+2})$ . Let us now fix  $c > 0$  and let  $N, M > 0$  be such that for all  $\xi \in \mathbb{R}^d$ ,  $|\lambda(\xi)| \leq M(1 + |\xi|^N)$ . Because of the  $\varepsilon$ -oscillation of  $(u^\varepsilon(t))$  for all  $t \in \mathbb{R}$ , we can write for  $R, c > 0$ ,

$$\begin{aligned} (1 - \chi) \left( \frac{|\varepsilon^2 D_t| + |\varepsilon D_x|^N}{R} \right) (\phi u^\varepsilon) \\ = (1 - \chi) \left( \frac{|\varepsilon^2 D_t| + |\varepsilon D_x|^N}{R} \right) \chi \left( \frac{|\varepsilon D_x|^N}{cR} \right) (\phi u^\varepsilon) + \theta_{\varepsilon,R}, \end{aligned}$$

with  $\theta_{\varepsilon,R} = \left( 1 - \chi \left( \frac{|\varepsilon D_x|^N}{cR} \right) \right) (\phi u^\varepsilon)$ . Clearly,

$$\|\theta_{\varepsilon,R}\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)}^2 \leq \int_{\mathbb{R}} |\phi(t)|^2 \|u_{(cR)^{(1/N)}(t)}^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 dt,$$

and by (2.6), we deduce  $\limsup_{\varepsilon \rightarrow 0^+} \|\theta_{\varepsilon,R}\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} \xrightarrow{R \rightarrow \infty} 0$ . Therefore, using (2.7), we obtain in  $L^2(\mathbb{R}_t \times \mathbb{R}_x^d)$ ,

$$\begin{aligned} (1 - \chi) \left( \frac{|\varepsilon^2 D_t| + |\varepsilon D_x|^N}{R} \right) (\phi u^\varepsilon) \\ = (1 - \chi) \left( \frac{|\varepsilon^2 D_t| + |\varepsilon D_x|^N}{R} \right) \chi \left( \frac{|\varepsilon D_x|^N}{cR} \right) \chi (\varepsilon^2 D_t + \lambda(\varepsilon D_x)) (\phi u^\varepsilon) + o_{R \rightarrow \infty}(1). \end{aligned}$$

To conclude the proof, we observe that choosing  $c(M + 1) < 1/4$ , then, for every  $R > 2(2 + M)$  the support of

$$(\tau, \xi) \longmapsto (1 - \chi) \left( \frac{|\tau| + |\xi|^N}{R} \right) \chi \left( \frac{|\xi|^N}{cR} \right) \chi(\tau + \lambda(\xi))$$

is empty. This shows that for every  $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ ,

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{|\varepsilon^2 \tau| + |\varepsilon \xi|^N > R} |\widehat{\phi u^\varepsilon}(\tau, \xi)|^2 d\tau d\xi \xrightarrow{R \rightarrow \infty} 0$$

holds, which is sufficient to prove the  $(u_\varepsilon)$  satisfies (2.4).  $\square$

**2.4. Localisation of Wigner measures on the critical set.** It turns out that the fact that  $(u^\varepsilon)$  be a sequence of solutions to (2.5) imposes restrictions on the measures  $\mu_t$  that can be attained as a limit of their Wigner functions. In the region in the phase space  $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$  where equation (2.5) is dispersive (*i.e.* away from the critical points of  $\lambda$ ) the energy of the sequence  $(u^\varepsilon)$  is dispersed at infinite speed to infinity. These heuristics are made precise in the following result.

**Proposition 2.6.** *Let  $(u^\varepsilon)$  be a sequence of solutions to (2.5) issued from an  $L^2(\mathbb{R}^d)$ -bounded and  $\varepsilon$ -oscillating sequence of initial data  $(u_0^\varepsilon)$ , and suppose that the measures*

$\mu_t$  are given by Proposition 2.3. Then, for almost every  $t \in \mathbb{R}$  the measure  $\mu_t$  is supported above the set of critical points of  $\lambda$  :

$$\text{supp } \mu_t \subset \Lambda = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : \nabla \lambda(\xi) = 0\}.$$

The result of Proposition 2.6 follows from a geometric argument : the fact that  $u^\varepsilon$  are solutions to (2.5) translates in an invariance property of the measures  $\mu_t$ .

**Lemma 2.7.** *For almost every  $t \in \mathbb{R}$ , the measure  $\mu_t$  is invariant by the flow*

$$\phi_s^1 : \mathbb{R}^d \times \mathbb{R}^d \ni (x, \xi) \longmapsto (x + s\nabla \lambda(\xi), \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad s \in \mathbb{R}.$$

This means that for every function  $a$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that is Borel measurable one has:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} a \circ \phi_s^1(x, \xi) \mu_t(dx, d\xi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} a(x, \xi) \mu_t(dx, d\xi), \quad s \in \mathbb{R}.$$

This result is part of Theorem 2 in [36]. We reproduce the argument here for the reader's convenience, since we are going to use similar techniques in the sequel.

*Proof of Lemma 2.7.* It is enough to show that, for all  $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  and  $\Xi \in C_0^\infty(\mathbb{R})$ , the quantity

$$R^\varepsilon(\Xi, a) := \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} \Xi(t) \frac{d}{ds} (a \circ \phi_s^1(x, \xi)) \Big|_{s=0} W_{u^{\varepsilon_k}(t)}^{\varepsilon_k}(x, \xi) dx d\xi dt$$

tends to 0 for the subsequence  $\varepsilon_k$  of Proposition 2.3. Note that  $\frac{d}{ds} (a \circ \phi_s^1) \Big|_{s=0} = \nabla_\xi \lambda \cdot \nabla_x a = \{\lambda, a\}$ ; therefore, by the symbolic calculus of semiclassical pseudodifferential operators, Proposition A.1:

$$\text{op}_\varepsilon \left( \frac{d}{ds} (a \circ \phi_s^1) \Big|_{s=0} \right) = \frac{i}{\varepsilon} [\lambda(\varepsilon D), \text{op}_\varepsilon(a)] + O_{\mathcal{L}(L^2(\mathbb{R}^d))}(\varepsilon)$$

and, using the fact that  $(u^\varepsilon)$  solves (2.5):

$$\begin{aligned} & \frac{i}{\varepsilon} \int_{\mathbb{R}} \Xi(t) ([\lambda(\varepsilon D), \text{op}_\varepsilon(a)] u^\varepsilon(t), u^\varepsilon(t)) dt + O(\varepsilon) \\ &= -\varepsilon \int_{\mathbb{R}} \Xi(t) \frac{d}{dt} (\text{op}_\varepsilon(a) u^\varepsilon(t), u^\varepsilon(t)) dt = \varepsilon \int_{\mathbb{R}} \Xi'(t) (\text{op}_\varepsilon(a) u^\varepsilon(t), u^\varepsilon(t)) dt = O(\varepsilon). \end{aligned}$$

This estimate together with identity (2.1) show that  $R^\varepsilon(\Xi, a) = O(\varepsilon)$ , which gives the result that we wanted to prove.  $\square$

Proposition 2.6 follows easily from Lemma 2.7 and the following elementary fact.

**Lemma 2.8.** *Let  $\Omega \subset \mathbb{R}^d$  and  $\Phi_s : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d \times \Omega$  a flow satisfying: for every compact  $K \subset \mathbb{R}^d \times \Omega$  such that  $K$  contains no stationary points of  $\Phi$  there exist constants  $\alpha, \beta > 0$  such that:*

$$\alpha|s| - \beta \leq |\Phi_s(x, \xi)| \leq \alpha|s| + \beta, \quad \forall (x, \xi) \in K.$$

Let  $\mu$  be a finite, positive Radon measure on  $\mathbb{R}^d \times \Omega$  that is invariant by the flow  $\Phi_s$ . Then  $\mu$  is supported on the set of stationary points of  $\Phi_s$ .

*Proof.* It suffices to show that  $\mu(K) = 0$  for every compact set  $K \subset \mathbb{R}^d \times \Omega$  as in the statement of the lemma. By the assumption made on  $\Phi_s$ , it is possible to find  $s_k \rightarrow +\infty$  such that  $\Phi_{s_k}(K)$ ,  $k \in \mathbb{N}$ , are mutually disjoint. The invariance property of  $\mu$  implies that  $\mu(\Phi_{s_k}(K)) = \mu(K)$  and therefore, for every  $N > 0$ :

$$\mu \left( \bigcup_{k=1}^N \Phi_{s_k}(K) \right) = N\mu(K).$$

Since  $\mu$  is finite, we must have  $\mu(K) = 0$ .  $\square$

### 3. TWO-MICROLOCAL ANALYSIS OF DISPERSIVE EQUATIONS

The localization result for semiclassical measures that we obtained in the preceding section is still very far from the conclusions of Theorems 1.1 and 1.2. In particular, Proposition 2.6 does not explain how the measures  $\mu_t$  depend on the sequence of initial data of the sequence of solutions  $(u^\varepsilon)$ . For obtaining more information, we use two-microlocal tools that we introduce in a rather general framework in this section and that we use for analyzing dispersion effects and proving Theorems 1.1 and 1.2.

**3.1. Two-microlocal Wigner distributions.** From now on, we assume that  $X$  is a connected, closed embedded submanifold of  $(\mathbb{R}^d)^*$  with codimension  $p > 0$ . Given any  $\sigma \in X$ ,  $T_\sigma X$  and  $N_\sigma X$  will stand for the cotangent and normal spaces of  $X$  at  $\sigma$  respectively. Denote:

$$NX = \{(\sigma, v) : \sigma \in X, v \in N_\sigma X\}, \quad T^*X = \{(\sigma, v) : \sigma \in M, v \in T_\sigma^*X\}.$$

The tubular neighborhood theorem (see for instance [28]) ensures that there exists an open neighborhood  $U$  of  $\{(\sigma, 0) : \sigma \in X\} \subseteq NX$  such that the map:

$$U \ni (\sigma, v) \longmapsto \sigma + v \in (\mathbb{R}^d)^*,$$

is a diffeomorphism onto its image  $V$ . Its inverse is given by:

$$V \ni \xi \longmapsto (\sigma(\xi), \xi - \sigma(\xi)) \in U,$$

for some smooth map  $\sigma : V \rightarrow X$ . When  $X = \{\xi_0\}$  consists of a single point, the function  $\sigma$  is constant, identically equal to  $\xi_0$ .

We extend the phase space  $T^*\mathbb{R}^d := \mathbb{R}_x^d \times (\mathbb{R}^d)_\xi^*$  with a new variable  $\eta \in \overline{\mathbb{R}^d}$ , where  $\overline{\mathbb{R}^d}$  is the compactification of  $\mathbb{R}^d$  obtained by adding a sphere  $\mathbf{S}^{d-1}$  at infinity. The test functions associated with this extended phase space are functions  $a \in \mathcal{C}^\infty(T^*\mathbb{R}_{x,\xi}^d \times \mathbb{R}_\eta^d)$  which satisfy the two following properties:

- (1) there exists a compact  $K \subset T^*\mathbb{R}^d$  such that, for all  $\eta \in \mathbb{R}^p$ , the map  $(x, \xi) \mapsto a(x, \xi, \eta)$  is a smooth function compactly supported in  $K$ ;
- (2) there exists a smooth function  $a_\infty$  defined on  $T^*\mathbb{R}^d \times \mathbf{S}^{d-1}$  and  $R_0 > 0$  such that, if  $|\eta| > R_0$ , then  $a(x, \xi, \eta) = a_\infty(x, \xi, \eta/|\eta|)$ .

We denote by  $\mathcal{A}$  the set of such functions and for  $a \in \mathcal{A}$  we write:

$$a_\varepsilon(x, \xi) := a\left(x, \xi, \frac{\xi - \sigma(\xi)}{\varepsilon}\right).$$

Given  $f \in L^2(\mathbb{R}^d)$ , we define the two-microlocal Wigner distribution  $W_f^{X, \varepsilon} \in \mathcal{D}'(\mathbb{R}^d \times V \times \overline{\mathbb{R}^d})$  by:

$$(3.1) \quad \left\langle W_f^{X, \varepsilon}, a \right\rangle := (\text{op}_\varepsilon(a_\varepsilon)f|f)_{L^2(\mathbb{R}^d)}, \quad \forall a \in \mathcal{C}_c^\infty(\mathbb{R}^d \times V \times \overline{\mathbb{R}^d}).$$

This family of operators  $(\text{op}_\varepsilon(a_\varepsilon))_{\varepsilon>0}$  is uniformly bounded in  $L^2(\mathbb{R}^d)$ , by the Calderón-Vaillancourt theorem (see Appendix A). In addition, any function  $a \in \mathcal{C}_c^\infty(\mathbb{R}^d \times V)$  can be naturally identified to a function in  $\mathcal{C}_c^\infty(\mathbb{R}^d \times V \times \overline{\mathbb{R}^d})$  which does not depend on the last variable. For such  $a$ , one clearly has

$$\left\langle W_f^{X, \varepsilon}, a \right\rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} a(x, \xi) W_f^\varepsilon(x, \xi) dx d\xi.$$

Putting the above remarks together, one obtains the following.

**Proposition 3.1.** *Let  $(f^\varepsilon)_{\varepsilon>0}$  be bounded in  $L^2(\mathbb{R}^d)$ ; suppose in addition that this sequence has a semiclassical measure  $\mu$ . Then,  $(W_{f^\varepsilon}^{X, \varepsilon})_{\varepsilon>0}$  is a bounded sequence in  $\mathcal{D}'(\mathbb{R}^d \times V \times \overline{\mathbb{R}^d})$  whose accumulation points  $\mu^X$  satisfy:*

$$\left\langle \mu^X, a \right\rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} a(x, \xi) \mu(dx, d\xi), \quad \forall a \in \mathcal{C}_c^\infty(\mathbb{R}^d \times V).$$

The distributions  $\mu^X$  turn out to have additional structure (they are not positive measures on  $\mathbb{R}^d \times V \times \overline{\mathbb{R}^d}$ , though) and can be used to give a more precise description of the restriction  $\mu|_{\mathbb{R}^d \times X}$  of semiclassical measures. The measure  $\mu^X$  decomposes into two parts: a *compact* part, which is essentially the restriction of  $\mu^X$  to the interior  $\mathbb{R}^d \times V \times \mathbb{R}^d$  of  $\mathbb{R}^d \times V \times \overline{\mathbb{R}^d}$ , and a *part at infinity*, which corresponds to the restriction to the sphere at infinity  $\mathbb{R}^d \times V \times \mathbf{S}^{d-1}$ .

