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Bruno Courcelle

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On quasi-planar graphs : clique-width and logical description.

Bruno Courcelle
Labri, CNRS and Bordeaux University*
33405 Talence, France
email: courcell@labri.fr

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Abstract

Motivated by the construction of FPT graph algorithms parameterized by clique-width or tree-width, we study graph classes for which tree-width and clique-width are linearly related. This is the case for all graph classes of bounded expansion, but in view of concrete applications, we want to have "small" constants in the comparisons between these width parameters.

We focus our attention on graphs that can be drawn in the plane with limited edge crossings, for an example, at most p crossings for each edge. These graphs are called *p-planar*. We consider a more general situation where the graph of edge crossings must belong to a fixed class of graphs $\mathcal{D}$. For p-planar graphs, $\mathcal{D}$ is the class of graphs of degree at most p. We prove that tree-width and clique-width are linearly related for graphs drawable with a graph of crossings of bounded average degree.

We prove that the class of 1-planar graphs, although conceptually close to that of planar graphs, is not characterized by a monadic second-order sentence. We identify two subclasses that are.

Introduction

Most fixed-parameter tractable (FPT) algorithms parameterized by tree-width or clique-width need a tree-decomposition of the input graph, or a clique-width

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term[1] defining it. This observation concerns in particular the linear-time verification of a graph property expressed in monadic second-order logic (an MS property) [20, 21, 26] for graphs of tree-width or clique-width bounded by some fixed value. This method, based on automata[2], is implemented in the running system AUTOGRAPH[3]. Unfortunately, tree-width and clique-width (and the corresponding optimal decompositions and terms) are difficult to compute[4] [3, 30].

Motivated by the construction of FPT graph algorithms parameterized by clique-width or tree-width, and based on automata [18, 19], we study graph classes $\mathcal{C}$ for which clique-width is linearly bounded in tree-width, and we obtain a usable method to construct clique-width terms from tree-decompositions.

We recall that the clique-width of an undirected graph G, denoted by $cwd(G)$, is bounded by $3 \cdot 2^{twd(G)-1}$ where $twd(G)$ denotes its tree-width. We are interested in cases where $cwd(G) \leq a \cdot twd(G)$ and a is a "small" constant making it possible to use the algorithms developped in [18, 19], that are based on graph decompositions witnessing "small" clique-width. There are good approximation algorithms for constructing tree-decompositions[5] [7] but presently none for approximating clique-width, without exponential jumps. The cubic-time approximation algorithm of [44] produces a clique-width term of width at most 8^k for given k and an input graph of clique-width at most k. Another similar one is in [45].

A linear-time algorithm presented in [17] transforms a tree-decomposition (T, f) of a graph G into a *clique-width term* (an algebraic term written with the graph operations upon which clique-width is based) defining the same graph G. If G is in one of the "good" classes we will consider, and the width of (T, f) is k, then the produced clique-width term has width at most $a \cdot k$. The construction of automata for checking monadic second-order properties is actually easier for clique-width terms than for those encoding tree-decompositions (cf. [16, 18, 20]) and having $cwd(G) \leq a \cdot twd(G)$ makes such constructions and the algorithm of [17] usable. For graph classes of bounded expansion [41], we have $cwd(G) = O(twd(G))$ but the hidden constants arising from the proofs are frequently huge.

In this article, we focus our attention on graphs that can be drawn in the plane with limited edge crossings, for an example, at most p crossings for each edge : these graphs are called *p-planar*. We consider the more general

[1] Clique-width is defined from algebraic terms that are based on a tree-shaped decomposition of the considered graph.

[2] called *fly-automata*, because they compute their transitions instead of storing them in huge, unmanageable tables.

[3] AUTOGRAPH can even compute values associated with graphs [19], for an example, the number of 3-colorings. It is written in Common Lisp by I. Durand. See http://dept-info.labri.u-bordeaux.fr/~idurand/autograph. An online version, currently in development, is in http://trag.labri.fr.

[4] It is possible to decide in linear time if a graph G of tree-width k has clique-width at most m, for fixed k and m [28], but the complicated algorithm does not highlight the structural properties of G ensuring that $cwd(G) \leq m$.

[5] The recent algorithm of [6] is not practically usable. Efficient algorithms can be obtained from the PACE challenge: see https://pacechallenge.wordpress.com/pace-2017/track-a-treewidth/

situation where the graph of edge crossings must belong to a fixed monotone[6] class $\mathcal{D}$. For p-planar graphs, $\mathcal{D}$ is the class of graphs of degree at most p. If $\mathcal{D}$ is the class of graphs having clique number at most $p-1$ (no p vertices induce a clique), we get the notion of *p-quasi-planar* graph [1, 32, 33].

We will use the term *quasi-planar* in a wider sense, where this notion depends on a fixed monotone class $\mathcal{D}$ that specifies the allowed types of crossings. We will prove that tree-width and clique-width are linearly related for $\mathcal{D}$-quasi-planar graphs if the graphs in $\mathcal{D}$ have bounded average degree. This result does not apply to p-quasi-planar graphs, that raise difficult open questions.

We also prove that the class of 1-planar graphs, although conceptually close to that of planar graphs, is not characterized by a monadic second-order sentence. However the classes of outer and optimal 1-planar graphs are. (Definitions are in Section 3).

Summary : In Section 1 we review definitions and known results, in particular those concerning nowhere dense and bounded expansion graph classes. In Section 2, we compare clique-width to tree-width for quasi-planar graphs. In Section 3, we review some basic notions of monadic second-order logic (MS logic), we prove that 1-planarity is not MS-expressible and we consider two particular classes of 1-planar graphs that are MS definable. We list some open questions in the conclusion.

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1 Definitions and basic facts

Most definitions are well-known, we review notation and a few results. We denote by $\uplus$ the union of two disjoint sets, by $[k]$ the set $\{1, ..., k\}$, by $|X|$ the cardinality of a set X and by $\mathcal{P}(X)$ its powerset.

Graphs

All graphs are nonempty and finite. We will compare tree-width and clique-width for undirected *simple graphs*, *i.e.*, that are loop-free and without parallel edges. The extension of our results to directed graphs is easy, by modifying the proofs of [17]. Undefined notions are as in [25].

A graph[7] G has vertex set V_G and edge set E_G. An edge linking vertices u and v is designated by uv or vu. We denote by $G[X]$ the induced subgraph of

[6] As in [41], we call *monotone* a class of graphs closed under taking subgraphs and *hereditary* a class closed under taking induced subgraphs.

[7] We review the representations of graphs by logical structures and monadc second-order logic in Section 3.1.

G with vertex set $X \cap V_G$, where X need not be a subset of V_G : this convention allows to deal with cases where $X \subseteq V_H$ and G is a subgraph of H. Similarly, if X and Y are disjoint sets, then $G[X,Y]$ is the bipartite graph with vertex set $V_G \cap (X \cup Y)$ and whose edges are those of G between X and Y.

If $x \in V_G$ and $r \geq 0$, then $N_G^r(x)$ denotes the set of *neighbours* at distance at most r of x, where the distance between two vertices is the minimal number of edges of a path connecting them. We write $N_G(x)$ for $N_G^1(x)$; G has *radius at most* r if $V_G \subseteq N_G^r(x)$ for some vertex x.

If X, Y are disjoint sets, we define $\Omega_G(X,Y) := \{N_G(x) \cap Y \mid x \in X \cap V_G\} \subseteq \mathcal{P}(Y \cap V_G)$. As in [31], we denote by $\lambda_G(k)$ the maximum cardinality of $\Omega_G(V_G - Y, Y)$ for Y of cardinality at most k. Hence $\lambda_G(k) \leq 2^k$. If, furthermore, $1 \leq m \leq k$ and $|N_G(x)| \leq m$ for each $x \in X$, then $\lambda_G(k) = O(k^m)$ for fixed m (see [17]).

A class of graphs is *monotone* (resp. *hereditary*) if it is closed under taking subgraphs (resp. induced subgraphs). By an *s-coloring* of a graph, we mean a *proper* coloring of the vertices that uses colors in $[s]$; "proper" means that adjacent vertices have different colors. For other types of colorings, we will specify the requirements.

The incidence graph of a graph G, denoted by $Inc(G)$, is the bipartite graph defined from G by inserting an additional vertex on each edge ; this vertex represents the corresponding edge. (This is useful for expressing graph properties with edge set quantifications, cf. Section 3).

We will use several times as counter-example the set $\mathcal{SC}$ of *subdivided cliques,* defined as the set of graphs $Inc(K_r)$, for $r \geq 3$.

1.1 Sparse graphs

We recall that, unless otherwise specified, graphs are simple and undirected. We review some definitions and facts related to sparseness.

Definition 1.1 *Sparseness and degree bounds.*

(a) A graph is *p-degenerate* if each of its subgraphs has a vertex of degree at most p. We denote by $\mathcal{D}_q$ the class of p-degenerate graphs.

(b) A graph G is *uniformly q-sparse* if $|E_H| \leq q.|V_H|$ for each of its subgraphs H. We denote by $\mathcal{U}_q$ the class of such graphs. In the terminology of [41], these subgraphs H have *edge density* at most q and $\nabla_0(G)$ denotes $\max\{|E_H|\,/\,|V_H| \mid H \subseteq G\}$. For a class of graphs $\mathcal{C}$, $\nabla_0(\mathcal{C})$ denotes $\sup\{\nabla_0(G) \mid G \in \mathcal{C}\}$.

Every (simple) planar graph G is uniformly 3-sparse because $|E_G| \leq 3\,|V_G| - 6$. A graph is uniformly $\lceil d/2 \rceil$-sparse if its maximum degree is d.

These classes are related by the strict inclusions : $\mathcal{U}_q \subset \mathcal{D}_{2q} \subset \mathcal{U}_{2q}$ (cf. [41], Section 3.2).

Proposition 1.2 : (1) A graph is in $\mathcal{D}_p$ if and only if it has an acyclic orientation of indegree at most p. Every graph in $\mathcal{D}_p$ has a $(p+1)$-coloring.

(2) A graph is in $\mathcal{U}_q$ if and only if it has an orientation of indegree at most q. Every graph in $\mathcal{U}_q$ has a $(2q+1)$-coloring.

Proof: (1) See [41], Proposition 3.2.
(2) See [34], Theorem 6.13 or Proposition 9.40 of [20]. □

We denote by $\mathcal{S}_r$ (resp. $\mathcal{N}_r$) the class of graphs G that have no subgraph isomorphic to the complete bipartite graph $K_{r,r}$ (resp. to the r-clique K_r, hence, whose *clique number* $\omega(G)$ is at most $r-1$). Hence, $\mathcal{S}_r \subseteq \mathcal{N}_{2r}$. We have $\mathcal{U}_q \subset \mathcal{S}_{2q+1}$. For every r and q, there are graphs in $\mathcal{S}_r$ that are not uniformly q-sparse (because there is a constant c such that, if $r \geq 3$, there is a graph having n vertices and at least $c \cdot n^{2-2/(r+1)}$ edges, a result by Erdős and Stone, see [25], Section 7.1). If $s \geq 3$, then $\mathcal{N}_s$ contains the graphs $K_{r,r}$, hence is not included in any class $\mathcal{U}_q$.