**3.2. The compact part.** For  $\sigma \in X$ , we define functions of  $L^2(N_\sigma X)$  as functions  $\mathbb{R}^p \ni z \mapsto f(z)$  where  $z$  is the parameter of a parametrization of  $N_\sigma X$ . These parametrizations depend on the system of equations of  $X$  that we choose in a neighborhood of the point  $\sigma$ . Let  $\varphi(\xi) = 0$  be such a system in an open set  $\Omega$ . Then, a parametrization of  $N_\sigma X$  associated to this system of equations is

$$N_\sigma X = \{ {}^t d\varphi(\sigma)z, \quad z \in \mathbb{R}^p \}.$$

Besides, one associate with the system  $\varphi(\xi) = 0$  a smooth map from the neighborhood  $\Omega$  of  $\sigma$  into the set of  $d \times p$  matrices such that

$$(3.2) \quad \xi - \sigma(\xi) = B(\xi)\varphi(\xi), \quad \xi \in \Omega.$$

Given a function  $a \in \mathcal{C}_c^\infty(\mathbb{R}^d \times \Omega \times \mathbb{R}^d)$  and a point  $(\sigma, v) \in TX$ , we can use the system of coordinates  $\varphi(\xi) = 0$  to define an operator acting on  $f \in L^2(N_\sigma X)$  given by:

$$Q_a^\varphi(\sigma, v)f(z) = \int_{\mathbb{R}^p \times \mathbb{R}^p} a\left(v + {}^t d\varphi(\sigma) \frac{z+y}{2}, \sigma, B(\sigma)\eta\right) f(y) e^{i\eta \cdot (z-y)} \frac{d\eta dy}{(2\pi)^p}.$$

In other words,  $Q_a^\varphi(\sigma, v)$  is obtained from  $a$  by applying the non-semiclassical Weyl quantization to the symbol  $a(v + {}^t d\varphi(\sigma) \cdot, \sigma, B(\sigma) \cdot) \in \mathcal{C}_c^\infty(\mathbb{R}^p \times \mathbb{R}^p)$ ,

$$Q_a^\varphi(\sigma, v) = a^W(v + {}^t d\varphi(\sigma)z, \sigma, B(\sigma)D_z).$$

If one changes the system of coordinates into  $\tilde{\varphi}(\xi) = 0$  on some open neighborhood  $\tilde{\Omega}$  of  $\sigma$ , then, there exists a smooth map  $R(\xi)$  defined on the open set  $\Omega \cap \tilde{\Omega}$  (where both system of coordinates can be used), and valued in the set of invertible  $p \times p$  matrices, such that  $\tilde{\varphi}(\xi) = R(\xi)\varphi(\xi)$ . One then observe that the matrix  $\tilde{B}(\xi)$  associated with the choice of  $\tilde{\varphi}$  is given by  $\tilde{B}(\xi) = B(\xi)R(\xi)^{-1}$ . Besides, for  $a \in \mathcal{C}_c^\infty(\mathbb{R}^d \times (\Omega \cap \tilde{\Omega}) \times \mathbb{R}^d)$ ,

$$\begin{aligned} Q_a^{\tilde{\varphi}}(\sigma, v) &= \int_{\mathbb{R}^p \times \mathbb{R}^p} a\left(v + {}^t d\tilde{\varphi}(\sigma) \frac{z+y}{2}, \sigma, \tilde{B}(\sigma)\eta\right) f(w) e^{i\eta \cdot (z-y)} \frac{d\eta dy}{(2\pi)^p} \\ &= \int_{\mathbb{R}^p \times \mathbb{R}^p} a\left(v + {}^t d\varphi(\sigma) {}^t R(\sigma) \frac{z+y}{2}, \sigma, B(\sigma)R(\sigma)^{-1}\eta\right) f(w) e^{i\eta \cdot (z-y)} \frac{d\eta dy}{(2\pi)^p}. \end{aligned}$$

We obtain

$$Q_a^{\tilde{\varphi}}(\sigma, v) = U(\sigma)Q_a^\varphi(\sigma, v)U^*(\sigma),$$

where  $U(\sigma)$  is the unitary operator of  $L^2(N_\sigma X) \sim L^2(\mathbb{R}^p)$  associated with the linear map from  $\mathbb{R}^p$  into itself:  $z \mapsto {}^t R(\sigma)z$ . More precisely,

$$\forall f \in L^2(\mathbb{R}^p), \quad U(\sigma)f(z) = |\det R(\sigma)|^{\frac{p}{2}} f({}^t R(\sigma)z).$$

This map is the one associated with the change of parametrization on  $N_\sigma X$  induced by turning  $\varphi$  into  $\tilde{\varphi}$ , and the map  $(z, \zeta) \mapsto ({}^t R(\sigma)z, R(\sigma)^{-1}\zeta)$  is a symplectic transform of the cotangent of  $\mathbb{R}^p$ . This is the standard rule of transformation of pseudodifferential operators through linear change of variables (see [1] for an example or any textbook about pseudodifferential calculus).

Because of this invariance property with respect to the change of system of coordinates, we shall say that  $a$  defines an operator  $Q_a$  on  $L^2(N_\sigma X)$ . Clearly,  $Q_a(\sigma, v)$  is smooth and compactly supported in  $(\sigma, v)$ ; moreover,  $Q_a(\sigma, v) \in \mathcal{K}(L^2(N_\sigma X))$ , for every  $(\sigma, v) \in TX$ , where  $\mathcal{K}(L^2(N_\sigma X))$  stands for the space of compact operators on  $L^2(N_\sigma X)$ .

**Proposition 3.2.** *Let  $\mu^X$  be given by Proposition 3.1. Then there exist a positive measure  $\nu$  on  $T^*X$  and a measurable family:*

$$M : T^*X \ni (\sigma, v) \longmapsto M(\sigma, v) \in \mathcal{L}_+^1(L^2(N_\sigma X)),$$

satisfying

$$\mathrm{Tr}_{L^2(N_\sigma X)} M(\sigma, v) = 1, \quad \text{for } \nu\text{-a.e. } (\sigma, v) \in T^*X.$$

such that, for every  $a \in C_c^\infty(\mathbb{R}^d \times V \times \mathbb{R}^d)$  one has:

$$\langle \mu^X, a \rangle = \int_{T^*X} \text{Tr}_{L^2(N_{\sigma X})}(Q_a(\sigma, v)M(\sigma, v))\nu(d\sigma, dv).$$

*Proof.* We suppose that we are given a local system of  $p$  equations of  $X$  by  $\varphi(\xi) = 0$ . Without loss of generality, we may assume that  $d_{\xi'}\varphi(\xi)$  is invertible. We consider the smooth valued function  $B$  satisfying  $\xi - \sigma(\xi) = B(\xi)\varphi(\xi)$  and we introduce the local diffeomorphism

$$\Phi : (\varphi(\xi), \xi'') \mapsto \xi.$$

Note that if  $\xi = \Phi(\zeta)$ ,  $\zeta = (\zeta', \zeta'')$ , we have  $\zeta' = \varphi(\xi) = \varphi(\Phi(\zeta))$ . We use this diffeomorphism according to the next lemma.

**Lemma 3.3.** *For all  $f \in L^2(\mathbb{R}^d)$  and  $a \in \mathcal{A}$ ,*

$$(\text{op}_\varepsilon(a_\varepsilon)f, f) = \left( \text{op}_\varepsilon \left( a \left( {}^t d\Phi(\xi)^{-1}x, \Phi(\xi), B(\Phi(\xi)) \frac{\xi'}{\varepsilon} \right) K_\varepsilon f, K_\varepsilon f \right) + O(\varepsilon)\|f\|^2 \right)$$

where  $f \mapsto K_\varepsilon f$  is an isometry of  $L^2(\mathbb{R}^d)$ .

The proof of this lemma is in the Appendix C. This lemma and the results of [15] on vector spaces give that up to a subsequence, there exist a positive measure  $\nu_0$  on  $T^*\mathbb{R}^{d-p}$  and a measurable family of trace 1 operators:

$$M_0 : T^*\mathbb{R}^{d-p} \ni (\sigma, v) \mapsto M_0(\sigma, v) \in \mathcal{L}_+^1(L^2(\mathbb{R}^p)),$$

satisfying

$$\begin{aligned} & (\text{op}_\varepsilon(a_\varepsilon)f, f) \\ &= \int_{\mathbb{R}^{d-p} \times \mathbb{R}^{d-p}} \text{Tr}_{L^2(\mathbb{R}^p)} \left( a^W \left( {}^t d\Phi(0, \theta'')^{-1}(z, u''), \Phi(0, \theta''), B(\Phi(0, \theta''))D_z \right) \right. \\ & \quad \left. \times M_0(u'', \theta'') \right) d\nu_0(du'', d\theta''). \end{aligned}$$

The reader will find in Appendix B comments on these operator-valued families. Note that the map  $\theta'' \mapsto \sigma = \Phi(0, \theta'')$  is a parametrization of  $X$  with associated parametrization of  $T^*X$ ,

$$(\theta'', u'') \mapsto (\sigma, v) = (\Phi(0, \theta''), {}^t d\Phi(0, \theta'')^{-1}(0, u'')).$$

Since the Jacobian of this mapping is 1, after a change of variable, we obtain an operator valued measurable family  $M$  on  $T^*X$  and a measure  $\nu$  on  $T^*X$  such that

$$\begin{aligned} & (\text{op}_\varepsilon(a_\varepsilon)f, f) \\ &= \int_{T^*X} \text{Tr}_{L^2(\mathbb{R}^p)} \left( a^W \left( {}^t d\Phi(0, \theta''(\sigma))^{-1}(z, 0) + v, \sigma, B(\sigma)D_z \right) M(\sigma, v) \right) d\nu(d\sigma, dv). \end{aligned}$$

We now take advantage of the fact that  $\varphi(\Phi(\zeta)) = \zeta'$  for all  $\zeta \in \mathbb{R}^d$  in order to write

$$d_\xi \varphi(\Phi(\zeta))d\Phi(\zeta) = (\text{Id}, 0).$$

We deduce

$$\forall z \in \mathbb{R}^p, \quad {}^t d\Phi(\zeta) {}^t d\varphi(\Phi(\zeta))z = (z, 0),$$

which implies

$$\forall z \in \mathbb{R}^p, \quad {}^t d\varphi(\Phi(\zeta))z = {}^t d\Phi(\zeta)^{-1}(z, 0).$$

Therefore,

$$\begin{aligned} (\text{op}_\varepsilon(a_\varepsilon)f, f) &= \int_{T^*X} \text{Tr}_{L^2(\mathbb{R}^p)} \left( a^W \left( {}^t d\varphi(\sigma)z + v, \sigma, B(\sigma)D_z \right) M(\sigma, v) \right) d\nu(d\sigma, dv) \\ &= \int_{T^*X} \text{Tr}_{L^2(N_\sigma X)} \left( Q_a(\sigma)M(\sigma, v) \right) d\nu(d\sigma, dv). \end{aligned}$$

□

**3.3. Measure structure of the part at infinity.** To analyze the part at infinity, we use a cut-off function  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  such that  $0 \leq \chi \leq 1$ ,  $\chi(\eta) = 1$  for  $|\eta| \leq 1$  and  $\chi(\eta) = 0$  for  $|\eta| \geq 2$ , and we write

$$\left\langle W_f^{X,\varepsilon}, a \right\rangle = \left\langle W_f^{X,\varepsilon}, a_R \right\rangle + \left\langle W_f^{X,\varepsilon}, a^R \right\rangle,$$

with

$$(3.3) \quad a_R(x, \xi, \eta) := a(x, \xi, \eta)\chi\left(\frac{\eta}{R}\right) \quad \text{and} \quad a^R(x, \xi, \eta) := a(x, \xi, \eta)\left(1 - \chi\left(\frac{\eta}{R}\right)\right).$$

Observe that  $a_R$  is compactly supported in all variables. We thus focus on the second part, and more precisely on the quantity

$$\limsup_{R \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \left\langle W_f^{X,\varepsilon}, a^R \right\rangle.$$

We denote by  $S\Lambda$  the compactified normal bundle to  $\Lambda$ , the fiber of which above  $\sigma$  is  $T_\sigma^*\mathbb{R}^d \times S_\sigma\Lambda$  with  $S_\sigma\Lambda$  being obtained by taking the quotient of  $N_\sigma\Lambda$  by the action of  $\mathbb{R}_+^*$  by homotheties.

**Proposition 3.4.** *Let  $(f^\varepsilon)$  be a bounded family of  $L^2(\mathbb{R}^d)$ . There exists a subsequence  $\varepsilon_k$  and a measure  $\gamma$  on  $S_\Lambda$  such that for all  $a \in \mathcal{A}$ ,*

$$\begin{aligned} \lim_{R \rightarrow \infty} \lim_{k \rightarrow +\infty} \left\langle W_{f^{\varepsilon_k}}^{X,\varepsilon_k}, a^R \right\rangle &= \int_{\mathbb{R}^d \times X \times \mathbf{S}^{d-1}} a_\infty(x, \sigma, \omega) \gamma(dx, d\sigma, d\omega) \\ &\quad + \int_{\mathbb{R}^d \times X^c \times \mathbf{S}^{d-1}} a_\infty\left(x, \xi, \frac{\xi - \sigma(\xi)}{|\xi - \sigma(\xi)|}\right) \mu(dx, d\xi). \end{aligned}$$

*Proof.* We begin by recalling the arguments that prove the existence of the measure  $\gamma$ , which are the same that the one developed in the vector case in [15]. Since  $a = a_\infty$  for  $|\eta|$  large enough, we have  $a^R = a_\infty^R$  as soon as  $R$  is large enough and the quantity

$$\limsup_{R \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \left\langle W_{f^\varepsilon}^{X,\varepsilon}, a^R \right\rangle$$

will only depend on  $a_\infty$ . Therefore, by considering a dense subset of  $\mathcal{C}_c(T^*\mathbb{R}^d \times \mathbf{S}^{d-1})$ , we can find a subsequence  $(\varepsilon_k)$  by a diagonal extraction process such that the following linear form on  $\mathcal{C}_c(T^*\mathbb{R}^d \times \mathbf{S}^{d-1})$  is well-defined

$$\ell : a_\infty \mapsto \lim_{R \rightarrow \infty} \lim_{k \rightarrow +\infty} \left\langle W_{f^{\varepsilon_k}}^{X, \varepsilon_k}, a^R \right\rangle.$$

We then observe that

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \exists C_{\alpha, \beta} > 0, \quad \sup_{\mathbb{R}^{2d}} \left| \partial_x^\alpha \partial_\xi^\beta (a^R)_\varepsilon \right| \leq C_{\alpha, \beta} (\varepsilon^{|\beta|} + R^{-|\beta|}).$$

This implies that the symbolic calculus on symbols  $(a^R)_\varepsilon$  is semiclassical with respect to the small parameter  $\sqrt{\varepsilon^2 + R^{-2}}$ . To be precise, one has the following weak Garding inequality: if  $a \geq 0$ , then, for all  $\kappa > 0$ , there exists a constant  $C_\kappa$  such that

$$\left\langle W_{f^\varepsilon}^{X, \varepsilon}, a^R \right\rangle \geq - \left( \kappa + C_\kappa \left( \varepsilon + \frac{1}{R} \right) \right) \|f^\varepsilon\|_{L^2(\mathbb{R}^d)}^2.$$

We then conclude that the linear form  $\ell$  defined above is positive and defines a positive Radon measure  $\tilde{\rho}$ . It remains to decompose  $\tilde{\rho}$  above  $X^c$ . In this purpose, we set

$$a^{R, \delta}(x, \xi, \eta) = a^R(x, \xi, \eta) \left(1 - \chi\left(\frac{\xi - \sigma(\xi)}{\delta}\right)\right)$$

and we observe that, by the definition of  $\mu$ :

$$\lim_{\delta \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \left\langle W_{f^{\varepsilon_k}}^{X, \varepsilon_k}, a^{R, \delta} \right\rangle = \int_{\mathbb{R}^d \times X^c \times \mathbf{S}^{d-1}} a_\infty \left( x, \xi, \frac{\xi - \sigma(\xi)}{|\xi - \sigma(\xi)|} \right) \mu(dx, d\xi),$$

which concludes the proof of the existence of the measure  $\gamma$ .

Let us now analyze the geometric properties of this measure. We choose a system of local coordinates of  $\Lambda$  and introduce the matrix  $B$  as in (3.2). By Lemma 3.3 and the result of [15] for vector spaces: up to a subsequence, there exists a measure  $\tilde{\gamma}_0$  on  $\mathbb{R}^d \times \mathbb{R}^{d-p} \times \mathbf{S}^{p-1}$  such that

$$\begin{aligned} & \lim_{\delta \rightarrow 0^+} \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \left\langle W_f^{X, \varepsilon}, a^{R, \delta} \right\rangle \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^{d-p} \times \mathbf{S}^{p-1}} a_\infty \left( {}^t d\Phi(0, \xi'')^{-1} x, \Phi(0, \xi''), \frac{B(\Phi(0, \xi'')) \omega}{|B(\Phi(0, \xi'')) \omega|} \right) \tilde{\gamma}_0(dx, d\xi, d\omega). \end{aligned}$$

The mapping  $\xi'' \mapsto \Phi(0, \xi'')$  is a parametrization of  $X$  and the mapping  $(x, \xi) \mapsto ({}^t d\Phi(0, \xi'')^{-1} x, \Phi(0, \xi''))$  is the associated mapping of  $T_X^* \mathbb{R}^d$ . Therefore, this relation defines a measure  $\tilde{\gamma}$  on  $T^*X \times \mathbf{S}^{p-1}$  such that

(3.4)

$$\lim_{\delta \rightarrow 0^+} \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \left\langle W_f^{X, \varepsilon}, a^{R, \delta} \right\rangle = \int_{T^*X \times \mathbf{S}^{p-1}} a_\infty \left( x, \sigma, \frac{B(\sigma) \omega}{|B(\sigma) \omega|} \right) \tilde{\gamma}(dx, d\xi, d\omega).$$

Besides, using that  $\text{Id} = d\sigma(\sigma_0) + B(\sigma_0)d\varphi(\sigma_0)$  for any  $\sigma_0 \in X$ , we deduce that for any  $\zeta \in T_{\sigma_0} \mathbb{R}^d$ , we have the decomposition

$$\zeta = d\sigma(\sigma_0)\zeta + B(\sigma_0)d\varphi(\sigma_0)\zeta, \quad \text{with } d\sigma(\sigma_0)\zeta \in T_{\sigma_0} X \quad \text{and} \quad B(\sigma_0)d\varphi(\sigma_0)\zeta \in N_{\sigma_0} X.$$



Now, since  $d\varphi$  is of rank  $p$ , one can write any  $\omega \in \mathbf{S}^{p-1}$  as  $\omega = d\varphi(\sigma_0)\zeta$  and the points  $B(\sigma_0)\omega$  are in  $N_{\sigma_0}X$ . By identification of  $\gamma$  in (3.4), we deduce that  $\gamma(x, \sigma, \cdot)$  is a measure on the set

$$\left\{ \frac{B(\sigma)\omega}{|B(\sigma)\omega|}, \omega \in \mathbf{S}^{p-1} \right\} = N_{\sigma}X / \mathbb{R}_+^* = S_{\sigma}X,$$

which completes the proof of the proposition.  $\square$

**3.4. Two microlocal Wigner measures and families of solutions to dispersive equations.** We now consider families of solutions to equation (2.5). As proved in Proposition 2.6, the Wigner measure of the family  $(u^{\varepsilon}(t))$  concentrates on the set  $\Lambda = \{\nabla\lambda(\xi) = 0\}$ . To analyze  $\mu^t$  above  $\Lambda$ , we perform a second microlocalization above the set  $X = \Lambda$ , with average in time. We consider for  $\theta \in L^1(\mathbb{R})$  the quantities

$$\int_{\mathbb{R}} \theta(t) \left\langle W_{u^{\varepsilon}(t)}^{\Lambda, \varepsilon}, a \right\rangle dt$$

for symbols  $a \in \mathcal{A}$ . Up to extracting a subsequence  $\varepsilon_k$ , we construct  $L^{\infty}$  maps

$$t \mapsto \gamma_t(dx, d\sigma, d\omega), \quad t \mapsto \nu_t(d\sigma, dv), \quad t \mapsto M_t(\sigma, v)$$

valued respectively on the set of positive Radon measures on  $\mathbb{R}^d \times \Lambda \times \mathbf{S}^{d-1}$ , on the set of positive Radon measures on  $T^*\Lambda$  and finally on the set of measurable families from  $T^*\Lambda$  onto the set of trace class operators on  $L^2(N\Lambda)$ , such that for all  $\theta \in L^1(\mathbb{R})$  and for all  $a \in \mathcal{S}^0(d)$ :

$$\begin{aligned} \int_{\mathbb{R}} \theta(t) \left\langle W_{u^{\varepsilon_k}(t)}^{\Lambda, \varepsilon_k}, a \right\rangle dt &\xrightarrow{k \rightarrow +\infty} \int_{\mathbb{R}} \int_{\mathbb{R}^d \times \Lambda \times \mathbf{S}^{d-1}} \theta(t) a_{\infty}(x, \sigma, \omega) \gamma_t(dx, d\sigma, d\omega) dt \\ &\quad + \int_{\mathbb{R}} \int_{T^*\Lambda} \theta(t) \text{Tr}_{L^2(N_{\sigma}\Lambda)}(Q_a(\sigma, v) M_t(\sigma, v) \nu_t(d\sigma, dv)) dt. \end{aligned}$$

The measures  $\gamma^t$  and  $\nu^t$ , and the map  $M^t$  satisfy additional properties coming from the fact that the family  $(u^{\varepsilon}(t))$  solves a time-dependent equation. These properties are discussed in the next two sections. We shall see that we can choose the sequence  $\varepsilon_k$  such that the map  $t \mapsto M_t$  is continuous (and even  $\mathcal{C}^1$ ), and the measures  $\gamma_t$  are invariant under a linear flow. These properties are discussed in the next two sections.

**3.5. Transport properties of the compact part.** Since  $\Lambda$  is the set of critical points of  $\lambda$ , the matrix  $d^2\lambda$  is intrinsically defined above points of  $\Lambda$ . Thus, using the formalism of the preceding sections, it defines an order two differential operator  $Q_{d^2\lambda(\sigma)\eta\cdot\eta}$  which acts on functions on  $\mathcal{S}(N_{\sigma}\Lambda)$ . We write abusively

$$Q_{d^2\lambda(\sigma)\eta\cdot\eta} = d^2\lambda(\sigma) D_z \cdot D_z.$$

**Proposition 3.5.** *The map  $t \mapsto \nu_t$  is constant and the map*

$$t \mapsto M_t(\sigma, v) \in \mathcal{C}(\mathbb{R}; \mathcal{L}_+^1(L^2(N_{\sigma}\Lambda)))$$

*solves the Heisenberg equation (1.16).*

*Proof.* We analyze for  $a \in \mathcal{C}_0^\infty(\mathbb{R}^{3d})$  the time evolution of the quantity  $\left\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, a \right\rangle$ . We have

$$\frac{d}{dt} \left\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, a \right\rangle = \frac{1}{i\varepsilon^2} ([\text{op}_\varepsilon(a_\varepsilon), \lambda(\varepsilon D)] u^\varepsilon(t), u^\varepsilon(t)) + \frac{1}{i} ([\text{op}_\varepsilon(a_\varepsilon), V_{\text{ext}}] u^\varepsilon(t), u^\varepsilon(t)).$$

By standard symbolic calculus for Weyl quantization, we have in  $\mathcal{L}(L^2(\mathbb{R}^d))$

$$\frac{1}{i\varepsilon^2} [\text{op}_\varepsilon(a_\varepsilon), \lambda(\varepsilon D)] = \frac{1}{\varepsilon} \text{op}_\varepsilon(\nabla \lambda(\xi) \cdot \nabla_x a_\varepsilon) + O(\varepsilon).$$

Besides, by Taylor formula and by use of  $\nabla \lambda(\sigma(\xi)) = 0$ , we have

$$(3.5) \quad \nabla \lambda(\xi) = d^2 \lambda(\sigma(\xi)) (\xi - \sigma(\xi)) + \Gamma(\xi) (\xi - \sigma(\xi)) \cdot (\xi - \sigma(\xi)),$$

where  $\Gamma$  is a smooth matrix. This yields

$$\frac{1}{\varepsilon} \nabla \lambda(\xi) \cdot \nabla_x a_\varepsilon(x, \xi) = b_\varepsilon(x, \xi)$$

with

$$b(x, \xi, \eta) = d^2 \lambda(\sigma(\xi)) \eta \cdot \nabla_x a(x, \xi, \eta) + \Gamma(\xi) (\xi - \sigma(\xi)) \cdot \eta \nabla_x a(x, \xi, \eta).$$

At this stage of the proof, we see that  $\frac{d}{dt} \left\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, a \right\rangle$  is uniformly bounded in  $\varepsilon$ , thus using a suitable version of Ascoli's theorem and a standard diagonal extraction argument, we can find a sequence  $(\varepsilon_k)$  such that the limit exists for all  $a \in \mathcal{C}_0^\infty(\mathbb{R}^{3d})$  and all time  $t \in [0, T]$  (for some  $T > 0$  fixed) with a limit that is a continuous map in time. The transport equation that we are now going to prove shall guarantee the unicity of the limit.

We observe that for any local system of equations of  $\Lambda$ ,  $\varphi(\xi) = 0$ , the operator  $Q_b^\varphi$  satisfies for  $(\sigma, v) \in T\Lambda$ ,

$$\begin{aligned} Q_b^\varphi(\sigma, v) &= b^W(v + {}^t d\varphi(\sigma)z, \sigma, B(\sigma)D_z) \\ &= \text{op}_1(d^2 \lambda(\sigma)B(\sigma)\eta \cdot \nabla_x a(v + {}^t d\varphi(\sigma)z, \sigma, B(\sigma)\eta)). \end{aligned}$$

On the other hand, we observe that, setting

$$\theta(\xi, \eta) = \frac{1}{2} d^2 \lambda(\xi) \eta \cdot \eta,$$

we have

$$(3.6) \quad \begin{aligned} i [Q_\theta^\varphi(\sigma), Q_b^\varphi(\sigma, v)] &= i [{}^t B(\sigma) d^2 \lambda(\sigma) B(\sigma) D_z \cdot D_z, Q_a^\varphi(\sigma, v)] \\ &= \text{op}_1({}^t d\varphi(\sigma) {}^t B(\sigma) d^2 \lambda(\sigma) B(\sigma) \eta \cdot \nabla_x a(v + {}^t d\varphi(\sigma)z, \sigma, B(\sigma)\eta)), \end{aligned}$$

and we now focus on the matrix  ${}^t d\varphi(\sigma) {}^t B(\sigma) d^2 \lambda(\sigma) B(\sigma)$ , and thus on the properties of the hessian  $d^2 \lambda(\sigma)$ .