Definitions 1.3 : *Shallow minors, bounded expansion and nowhere dense classes.*

We review notions developped by Nešetřil and Ossona de Mendez in [41] and previous articles.

(a) A *minor* H of a graph G is obtained by choosing pairwise disjoint non-empty sets of vertices $V_1, ..., V_p$ such that each graph $G[V_i]$ is connected ; the vertices of H are $v_1, ..., v_p$ and there is an edge in H between v_i and v_j, where $i \neq j$, only if there is an edge in G between V_i and V_j.

Then, H is a *d-shallow minor*, if each graph $G[V_i]$ has radius at most d. A 0-shallow minor is just a subgraph.

(b) For a class of graphs $\mathcal{C}$, we denote by $\mathcal{C}\nabla d$, the class of d-shallow minors of its graphs, and by $\nabla_d(\mathcal{C})$ the value $\nabla_0(\mathcal{C}\nabla d)$. Then, $\mathcal{C}$ has *bounded expansion* if, for each d, $\nabla_d(\mathcal{C})$ is finite, equivalently, if for each d, there is an integer q such that every d-shallow minor of a graph in $\mathcal{C}$ is in $\mathcal{U}_q$ (is uniformly q-sparse).

(c) A class of graphs $\mathcal{C}$ is *nowhere dense* if, for each integer d, there is an integer q such that every d-shallow minor of a graph in $\mathcal{C}$ is in $\mathcal{N}_{q+1}$, hence, has clique number at most q.

Examples 1.4 : By [41, 42] the following graph classes have bounded expansion : graphs of bounded degree, minor-closed classes, classes that exclude a topological minor, and for each p, the classes of graphs whose crossing number is at most p and of those that are p-planar,

Bounded expansion for a class $\mathcal{C}$ implies p-colorability for some $p \leq 1 + 2\nabla_0(\mathcal{C})$ (by Proposition 1.2) but not vice-versa : the subdivided cliques (in $\mathcal{SC}$) are 2-colorable but do not have bounded expansion as each K_r is a 1-shallow minor of $Inc(K_r)$. For the same reason, $\mathcal{SC}$ is not nowhere dense.

Classes having locally bounded tree-width or locally bounded expansion ([41], Section 5.6) are nowhere dense. The class of graphs that have maximal degree no larger than their *girth* (the minimal size of an induced cycle) is

nowhere dense but not uniformly q-sparse for any q, hence has not bounded expansion (Example 5.1[8] in [41]).

Definition 1.5 : *Neighbourhood complexity.*

Let G be a graph, Y a set of vertices and $r \geq 1$. We denote by $\mu^r_G(Y)$ the cardinality of the set $\{N^r_G(x) \cap Y \mid x \in V_G\}$. Clearly, $\lambda_G(Y) \leq \mu^1_G(Y)$ (because in the definition of $\mu^1_G(Y)$, we may have $x \in Y$). We define $\nu_r(G) := \max\{\mu^r_G(Y)/|Y| \mid \emptyset \neq Y \subseteq V_G, \}$, and, for a class $\mathcal{C}$, $\nu_r(\mathcal{C}) := \sup\{\nu_r(G) \mid G \in \mathcal{C}\}$.

Theorem 1.6 : (1) A class of graphs $\mathcal{C}$ has bounded expansion if and only if $\nu_r(\mathcal{C})$ is finite for each r.

(2) A class of graphs $\mathcal{C}$ is nowhere dense if and only if :

$$\forall r \in \mathbb{N}, \forall \varepsilon \in R, \exists c \in \mathbb{R}, \forall G \in \mathcal{C}, \forall Y \subseteq V_G, \mu^r_G(Y) \leq c\,|Y|^{1+\varepsilon}\,.$$

Assertion (1) is proved in [47] and Assertion (2) in [27]. The proof of (1) shows in particular that $\nu_r(\mathcal{C}) \leq f(r, \nabla_r(\mathcal{C}))$ for some function f. For our comparison of tree-width and clique-width, we need only bound $\nu_1(\mathcal{C})$. However, the function f is so large that we cannot obtain any usable bound on clique-width. Theorem 1.6 remains of great interest for studies in graph structure.

1.2 Tree-width and clique-width

Tree-decompositions and tree-width are well-known [5, 20, 25, 26], hence we do not repeat the definitions. (We will not manipulate tree-decompositions.) We denote by $twd(G)$ the tree-width of a graph G. Similarly, for clique-width[9], denoted by $cwd(G)$, we refer the reader to [13, 17, 20] (and [18, 19, 21] for the associated FPT algorithms to check monadic second-order graph properties).

Here are a few facts ([20], Example 2.56 and Proposition 2.106) : if $r \geq 2$, we have $twd(K_r) = r - 1, twd(K_{r,r}) = r$ and $cwd(K_r) = cwd(K_{r,r}) = 2$; if $G_{r,s}$ is the rectangular $r \times s$-grid and $r \leq s$, then $twd(G_{r,s}) = r$ and $r + 1 \leq cwd(G_{r,s}) \leq r + 2$.

We recall that $\mathcal{S}_r$ is the class of graphs without subgraphs isomorphic to $K_{r,r}$. The following results motivate our study.

Theorem 1.7 : (1) For every graph G, we have $cwd(G) \leq 3 \cdot 2^{twd(G)-1}$.

[8] And the existence, for each p of graphs of maximal degree p, chromatic number larger that $\sqrt{p}/2$ and of unbounded girth, see [40]. These graphs are not all in any class $\mathcal{U}_q$ by Proposition 1.2(2).

[9] Vertex labelled graphs are constructed with disjoint union, relabellings (a relabelling replaces everywhere a label a by label b), edge addition operations ($add_{a,b}$ adds an edge between each a-labelled vertex and each b-labelled vertex, unless there is already one, as it is intended to build a simple graph). The basic graphs are isolated labelled vertices. Every graph is defined by a term over these operations. Its clique-width is the minimum number of labels in a term that defines it.

(2) There is no constant $c \geq 1$ such that $cwd(G) \leq O(twd(G)^c)$ for all graphs G.

(3) If $G \in \mathcal{S}_r$, then $twd(G) \leq 3(r-1)cwd(G) - 1$.

Proof : Assertions (1) and (2) are proved in [13]. For proving (2), the authors construct graphs of tree-width $2k$ and clique-width larger than 2^{k-1}. Assertion (3) is proved in [36] (also [20], Proposition 2.115). □

Our constructions will exploit the following result from [17] (Theorem 11).

Theorem 1.8 : For every graph G, we have $cwd(G) \leq \lambda_G(twd(G)+1)+1$.

Corollary 1.9 : (1) If $\mathcal{C}$ is a class of graphs having bounded expansion, then $cwd(G) \leq \nu_1(\mathcal{C}) \cdot (twd(G)+1)+1$ for every graph G in $\mathcal{C}$. Hence, for such graphs, clique-width and tree-width are linearly related.

(2) If $\mathcal{C}$ is nowhere dense and $\varepsilon > 0$, we have $cwd(G) = O(twd(G)^{1+\varepsilon})$ for every graph G in $\mathcal{C}$. Hence, for such graphs, clique-width and tree-width are almost linearly related.

Proof : As in both cases, $\mathcal{C}$ excludes some $K_{r,r}$ as a subgraph because K_r is a 1-shallow minor of $K_{r,r}$, Theorem 1.7(3) is applicable. The first parts of the two assertions follow from Theorem 1.8 and Theorem 1.6 with $r = 1$. □

Remarks 1.10

(1) For a comparison, we have $cwd(G) = O(twd(G)^q)$ if G is uniformly q-sparse ([17], Theorem 19). No better bound is known[10], which shows a gap between nowhere density and uniform sparseness. (However, if G is the incidence graph of a hypergraph of tree-width k whose edges have at most q vertices, then $cwd(G) = O(k^{q-1})$ for fixed q by Theorem 22 of [17]).

(2) The bounds on $\nu_1(\mathcal{C})$ in terms of $\nabla_1(\mathcal{C})$ derived from the existing proof of Theorem 1.6(1) are extremely large. We are thus motivated to bound directly the ratios $cwd(G)/twd(G)$ for G in particular classes having bounded expansion. That $cwd(G) = O(twd(G))$ for G in a class having bounded expansion follows also from Theorem 18 of [31], see Table 2 in Section 2.4.

(3) Graph classes of bounded clique-width are studied in several articles [9, 10, 11, 23, 24, 38]. It would be interesting to have classes of unbounded clique-width for which $cwd(G) = O(twd(G)^\alpha)$ where $0 < \alpha < 1$. However, we have no tools for obtaining such results.

(4) The converse of Corollary 1.9 does not hold. Consider the class $\mathcal{SC}$ of subdivided cliques. For each r, we have $twd(Inc(K_r)) \geq r-1$ and $cwd(Inc(K_r)) \leq r+3$ (see [8, 17]). Hence, $cwd(Inc(K_r)) \leq twd(Inc(K_r))+4$. But $\mathcal{SC}$ has not bounded expansion and is not nowhere dense as observed in Examples 1.4. □

[10] Looking for c such that $cwd(G) = O(twd(G)^c)$ may be formulated as bounding $\log(cwd(G))/\log(twd(G))$. This type of formulation is used in [41].

We will be interested by graph classes that have no $K_{r,r}$ as a subgraph, and such that λ_G is linear with a "small" constant, so that $twd(G) = O(cwd(G))$ and the corresponding bounding is usable.

Remark 1.11 : *About tree-decompositions and Theorem 1.8.*

Its proof consists in an algorithm that transforms a tree-decomposition (T, f) of a graph G into a clique-width term t that defines the same graph. If (T, f) has width k, then t has width m (is built with m labels) where $m \leq \lambda_G(k+1)+1$. Hence $cwd(G) \leq m$. The computation time is linear in the number n of vertices of G for fixed k. More precisely, it is $O(n \cdot k(\log(k)+m\log(m)))$ by using standard data structures. The values m and k are determined during the computation of t. From λ_G we get an upperbound to the computation time, but the algorithm can be used even if $\lambda_G(k+1)$ is not known or is bounded by a huge value.

The tree-decomposition (T, f) is given by a *normal tree* T for G, which means that V_G is the set of nodes of T, that T is rooted and any two adjacent vertices[11] of G are comparable for the ancestor relation of T, denoted by $\leq_T$ ($u <_T v$ if and only if v is an ancestor of u, so that the root is the maximal element). The "box" function f of the tree-decomposition is then defined by :

$$f(u) := \{u\} \cup \{v \in V_G \mid u \leq_T v \text{ and } \ wv \in E_G \text{ for some } w \leq_T u\}.$$

Hence, (T, f) is encoded in a very compact way[12], just by the function that specifies the father of any node that is not the root.