For  $\xi \in \Lambda$ , the bilinear form  $d^2\lambda(\xi)$  is defined intrinsically on  $T_\xi\mathbb{R}^d$  and  $d^2\lambda(\xi) = 0$  on  $T_\xi\Lambda$ . We can associate to the function  $\xi \mapsto \sigma(\xi)$  a decomposition of  $T\mathbb{R}^d$  above points  $\xi \in \mathbb{R}^d$  as

$$\begin{aligned} T_\xi\mathbb{R}^d &\rightarrow T_{\sigma(\xi)}\Lambda \times N_{\sigma(\xi)}\Lambda \\ (\xi, \zeta) &\mapsto (d\sigma(\xi)\zeta, (\text{Id} - d\sigma(\xi))\zeta) \end{aligned}$$

By construction, if  $\xi \in \Lambda$ , we have

$$\text{Id} - d\sigma(\xi) = B(\xi)d\varphi(\xi).$$

We deduce that any  $\zeta \in T_\xi\mathbb{R}^d$  satisfies

$$\zeta = d\sigma(\xi)\zeta + B(\xi)d\varphi(\xi)\zeta \quad \text{with } d\sigma(\xi)\zeta \in T_\sigma\Lambda.$$

Therefore,

$$\forall \xi \in \Lambda, \quad d^2\lambda(\xi) = d^2\lambda(\xi)B(\xi)d\varphi(\xi).$$

Taking into account this information, equation (3.6) becomes

$$i[Q_\theta^\varphi(\sigma), Q_b^\varphi(\sigma, v)] = \text{op}_1(d^2\lambda(\sigma)B(\sigma)\eta \cdot \nabla_x a(v + {}^t d\varphi(\sigma)z, \sigma, B(\sigma)\eta)).$$

We conclude

$$Q_b^\varphi(\sigma, v) = i[Q_\theta^\varphi(\sigma), Q_b^\varphi(\sigma, v)].$$

This implies that

$$i\partial_t(M_t(\sigma, v)\nu_t(d\sigma, dv)) = \left[ \frac{1}{2}d^2\lambda(\sigma)D_z \cdot D_z + m_{V_{\text{ext}}(t, \cdot)}(v, \sigma), M_t(\sigma, v) \right] \nu_t(d\sigma, dv).$$

Taking the trace, we get  $\partial_t\nu_t = 0$ , thus  $\nu_t$  is equal to some constant measure  $\nu$  and  $M_t$  satisfies equation (1.16), which proves the proposition.  $\square$

**3.6. Invariance and localization of the measure at infinity.** We are concerned with the property of the  $L^\infty$ -map  $t \mapsto \gamma^t(dx, d\sigma, d\omega)$  valued in the set of positive Radon measures on  $\mathbb{R}^d \times \Lambda \times \mathbf{S}^{d-1}$ . We now define a flow on  $\mathbb{R}^d \times \Lambda \times \mathbf{S}^{d-1}$  by setting for  $s \in \mathbb{R}$

$$\phi_2^s : (x, \sigma, \omega) \mapsto (x + s d^2\lambda(\sigma)\omega, \sigma, \omega).$$

**Proposition 3.6.** *The measure  $\gamma^t$  is invariant by the flow  $\phi_2^s$ .*

*Proof.* We use the cut-off function  $\chi$  introduced before and set

$$a^{R, \delta}(x, \xi, \eta) = a(x, \xi, \eta) \chi\left(\frac{\xi - \sigma(\xi)}{\delta}\right) \left(1 - \chi\left(\frac{\eta}{R}\right)\right);$$

we introduce the smooth symbol

$$b_s^{R, \delta}(x, \xi, \eta) = a^{R, \delta}\left(x + s d^2\lambda(\xi)\frac{\eta}{|\eta|}, \xi, \eta\right),$$

which satisfies  $(b_s^R)_\infty = a_\infty \circ \phi_2^s$ . Using equation (3.5), we obtain

$$(b_s^{R, \delta})_\varepsilon(x, \xi) = a^{R, \delta}\left(x + \frac{s}{|\xi - \sigma(\xi)|} \nabla\lambda(\xi), \xi, \frac{\xi - \sigma(\xi)}{\varepsilon}\right) + \delta r_\varepsilon^{R, \delta}(x, \xi)$$

where for all multi-index  $\alpha, \beta \in \mathbb{N}^d$ , there exists a constant  $C_{\alpha, \beta} > 0$  such that  $r_\varepsilon^{R, \delta}$  satisfies:

$$\sup_{x, \xi \in \mathbb{R}^d} \left| \partial_x^\alpha \partial_\xi^\beta r_\varepsilon^{R, \delta} \right| \leq C_{\alpha, \beta}.$$

As a consequence,  $\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, r_\varepsilon^{R, \delta} \rangle$  is uniformly bounded in  $R, \delta, \varepsilon$  and:

$$\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, b_s^{R, \delta} \rangle = \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_s^{R, \delta} \rangle + O(\delta),$$

uniformly with respect to  $R$  and  $\varepsilon$ , with

$$\tilde{b}_s^{R, \delta}(x, \xi, \eta) = a^{R, \delta} \left( x + \frac{s}{|\xi - \sigma(\xi)|} \nabla \lambda(\xi), \xi, \eta \right).$$

Note that this symbol is smooth because  $|\xi - \sigma(\xi)| > R\varepsilon$  on the support of  $a^{R, \delta}$ . We are going to prove that for all  $\theta \in C_0^\infty(\mathbb{R})$ ,

$$\lim_{\delta \rightarrow 0^+} \lim_{R \rightarrow \infty} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \theta(t) \frac{d}{ds} \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_s^{R, \delta} \rangle dt = 0.$$

Indeed, by the calculus of the preceding section, we have

$$\frac{d}{ds} \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_s^{R, \delta} \rangle = \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \nabla \lambda \cdot \nabla_x c_s^{R, \delta} \rangle$$

with

$$c_s^{R, \delta}(x, \xi, \eta) = \frac{1}{|\xi - \sigma(\xi)|} a^{R, \delta} \left( x + \frac{s}{|\xi - \sigma(\xi)|} \nabla \lambda(\xi), \xi, \eta \right).$$

The symbol  $c_s^{R, \delta}$  is such that for all all multi-index  $\alpha \in \mathbb{N}^d$ , there exists a constant  $C_\alpha > 0$  for which:

$$\sup_{x, \xi \in \mathbb{R}^d} \left| \partial_x^\alpha (c_s^{R, \delta})_\varepsilon \right| \leq C_\alpha (R\varepsilon)^{-1}.$$

This implies in particular:

$$\left\| \text{op}_\varepsilon((c_s^{R, \delta})_\varepsilon) \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \frac{C}{R\varepsilon}.$$

By symbolic calculus, we have

$$\frac{1}{i\varepsilon} \left[ \text{op}_\varepsilon((c_s^{R, \delta})_\varepsilon), \lambda(\varepsilon D) \right] = \text{op}_\varepsilon(\nabla \lambda(\xi) \cdot \nabla_x (c_s^{R, \delta})_\varepsilon) + O\left(\frac{\varepsilon}{R}\right).$$

We deduce that for all  $\theta \in C_0^\infty(\mathbb{R})$ ,

$$\begin{aligned}
& \int_{\mathbb{R}} \theta(t) \frac{d}{ds} \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_s^{R, \delta} \rangle dt \\
&= \int_{\mathbb{R}} \theta(t) \left( \frac{1}{i\varepsilon} [\text{op}_\varepsilon((c_s^{R, \delta})_\varepsilon), \lambda(\varepsilon D)] u^\varepsilon(t), u^\varepsilon(t) \right) dt + O\left(\frac{\varepsilon}{R}\right) \\
&= \int_{\mathbb{R}} \theta(t) \left( \frac{1}{i\varepsilon} [\text{op}_\varepsilon((c_s^{R, \delta})_\varepsilon), \lambda(\varepsilon D) + \varepsilon^2 V_{\text{ext}}(t, x)] u^\varepsilon(t), u^\varepsilon(t) \right) dt + O\left(\frac{1}{R}\right) \\
&= -\varepsilon \int_{\mathbb{R}} \theta(t) \frac{d}{dt} (\text{op}_\varepsilon((c_s^{R, \delta})_\varepsilon) u^\varepsilon(t), u^\varepsilon(t)) dt + O\left(\frac{1}{R}\right) \\
&= O(\varepsilon) + O\left(\frac{1}{R}\right).
\end{aligned}$$

As a conclusion,

$$\begin{aligned}
\langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, b_s^{R, \delta} \rangle &= \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_s^{R, \delta} \rangle + O(\delta) \\
&= \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, \tilde{b}_0^{R, \delta} \rangle + O(|s|\varepsilon) + O(|s|R^{-1}) + O(\delta) \\
&= \langle W_{u^\varepsilon(t)}^{\Lambda, \varepsilon}, b_0^{R, \delta} \rangle + O(|s|\varepsilon) + O(|s|R^{-1}) + O(\delta),
\end{aligned}$$

which implies the Proposition.  $\square$

**3.7. Proofs of Theorems 1.1 and 1.2.** Remind that, as explained in Remark 1.3, Theorem 1.2 implies Theorem 1.1. Indeed, if  $\Lambda$  reduces to a set of isolated points, the operator  $M_t$  is the orthogonal projector of  $L^2(\mathbb{R}^d)$  on the function  $u_\xi(t)$  and the equations (1.14) and (1.16) are equivalent.

Let us now prove Theorem 1.2. We first observe that the measure  $\gamma_t$  is zero. Indeed, by **H2**; for  $\sigma \in \Lambda$ ,  $d^2\lambda(\sigma)$  is one to one on  $N_\sigma\Lambda$ . Therefore, since  $\gamma_t$  is a measure on  $S\Lambda$ , the invariance property of Proposition 3.6 and an argument similar to the one of Lemma 2.8 yields that  $\gamma_t = 0$ . As a consequence, the semi-classical measure  $\mu_t$  is only given by the compact part and one has for any  $a \in C_0^\infty(\mathbb{R}^{2d})$  and  $\phi \in L^1(\mathbb{R})$ ,

$$\int_{\mathbb{R}} \phi(t) \int_{\mathbb{R}^{2d}} a(x, \xi) \mu^t(dx, d\xi) = \int_{\mathbb{R}} \phi(t) \int_{T^*\Lambda} \text{Tr}_{L^2(N_\sigma\Lambda)} (Q_a(\sigma, v) M_t(d\sigma, dv)) dt.$$

Then, taking  $\phi = \mathbf{1}_{[a, b]}$  for  $a, b \in \mathbb{R}$ ,  $a < b$ , and in view of Proposition 2.3 and of Lemma 2.5, we deduce that for every every  $\phi \in C_c(\mathbb{R}^d)$  one has for the subsequence defining  $M_t$  and  $\nu_t$ :

$$\lim_{\varepsilon \rightarrow 0} \int_a^b \int_{\mathbb{R}^d} \phi(x) |u^\varepsilon(t, x)|^2 dx dt = \int_a^b \int_{T^*\Lambda} \text{Tr}_{L^2(N_\xi\Lambda)} [Q_\phi(v, \xi) M_t(v, \xi)] \nu(dv, d\xi) dt,$$

where  $M_t$  satisfies (1.16). This concludes the proof of Theorem 1.2. We emphasize that the measure  $\nu$  and the operator valued family  $M_0$  are utterly determined by the initial data.

#### 4. SEMICLASSICAL MEASURES FOR THE BLOCH DECOMPOSITION

In this section we prove Theorem 1.6, which requires in particular to sum series induced by the decomposition of the data in equation (1.19). The proof combines three ingredients. We first analyse the high-frequency behavior of the spectral projectors associated to the Bloch decomposition  $\Pi_{\lambda_j}^\varepsilon$  (see (1.18)) and of the operator of restriction to the diagonal  $y = \frac{x}{\varepsilon}$ . Second, we prove *a priori* estimates for solutions of equation (1.7) and we study the decomposition of  $\psi^\varepsilon$  to (1.2) as:

$$\psi^\varepsilon = \sum_{n=1}^{\infty} \psi_n^\varepsilon, \quad \text{where} \quad \psi_n^\varepsilon = \Pi_{\lambda_n}^\varepsilon U^\varepsilon|_{y=x/\varepsilon},$$

$U^\varepsilon$  being the solution to (1.7) whose initial datum is  $\psi^\varepsilon|_{t=0} \otimes 1$ . Finally, the last step of the proof consists in computing the Wigner measure of  $(\psi^\varepsilon)$ . We shall prove that it is the sum of the Wigner measures of some solutions to dispersive equations with initial data  $\psi_n^\varepsilon|_{t=0}$ .

A function  $U \in L^2(\mathbb{R}_x^d \times \mathbb{T}_y^d)$  can be written as:

$$U(x, y) = \sum_{k \in \mathbb{Z}^d} U_k(x) e^{i2\pi k \cdot y},$$

with:

$$\|U\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 = \sum_{k \in \mathbb{Z}^d} \|U_k\|_{L^2(\mathbb{R}^d)}^2.$$

We denote by  $H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)$ , for  $s \geq 0$ , the Sobolev space consisting of those functions  $U \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$  such that:

$$\|U\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}^2 := \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} (1 + |\varepsilon \xi|^2 + |k|^2)^s |\widehat{U}_k(\xi)|^2 d\xi.$$

We will repeatedly use the following remark.

*Remark 4.1.* There exists a constant  $c > 0$  such that:

$$c^{-1} \|U\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} \leq \| \langle \varepsilon D_x \rangle^s U \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + \| P(\varepsilon D_x)^{s/2} U \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq c \|U\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)},$$

for every  $U \in L^2(\mathbb{R}^d \times \mathbb{T}^d)$  and  $\varepsilon > 0$ , where, as usual,

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2}.$$

**4.1. High frequency behavior of Bloch-band projectors.** Here we shall gather some results describing how the  $\varepsilon$ -oscillation property behaves under the action of the operators  $P_{\varphi_n}^\varepsilon$  that are used in the proof of Theorem 1.6.

We start by analysing the boundedness properties of the spectral projectors. We shall use the following remark

*Remark 4.2.* Let  $j \in \mathbb{N}$ . Any  $\xi \in \mathbb{R}^d$  writes  $\xi = \eta + 2\pi k$  with  $k \in \mathbb{Z}^d$  and  $\eta$  in the Brillouin zone. Therefore, the relation  $\varphi_j(y, \eta + 2k\pi) = e^{-i2\pi k \cdot y} \varphi_j(y, \eta)$  implies the existence of a constant  $C > 0$  such that for all  $(\xi, y) \in \mathbb{R}^d \times \mathbb{T}^d$  and  $s \geq 0$ ,

$$|D_y^s \varphi_j(y, \xi)| \leq C \langle \xi \rangle^s \quad \text{and} \quad |D_\xi^s \varphi_n(y, \xi)| \leq C.$$

**Lemma 4.3.** *Given  $s \geq 0$ , there exists  $C_s > 0$  such that:*

$$\|P_{\varphi_j}^\varepsilon U\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} \leq C_s \|U\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)},$$

for every  $j \in \mathbb{N}$ ,  $\varepsilon > 0$  and  $U \in H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)$ .

*Proof.* Since  $[P(\varepsilon D_x)^{s/2}, P_{\varphi_j}^\varepsilon] = 0$ , one has:

$$\|P(\varepsilon D_x)^{s/2} P_{\varphi_j}^\varepsilon U\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \|P(\varepsilon D_x)^{s/2} U\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}.$$

A direct calculation gives, for every  $u \in L^2(\mathbb{R}^d)$ ,

$$\|\varphi_j(\cdot, \varepsilon D_x) u\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}.$$

On the other hand, as  $\|\varphi_j(\cdot, \xi)\|_{L^2(\mathbb{T}^d)} = 1$  for any  $\xi \in \mathbb{R}^d$ , one has for any  $U \in H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)$ :

$$\left\| \langle \varepsilon D_x \rangle^s \int_{\mathbb{T}^d} \overline{\varphi_j(z, \varepsilon D_x)} U(\cdot, z) dz \right\|_{L^2(\mathbb{R}^d)} \leq \| \langle \varepsilon D_x \rangle^s U \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}.$$

Combining this fact with the estimate valid for any  $u \in L^2(\mathbb{R}^d)$ ,

$$\begin{aligned} \|\langle \varepsilon D_x \rangle^s \varphi_j(\cdot, \varepsilon D_x) u\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} &= \|\varphi_j(\cdot, \varepsilon D_x) \langle \varepsilon D_x \rangle^s u\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \\ &\leq \| \langle \varepsilon D_x \rangle^s u \|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

gives the bound:

$$\| \langle \varepsilon D_x \rangle^s P_{\varphi_j}^\varepsilon U \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \| \langle \varepsilon D_x \rangle^s U \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}.$$

The conclusion follows by Remark 4.1. □

Strongly  $\varepsilon$ -oscillating sequences enjoy good frequency localization properties with respect to the Bloch decomposition. This is the content of the following lemma, that will be used to reduce our analysis to that of sequences of initial data that are a superposition of a finite number of Bloch waves. Set, for a Bloch energy  $\lambda_n$ , with  $n \in \mathbb{N}$ :

$$I_n := \{j \in \mathbb{N} : P(\xi) \varphi_j(\xi, \cdot) = \lambda_n(\xi) \varphi_j(\xi, \cdot)\};$$

Clearly  $\Pi_{\lambda_n}^\varepsilon := \sum_{j \in I_n} P_{\varphi_j}^\varepsilon$ .