The notion of *tree-depth* is based on normal trees. The tree-depth of a connected graph G, denoted by $td(G)$, is the minimum height[13] of a normal tree for G. If G is not connected, its tree-depth is the maximum of those of its connected components. For G with n vertices, we have ([41], Section 6.4) :

$$twd(G) + 1 \leq td(G) \leq (twd(G) + 1)\log(n).$$

2 Quasi-planar graphs

We define and study different notions of quasi-planarity.

Definition 2.1 : *The crossing graph of a drawing.*

Let D be a drawing in the plane of a graph G. The curve segments representing edges – we will call them frequently *edges* – may cross but not touch. No three edges can cross at a same point, and two edges intersect either at a crossing point or at an end point of both edges. An edge does not cross itself.

[11] Adjacent nodes in T need not be adjacent in G.

[12] Assuming that the graph G is also given.

[13] The *height* of a rooted tree is the maximum number of nodes on a path between the root and any node.

(Touching points and self-crossings can easily be removed and they have no use in drawings intended to minimize the number of intersections of edges). A drawing is *simple* if any two edges cross at most once[14].

If H is a subgraph of G, then $D[H]$ is the drawing of H, *inherited from* D, obtained by removing the points and curve segments corresponding to vertices and edges not in H.

We define the *crossing graph* of D, denoted by $\Xi(D)$, as the graph whose vertex set is E_G and two vertices are adjacent if and only if the corresponding edges cross. It is the intersection graph of the open curve segments representing the edges.

Table 1 shows how some existing definitions can be expressed in terms of crossing graphs. The column "Some crossing graph has" means : "there exists a drawing whose crossing graph has" this property.

A graph is *p-planar* if it has a drawing D such that each edge is crossed by at most p others (two edges can cross several times), hence, whose crossing graph has maximum degree at most p. It is *simply* p-planar if the same holds for a simple drawing. It is clear that a 1-planar drawing can be transformed into a simple 1-planar drawing with no more crossings, but it is not clear whether a similar property holds for p-planar drawings, $p \geq 2$.

A graph is *p-quasi-planar* if it has a drawing whose crossing graph has no p-clique. The 2-quasi-planar graphs are nothing but the planar graphs. References for these definitions are [33, 39, 46, 48]. Every p-planar graph is $(p+2)$-quasi-planar. Furthermore, if $p \geq 3$, every simply p-planar graph is $(p+1)$-quasi-planar ([2], the proof is difficult).

Skewness at most p means that we obtain a planar graph by deleting p edges.

The *crossing number* is defined as the minimal number of crossings, and the *pairwise crossing number* is the minimal number of pairs of edges that cross. Whether it is always equal to the crossing number is an open question (see [49] for detailed definitions and a survey of results).

All these classes, except for p-quasi planar graphs, are known to have bounded expansion [41], Section 14.2.

Graph property	*Some crossing graph has:*
Planarity	No edge
Pairwise crossing number $\leq p$	At most p edges
Skewness $\leq p$	No edge after removing p vertices
p-planarity	Degree at most p
p-quasi-planarity	Clique number at most $p-1$

Table 1

Definition 2.2 : *Quasi planarity.*

[14] Graphs are always simple, without loops and parallel edges, but their drawings may not be simple.

Let $\mathcal{D}$ be a monotone class of graphs (*i.e.,* that is closed under taking subgraphs). We say a graph G is $\mathcal{D}$-*quasi-planar* if it has a drawing whose crossing graph is in $\mathcal{D}$. We denote by $QP(\mathcal{D})$ the class of $\mathcal{D}$-quasi-planar graphs.

Let us review some results and open questions relevant to our concern.

The simply p-planar graphs are uniformly q-sparse where $q = \lceil 4.108\sqrt{p} \rceil$ by [46]. They form a class of bounded expansion (Proposition 2.11). A 1-planar graph has at most $4n-8$ edges for n vertices, and 1-planarity is an NP-complete property[15] [12, 22].

The class of p-quasi-planar graphs, studied in [1, 32, 33], is $QP(\mathcal{N}_p)$. The number of edges of a p-quasi-planar graph with n vertices is conjectured to be $O(n)$ for each fixed p. It is bounded by $8n$ if $p \leq 4$. Otherwise, it is $O(n(\log(n))^{4p-16})$ by [1].

2.1 Bounds on clique-width.

We recall from [31] and [17] the proof of the following fact because its argument will be used below in related cases.

Proposition 2.3 : Let $k \geq 3$. If G is planar, then $\lambda_G(k) \leq 6k - 9$.

Proof: We consider a planar graph G, a set Y of k vertices and $X := V_G - Y$. We will bound the number $|\Omega_G(X,Y)|$, *i.e.,* the number of sets of the form $N_G(x) \cap Y$ for some $x \in X$. We will write for Ω for Ω_G.

We can do that for $G[X,Y]$ instead of G because removing edges in $G[X]$ or in $G[Y]$ preserves planarity and does not modify $\Omega(X,Y)$.

We denote by X_1, X_2 and X_3 the sets of vertices of X having degree, respectively, at most 1, exactly 2 and at least 3 in $G[X,Y]$. We have $|\Omega(X_1,Y)| \leq k+1$. Next we consider $\Omega(X_2,Y)$. The bipartite graph $G[X_2,Y]$ is planar. For each vertex in X_2, we link its two neighbours (they are both in Y). We obtain a planar graph H with vertex set Y of cardinality k. Each edge of H corresponds to a set in $\Omega(X_2,Y)$. Hence, $|\Omega(X_2,Y)| = |E_H| \leq 3k - 6$.

We now consider the bipartite planar graph $K := G[X_3,Y]$. As each vertex in X_3 has degree at least 3 in K, we have $3\,|X_3| \leq |E_K|$. As K is planar and bipartite, $|E_K| \leq 2\,|V_K| - 4$. Hence, $3\,|X_3| \leq |E_K| \leq 2(|X_3| + k) - 4$ which gives $|X_3| \leq 2k - 4$, and so, $|\Omega(X_3,Y)| \leq |X_3| \leq 2k - 4$.

Hence, $|\Omega(X,Y)| = |\Omega(X_1,Y)| + |\Omega(X_2,Y)| + |\Omega(X_3,Y)| \leq k + 1 + 3k - 6 + 2k - 4 = 6k - 9$. □

Corollary 2.4 : If G is planar with at least one edge, then $cwd(G) \leq 6\ twd(G) - 2$.

Proof: If $twd(G) \geq 2$, we get the result by Theorem 1.8 and Proposition 2.3, because $6(k+1) - 9 + 1 = 6k - 2$. Otherwise, $twd(G) = 1$, G is a forest and $cwd(G) \leq 3$. The inequality also holds. □

[15] Even if one adds a single edge to a planar graph.

The class of graphs whose crossing number is at most p is contained in a minor-closed class and has bounded expansion (see [41], Chapter 5).

Corollary 2.5 : If G has crossing number p, then $cwd(G) \leq 6\ twd(G) - 2 + \lceil p/2 \rceil$.

Proof : First an easy observation.

Claim : If G is obtained from a graph H by the addition of m edges (and possibly of vertices as ends of these new edges), then, for each k, $\lambda_G(k) \leq \lambda_H(k) + m$.

Proof : Because at most m sets $N_G(x) \cap Y$, $x \in V_G$, are not in $\Omega_H(V_H - Y, Y)$ where Y is a set of k vertices of H. □

If G has crossing number p, it has a drawing D such that $\Xi(D)$ has at most p edges. By removing at most $\lceil p/2 \rceil$ vertices of $\Xi(D)$ and their incident edges, one can get a graph without edges. Hence, by removing at most $\lceil p/2 \rceil$ edges of G, one can get a graph H whose drawing $D[H]$ has no crossings. Hence, H is planar, and by Proposition 2.3 and the claim, we have $\lambda_G(k) \leq 6\ k - 9 + \lceil p/2 \rceil$. As in Corollary 2.4, we get $cwd(G) \leq 6\ twd(G) - 2 + \lceil p/2 \rceil$. □

Remark about the claim : The survey article [35] states that if one adds or deletes an edge to a graph, one can increase or decrease its clique-width by at most[16] 2 (Theorem 9). Hence, if one adds m edges to a graph, one can increase its clique-width by at most $2m$. However, Claim 2.5.1 shows that the bound to clique-width expressed in terms of tree-width increases by at most m. There is no contradiction because Theorem 1.8 and Corollary 2.4 yield upperbounds and no exact values.

Let us digress a little, and examine unions of graphs.

Unions of graphs.

Let H and K be concrete graphs (not graphs up to isomorphism). Their union $H \cup K$ is defined by $V_{H \cup K} := V_H \cup V_K$ and $E_{H \cup K} := E_H \cup E_K - F$ where incidences as in H and K, and F is the set of edges of K that have the same two ends as an edge of H (hence $H \cup K$ is simple). For example, a rectangular grid is the union of two trees.

Proposition 2.6 : For any two graphs H and K, and $k \geq 2$, we have $\lambda_{H \cup K}(k) \leq \lambda_H(k) \cdot \lambda_K(k)$. If H and K are disjoint, then $\lambda_{H \cup K}(k) \leq \lambda_H(k) + \lambda_K(k)$.

Proof: The first assertion follows from the fact :

$$\Omega_{H \cup K}(X, Y) = \{(N_H(x) \cap Y) \cup (N_K(x) \cap Y) \mid x \in X\}.$$

If H and K are disjoint, we have :

[16] It is an open question whether the number 2 can be replaced by 1.

$$\Omega_{H\cup K}(X,Y) = \{N_H(x)\cap Y \mid x \in X\cap V_H\}\cup\{N_K(x)\cap Y \mid x \in X\cap V_K\}$$

which yields the second assertion. □

However, we can get better upper bounds in some cases.

Example 2.7 : If $G = H \cup K$, where H and K are planar, then $\lambda_G(k) \leq 9(2k-3)^2$ by Propositions 2.3 and 2.6. However, by going back to the proof of Proposition 2.3, we get $\lambda_G(k) < 16k^2$. We sketch the proof, by using the notation of that proposition. Without loss of generality, we assume that H and K are edge disjoint. We have $|\Omega_G(X_1,Y)| \leq k+1$ and $|\Omega_G(X_2,Y)| \leq k(k-1)/2$.

We have $X_3 = X_H \cup X_K \cup X_{2,2} \cup X_{1,2} \cup X_{2,1}$ where :

X_H is the set of vertices incident with at least 3 edges of H, and similarly for X_K,

$X_{2,2}$ is the set of vertices incident with 2 edges of H and two edges of K,

$X_{1,2}$ is set of vertices incident with one edge of H and 2 edges of K,

$X_{2,1}$ is similar by exchanging H and K.