**Lemma 4.4.** *Let  $(v^\varepsilon)$  be a bounded sequence in  $L^2(\mathbb{R}^d)$  that is strongly  $\varepsilon$ -oscillating. Then, for every  $s \geq 0$ ,*

$$\limsup_{\varepsilon \rightarrow 0^+} \sum_{n > N} \|\Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}^2 \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* By Remark 4.1, it is enough to prove the result for

$$\|P(\varepsilon D_x)^k \Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \quad \text{and} \quad \|\langle \varepsilon D_x \rangle^{2k} \Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2$$

for all  $k \in \mathbb{N}$  and the result will follow for

$$\|\Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{H_x^k(\mathbb{R}^d \times \mathbb{T}^d)}^2.$$

One shall then conclude by standard interpolation. We write

$$(4.1) \quad \widehat{\Pi_{\lambda_n}^\varepsilon v^\varepsilon}(\xi, y) = \sum_{j \in I_n} \varphi_j(y, \varepsilon \xi) \left( \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \varepsilon \xi) dz \right) \widehat{v}^\varepsilon(\xi).$$

Therefore, for any  $k \in \mathbb{N}$  and  $y \in \mathbb{T}^d$ ,

$$(4.2) \quad P(\varepsilon \xi)^k \widehat{\Pi_{\lambda_n}^\varepsilon v^\varepsilon}(\xi, y) = \lambda_n(\varepsilon \xi)^k \sum_{j \in I_n} \varphi_j(y, \varepsilon \xi) \left( \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \varepsilon \xi) dz \right) \widehat{v}^\varepsilon(\xi).$$

For every  $\xi \in \mathbb{R}^d$  one has

$$\sum_{n \in \mathbb{N}} \left| \lambda_n(\xi)^k \sum_{j \in I_n} \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \xi) dz \right|^2 = \|P(\xi)^k 1\|_{L^2(\mathbb{T}^d)}^2 = p_{4k}(\xi),$$

where  $p_{4k}$  is a non-negative polynomial of degree at most  $4k$ . Dini's theorem ensures that for every  $R > 0$  the following holds:

$$D_N(R) := \sup_{|\xi| \leq R} \sum_{n > N} \left| \lambda_n(\xi)^k \sum_{j \in I_n} \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \xi) dz \right|^2 \xrightarrow{N \rightarrow \infty} 0.$$

Let  $\delta > 0$  and choose  $R > 0$  such that for  $0 \leq j \leq k$ :

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{|\varepsilon \xi| > R} p_{4j}(\varepsilon \xi) |\widehat{v}^\varepsilon(\xi)|^2 d\xi < \delta$$

(this is where the strong  $\varepsilon$ -oscillation hypothesis comes into play). Then, using (4.1) and (4.2) we can estimate:

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0^+} \sum_{n > N} \|P(\varepsilon D_x)^k \Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2 \\ & \leq C \limsup_{\varepsilon \rightarrow 0^+} \sum_{n > N} \int_{|\varepsilon \xi| \leq R} \left| \lambda_n(\varepsilon \xi)^k \sum_{j \in I_n} \int_{\mathbb{T}^d} \overline{\varphi_j}(z, \varepsilon \xi) dz \right|^2 |\widehat{v}^\varepsilon(\xi)|^2 \frac{d\xi}{(2\pi)^d} + C\delta \\ & \leq C D_N(R) \limsup_{\varepsilon \rightarrow 0^+} \|v^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 + C\delta \xrightarrow{N \rightarrow \infty} C\delta. \end{aligned}$$

Since  $\delta > 0$  is arbitrary, this proves the result when  $s \geq 0$  is an even integer. For dealing



with  $\|\langle \varepsilon D_x \rangle^{2k} \Pi_{\lambda_n}^\varepsilon v^\varepsilon\|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)}^2$ , we observe that

$$\langle \varepsilon D_x \rangle^{2k} \widehat{\Pi_{\lambda_n}^\varepsilon v^\varepsilon}(\xi, y) = \langle \varepsilon \xi \rangle^{2k} \sum_{j \in I_n} \varphi_j(y, \varepsilon \xi) \left( \int_{\mathbb{T}^d} \overline{\varphi_j(z, \varepsilon \xi)} dz \right) \widehat{v}^\varepsilon(\xi),$$

and we argue as before to conclude the proof.  $\square$

*Remark 4.5.* If one only assumes that  $(v^\varepsilon)$  is  $\varepsilon$ -oscillatory then one gets the conclusion of Lemma 4.4 in the particular case  $s = 0$ .

**4.2. High frequency behavior of the operator of restriction to the diagonal.** We now focus on the properties of the operator of restriction to the diagonal  $y = x/\varepsilon$ , i.e. under the action of the operator  $L^\varepsilon$  that maps a function  $U$  defined on  $\mathbb{R}^d \times \mathbb{T}^d$  to:

$$L^\varepsilon U(x) := U\left(x, \frac{x}{\varepsilon}\right).$$

**Lemma 4.6.** *Suppose  $s > d/2$ , then operator*

$$L^\varepsilon : L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d)) \longrightarrow L^2(\mathbb{R}^d)$$

*is uniformly bounded in  $\varepsilon$ .*

*Moreover, if  $U^\varepsilon \in L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d))$  satisfies the estimate:*

$$(4.3) \quad \limsup_{\varepsilon \rightarrow 0^+} \|\mathbb{1}_R(\varepsilon D_x) U^\varepsilon\|_{L^2(\mathbb{R}^d; H^s(\mathbb{T}^d))} \xrightarrow{R \rightarrow \infty} 0,$$

*where  $\mathbb{1}_R$  is the characteristic function of  $\{|\xi| > R\}$ , then the sequence  $(L^\varepsilon U^\varepsilon)$  is bounded in  $L^2(\mathbb{R}^d)$  and  $\varepsilon$ -oscillating.*

*Proof.* Let  $U^\varepsilon \in L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d))$  and write it as

$$U^\varepsilon(x, y) = \sum_{k \in \mathbb{Z}^d} U_k^\varepsilon(x) e^{i2\pi k \cdot y},$$

and

$$\|U\|_{L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d))}^2 = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2s} \|U_k\|_{L^2(\mathbb{R}^d)}^2.$$

Then there exist constants  $C, C_{d,s} > 0$  such that

$$\sum_{k \in \mathbb{Z}^d} \|U_k^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C \left( \sum_{k \in \mathbb{Z}^d} |k|^{2s} \|U_k^\varepsilon\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} \leq C_{d,s} \|U^\varepsilon\|_{L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d))},$$

and therefore:

$$(4.4) \quad \|L^\varepsilon U^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \sum_{k \in \mathbb{Z}^d} \|U_k^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_{d,s} \|U^\varepsilon\|_{L^2(\mathbb{R}_x^d; H^s(\mathbb{T}_y^d))}.$$

Let us now show that, under the hypothesis of the proposition,  $v^\varepsilon := L^\varepsilon U^\varepsilon$  defines an  $\varepsilon$ -oscillating sequence. Given  $\delta > 0$ , since  $s > d/2$ , there exists  $N_\delta > 0$  such that

$$\sum_{|k| > N_\delta} |k|^{-2s} < \delta^2.$$

Define:

$$v_\delta^\varepsilon(x) = \sum_{|k| \leq N_\delta} U_k^\varepsilon(x) e^{i2\pi k \cdot \frac{x}{\varepsilon}}.$$

Clearly,

$$\|v^\varepsilon - v_\delta^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \delta \|U^\varepsilon\|_{L^2(\mathbb{R}^d; H^s(\mathbb{T}_y^d))}.$$

Therefore, it suffices to show that for any  $\delta > 0$  the sequence  $(v_\delta^\varepsilon)$  is  $\varepsilon$ -oscillating. The Fourier transform of  $v_\delta^\varepsilon$  is:

$$\widehat{v_\delta^\varepsilon}(\xi) = \sum_{|k| \leq N_\delta} \widehat{U_k^\varepsilon} \left( \xi - \frac{2\pi k}{\varepsilon} \right).$$

Therefore,

$$\|\mathbf{1}_R(\varepsilon D_x) v_\delta^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \sum_{|k| \leq N_\delta} \|\mathbf{1}_R(\varepsilon D_x + 2\pi k) U_k^\varepsilon\|_{L^2(\mathbb{R}^d)}.$$

If  $R > R_0$  for  $R_0 > 0$  large enough, one has  $\mathbf{1}_R(\cdot + 2\pi k) \leq \mathbf{1}_{R/2}$  for every  $|k| \leq N_\delta$ . This allows us to conclude that for  $R > R_0$ :

$$\|\mathbf{1}_R(\varepsilon D_x) v_\delta^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \sum_{|k| \leq N_\delta} \|\mathbf{1}_{R/2}(\varepsilon D_x) U_k^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_{d,s} \|\mathbf{1}_R(\varepsilon D_x) U^\varepsilon\|_{L^2(\mathbb{R}^d; H^s(\mathbb{T}^d))}$$

and the conclusion follows.  $\square$

Combining Lemmas 4.3 and 4.6 yields the following result.

**Corollary 4.7.** *If  $(\langle \varepsilon D_x \rangle^s v^\varepsilon)$  is bounded in  $L^2(\mathbb{R}^d)$  for some  $s > d/2$  then the sequence  $(w^\varepsilon)$  defined by:*

$$w^\varepsilon(x) := L^\varepsilon P_{\varphi_j}^\varepsilon v^\varepsilon(x) = P_{\varphi_j}^\varepsilon v^\varepsilon \left( x, \frac{x}{\varepsilon} \right), \quad j \in \mathbb{N}$$

is  $\varepsilon$ -oscillating.

**4.3. A priori estimates.** In the proof of Theorem 1.6, we shall use the following a priori estimates for the solutions of equation (1.7).

**Lemma 4.8.** *Given  $s \geq 0$ , there exists a constant  $C_s > 0$  such that any solution  $U^\varepsilon$  to (1.7) with initial datum  $U_0^\varepsilon \in H^s(\mathbb{R}^d \times \mathbb{T}^d)$  satisfies:*

$$(4.5) \quad \|U^\varepsilon(t, \cdot)\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} \leq \|U_0^\varepsilon\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} + C_s \varepsilon |t|,$$

uniformly in  $\varepsilon > 0$ .

*Proof.* In view of Remark 4.1, we are first going to study the families  $(\langle \varepsilon D_x \rangle U^\varepsilon)$  and  $(P(\varepsilon D_x)^{1/2} U^\varepsilon)$ . Start noticing that  $\langle \varepsilon D_x \rangle U^\varepsilon$  satisfies the equation

$$i\varepsilon^2 \partial_t (\langle \varepsilon D_x \rangle U^\varepsilon) = P(\varepsilon D_x) (\langle \varepsilon D_x \rangle U^\varepsilon) + \varepsilon^2 V_{\text{ext}} \langle \varepsilon D_x \rangle U^\varepsilon + \varepsilon^2 [V_{\text{ext}}, \langle \varepsilon D_x \rangle] U^\varepsilon.$$

As a consequence, using the boundedness of  $\nabla_x V_{\text{ext}}$  on  $\mathbb{R} \times \mathbb{R}^d$ , we obtain by the symbolic calculus of semiclassical pseudodifferential operators, that the source term can be estimated by:

$$\| [V_{\text{ext}}(t, \cdot), \langle \varepsilon D_x \rangle] U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq C\varepsilon \| U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)},$$

for some constant  $C >$  independent of  $\varepsilon > 0$  and  $t \in \mathbb{R}$ . Using standard energy estimates, we deduce the existence of a constant  $C_1 > 0$  such that for all  $t \in \mathbb{R}$ ,

$$\| \langle \varepsilon D_x \rangle U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \| \langle \varepsilon D_x \rangle U_0^\varepsilon \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + C_1 \varepsilon |t|.$$

A completely analogous argument yields the estimate:

$$\| P(\varepsilon D_x)^{1/2} U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \| P(\varepsilon D_x)^{1/2} U_0^\varepsilon \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + C_1 \varepsilon |t|.$$

A standard recursive argument gives, for all  $s \in \mathbb{N}$ , the existence of a constant  $C_s > 0$  such that for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} \| \langle \varepsilon D_x \rangle^s U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + \| P(\varepsilon D_x)^{s/2} U^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} \leq \\ \| \langle \varepsilon D_x \rangle^s U_0^\varepsilon \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + \| P(\varepsilon D_x)^{s/2} U_0^\varepsilon \|_{L^2(\mathbb{R}^d \times \mathbb{T}^d)} + C_s \varepsilon |t|, \end{aligned}$$

and the result follows for any  $s \in \mathbb{R}^+$  by interpolation.  $\square$

The proof of Lemma 4.8 shows that the following remark holds.