From the proof of Proposition 2.3, we have:

$|X_H|, |X_K| \leq 2k-4$,
$|\Omega(X_{2,2},Y)| \leq (3k-6)^2 = 9(k-2)^2$ and
$|\Omega(X_{1,2},Y)|, |\Omega(X_{2,1},Y)| \leq (3k-6)(k-2) = 3(k-2)^2$.

Hence,

$$|\Omega(X_3,Y)| \leq 4(k-2)+9(k-2)^2+6(k-2)^2 = 4(k-2)+15(k-2)^2,$$
$$|\Omega(X,Y)| \leq k+1+k(k-1)/2+4(k-2)+15(k-2)^2 \leq 16k^2-55k+53,$$

for $k \geq 3$. □

Remark : Answering a natural question, we observe that the class of graphs $H \cup K$ where H and K belong to classes having bounded expansion need not have bounded expansion: each subdivided clique is the union of two trees, but $\mathcal{SC}$ does not have bounded expansion as noted in Example 1.4.

2.2 Sparse crossing graphs

We now consider the graphs in $QP(\mathcal{U}_q)$, *i.e.*, those that are drawable with a crossing graph that is uniformly q-sparse.

Lemma 2.8 : (1) Let $H \in \mathcal{U}_q$. If $s \geq 2$ and $\{V_1, \ldots, V_m\}$ is a partition of V_H in *independent*[17] sets of cardinality at most s, then H has a $(2sq+1)$-coloring such that the vertices of each set V_i have the same color.

(2) If H has maximum degree p, then the same holds with $sp+1$ colors.

Proof : (1) We will use facts recalled in Proposition 1.2. The graph H has an orientation of indegree at most q. Let K be obtained from H by fusing, for each i, the vertices of V_i into a single vertex. This graph has an orientation of indegree at most sq hence, an $(2sq+1)$-coloring. As there is no edge between any two vertices of each V_i, we obtain a coloring of H as desired.

(2) Let now H have degree at most p. For each $i = 1, ..., m$, there is an $(sp+1)$-coloring of $H[V_1 \cup ... \cup V_i]$ such that the vertices of each set $V_j, j \leq i$, have the same color. The proof is by induction on i. This gives the result. □

Remark : If H has maximum degree p, then it is in $\mathcal{U}_q$ where $q := \lceil p/2 \rceil$. If p is even, then (2) gives the same result as (1). If $p = 2r+1$, then the coloring of (1) uses at most $2sr + 2s + 1$ colors whereas that of (2) uses only at most $2sr + s + 1$ colors.

Proposition 2.9 : (1) If $G \in QP(\mathcal{U}_q)$, then for $k \geq 3$, we have $\lambda_G(k) \leq 6k(4q+1) - 48q - 9$ and so, $cwd(G) \leq 6twd(G)(4q+1) - 24q - 2$.

(2) If G is p-planar, then for $k \geq 3$, we have $\lambda_G(k) \leq 6k(2p+1) - 18p - 9$ and so, $cwd(G) \leq 6twd(G)(2p+1) - 6p - 2$.

Proof : (1) Let $k \geq 3, q \geq 0$ and $G \in QP(\mathcal{U}_q)$. Let Y be a set of k vertices of G and $X := V_G - Y$. As in the proof of Proposition 2.3, we need only consider $G[X, Y]$.

We partition X into $X_1 \uplus X_2 \uplus X_3$ where X_1 is the set of vertices having at most one neighbour in Y, X_2 is the set of those having exactly two neighbours in Y, and X_3 the set of those having at least 3 neighbours in Y.

We have $|\Omega(X_1, Y)| \leq k+1$.We now bound $|\Omega(X_2, Y)|$.

Let X_2 be enumerated as $\{v_1, \ldots, v_m\}$. The bipartite graph $G[X_2, Y]$ has a drawing whose graph of crossings H is in $\mathcal{U}_q$. Let us partition the set V_H, *i.e.* the set $E_{G[X_2,Y]}$ into $V_1, V_2, ..., V_m$ where V_i is the set of two edges incident with v_i. Any such two edges do not cross, hence are not ajacent in H. By Lemma 2.8, there is a $(4q+1)$-coloring of H such that the two vertices of each V_i have the same color, call it c_i. Let then $X_{2,j}$ be the set of vertices v_i of X_2 such that $c_i = j$. In other words, $G[X_2, Y]$ has an edge coloring with colors in

[17] Also called *stable*: no two vertices are adjacent.

$[4q+1]$ such that the two edges incident with a vertex in X_2 have the same color, and no two edges with same color cross.

The set $\Omega(X_2, Y)$ is the union of the sets $\Omega(X_{2,j}, Y)$. Each graph $G[X_{2,j}, Y]$ is planar. As in the proof of Proposition 2.3, we get $|\Omega(X_{2,j}, Y)| \leq 3k-6 = 3(k-2)$. Hence $|\Omega(X_2, Y)| \leq 3(4q+1)(k-2)$.

Next, we bound the cardinality of X_3 that we enumerate as $\{v_1, \ldots, v_r\}$. We delete from the bipartite graph $G[X_3, Y]$ some edges so that each vertex in X_3 has degree exactly 3 in the resulting graph, that we denote by G'. It has a drawing D' inherited from some drawing D of G whose graph of crossings H is in $\mathcal{U}_q$. Hence $\Xi(D') \in \mathcal{U}_q$. We get a partition of the set $V_{\Xi(D')}$ into $V_1 \uplus ... \uplus V_r$ where V_i is the set of three edges incident with v_i. They do not cross, hence they are not ajacent in $\Xi(D')$. By Lemma 2.7, there is a proper coloring of $\Xi(D')$ with colors in $[6q+1]$ such that the three vertices of each set V_i have the same color, call it c_i. Let then $X_{3,j}$ be the set of vertices v_i of X_3 such that $c_i = j$. Hence, $G'[X_3, Y]$ has an edge coloring with at most $6q+1$ colors such that all edges incident with a vertex in X_3 have same color and no two edges with same color cross. Each graph $G[X_{3,j}, Y]$ is planar. As in the proof of Proposition 2.3, we get $|X_{3,j}| \leq 2k-4$. Hence $|\Omega(X_3, Y)| \leq |X_3| \leq 2(6q+1)(k-2)$.

Finally, we get

$$\begin{aligned} |\Omega(X,Y)| &\leq k+1+3(4q+1)(k-2)+2(6q+1)(k-2) \\ &= 6k(4q+1)-48q-9. \end{aligned}$$

(2) Assume now that H has degree at most p. By Lemma 2.7, we can use $2p+1$ and $3p+1$ colors for, respectively, X_2 and X_3, instead of $4q+1$ and $6q+1$. This gives :

$$\begin{aligned} |\Omega(X,Y)| &\leq k+1+3(2p+1)(k-2)+2(3p+1)(k-2) \\ &= 6k(2p+1)-18p-9. \end{aligned}$$

As observed after Lemma 2.8, this makes a difference with (1) for odd values of p. □

The next two propositions show some properties of the classes $QP(\mathcal{U}_q)$.

Proposition 2.10 : For each q, $QP(\mathcal{U}_q) \subseteq \mathcal{U}_{6q+3}$.

Proof : Let $G \in QP(\mathcal{U}_q)$ having n vertices. It has a drawing D whose crossing graph $\Xi(D)$ is in $\mathcal{U}_q$.

The graph $\Xi(D)$ has a $(2q+1)$-coloring. Hence G has a $(2q+1)$-edge coloring such that no two edges having the same color cross in D. Each graph G_c, defined as the subgraph of G whose edges have color c is planar, hence has at most $3n-6$ edges. Hence G has at most $(2q+1)(3n-6)$ edges. The same holds for all its subgraphs as they are in $QP(\mathcal{U}_q)$. Hence, $G \in \mathcal{U}_{6q+3}$. □

Remark: To prove that p-planar graphs defined from drawings that may not be simple (every edge is crossed by at most p edges) are uniformly $3(p+1)$-sparse, we use in the previous proof a $(p+1)$-coloring of $\Xi(D)$. The article [46]

proves that simply p-planar graphs (defined from simple drawings) are uniformly m-sparse where $m = \lceil 4.108\sqrt{p} \rceil$.

We need a definition and a lemma. A path in a graph G is *narrow* if it has length at least 2 and all its intermediate vertices have degree 2 in G. Two narrow paths are *disjoint* if no vertex of one is an intermediate vertex of the other. In a drawing, a *self-crossing* of a narrow path is a point where two edges of this path cross.

Lemma 2.11 : Let $\mathcal{P}$ be a set of pairwise disjoint narrow paths in a graph H. A drawing D of H can be transformed into a drawing D' of the same graph where no path of $\mathcal{P}$ has a self-crossing. The crossing graph $\Xi(D')$ is a subgraph of $\Xi(D')$, with same set of vertices.

Proof: We show how to eliminate one self-crossing without introducing new crossings. By repeating this step, one obtains a drawing as desired.

Let D be a drawing of H where a narrow path P from x to y has a self-crossing at point z of the plane (this point is not a vertex). Assume that P is the sequence of edges $f_1, ..., f_p$ where $f_1 = xu_1, f_i = u_{i-1}u_i$ and $f_p = u_{p-1}y$. Let z be the crossing point of, say[18], f_4 and f_8. On the curve segment u_3u_4, let v be the last crossing before z, and $v := u_3$ if there is no crossing between u_3 and z. On the curve segment u_7u_8, let w be the first crossing after z, and $w := u_8$ if there is no crossing between z and u_8. On the curve segment S from v to w that concatenates uz and zw, we can place $u_4, ..., u_7$ (they have degree 2), and so, S is not crossed. In particular, no edge among f_5,...,f_7 is now crossed. All crossings of D lying on the loop consisting of the curve segments zu_4, u_4u_5, ...,u_6u_7, u_7w have disappeared and no new crossing has been created. Hence, $\Xi(D')$ is a subgraph of $\Xi(D)$ having the same vertices. □

A graph Z is a *d-shallow topological minor* of a graph G if there is a subgraph H of G that is obtained from Z by edge subdivisions, such that each edge e of Z is replaced by a path P_e with at most $2d + 1$ edges. (Z is then a d-shallow minor.) The paths P_e of length at least 2 are pairwise disjoint narrow paths of H. By Corollary 4.1 in [41], a class $\mathcal{C}$ has bounded expansion if and only if, for each d, there is an integer q such that the d-shallow topological minors of the graphs in $\mathcal{C}$ are in $\mathcal{U}_q$

Proposition 2.12 : For each q, the class $QP(\mathcal{U}_q)$ has bounded expansion.

Proof : Let us fix integers q and d. Let $G \in QP(\mathcal{U}_q)$ and Z be a d-shallow topological minor of G, defined from some subgraph H of G, that is thus also in $QP(\mathcal{U}_q)$. It has a drawing D whose crossing graph is in $\mathcal{U}_q$.

This drawing yields a potential drawing of Z as follows : for each edge e of Z, the curve segments representing the edges of P_e, say $f_1, \ldots, f_p$ in this order, are merged into a single curve segment to represent e. If f_i and f_j cross, then this curve segment has a self-crossing. But self-crossings can be eliminated from

[18] The proof is the same if z is a crossing point of any f_i and f_j such that $i < j$.

D by Lemma 2.11, giving a drawing D' of H such that $\Xi(D')$ is a subgraph of $\Xi(D)$. Hence $\Xi(D')$ is in $\mathcal{U}_q$, and we can choose for it an orientation of indegree at most q.

We get a drawing D'' of Z by merging into a single curve segment intended to represent e all curve segments representing the edges of P_e. It may have pairs of edges that cross several times.

Let us enumerate as $(e,1),...,(e,p)$, where $1 \leq p \leq 2d+1$, the edges of P_e for an edge e of Z. If e is not subdivided, then $(e,1)$ denotes e, for the purpose of uniform notation.

The graph $\Xi(D')$ has now vertices of the form (e,i) for $e \in E_H$, and an edge between (e,i) and (e',j) if and only if (e,i) and (e',j) cross. The graph $\Xi(D'')$ is obtained from $\Xi(D')$ by fusing, for each $e \in E_Z$, the vertices[19] $(e,1),...,(e,p)$ into a single one, actually e. For each edge g in $\Xi(D'')$, say between e and f, we choose (e,i) and (f,j) that are adjacent in $\Xi(D')$ and we orient $g : e \to f$ if and only if $(e,i) \to (f,j)$ in the chosen orientation of $\Xi(D')$. We obtain an orientation of $\Xi(D'')$ of indegree at most $q(2d+1)$. Hence $Z \in QP(\mathcal{U}_{q(2d+1)})$ and $Z \in \mathcal{U}_{6q(2d+1)+3}$ by Proposition 2.10. Hence, $QP(\mathcal{U}_q)$ has bounded expansion. □

This proposition extends Theorem 14.4 of [41] establishing[20] that the class of simply p-planar graphs has bounded expansion.

Remark 2.13 : *Another notion of crossing graph.*

If D is a simple drawing of a graph G, then we define a graph $\Gamma(D)$ whose vertices are the (points of the plane representing the) crossings of edges and two crossings are adjacent if they are consecutive on some edge. This graph is planar of maximum degree 4. It has no edge if D is 1-planar. It can have cycles if D is 2-planar. It is easy to prove that $\Gamma(D)$ is a forest if and only if $\Xi(D)$ is a forest. This alternative notion gives a more visual approach of crossings. □

2.3 Rank-width

Rank-width [31, 43, 44, 45] is a graph complexity measure that is equivalent to clique-width in the sense that the same graph classes have bounded rank-width and bounded clique-width. It provides a polynomial-time approximation algorithm for computing clique-width and clique-width terms [45]. It is related to clique-width and tree-width as follows, where $rwd(G)$ denotes the rank-width of a graph :

$$rwd(G) \leq cwd(G) \leq 2^{rwd(G)+1} - 1 \quad (1)$$

$$rwd(G) \leq twd(G) + 1. \quad (2)$$

[19] They form an independent set in $\Xi(D')$.

[20] This theorem is stated for drawings where each edge has at most p crossings. Its proof is incorrect as it uses the result of [46] concerning simply p-planar graphs for drawings that need not be simple.

It is proved in [31] that for every graph G with at least one edge[21] :

$cwd(G) \leq 2\lambda_G(rwd(G)) - 1$.

Hence, all our results that are based on bounding λ_G give bounds of the same type (linear or quasi-linear) for clique-width in terms of rank-width, thus improving inequality (1). In particular, by Proposition 2.9, we have $cwd(G) < 12(rwd(G)+1)(4p+1)$ if G is in $QP(\mathcal{U}_p)$.

2.4 Summary of comparisons

Table 2 shows the main results.

Graph class	Bound on clique-width for G of tree-width k	Proof
$\mathcal{S}_r$	$O(k^r)$	[31], Theorem 21
$\mathcal{U}_q$	$O(k^q)$	[17], Theorem 19
Nowhere dense	$O(k^{1+\varepsilon})$ for each $\varepsilon > 0$	Corollary 1.9
Bounded expansion or just $\nabla_1(G) \leq b$	$2^{2b+1}(k+1)$	[31], Theorem 18
No K_r minor	$O(k)$	[31], Theorem 10
$QP(\mathcal{U}_q)$	$6(k+1)(4q+1) - 24q - 2$	Proposition 2.9
p-planar	$6(k+1)(2p+1) - 6p - 2$	Proposition 2.9
planar (= 0-planar)	$6k - 2$	Corollary 2.4
degree $\leq d$	$d(k+1)+2$	Remark below

Table 2.

Remark : For a graph G of degree at most d and a set Y of k vertices, each vertex of Y belongs to at most d sets $N_G(x) \cap Y$ for $x \notin Y$, because it has degree at most d. Hence, $\lambda_G(k) \leq kd + 1$, and $cwd(G) \leq d(twd(G)+1)+2$.

3 Descriptions in monadic second-order logic

The main objective is here to prove that 1-planarity is not monadic second-order expressible[22] (MS-expressible in short). Under the assumption that $P \neq NP$, this follows from the fact that 1-planarity is NP-complete for graphs of bounded tree-width [4], because otherwise, it would be decidable in linear-time[23] [20, 26].

[21] From (2), one gets $cwd(G) \leq 2\lambda_G(twd(G)+1) - 1$, to be compared with Theorem 1.8.

[22] Monadic second-order logic is reviewed in the next subection.

[23] Since every MS definable graph property is decidable in linear time on any class of bounded tree-width.

However, we think interesting to give a proof that does not depend on the $P \neq NP$ assumption. Furthermore, our construction shows that additional conditions like considering graphs of bounded degree do not make 1-planarity MS-expressible.

We will also consider particular classes of 1-planar graphs that are MS definable. A 1-planar graph is *optimal* if it has the maximum number of edges, that is $4n-8$, for n vertices. It is *u-1-planar,* which means *uniquely 1-planarly embeddable,* if any two 1-planar drawings are homeomorphic, as embeddings in the sphere. We denote by $\mathcal{U}1\mathcal{P}$ the class of u-1-planar graphs. An optimal 1-planar graph is u-1-planar unless it is isomorphic to one of particular graphs denoted by XW_{2k}, cf. [48].

We first review a few definitions about monadic second-order logic (only those needed). The reader knowing it (cf. [18, 19, 20, 21]) can skip the next subsection.

3.1 MS formulas and transductions from words to graphs.

Logical expression of graph properties.

For representing a graph G, we use the logical structure $\langle V_G, edg_G\rangle$ where edg_G is the binary symmetric ajacency relation. We identify G and $\langle V_G, edg_G\rangle$.

Monadic second-order logic (*MS logic* in short ; see [20] for a thorough study) allows set quantifications (but no quantifications on relations, such as subrelations of edg_G). Set variables are capital letters ; they denote sets of vertices. The following MS sentence[24] φ :

$$\exists X, Y.(X \cap Y = \emptyset \wedge \forall u, v.\{edg(u,v) \Longrightarrow [\neg(u \in X \wedge v \in X) \wedge \neg(u \in Y \wedge v \in Y) \wedge \neg(u \notin X \cup Y \wedge v \notin X \cup Y)]\})$$

expresses that G is 3-colorable (X, Y and $V_G - (X \cup Y)$ are the three color classes). Formally, G is 3-colorable if and only if $G \models \varphi$. Hence, 3-colorability is *MS-expressible.*

For expressing that G is a cycle with at least 3 vertices, we use :

$$3vertices \wedge degree2 \wedge connectivity.$$

Connectivity is expressed by :

$$\neg\exists X.(X \neq \emptyset \wedge (\exists x.x \notin X) \wedge \forall u, v.\{edg(u,v) \Longrightarrow (u \in X \Longrightarrow v \in X)\}).$$

[24] A *sentence* is a logical formula without free variables.

The reader will easily write the sentences $3vertices$ expressing that the graph has at least 3 vertices and $degree2$ expressing that all its vertices have degree 2.

Edge set quantifications.

We already defined[25] $Inc(G)$, the *incidence graph* of $G = \langle V_G, edg_G \rangle$. Here, we consider it as the bipartite graph $\langle V_G \cup E_G, inc_G \rangle$ where E_G is the set of edges and inc_G is the incidence relation : $inc_G(e, u)$ holds if and only if u is an end of edge e.

An edge of G becomes a vertex in $Inc(G)$; it is no longer defined as a pair of vertices. The edges are the elements that occur as first components of pairs in inc_G. Hence, an MS formula over a structure $\langle W, inc \rangle$ intended to be some $Inc(G)$ can distinguish the edges from the vertices of the potential graph G and check that it is actually an incidence graph. An MS formula over $\langle V_G \cup E_G, inc_G \rangle$ can use edge set quantifications to express a property of G. An *MS_2 graph property* is a property that is expressed on incidence graphs by an MS sentence. An example of an MS_2 property that is not MS-expressible is the existence of a Hamiltonian cycle. It is expressed in $Inc(G)$ by :

> "there exists a set $X \subseteq E_G$ such that the graph $Inc(G)[X \cup V_G]$ is a cycle"

However, for each q, the same properties of graphs in $\mathcal{U}_q$ are MS_2 and MS-expressible. Formally, every MS sentence φ written with inc can be translated into an MS one $\varphi^{[q]}$, written with edg, such that, for every graph G in $\mathcal{U}_q$ we have $G \models \varphi^{[q]}$ if and only if $Inc(G) \models \varphi$ (Chapter 9 of [20]).

Properties of words.

Let A be a finite alphabet. A nonempty word[26] w over A of length n is represented by the logical structure $S(w) := \langle [n], \leq, (lab_a)_{a \in A} \rangle$ where each $i \in [n]$ is a *position*, *i.e.*, an occurrence of some letter. The binary relation $\leq$ is the order of positions and the unary relations lab_a indicate where letters occur : $lab_a(u)$ is true if and only if a occurs at position u. Formulas of MS logic use quantified variables denoting here sets of positions of the considered word represented by $S(w)$.

For an example, the formula

$$\exists X \forall u.(u \in X \Longrightarrow (lab_a(u) \vee \exists v.(v \notin X \wedge u < v \wedge lab_b(u))))$$

says that there is a set X of positions that are either occurrences of a, or are before a occurrences of b not in X. Note the use of $u < v$ abreviating $u \leq v \wedge \neg(u = v)$.

[25] in a more concrete way.