*Remark 4.9.* Given  $s \geq 0$ , let  $(V^\varepsilon(t, \cdot))$  solve

$$i\varepsilon^2 \partial_t V^\varepsilon = P(\varepsilon D_x) V^\varepsilon + V_{\text{ext}} V^\varepsilon + \varepsilon^3 G^\varepsilon,$$

with initial data  $V_0^\varepsilon$  and  $\sup_{t \in \mathbb{R}} \| G^\varepsilon(t, \cdot) \|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}$  uniformly bounded in  $\varepsilon$ . Then, there exists a constant  $\tilde{C}_s > 0$  such that

$$\| V^\varepsilon(t, \cdot) \|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} \leq \| V_0^\varepsilon \|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} + \tilde{C}_s \varepsilon |t|.$$

**4.4. Decomposition of  $\psi^\varepsilon$ .** Let  $(\psi_0^\varepsilon)$  be a strongly  $\varepsilon$ -oscillating sequence satisfying the assumptions of Theorem 1.6. Denote by  $U^\varepsilon$  the solution to (1.7) whose initial datum is  $U_0^\varepsilon := \psi_0^\varepsilon \otimes 1$ . Clearly,  $(U_0^\varepsilon)$  is bounded in  $H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)$  for every  $s \geq 0$  and, by Lemma 4.8, generates a bounded family  $(U^\varepsilon(t, \cdot))$  in  $H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)$ . By uniqueness of solutions to (1.2), the solution  $\psi^\varepsilon$  to (1.2) whose initial datum is  $\psi_0^\varepsilon$  must satisfy:

$$\psi^\varepsilon(t, \cdot) = L^\varepsilon U^\varepsilon(t, \cdot), \quad t \in \mathbb{R}.$$

Define

$$(4.6) \quad \psi_n^\varepsilon(t, \cdot) := L^\varepsilon \Pi_{\lambda_n}^\varepsilon U^\varepsilon(t, \cdot) = \sum_{j \in I_n} L^\varepsilon P_{\varphi_j}(y, \varepsilon D_x) U^\varepsilon(t, \cdot).$$

**Lemma 4.10.** For every  $n \in \mathbb{N}$ ,  $\psi_n^\varepsilon$  solves an equation of the form

$$(4.7) \quad i\varepsilon^2 \partial_t \psi_n^\varepsilon(t, x) = \lambda_n(\varepsilon D_x) \psi_n^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) \psi_n^\varepsilon(t, x) + \varepsilon^2 f_n^\varepsilon(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where for all  $a \in C_0^\infty(\mathbb{R}^{2d})$ , uniformly in  $t \in \mathbb{R}$ , we have

$$\| \text{op}_\varepsilon(a) f_n^\varepsilon(t, \cdot) \|_{L^2(\mathbb{R}^d)} \leq C_n(a) \varepsilon$$

for some constant  $C_n(a) > 0$ . Besides, for all  $t \in \mathbb{R}$ ,  $(\psi_n^\varepsilon(t, \cdot))$  is  $\varepsilon$ -oscillating.

*Proof.* Define for  $j \in I_n$ ,

$$u_j^\varepsilon(t, x) := \int_{\mathbb{T}^d} \overline{\varphi}_j(y, \varepsilon D_x) U^\varepsilon(t, x, y) dy, \quad \psi_n^\varepsilon(t, x) = \sum_{j \in I_n} \varphi_j\left(\frac{x}{\varepsilon}, \varepsilon D_x\right) u_j^\varepsilon(t, x).$$

Since  $(U^\varepsilon)$  solves (1.7) and

$$U^\varepsilon(t, x, y) = \sum_{n \in \mathbb{N}} \sum_{j \in I_n} \varphi_j(y, \varepsilon D_x) u_j^\varepsilon(t, x),$$

it turns out that  $u_j^\varepsilon$  solves:

$$i\varepsilon^2 \partial_t u_j^\varepsilon(t, x) = \lambda_n(\varepsilon D_x) u_j^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) u_j^\varepsilon(t, x) + \varepsilon^2 g_j^\varepsilon(t, x),$$

where:

$$g_j^\varepsilon := \int_{\mathbb{T}^d} [\overline{\varphi}_j(y, \varepsilon D_x), V_{\text{ext}}] U^\varepsilon(\cdot, y) dy.$$

Since  $\lambda_j(\xi)$  is  $2\pi\mathbb{Z}^d$ -periodic, it is easy to check that:

$$[L^\varepsilon \varphi_j(\cdot, \varepsilon D_x), \lambda_j(\varepsilon D_x)] = 0.$$

This implies that:

$$i\varepsilon^2 \partial_t \psi_n^\varepsilon(t, x) = \lambda_n(\varepsilon D_x) \psi_n^\varepsilon(t, x) + \varepsilon^2 V_{\text{ext}}(t, x) \psi_n^\varepsilon(t, x) + \varepsilon^2 f_n^\varepsilon(t, x),$$

with

$$f_n^\varepsilon := \sum_{j \in I_n} \int_{\mathbb{T}^d} \left[ \varphi_j\left(\frac{\cdot}{\varepsilon}, \varepsilon D_x\right) \overline{\varphi}_j(y, \varepsilon D_x), V_{\text{ext}} \right] U^\varepsilon(\cdot, y) dy.$$

Arguing as in the proof of Lemma 4.3 and using symbolic calculus for semiclassical pseudodifferential operators, we obtain that for all  $a \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ ,

$$\|\text{op}_\varepsilon(a) f_n^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C_a \varepsilon,$$

uniformly in  $t \in \mathbb{R}$ .

Finally, note that the initial data of equation (4.7) are  $\varepsilon$ -oscillating by Corollary 4.7 and the same holds for  $(\psi_n^\varepsilon(t, \cdot))$  for all  $t \in \mathbb{R}$  by the argument used at the beginning of the proof of Remark 2.5.  $\square$

**Lemma 4.11.** *For all  $t \in \mathbb{R}$  and all  $N \in \mathbb{N}$ , there exists  $C_N > 0$  such that for all  $t \in \mathbb{R}$ ,*

$$(4.8) \quad \left\| \sum_{n > N} \psi^\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} \leq r_N^\varepsilon + C_N \varepsilon |t|, \quad \limsup_{\varepsilon \rightarrow 0^+} r_N^\varepsilon \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* Lemma 4.6 gives that for  $s > \frac{d}{2}$ , there exists  $C > 0$  such that

$$\left\| \sum_{n > N} \psi^\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} = \left\| L \left( \sum_{n > N} \Pi_{\lambda_n}^\varepsilon U^\varepsilon(t, \cdot) \right) \right\|_{L^2(\mathbb{R}^d)} \leq C \left\| \sum_{n > N} \Pi_{\lambda_n}^\varepsilon U^\varepsilon(t, \cdot) \right\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}.$$

We then apply Remark 4.9 to the family  $V^\varepsilon(t, \cdot) = \sum_{n>N} \Pi_{\lambda_n}^\varepsilon U^\varepsilon(t, \cdot)$  for which

$$G^\varepsilon(t, x) = \left[ \sum_{n>N} \Pi_{\lambda_n}^\varepsilon, V_{\text{ext}}(t, x) \right] U^\varepsilon(t, x) = - \sum_{n \leq N} [\Pi_{\lambda_n}^\varepsilon, V_{\text{ext}}(t, x)] U^\varepsilon(t, x).$$

We deduce the existence of  $C_N > 0$  such that

$$\left\| \sum_{n>N} \Pi_{\lambda_n}^\varepsilon U^\varepsilon(t, \cdot) \right\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)} \leq C_N \varepsilon |t| + \left\| \sum_{n>N} \Pi_{\lambda_n}^\varepsilon U^\varepsilon(0, \cdot) \right\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}.$$

For concluding, we observe that  $U^\varepsilon(0, \cdot) = \psi_0^\varepsilon \otimes 1$  is constant in the variable  $y$  and, by Remark 4.1 and Lemma 4.4, we can write for every  $s > 0$ ,

$$\limsup_{\varepsilon \rightarrow 0^+} \left\| \sum_{n>N} \Pi_{\lambda_n}^\varepsilon (\psi_0^\varepsilon \otimes 1) \right\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}^2 \leq C \limsup_{\varepsilon \rightarrow 0^+} \sum_{n>N} \left\| \Pi_{\lambda_n}^\varepsilon (\psi_0^\varepsilon \otimes 1) \right\|_{H_\varepsilon^s(\mathbb{R}^d \times \mathbb{T}^d)}^2 \xrightarrow{N \rightarrow \infty} 0,$$

where we have used the orthogonality of the eigenprojectors  $\Pi_{\lambda_n}^\varepsilon$ , and the commutation of the operators  $P(\varepsilon D)$  and  $\langle \varepsilon D_x \rangle^s$  with  $\Pi_{\lambda_n}^\varepsilon$ .  $\square$

**Corollary 4.12.** *As a consequence*

$$\lim_{N \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \left\| \psi^\varepsilon(t, \cdot) - \sum_{0 \leq n \leq N} \psi_n^\varepsilon(t, \cdot) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

**4.5. The Wigner measure of  $\psi^\varepsilon$  and the accumulation points of  $|\psi^\varepsilon|^2$ .** Our first result gives a description of the time and space Wigner measure of a sequence of solutions to (1.2) in terms of the Bloch decomposition.

**Proposition 4.13.** *Let  $(\psi^\varepsilon)$  be a sequence of solutions to (1.2) issued from an initial data satisfying the assumptions of Theorem 1.6. Then there exist a subsequence such that, for every  $\phi \in C_0^\infty(\mathbb{R}_t)$  and  $a \in C_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$ :*

$$\lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k}) \phi \psi^{\varepsilon_k}, \phi \psi^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = \sum_{n \in \mathbb{N}} \int_{\mathbb{R} \times \mathbb{R}^{2d}} a(t, \lambda_n(\xi), x, \xi) |\phi(t)|^2 \mu_t^n(dx, d\xi) dt,$$

where, for each  $n \in \mathbb{N}$ , the measures  $\mu_t^n \in \mathcal{M}_+(\mathbb{R}^d \times \mathbb{R}^d)$  are supported on  $\{(x, \xi) : \nabla \lambda_n(\xi) = 0\}$  and, for every  $b \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_{x,\xi}^{2d})$ :

$$(4.9) \quad \lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(b) \psi_n^{\varepsilon_k}, \psi_n^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = \int_{\mathbb{R} \times \mathbb{R}^{2d}} b(t, x, \xi) \mu_t^n(dx, d\xi) dt.$$

*Proof.* We apply Proposition 2.1 to obtain a subsequence such that, for every  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_t)$  and  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$ :

$$(4.10) \quad \lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi^{\varepsilon_k}, \phi\psi^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} a(t, \tau, x, \xi) |\phi(t)|^2 \tilde{\mu}(dt, d\tau, dx, d\xi),$$

for some positive measure  $\tilde{\mu} \in \mathcal{M}_+(\mathbb{R}^2 \times \mathbb{R}^{2d})$ . From the decomposition (4.6), one also has:

$$(4.11) \quad (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi^{\varepsilon_k}, \phi\psi^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = \sum_{m,n \in \mathbb{N}} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi_n^{\varepsilon_k}, \phi\psi_m^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} + o(\varepsilon).$$

We claim the following:

i) If  $n \neq m$  then for all  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$

$$(4.12) \quad \lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi_n^{\varepsilon_k}, \phi\psi_m^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = 0.$$

ii) If  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  is such that  $a(t, \tau, x, \xi) = 0$  in a neighbourhood of  $\{\tau = \lambda_n(\xi)\}$  then:

$$(4.13) \quad \lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi_n^{\varepsilon_k}, \phi\psi_n^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = 0.$$

Before proving these claims, we show how the result follows from i) and ii). Let us denote

$$D_m := \{(\tau, \xi) : \tau = \lambda_m(\xi)\}.$$

Notice that if  $\tau$  varies in a compact set, then there exists a finite number of eigenvalues  $\lambda_n(\xi)$  such that  $\tau = \lambda_n(\xi)$ . Let  $n \in \mathbb{N}$ ; by hypothesis **H0'**, there exists a neighbourhood  $U_n$  of  $D_n$  such that  $D_m \cap U_n = \emptyset$  for every  $m \in \mathbb{N}$ ,  $n \neq m$ . Let  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  such that  $a(t, \tau, x, \xi) = 0$  if  $(\tau, \xi) \notin U_n$ . Then (4.11) together with i) and ii) imply:

$$\lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi^{\varepsilon_k}, \phi\psi^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)} = \lim_{k \rightarrow \infty} (\text{op}_{\varepsilon_k}(a^{\varepsilon_k})\phi\psi_n^{\varepsilon_k}, \phi\psi_n^{\varepsilon_k})_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)}.$$

Since, by Corollary 4.7,  $(\psi_n^\varepsilon|_{t=0})$  is  $\varepsilon$ -oscillating, Lemma 4.10 combined with Propositions 2.2, 2.3 and 2.6 imply that the limits satisfy:

$$\begin{aligned} \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} a(t, \tau, x, \xi) |\phi(t)|^2 \tilde{\mu}(dt, d\tau, dx, d\xi) \\ = \int_{\mathbb{R} \times \mathbb{R}^{2d}} a(t, \lambda_n(\xi), x, \xi) |\phi(t)|^2 \mu_t^n(dx, d\xi) dt, \end{aligned}$$

where the measures  $\mu_t^n$  are supported over the set of critical points of  $\lambda_n$ . If  $a \in \mathcal{C}_0^\infty(\mathbb{R}_{t,\tau}^2 \times \mathbb{R}_{x,\xi}^{2d})$  is arbitrary, one covers the support of  $a$  by a finite union of neighbourhoods  $U_n$  of  $D_n$  such that  $U_n \cap D_m = \emptyset$  for  $m \neq n$  and decomposes  $a$  as a sum of functions supported on each  $U_n$ . The result then follows from the previous computation.

Claim ii) simply follows from Proposition 2.2. To prove claim i) consider the sequence of distributions:

$$\left\langle \tilde{W}_{m,n}^\varepsilon, a \right\rangle := (\text{op}_{\varepsilon_k}(a^\varepsilon) \phi \psi_n^\varepsilon, \phi \psi_m^\varepsilon)_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d)},$$

where  $n \neq m$ . All accumulation points of  $(\tilde{W}_{\phi \psi_n^\varepsilon}^\varepsilon)$  and  $(\tilde{W}_{\phi \psi_m^\varepsilon}^\varepsilon)$  are measures supported on disjoint sets, by Proposition 2.2. It follows from Proposition 3.3 in [24] that along any convergent subsequence  $(\tilde{W}_{m,n}^{\varepsilon_k})$  converges to zero. Therefore,  $(\tilde{W}_{m,n}^\varepsilon)$  itself converges to zero.  $\square$

The next result shows how Wigner measures can be used to obtain the limit of the position densities  $|\psi^\varepsilon|^2$ .

**Proposition 4.14.** *Let  $(\psi^\varepsilon)$  be a sequence of solutions to (1.2) issued from a strongly  $\varepsilon$ -oscillating sequence of initial data. Let  $(\varepsilon_k)$  and  $\mu^n$  be as in Proposition (4.13). Suppose in addition that  $(|\psi^{\varepsilon_k}|^2)$  converges in  $\mathcal{M}_+(\mathbb{R}_t \times \mathbb{R}_x^d)$  to a measure  $\nu$ . Then,*

$$\int_{\mathbb{R} \times \mathbb{R}^d} \phi(t, x) \nu(dt, dx) = \sum_{n \in \mathbb{N}} \int_{\mathbb{R} \times \mathbb{R}^{2d}} \phi(t, x) \mu_t^n(dx, d\xi) dt.$$

*Proof.* Set  $g_N^\varepsilon = \sum_{n=1}^N \psi_n^\varepsilon$ . Since the  $\psi_n^\varepsilon$  are  $\varepsilon$ -oscillating by Lemma 4.10, Remark 2.5, Proposition 2.1 and equation (4.13) give

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R} \times \mathbb{R}^d} \phi(t, x) |g_N^{\varepsilon_k}(t, x)|^2 dx dt = \sum_{n \in \mathbb{N}, n \leq N} \int_{\mathbb{R} \times \mathbb{R}^{2d}} \phi(t, x) \mu_t^n(dx, d\xi) dt.$$

Then Corollary 4.12 implies:

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R} \times \mathbb{R}^d} \phi(t, x) |\psi^{\varepsilon_k}(t, x)|^2 dx dt = \lim_{N \rightarrow \infty} \sum_{n \in \mathbb{N}, n \leq N} \int_{\mathbb{R} \times \mathbb{R}^{2d}} \phi(t, x) \mu_t^n(dx, d\xi) dt.$$

$\square$

**4.6. Proofs of Theorems 1.6 and 1.7, and of Proposition 1.8.** Let us begin with Theorems 1.6 and 1.7

*Proof.* Assume we have **H0'**. Then, by Proposition 4.14, there exists a subsequence  $\varepsilon_k$  (eventually extracted from the preceding  $\varepsilon_k$ ) such that for any  $\phi \in C_0^\infty(\mathbb{R}_x^d)$  and for any real numbers  $a < b$ ,

$$\lim_{k \rightarrow \infty} \int_a^b \int_{\mathbb{R}^d} \phi(x) |\psi^{\varepsilon_k}(t, x)|^2 dx = \sum_{n \in \mathbb{N}} \int_a^b \int_{\mathbb{R}_x^d} \phi(x) \mu_t^n(dx, d\xi).$$

By equation (4.9), the  $\varepsilon$ -oscillation of  $\psi_n^\varepsilon$ , equation (4.7) and Remark 2.5,  $\mu^n$  is a Wigner measure of the family  $(\psi_n^\varepsilon)$ . The analysis of Section 3 has precisely described the structure of Wigner measures associated with families of  $\varepsilon$ -oscillating families of solutions

to equations of the form (4.7): there exist positive measures  $\nu_n \in \mathcal{M}_+(T\Lambda_n)$ , and a measurable family of self-adjoint, positive, trace-class operators

$M_n(t) : T_\xi \Lambda_n \ni (z, \xi) \mapsto M_n(t)(z, \xi) \in \mathcal{L}_+^1(L^2(N_\xi \Lambda_n))$ ,  $\text{Tr}_{L^2(N_\xi \Lambda_n)} M_n(t, z, \xi) = 1$ , satisfying the Heisenberg equation (1.21) and such that for all  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_x^d)$ ,

$$\int_a^b \int_{\mathbb{R}_x^d} \phi(x) \mu_t^n(dx, d\xi) dt = \int_a^b \int_{\mathbb{R}_x^d} \text{Tr}_{L^2(N_\xi \Lambda_n)} (m_\phi(z, \xi) M_n(t, z, \xi)) \nu_n(dz, dt)$$

with the notations of the Introduction. This concludes the proof of Theorem 1.6.

Theorem 1.7 comes from the fact that the operator  $M_n$  is of rank one as soon as the set  $\Lambda_n$  consists of isolated points as described in Section 3.  $\square$

Proposition 1.8 comes from computing the weak limits appearing in Theorem 1.1 in the context of Bloch waves, which are those appearing in Theorem 1.7.

**Proposition 4.15.** *Let  $\xi \in \mathbb{R}^d$  and suppose that for every  $k \in \mathbb{Z}^d$  the functions*

$$x \mapsto \psi_0^\varepsilon(x) e^{-i \frac{\xi - 2\pi k}{\varepsilon} \cdot x}$$

*converge weakly, as  $\varepsilon \rightarrow 0^+$ , to some limit  $\psi_0^k$ . Then  $e^{-i \frac{\xi}{\varepsilon} \cdot x} L^\varepsilon P_{\varphi_j}^\varepsilon (\psi_0^\varepsilon \otimes 1)$  converge weakly, as  $\varepsilon \rightarrow 0^+$ , to:*

$$c^j(\xi) \sum_{k \in \mathbb{Z}^d} \overline{c^j(\xi - 2\pi k)} \psi_0^k, \quad c^j(\xi) := \int_{\mathbb{T}^d} \varphi_j(z, \xi) dz.$$

*Proof.* If  $\varphi_j(y, \xi)$  is a Bloch wave then it satisfies the Bloch periodicity condition:

$$(4.14) \quad \varphi_j(y, \xi + 2\pi k) = e^{-i 2\pi k \cdot y} \varphi_j(y, \xi).$$

Integrating (4.14) with respect to  $y$  on  $\mathbb{T}^d$  gives an expression for the Fourier coefficients of  $\varphi_j(\cdot, \xi)$ . Therefore:

$$(4.15) \quad \varphi_j(y, \xi) = \sum_{k \in \mathbb{Z}^d} c^j(\xi + 2\pi k) e^{i 2\pi k \cdot y}.$$

Recall that

$$L^\varepsilon P_{\varphi_j}^\varepsilon (\psi_0^\varepsilon \otimes 1) = \varphi_j\left(\frac{x}{\varepsilon}, \varepsilon D_x\right) \int_{\mathbb{T}^d} \overline{\varphi_j(z, \varepsilon D_x)} \psi_0^\varepsilon(x) dz$$

Using formulas (4.14) and (4.15) we deduce:

$$L^\varepsilon P_{\varphi_j}^\varepsilon (\psi_0^\varepsilon \otimes 1) = c^j(\varepsilon D_x) \sum_{k \in \mathbb{Z}^d} \overline{c^j(\varepsilon D_x - 2\pi k)} e^{i 2\pi k \cdot \frac{x}{\varepsilon}} \psi_0^\varepsilon(x),$$

and therefore:

$$e^{-i \frac{\xi}{\varepsilon} \cdot x} L^\varepsilon P_{\varphi_j}^\varepsilon (\psi_0^\varepsilon \otimes 1) = c^j(\varepsilon D_x + \xi) \sum_{k \in \mathbb{Z}^d} \overline{c^j(\varepsilon D_x - 2\pi k + \xi)} e^{-i \frac{\xi - 2\pi k}{\varepsilon} \cdot x} \psi_0^\varepsilon(x).$$

The result then follows by taking limits.  $\square$



## APPENDIX A. SEMICLASSICAL PSEUDODIFFERENTIAL OPERATORS

In this appendix we recall a few basic notions on the theory of pseudodifferential operators that we use through this article. The reader can consult the references [1, 17, 20, 41, 47] additional background and for proofs of the results that follow.

Recall that given a function  $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  that is bounded together with its derivatives (we denote the space of all such functions by  $S$ ), one defines the semiclassical pseudodifferential operator of symbol  $a$  obtained through the Weyl quantization rule to be the operator  $\text{op}_\varepsilon(a)$  that acts on functions  $f \in \mathcal{S}(\mathbb{R}^d)$  by:

$$\text{op}_\varepsilon(a)f(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} a\left(\frac{x+y}{2}, \varepsilon\xi\right) e^{i\xi \cdot (x-y)} f(y) dy \frac{d\xi}{(2\pi)^d}.$$

These operators are bounded in  $L^2(\mathbb{R}^d)$ . The Calderón-Vaillancourt theorem [13] ensures the existence of a constant  $C_d > 0$  such that for every  $a \in S$  one has

$$(A.1) \quad \|\text{op}_\varepsilon(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_d N_d(a),$$

where

$$N_d(a) := \sum_{\alpha \in \mathbb{N}^{2d}, |\alpha| \leq 6d} \sup_{\mathbb{R}^d \times \mathbb{R}^d} |\partial_{x,\xi}^\alpha a|.$$

We make use repeatedly of the following result, known as the symbolic calculus for pseudodifferential operators.

**Proposition A.1.** *Let  $a, b \in S$ , then*

$$\text{op}_\varepsilon(a)\text{op}_\varepsilon(b) = \text{op}_\varepsilon(ab) + \frac{\varepsilon}{2i}\text{op}_\varepsilon(\{a, b\}) + \varepsilon^2 R_\varepsilon^{(2)},$$

with  $\{a, b\} = \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b$  and

$$[\text{op}_\varepsilon(a), \text{op}_\varepsilon(b)] = \frac{\varepsilon}{i}\text{op}_\varepsilon(\{a, b\}) + \varepsilon^3 R_\varepsilon^{(3)},$$

$$\|R_\varepsilon^{(j)}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \sup_{|\alpha|+|\beta|=j} N_d(\partial_\xi^\alpha \partial_x^\beta a) N_d(\partial_\xi^\beta \partial_x^\alpha b), \quad j = 1, 2,$$

for some constant  $C > 0$  independent of  $a, b$  and  $\varepsilon$ .

We also need the formula for the conjugation of a pseudodifferential operator with a linear oscillating phase:

$$e_{\xi_0}^\varepsilon(x) := e^{i\frac{\xi_0}{\varepsilon} \cdot x}.$$

For every  $f \in L^2(\mathbb{R}^d)$  and  $a \in S$  one has:

$$(A.2) \quad \text{op}_\varepsilon(a)(e_{\xi_0}^\varepsilon f) = e_{\xi_0}^\varepsilon \text{op}(a_\varepsilon(\cdot, \cdot + \xi_0))f.$$

APPENDIX B. TRACE OPERATOR-VALUED MEASURES

In this appendix we recall general considerations on operator-valued measures. Let  $X$  be a complete metric space and  $(Y, \sigma)$  a measure space; write  $\mathcal{H} := L^2(Y, \sigma)$  and denote by  $\mathcal{L}^1(\mathcal{H})$ ,  $\mathcal{K}(\mathcal{H})$  and  $\mathcal{L}(\mathcal{H})$  the spaces of trace-class, compact and bounded operators on  $\mathcal{H}$  respectively. A trace-operator valued Radon measure on  $X$  is a linear functional:

$$M : \mathcal{C}_0(X) \longrightarrow \mathcal{L}^1(\mathcal{H})$$

satisfying the following boundedness condition. For every compact  $K \subset X$  there exist a constant  $C_K > 0$  such that:

$$\mathrm{Tr} |M(\phi)| = C_K \sup_K |\phi|, \quad \forall \phi \in \mathcal{C}_0(X).$$

Such an operator-valued measure is positive if for every  $\phi \geq 0$ ,  $M(\phi)$  is an Hermitian positive operator. Let  $M$  be a positive trace operator-valued measure on  $X$ , denote by  $\nu \in \mathcal{M}_+(X)$  the positive real measure defined by:

$$\int_X \phi(x) \nu(dx) = \mathrm{Tr} M(\phi), \quad \forall \phi \in \mathcal{C}_0(X).$$

The Radon-Nikodym theorem for operator valued measures (see, for instance, the appendix in [25]) ensures the existence of a  $\nu$ -locally integrable function:

$$Q : X \longmapsto \mathcal{L}^1(\mathcal{H}), \quad \mathrm{Tr} Q(x) = 1, \quad Q(x) \text{ positive Hermitian for } \nu\text{-a.e. } x \in X,$$

such that:

$$M(\phi) = \int_X \phi(x) Q(x) \nu(dx), \quad \forall \phi \in \mathcal{C}_0(X).$$

Note that this formula implies that  $M$  can be identified to a positive element of the dual of  $\mathcal{C}_0(X; \mathcal{K}(\mathcal{H}))$  via:

$$\langle M, T \rangle \equiv \int_X \mathrm{Tr}[T(x)M(dx)] := \int_X \mathrm{Tr}(T(x)Q(x))\nu(dx), \quad T \in \mathcal{C}_0(X; \mathcal{K}(\mathcal{H})).$$

It can be also shown that every such positive functional arises in this way. Let  $(e_j(x))_{j \in \mathbb{N}}$  denote an orthonormal basis of  $\mathcal{H}$  consisting of eigenfunctions of  $Q(x)$ :

$$Q(x)e_j(x) = \varrho_j(x)e_j(x), \quad \sum_{j=1}^{\infty} \varrho_j(x) = 1, \quad \nu\text{-a.e.}$$

Clearly, both  $\varrho_j$  and  $e_j$ ,  $j \in \mathbb{N}$ , are locally  $\nu$ -integrable and

$$Q(x) = \sum_{j=1}^{\infty} \varrho_j(x) |e_j(x)\rangle \langle e_j(x)|, \quad \nu\text{-a.e.},$$

where, as usual,  $|e_j(x)\rangle \langle e_j(x)|$  denotes the orthogonal projection in  $\mathcal{H}$  onto  $e_j(x)$ . Moreover, as a consequence of the monotone convergence theorem, the following result easily follows.