[26] A^+ denotes the set of nonempty words over A, and A^* denotes A^+ together with the empty word.

A well-known result [50] says that a language $L \subseteq A^+$ is regular if and only if it is MS definable, which means that there exists an MS sentence φ such that $w \in L$ if and only if $S(w) \models \varphi$.

Languages of the form $L_0 := \{(ab)^n c^n \mid n \geq 1\}$ or $L_1 := \{(ab)^n c^m \mid n \geq 3m+4\}$, to take two typical examples, are not MS definable because they are not regular. The latter fact is proved as follows. For every language L and word u, $L/u := \{v \in A^* \mid uv \in L\}$. If L is regular, there are only finitely many distinct languages L/u. But there are infinitely many languages $L_0/(ab)^n c = \{c^{n-1}\}$ for $n \geq 1$, and similarly, $L_1/(ab)^n c$. Hence, L_0 and L_1 are not regular.

Monadic second-order transductions.

Monadic second-order transductions are transformations of logical structures specified by MS formulas. We only review the very particular ones that will be used in the proof of Theorem 3.4. They transform words into graphs.

Let us fix A as above and two MS formulas α and $\eta(x, y)$ written with $\leq$ and the unary relation symbols $lab_a(u)$ (x, y are free first-order variables in η).

Let τ be the partial mapping from words in A^+ to graphs, defined as follows : $\tau(w) = G$ if and only if

$$S(w) = \langle V, \leq, (lab_a)_{a \in A} \rangle \models \alpha \text{ and, if this is true,}$$
$$G := \langle V, edg \rangle \text{ where the edge relation } edg \text{ is defined by:}$$
$$edg(x, y) :\Longleftrightarrow S(w) \models \eta(x, y).$$

The positions in w are made into vertices of G. The formula $\eta(x, y)$ must be written so that the relation it defines is symmetric and irreflexive[27] (as τ defines undirected and loop-free graphs).

The main fact we will use about transductions is the following lemma, a special case of the Backwards Translation Theorem, Theorem 7.10 of [20].

Lemma 3.0 : If τ is an MS transduction and φ is an MS sentence, then, the set of words w such that $\tau(w) \models \varphi$ is MS definable and is thus a regular language.

Proof sketch: We let ψ be obtained from φ by replacing each atomic formula $edg(u, v)$ by $\eta(u, v)$. Then, the words w such that $\tau(w) \models \varphi$ are those such that $S(w) \models \alpha \wedge \psi$. $\square$

To prove that a graph property P is *not MS definable*, it suffices to construct τ such that the set of words w such that $\tau(w)$ satisfies P is not regular. We will do that for proving Theorem 3.4.

[27] To ensure this, one can take $\eta(x, y)$ of the form $(\eta'(x, y) \vee \eta'(y, x)) \wedge x \neq y$ for some MS formula $\eta'(x, y)$.

3.2 1-planarity is not MS definable

We need some definitions and notation.

Definitions 3.1.

(a) Let G be a graph. We denote by $P(x_1, x_2, ..., x_n)$ where $n \geq 2$, a path from x_1 to x_n, with vertices $x_1, x_2, ..., x_n$ in this order, and by $C(x_1, x_2, ..., x_n)$ where $n \geq 3$, a cycle with vertices $x_1, x_2, ..., x_n$, such that we have $P(x_1, x_2, ..., x_n)$ and an edge x_1x_n.

Consider a drawing D of G in the plane, with possible edge crossings. A cycle $C(x_1, x_2, ..., x_n)$ without any *self-crossing* (no two of its edges cross) induces two open regions of the plane: the bounded one is denoted by $R(x_1, x_2, ..., x_n)$ and the unbounded one by $R^\infty(x_1, x_2, ..., x_n)$. The edges, *i.e.*, the curve segments representing the edges of $C(x_1, x_2, ..., x_n)$, are not in $R(x_1, x_2, ..., x_n) \cup R^\infty(x_1, x_2, ..., x_n)$. If the cycle has self-crossings, it determines at least three open regions of the plane. It *separates two vertices* u and v if these vertices are (that is, the corresponding points are) in different regions, and then, any path between u and v must cross some edge of the cycle or go through one of $x_1, x_2, ..., x_n$.

Two drawings are *homeomorphic* if they are so as embeddings in the sphere.

(b) For $n \geq 4$, let G_n be the graph with vertices $a_i, b_i, c_i, d_i, e_i, f_i$ for $i = 1, ..., n$. Figure 1 shows G_6. A cross in a quandrangular face indicates two edges that cross, for instance a_1b_2 and a_2b_1.

The graph G_n has $6n$ vertices and $24n - 8 - (n - 3) = 23n - 5$ edges. It is 1-planar but not optimal because of the $n - 3$ missing edges d_ic_{i+1} for $i = 2, ..., n - 2$. It has 8 vertices of degree 6, $2n - 6$ of degree 7 and all others have degree 8.

We let Q_n be the planar subgraph induced by the vertices c_i and d_i for $i = 2, ..., n - 1$.

Proposition 3.2 : Each graph G_n has a unique 1-planar drawing.

Proof: We will compare the (natural) 1-planar drawing $\overline{D}$ of G_n (Figure 1 shows $\overline{D}$ for G_6, from which the general case is easily understood) to an arbitrary 1-planar drawing D. Without loss of generality (since we consider graph embeddings in the sphere) we can assume that all vertices (except a_1, a_2, f_1) are in the bounded region $R(a_1, a_2, f_1)$ of D, as in Figure 1 for $\overline{D}$. In this figure, the edges of $C(a_1, a_2, f_1)$ are the thickest ones. Note that a_2f_1 is crossed by the "thin" edge a_1f_2; hence the unbounded region $R^\infty(a_1, a_2, f_1)$ contains half of the edge a_1f_2 and no vertex. We will prove that D it is homeomorphic to $\overline{D}$. In our discussion, points (vertices), edges, triangles (3-cycles), cycles, regions *etc.* will refer to D.

First, observe that in any 1-planar drawing, a 4-cycle has at most one crossing.

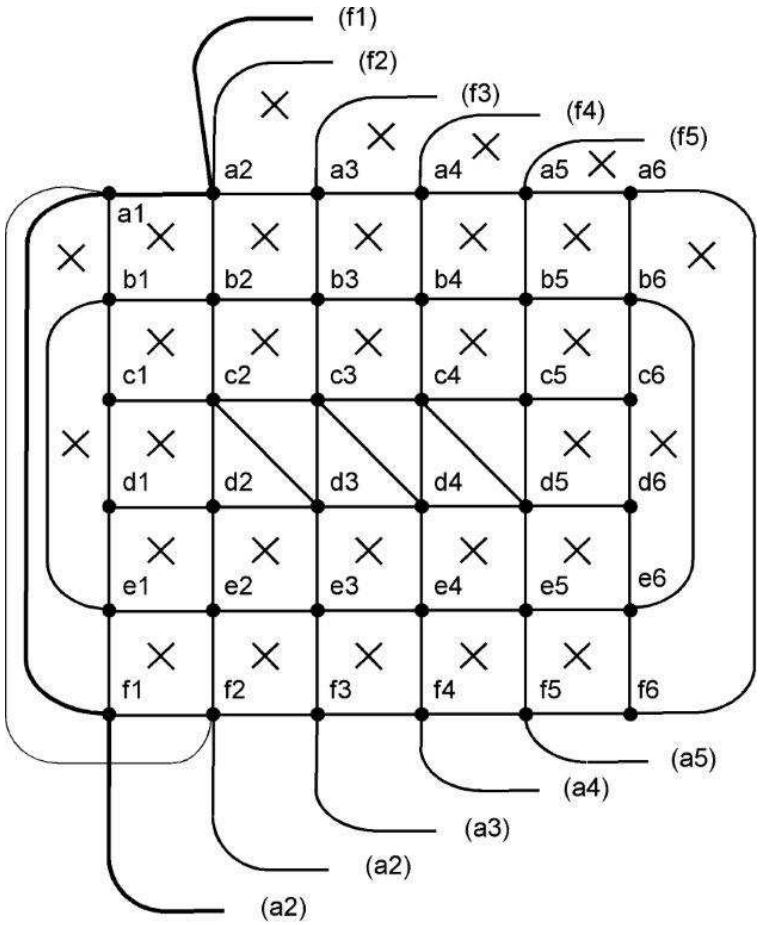


Figure 1: The graph G_6.

Claim 1 : Let $C(x,y,z)$ be a triangle in G_n and u,v be distinct vertices not in $\{x,y,z\}$. There are at least four edge-disjoint paths between u and v that avoid the vertices x,y,z. The triangle $C(x,y,z)$ does not separate u and v.

Proof: Let $H := G_n - \{x,y,z\}$. Removing any 3 edges of H keeps it connected. Hence, by Menger's Theorem ([25], Section 3.3), there are at least 4 edge-disjoint paths between u and v in H. These paths are in G_n and they avoid x,y,z. Let us give an example:

$$u = d_1, v = d_i, x = c_i, y = c_{i+1}, z = d_{i+1}, 2 \leq i \leq n-1.$$

We have the four edge-disjoint paths:

$P(d_1, d_2, ..., d_i)$, $P(d_1, e_1, e_2, ..., e_{i-1}, d_i)$,
$P(d_1, b_1, b_2, ..., b_n, e_n, e_{n-1}, ..., e_{i+1}, d_i)$ and
$P(d_1, c_1, b_1, a_1, a_2, ..., a_n, f_n, f_{n-1}, ..., f_i, e_i, d_i)$.

Going back to the general case, if the triangle $C(x,y,z)$ separates u and v, then one of its edges must be crossed twice. This is not possible. □

Claim 2 : Let $C(x,y,z) \neq C(a_1,a_2,f_1)$. Then $R(x,y,z)$ does not contain any vertex.

Proof: We have $R(x,y,z) \subset R(a_1,a_2,f_1)$. Assume $a_1 \notin \{x,y,z\}$. If $u \in R(x,y,z)$, then $C(x,y,z)$ separates a_1 and u which contradicts Claim 1. The proof is the same with a_2 or f instead of a_1. □

Claim 3 : Let $C(x,y,z)$ be as in Claim 2. If one edge of $C(x,y,z)$, say xy, is crossed by an edge uv, then u or v is z and the two edges, xz and yz are not crossed.

Proof : By Claim 2, no end of uv is in $R(x,y,z)$. Hence, u or v is z. If another edge would cross xz, it should have an end equal to y, but it would cross also uz (or vz). This contradicts 1-planarity. □

Claim 4 : Let $C(x,y,z)$ and $C(x,y,u)$ be triangles such that $\{x,y,z,u\} \cap \{a_1,a_2,f_1\} = \emptyset$. Either $R(x,y,z)$ and $R(x,y,u)$ are disjoint, and we may have an edge zu crossing xy, or they overlap (i.e., $R(x,y,z) - R(x,y,u) \neq \emptyset$ and $R(x,y,u) - R(x,y,z) \neq \emptyset$), xy is not crossed, and either xz crosses yu or yz crosses xu. If $C(x,y,v)$ is a third triangle such that $v \notin \{a_1,a_2,f_1\}$ then xy is not crossed.

Proof: In a planar drawing, either $R(x,y,z)$ and $R(x,y,u)$ are disjoint, or one is included in the other, that is, either $z \in R(x,y,u)$ or $u \in R(x,y,z)$. As D is 1-planar, we may have in the first case edge zu crossing xy. We cannot have $z \in R(x,y,u)$ or $u \in R(x,y,z)$ by Claim 2, hence the second case cannot happen. As D is 1-planar, we may also have xz crossing yu or yz crossing xu, but not both. By Claim 3, xy is not crossed.

If we have three triangles sharing the edge xy, two of them overlap. Hence, xy is not crossed. □

Claim 5 : Let $C(x,y,z)$ and $C(u,v,w)$ be such that $\{x,y,z,u,v,w\} \cap \{a_1,a_2,f_1\} = \emptyset$. If $R(x,y,z)$ and $R(u,v,w)$ overlap, then these two triangles share an edge.

Proof: By Claim 2, the two triangles share a vertex, say $x = u$. Assume for a contradiction that $\{y, z\} \cap \{v, w\} = \emptyset$. If xy and xz do not cross vw, then, again by Claim 2, $\{y, z\} \cap R(u, v, w) = \emptyset$, hence, $R(x, y, z) \cap R(u, v, w) = \emptyset$ contradicting the hypothesis. Assume xy crosses vw, then xz does not (otherwise vw has two crossings) and yz does not cross any edge of $C(u, v, w)$. Hence, either v or w is in $R(x, y, z)$ which contradicts Claim 2. Hence, $\{y, z\} \cap \{v, w\} \neq \emptyset$ which proves the statement. The two triangles are thus as in Claim 4. □

Next we consider $D[Q_n]$. This drawing is a union of triangles that contain no vertex. We will prove that it is planar. (The graph Q_n is planar but this does not imply that the 1-planar drawing $D[Q_n]$ is because c_3c_4 might cross d_3d_4.) We consider a 4-cycle $C(c_i, d_i, d_{i+1}, c_{i+1})$ where $2 \leq i \leq n-2$. Its vertices are denoted for simplicity and respectively by c, d, d', c'. We will also use b denoting b_i and d'' denoting d_{i+2}.

Claim 6 : The edge cc' does not cross dd' and the edge cd does not cross $c'd'$.

Proof: Assume for getting a contradiction that cc' crosses dd', so that cd does not cross $c'd'$.

Consider the triangle $C(b, c, c')$. The edge dd' crosses cc', and thus cannot cross edge bc or bc'. Hence, $C(b, c, c')$ separates d and d'. This is impossible by Claim 1 as there are 4 edge-disjoint paths between d and d' that avoid b, c, c' (one of them can be dd').

Assume now similarly that cd crosses $c'd'$, so that cc' does not cross dd'.The vertex d'' will play the role of b in the previous proof. The triangle $C(c', d', d'')$ separates c and d, which contradicts Claim 1. □

Hence, the cycle $C(c, d, d', c')$ is not self-crossing.

Claim 7 : No edge of Q_n is crossed by any edge of G_n.

Proof : Each edge of the Hamiltonian cycle $C(c_2, c_3, ..., c_{n-1}, d_{n-1}, ..., d_3, d_2)$ of Q_n is incident with three triangles, hence, is not crossed by Claim 4. An edge crossing c_id_{i+1} should link d_i and c_{i+1}, and an edge crossing c_id_i should link c_{i-1} and d_i. But the graph G_n has no such edges. □

As there are no crossings between edges of Q_n, the drawing $D[Q_n]$ is planar. Furthermore, by Claim 2, none of its triangles contains any vertex.

Hence, $D[Q_n]$ is as in Figure 1: the drawing is outerplanar with Hamiltonian (external) cycle $C(c_2, c_3, ..., c_{n-1}, d_{n-1}, ..., d_3, d_2)$. All other edges of Q_n are in $R(c_2, c_3, ..., c_{n-1}, d_{n-1}, ..., d_3, d_2)$.

We now prove the main statement. We consider a 1-planar drawing D of G_n and $\overline{D}$ as in Figure 1. For both of them, all vertices except a_1, a_2, f_1 are in the bounded region $R(a_1, a_2, f_1)$. Claims 1 to 7 hold for D and $\overline{D}$.

Let $\widehat{G_n}$ be obtained from G_n by adding the edges d_ic_{i+1} for $i = 2, ..., n-2$. It is an optimal 1-planar graph because it has a 1-planar drawing and its number of edges is $23n - 5 + n - 3 = 4 \cdot 6n - 8$.

By Claim 7, we can transform D and $\overline{D}$ into 1-planar drawings of $\widehat{G_n}$ by putting each additional edge d_ic_{i+1} inside $R(d_i, d_{i+1}, c_{i+1}, c_i)$ that contains already a single edge and no vertex. We obtain two 1-planar drawings of $\widehat{G_n}$. They are homeomorphic because $\widehat{G_n}$ is optimal and is not one of the special graphs XW_{2k} (cf. [48]). Hence D and $\overline{D}$ are also homeomorphic, by the same homeomorphism. □

For proving Theorem 3.4, we define, for $n \geq 4$ and $m \geq 0$, the graph $H_{n,m}$ as G_n augmented with m new vertices $g_1, ..., g_m$ and edges forming the path $P(d_2, g_1, g_2, ..., g_m, c_{n-1})$.

Lemma 3.3 : $H_{n,m}$ is 1-planar if and only if $m \geq 2n - 8$. It is u-1-planar if and only if $m = 2n - 8$.

Proof: If $m \geq 2n - 8$, we obtain a 1-planar drawing of $H_{n,m}$ by putting :

g_1 in $R(c_2, c_3, d_3)$, g_2 in $R(c_3, d_3, d_4)$, g_3 in $R(c_3, c_4, d_4)$,...,

g_{2n-9} in $R(c_{n-3}, c_{n-2}, d_{n-2})$,

and the remaining vertices, $g_{2n-8}, ..., g_m$, in $R(c_{n-2}, d_{n-2}, d_{n-1})$.

We now prove that, if $m < 2n - 8$, we cannot do any similar construction. Let us fix n. Assume D is a 1-planar drawing of $H_{n,m}$ where $m < 2n - 8$ and m is minimal with this property. It induces a drawing of G_n that must be homeomorphic to that of Figure 1, by Proposition 3.2.

The path $P := P(d_2, g_1, g_2, ..., g_m, c_{n-1})$ has no self-crossing, otherwise, we can shorten it and obtain a 1-planar drawing of $H_{n,m'}$ where $m' < m$.

No edge of G_n apart from c_id_{i+1} for $i = 2, ..., n-2$, and c_id_i for $i = 2, ..., n-2$, can be crossed. Hence P must be drawn inside $R(c_2, c_3, ..., c_{n-1}, d_{n-1}, ..., d_3, d_2)$. It must cross $2n-7$ edges, hence have at least $2n-8$ intermediate vertices. We cannot have $m < 2n - 8$.

If $m > 2n - 8$, these intermediate vertices can be placed in different ways.

If $m = 2n - 8$, the way described above is the unique one. □

We will use the transductions described in Section 3.1.

Theorem 3.4 : The class $1\mathcal{P}$ of 1-planar graphs and the class $\mathcal{U}1\mathcal{P}$ of uniquely 1-planary embeddable graphs are not monadic second-order definable.

Proof: We define $H_{n,m}$ from a word w of the form $(abcdef)^n g^m$ over the alphabet $A := \{a, b, c, d, e, f, g\}$. Each position in the word w is a vertex of $H_{n,m}$. The i-th occurrence of letter a is a_i, and similarly for $b_i, c_i, d_i, e_i, f_i, g_i$. The edges of $H_{n,m}$ are described by a first-order formula relative to the structure $S(w) = \langle P, \leq, (lab_x)_{x \in A} \rangle$ where $P := [6n+m]$ is the set of positions of w. It says that there is an edge between an occurrence of a and the next one, between an

occurrence of a and the following occurrence of b, between the last occurrence of a and the last occurrence of f, *etc.*

Hence, $H_{n,m}$ is the image of $S(w)$ under an MS-transduction τ. This transduction maps the words of the regular language $W := \{(abcdef)^n g^m \mid n \geq 4, m \geq 0\}$ to the graphs $H_{n,m}$, in a bijective way.

If 1-planarity would be MS-expressible, then, by Lemma 3.0, the language $L := W \cap \tau^{-1}(1\mathcal{P})$ would be MS definable, hence regular. But $L = \{(abcdef)^n g^m \mid n \geq 4, m \geq 2n-8\}$ and this language is not regular (we recalled in Subsection 3.1 how such a fact can be proved). If the class $\mathcal{U}1\mathcal{P}$ would be MS definable, then the language $\{(abcdef)^n g^{2n-8} \mid n \geq 4\}$ would be regular, which is not the case either, by a similar argument. $\square$

It follows that 1-planar graphs are not characerized by finitely many forbidden configurations such as minors, subgraphs or induced subgraphs. This is not surprizing because 1-planarity is an NP-complete property [12, 22]. They are even not characterized by an infinite set of forbidden induced subgraphs that would be MS definable, as are comparability graphs and interval graphs [15].

A natural question is then : What additional conditions might may 1-planarity MS-expressible ?

Our proof yields a corollary for three classes of graphs. One of them is $\mathcal{H}$, the class of graphs having a Hamiltonian cycle and a 1-planar drawing where any two edges of this cycle do not cross.

Next, we recall that a *rotation system* for a graph G describes the circular ordering of the edges incident to each vertex u in some drawing in the plane, either planar or not (see [14]). In the logical setting, this circular order is defined as a ternary relation $Next(u, x, y)$ that means : ux and uy are edges, and uy follows ux in the circular order of edges, according to some fixed orientation of the plane. We have $Next(u, x, x)$ if ux is the unique edge incident with u. Each drawing of the graph (with possible crossings) yields a rotation system, but this drawing may not be reconstructible from the rotation system. A pair $(G, Next)$ of a graph and a rotation system is called a *map* (see [14]). A map $(G, Next)$ is 1-*planar* if G has a drawing whose associated rotation system is $Next$.

Corollary 3.5: The following classes of structures are not MS definable:

(1) for each $d \geq 8$, the class of 1-planar graphs of degree at most d or of path-width at most d,

(2) the class $\mathcal{H}$,

(3) the class of 1-planar maps.

Proof : (1) This is immediate because the graphs $H_{n,m}$ have maximal degree 8 and path-width at most 8.