**Lemma B.1.** *Let  $M$  be a positive trace operator-valued measure on  $X$ . Then there exist  $a$  and a non-negative function  $\rho \in L^1_{\text{loc}}(X, \nu; L^1(Y, \sigma))$  such that, for every  $a \in C_0(X; L^\infty(Y, \sigma))$  one has:*

$$\int_X \text{Tr}[m_a(x)M(dx)] = \int_X \int_Y a(x, y)\rho(x, y)\sigma(dy)\nu(dx),$$

where  $m_a(x)$  denotes the operator acting on  $\mathcal{H}$  by multiplication by  $a(x, \cdot)$ . The density  $\rho$  is given by:

$$\rho(x, y) = \sum_{j=1}^{\infty} \varrho_j(x) |e_j(x, y)|^2.$$

### APPENDIX C. PROOF OF LEMMA 3.3

We denote by  $\mathcal{F}_\varepsilon$  the semi-classical Fourier transform defined for  $f \in L^2(\mathbb{R}^d)$  by

$$\mathcal{F}_\varepsilon f(\xi) = (2\pi\varepsilon)^{-d/2} \widehat{f}\left(\frac{\xi}{\varepsilon}\right).$$

We consider a smooth cut-off function  $\chi$  which is equal to 1 on the support so that we have  $a(x, \xi)\chi(\xi) = a(x, \xi)$  and we can write

$$\begin{aligned} (\text{op}_\varepsilon(a_\varepsilon)f, f) &= \\ & (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon\left(x, \frac{\xi + \xi'}{2}\right) e^{i\varepsilon x \cdot (\xi - \xi')} \mathcal{F}_\varepsilon f(\xi') \overline{\mathcal{F}_\varepsilon f(\xi)} \chi(\xi) \chi(\xi') d\xi d\xi' dx + O(\varepsilon). \end{aligned}$$

The rest term  $O(\varepsilon)$  comes from Taylor formula close to  $\frac{\xi + \xi'}{2}$  and the observation that

$$(\xi_j - \xi'_j) e^{i\varepsilon x \cdot (\xi - \xi')} = \frac{\varepsilon}{i} \partial_{x_j} \left( e^{i\varepsilon x \cdot (\xi - \xi')} \right), \quad 1 \leq j \leq d,$$

and the use of integration by parts. Similarly, we just need to consider vectors  $(\xi, \xi')$  which are close to the diagonal and if we introduce a smooth function  $\Theta$  compactly supported on  $|\xi| \leq 1$  and equal to 1 close to 0, then for some  $\delta > 0$  (that will be chosen small enough later), we have

$$\begin{aligned} (\text{op}_\varepsilon(a_\varepsilon)f, f) &= (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon\left(x, \frac{\xi + \xi'}{2}\right) \\ & \times e^{i\varepsilon x \cdot (\xi - \xi')} \mathcal{F}_\varepsilon f(\xi') \overline{\mathcal{F}_\varepsilon f(\xi)} \Theta\left(\frac{\xi - \xi'}{\delta}\right) \chi(\xi) \chi(\xi') d\xi d\xi' dx + O(\varepsilon). \end{aligned}$$

We are left with the integral

$$\begin{aligned}
I_\varepsilon &= (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon \left( x, \frac{\xi + \xi'}{2} \right) e^{i\varepsilon x \cdot (\xi - \xi')} \mathcal{F}_\varepsilon f(\xi') \overline{\mathcal{F}_\varepsilon f(\xi)} \chi(\xi) \chi(\xi') \\
&\quad \times \Theta \left( \frac{\xi - \xi'}{\delta} \right) d\xi d\xi' dx \\
&= (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon \left( x, \frac{\Phi(\zeta) + \Phi(\zeta')}{2} \right) e^{i\varepsilon x \cdot (\Phi(\zeta) - \Phi(\zeta'))} \mathcal{F}_\varepsilon f(\Phi(\zeta')) \\
&\quad \times \overline{\mathcal{F}_\varepsilon f(\Phi(\zeta))} J_\Phi(\zeta) J_\Phi(\zeta') \chi \circ \Phi(\zeta) \chi \circ \Phi(\zeta') \Theta \left( \frac{\Phi(\zeta) - \Phi(\zeta')}{\delta} \right) d\zeta d\zeta' dx
\end{aligned}$$

where  $\zeta \mapsto J_\Phi(\zeta)$  is the Jacobian of the diffeomorphism  $\Phi$ . Setting

$$\zeta = \theta + \varepsilon \frac{v}{2} \quad \text{and} \quad \zeta' = \theta - \varepsilon \frac{v}{2},$$

we have for  $t \in \mathbb{R}$ ,

$$\Phi(\theta + \varepsilon tv) = \Phi(\theta) + \varepsilon t d\Phi(\theta)v + \varepsilon^2 \int_0^1 d^2\Phi(\theta + \varepsilon tsv)[v, v](1-s) ds,$$

whence

$$\begin{aligned}
\frac{1}{2} (\Phi(\zeta) + \Phi(\zeta')) &= \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v], \\
\Phi(\zeta) - \Phi(\zeta') &= \varepsilon d\Phi(\theta)v + \varepsilon^2 B_\varepsilon^-(\theta, v)[v, v],
\end{aligned}$$

with

$$B_\varepsilon^\pm(\theta, v) = \int_0^1 d^2 \left( \Phi \left( \theta + \varepsilon s \frac{v}{2} \right) \pm \Phi \left( \theta - \varepsilon s \frac{v}{2} \right) \right) (1-s) ds.$$

Note that the functions  $B_\varepsilon^\pm$  are smooth, bounded and with bounded derivatives, uniformly in  $\varepsilon$ , as soon as the variables  $\theta$  and  $\varepsilon v$  are in a compact. We obtain

$$\begin{aligned}
I_\varepsilon &= (2\pi)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon \left( x, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) e^{ix \cdot (d\Phi(\theta)v + \varepsilon B_\varepsilon^-(\theta, v)[v, v])} \\
&\quad \times \mathcal{F}_\varepsilon f \left( \Phi \left( \theta - \varepsilon \frac{v}{2} \right) \right) \overline{\mathcal{F}_\varepsilon f \left( \Phi \left( \theta + \varepsilon \frac{v}{2} \right) \right)} J_\Phi \left( \theta + \varepsilon \frac{v}{2} \right) J_\Phi \left( \theta - \varepsilon \frac{v}{2} \right) d\theta d\theta' dx,
\end{aligned}$$

where we have omitted the localization functions in  $\theta + \varepsilon \frac{v}{2}$  and  $\theta - \varepsilon \frac{v}{2}$ , which makes that the integral is compactly supported in  $\theta$  and  $\varepsilon v$ , with moreover  $\varepsilon|v| \leq \delta$  on the domain of integration. We shall crucially use this information later.

The change of variable  $x = {}^t d\Phi(\theta)^{-1}u$  gives

$$\begin{aligned} I_\varepsilon &= (2\pi)^{-d} \int_{\mathbb{R}^{3d}} a_\varepsilon \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) \\ &\quad \times e^{iu \cdot v + i\varepsilon u \cdot {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]} \mathcal{F}_\varepsilon f \left( \Phi \left( \theta - \varepsilon \frac{v}{2} \right) \right) \overline{\mathcal{F}_\varepsilon f} \left( \Phi \left( \theta + \varepsilon \frac{v}{2} \right) \right) \\ &\quad \times J_\Phi \left( \theta + \varepsilon \frac{v}{2} \right) J_\Phi \left( \theta - \varepsilon \frac{v}{2} \right) J_\Phi^{-1}(\theta) d\theta d\theta' du, \end{aligned}$$

with the same property on the domain of integration ( $\theta$  in a compact and  $\varepsilon|v| < \delta$ ). Note that

$$\begin{aligned} &a_\varepsilon \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) \\ &= a \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v], \right. \\ &\quad \left. \frac{B \left( \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) \varphi \left( \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right)}{\varepsilon} \right) \\ &= a \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta), B(\Phi(\theta)) \frac{\theta'}{\varepsilon} \right) + \varepsilon r_\varepsilon(\theta, u, v)[v, v]. \end{aligned}$$

The matrix  $r_\varepsilon$  is supported in a compact independent of  $\varepsilon$  in the variables  $(u, \theta)$ . Besides, the matrix  $r_\varepsilon$  is smooth, bounded, and with bounded derivatives, uniformly in  $\varepsilon$ , as soon as the variable  $\varepsilon v$  is in a compact, which is the case on the domain of integration of the integral  $I_\varepsilon$ . Using Taylor formula on the Jacobian terms, we write

$$\begin{aligned} &a_\varepsilon \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) J_\Phi \left( \theta + \varepsilon \frac{v}{2} \right) J_\Phi \left( \theta - \varepsilon \frac{v}{2} \right) \\ &= a \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta), B(\Phi(\theta)) \frac{\theta'}{\varepsilon} \right) J_\Phi(\theta)^2 + \varepsilon r_\varepsilon^{(2)}(\theta, u, v)[v, v] + \varepsilon r_\varepsilon^{(1)}(\theta, u, v) \cdot v, \end{aligned}$$

where the vector  $\tilde{r}_\varepsilon$  is supported in a compact independent of  $\varepsilon$  in the variables  $(u, \theta)$  and, as  $r_\varepsilon$ , is smooth, bounded, and with bounded derivatives, uniformly in  $\varepsilon$  on the domain of integration of the integral  $I_\varepsilon$  (where  $\theta$  is in a compact and  $\varepsilon|v| \leq \delta$ ,  $\delta$  to be chosen later).

Denote by  $K_\varepsilon$  the isometry of  $L^2(\mathbb{R}^d)$  :

$$f^\varepsilon \mapsto J_\Phi(\cdot)^{\frac{d}{2}} \mathcal{F}_\varepsilon f(\Phi(\cdot)),$$

then

$$(\text{op}_\varepsilon(a_\varepsilon)f, f) = (\text{op}_\varepsilon(\tilde{a}_\varepsilon) K_\varepsilon f, K_\varepsilon f) + \varepsilon (R_\varepsilon K_\varepsilon f, K_\varepsilon f),$$

with

$$\tilde{a}_\varepsilon(u, \theta) = a \left( {}^t d\Phi(\theta)^{-1}u, \Phi(\theta), B(\Phi(\theta)) \frac{\theta'}{\varepsilon} \right),$$

and where  $R_\varepsilon$  is the operator of kernel

$$(\theta, \theta') \mapsto (2\pi\varepsilon)^{-d} K_\varepsilon \left( \frac{\theta + \theta'}{2}, \frac{\theta - \theta'}{\varepsilon} \right),$$

with  $K_\varepsilon = K_\varepsilon^{(1)} + K_\varepsilon^{(2)}$ ,

$$\begin{aligned} K_\varepsilon^{(1)}(\theta, v) &= \int_{\mathbb{R}^d} (r_\varepsilon^{(1)}(\theta, u, v) \cdot v + r_\varepsilon^{(2)}(\theta, u, v)[v, v]) e^{iu \cdot v + i\varepsilon u \cdot {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]} du, \\ K_\varepsilon^{(2)}(\theta, v) &= \int_{\mathbb{R}^d} a_\varepsilon \left( {}^t d\Phi(\theta)^{-1} u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) e^{iu \cdot v} \\ &\quad \times \frac{1}{\varepsilon} \left[ e^{i\varepsilon u \cdot {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]} - 1 \right] du. \end{aligned}$$

The proof concludes by Schur lemma and the next result.

**Lemma C.1.** *Let us fix  $\delta$  small enough. Then, for any  $j \in \{1, 2\}$ , there exists a constant  $C_j > 0$  such that for all  $\varepsilon > 0$ ,*

$$\int_{\mathbb{R}^d} \sup_{\theta \in \mathbb{R}^d} |K_\varepsilon^{(j)}(\theta, v)| dv \leq C_j.$$

Indeed, by this Lemma, we obtain that for all  $\varepsilon > 0$

$$\begin{aligned} (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \sup_{\theta \in \mathbb{R}^d} \left| K_\varepsilon \left( \frac{\theta + \theta'}{2}, \frac{\theta - \theta'}{\varepsilon} \right) \right| d\theta' &= (2\pi)^{-d} \int_{\mathbb{R}^d} \sup_{\theta \in \mathbb{R}^d} |K_\varepsilon(\theta - \varepsilon v, v)| dv \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} \sup_{\theta \in \mathbb{R}^d} |K_\varepsilon(\theta, v)| dv \leq C_1 + C_2, \end{aligned}$$

and similarly

$$\begin{aligned} (2\pi\varepsilon)^{-d} \int_{\mathbb{R}^d} \sup_{\theta' \in \mathbb{R}^d} \left| K_\varepsilon \left( \frac{\theta + \theta'}{2}, \frac{\theta - \theta'}{\varepsilon} \right) \right| d\theta &= (2\pi)^{-d} \int_{\mathbb{R}^d} \sup_{\theta' \in \mathbb{R}^d} |K_\varepsilon(\theta' + \varepsilon v, v)| dv \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} \sup_{\theta' \in \mathbb{R}^d} |K_\varepsilon(\theta', v)| dv \leq C_1 + C_2, \end{aligned}$$

By Schur Lemma, these two inequalities yield the boundedness of  $R_\varepsilon$  uniformly in  $\varepsilon$  as an operator on  $L^2(\mathbb{R}^d)$ .

Let us now prove Lemma C.1.

*Proof.* Note first that the functions  $K_\varepsilon^{(j)}$  are compactly supported in the variable  $\theta$ , uniformly in  $\varepsilon$ . We are going to prove that for any  $N > 0$ , there exists a constant  $C_{N,j}$  such that, for  $|v| > 1$ ,

$$(1 + |v|^2)^N |K_\varepsilon^{(j)}(\theta, v)| \leq C_{N,j}.$$

These inequalities are enough to conclude as in the lemma. For proving these inequalities, we crucially use that the domain of integration in  $u$  is compact and we shall gain the decrease in  $v$  by using the oscillations inside the integral.

Let us first focus on  $K_\varepsilon^{(1)}$ . Since  $\theta$  is in a compact and  $B_\varepsilon^-$  is bounded, we have

$$|v + \varepsilon {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]| \geq |v| - M\delta|v|$$

for some constant  $M$ . Therefore, if  $\delta M < 1/2$ , we have

$$|v + \varepsilon {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]| > \frac{1}{2}|v|,$$

and, for  $|v| > 1$ , integration by parts give

$$K_\varepsilon^{(1)}(\theta, v) = \int_{\mathbb{R}^d} |v + \varepsilon {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]|^{-2N} (\Delta_u^N r_\varepsilon^{(1)}(\theta, u, v) \cdot v + \Delta_u^N r_\varepsilon^{(2)}[v, v]) \\ \times e^{iu \cdot v + i\varepsilon u \cdot {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]} du.$$

Since  $r_\varepsilon^{(1)}$  and  $r_\varepsilon^{(2)}$  have smooth compactly supported derivatives in  $u$ , uniformly bounded in  $\varepsilon$ , we obtain the existence of a constant  $C_{N,1}$  such that

$$|K_\varepsilon^{(1)}(\theta, v)| \leq |v|^{-2N} C_{N,1}.$$

Let us now study  $K_\varepsilon^{(2)}$  that we turn into

$$K_\varepsilon^{(2)}(\theta, v) = i \int_0^1 \int_{\mathbb{R}^d} u {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v) a_\varepsilon \left( {}^t d\Phi(\theta)^{-1} u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right) \\ \times e^{iu \cdot v + i\varepsilon u \cdot {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v)[v, v]} du dt.$$

Once written on this form, one can see that the arguments developed for  $K_\varepsilon^{(1)}$  apply again since the function

$$u \mapsto u {}^t d\Phi(\theta)^{-1} B_\varepsilon^-(\theta, v) a_\varepsilon \left( {}^t d\Phi(\theta)^{-1} u, \Phi(\theta) + \frac{\varepsilon^2}{2} B_\varepsilon^+(\theta, v)[v, v] \right)$$

is compactly supported in the variable  $u$ , smooth and bounded with derivatives that are bounded uniformly in  $\varepsilon$ . □

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