(2) Each graph $H_{n,m}$ where n is odd (and at least 5) has a Hamiltonian cycle. If $m \geq 2n-8$, then $H_{n,m}$ has a 1-planar drawing where such a cycle has no self-crossing. See Figure 2, where $n = 7$, from which the general case can

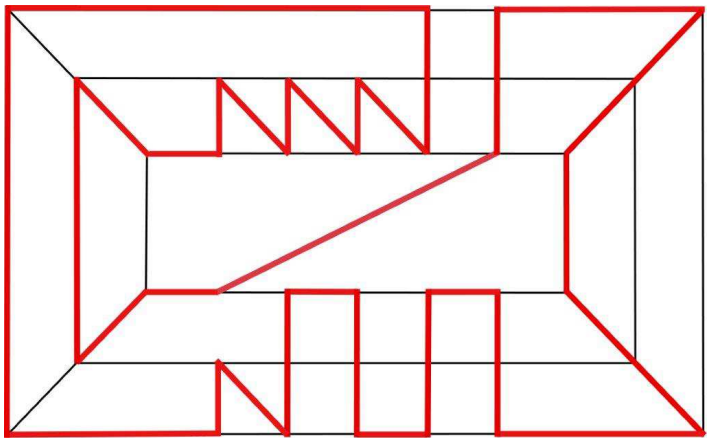


Figure 2: A non-self-crossing Hamiltonian cycle in a graph $H_{7,m}$ (for any m).

be inferred. (The Hamiltonian cycle $a_1, ..., a_5, b_5, c_5, b_4, c_4, ..., f_1$ is shown with bolder edges. We do not show all edges for clarity).

We replace the language W of the proof of Theorem 3.4 by the regular language

$$W' := \{(abcdef)^{2p+1}g^m \mid \ p \geq 2, m \geq 0\}.$$

If $\mathcal{H}$ would be MS definable, then the language

$$W' \cap \tau^{-1}(\mathcal{H}) = \{(abcdef)^{2p+1}g^m \mid \ p \geq 2, m \geq 2(2p+1) - 8\}$$

would be regular, which is not the case.

(3) Each graph $H_{n,m}$ can be equipped with a rotation system $Next_{n,m}$ such that the map $M_{n,m} := (H_{n,m}, Next_{n,m})$ is 1-planar if and only if $H_{n,m}$ is. The relation $Next_{n,m}$ is easily described by a first-order formula $\gamma(u, x, y)$.This formula will express that $Next_{n,m}(u, x, y)$ holds if, to take only a few clauses as examples :

u is an occurrence of letter c, x is the occurrence of c following u and y is the occurrence of letter d that follows x, or,

u is an occurrence of c, y is the occurrence of d following u and x is the occurrence of d that follows y, or,

y, u, x are three consecutive occurrences of letter g.

Hence, we have an MS transduction that construct $M_{n,m}$ from a word. The proof continues as in the other cases. □

Remark: An alternative construction.

Let us define $J_{n,m}$ as G_4 augmented with new vertices $g_1, ..., g_m, h_1, ..., h_n$, a path $P(d_2, g_1, ..., g_m, c_3)$ and n paths $P(c_2, h_i, d_3)$ for $i = 1, ..., n$. Then $J_{n,m}$

is 1-planar if and only if $n \leq m$. The proof of Theorem 3.4 is easily adapted. However, cf. Corollary 3.5, the graphs $J_{n,m}$ have unbounded degree, and no Hamiltonian cycle for large n; nevertheless, they have path-width at most 8 and a rotation system for $J_{n,m}$, as in the proof of Corollary 3.5 can be defined from a word $(abcdef)^4 g^m h^n$ that defines it.□

Edge-set quantifications do not help.

Theorem 3.4 deals with MS sentences that do not use edge-set quantifications. As 1-planar graphs are uniformly 4-sparse, MS_2 sentences are no more powerful than MS ones to express their properties (cf. Section 3.1). Hence, Theorem 3.4 also shows that the classes $1\mathcal{P}$ and $\mathcal{U}\ 1\mathcal{P}$ are not MS_2 definable.

Some positive monadic second-order expressibility results.

We denote by $\mathcal{O}p\mathcal{P}$ the class of *outer p-planar* graphs[28], that is, that have a Hamiltonian cycle and a simple p-planar drawing such that this cycle has no self-crossing and all other edges are inside the bounded region it defines. The class $\mathcal{O}1\mathcal{P}$ is included in $\mathcal{H}$ considered in Corollary 3.5.

An outer 1-planar graph is actually planar : consider a corresponding drawing; the edges not in the Hamiltonian cycle C can be put into two sets, say F and F' each of them having no two crossed edges; then the edges of F' can be redrawn outside of C, and we get a planar drawing. We prove in the appendix that its tree-width is at most 3.

Proposition 3.6: The class of optimal 1-planar graphs and the class of outer 1-planar graphs are MS definable.

Proof: We will use MS_2 sentences to define these classes.

Optimal graphs. By Theorem 11 of [48], a graph G is optimal (as 1-planar graph) if and only if it consists of a 3-connected quandragulated planar graph H and edges added in the following way : for each 4-cycle $C(x, y, z, u)$ of H, one adds the (crossing) edges xz and yu.

An MS_2 sentence for describing these graphs can be written of the form $\exists F.\varphi(F)$ where F is denotes a set of edges and $\varphi(F)$ expresses the following conditions relative to a graph G :

(a) Every vertex is the end of an edge in F,

(b) the graph $H := (V_G, F)$ is 3-connected and planar, it has no 3-cycle and every p-cycle for $p \geq 5$ has a chord,

(c) for every edge xz not in F, there is in H a 4-cycle of the form $C(x, y, z, u)$,

(d) for every 4-cycle $C(x, y, z, u)$ of H, the edges xz and yu are in $E_G - F$.

Outer 1-planar graphs. They are described by an MS_2 sentence of the form $\exists F.\psi(F)$ where F denotes a set of edges and $\psi(F)$ expresses the following conditions relative to a graph G :

(a) F is the set of edges of a Hamiltonian cycle C,

[28] Not to be confused with that of p-outer planar graphs that have tree-width at most $3p-1$ by [5].

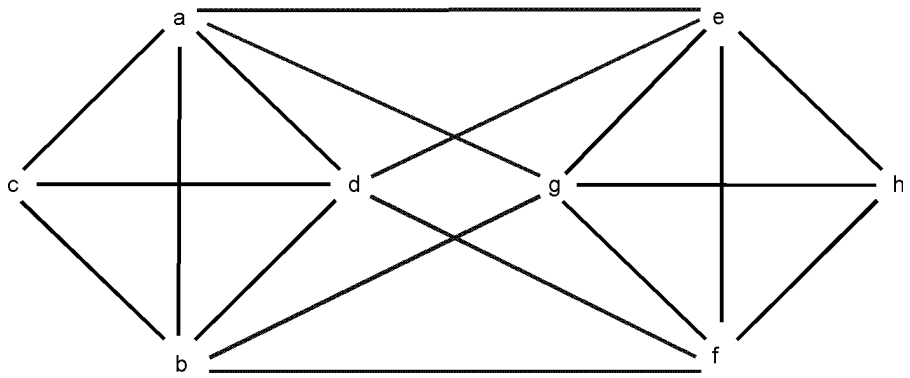


Figure 3: A 1-planar drawing of a planar graph H.

(b) there are no three edges e, e' and f in $E_G - F$ such that f crosses e and e' in the drawing of G such that C bounds the external face. As soon as C is fixed and drawn in the plane, the possible crossings of the edges in $E_G - F$ are imposed by the graph structure : consider $e = xy$ and $f = uv$, where x, y, u, v are pairwise distinct vertices. Then e crosses f if and only if there exist two disjoint paths P and P' between x and y that consist of edges from C and are such that u is in P and v in P'. These paths have only x and y in common and $C = P \cup P'$. This condition about e and f is MS_2-expressible. □

Let $\mathcal{A}$ be the class of 1-planar graphs G that are *apex*, which means that removing one vertex makes G planar.

Open question: Is membership in $\mathcal{A}$ MS-expressible ?
A difficulty comes from the following fact : if D is a 1-planar drawing of $G \in \mathcal{A}$, it is not necessarily the case that $D[G']$ is planar for some subgraph G' of G obtained by removing one vertex. This means that we may have to consider 1-planar drawings *with crossings* for planar graphs.

For an example, Figure 3 shows a 1-planar drawing of a planar graph H. This graph is 3-connected and has thus a unique planar drawing. Let G be H augmented with a vertex x and edges xa, xb, xc, xe, xh, xf: it is 1-planar and apex. However, G has no 1-planar drawing D such that $D[V_G - \{x\}]$ is planar, because it is not possible to insert x in the (unique) planar drawing of H so as to get a 1-planar drawing of G. Furthermore, removing from G any other vertex than x yields a graph that is not planar, because in each case, this graph has a K_5 minor.

Conjecture: For each $p \geq 2$, the class of p-planar graphs is not MS definable.

The p-planar graphs are obviously more complicated than the 1-planar ones, which motivates the conjecture. The graph $H_{n,m}$ of the proof of Theorem 3.4 is 2-planar if $n-4 \leq m$. If the converse holds, which we think, we get a proof of the conjecture for $p = 2$.

The same conjecture can be made for $QP(Forest)$ and the class of p-quasi-planar graphs.

4 Conclusion

We have exhibited graph classes for which clique-width and tree-width are linearly related. Apart from understanding graph structure, our motivation is to use tree-decompositions as intermediate steps for constructing clique-width terms, for graphs in classes of unbounded clique-width.

For some classes of bounded clique-width, "good" clique-terms can be constructed in polynomial time by using *modular decomposition*[29] instead of tree-decomposition as preliminary step, for example in [10].

More open questions

(1) Which planar graphs have no 1-planar drawing with crossings ? Likely, regarding our proof of Proposition 3.2, the triangulated graphs of high edge-connectivity are so.

(2) Which 1-planar graphs are u-1-planar ? In particular, which edges can be removed from an optimal[30] u-1-planar graph so that it remains u-1-planar ?

(3) Are the classes of p-quasi-planar graphs (for $p > 2$) nowhere dense ? Do they have bounded expansion ?

Independently of quasi-planarity, we can also ask :

(4) Does there exist a real number $\alpha < q$ such that $cwd(G) = O(twd(G)^{\alpha})$ for all G in $\mathcal{U}_q$?

5 Appendix: Outer 1-planar graphs

We recall that outerplanar graphs have tree-width at most 2, [5].

Proposition 5.1 : Outer 1-planar graphs have tree-width at most 3 and clique-width at most 6.

Proof sketch : We use induction on the number of edges to prove that every graph in $\mathcal{O}1\mathcal{P}$ has tree-width at most 3. In this proof, we allow parallel edges, which have no effect on tree-width.

The smallest nontrivial graph in $\mathcal{O}1\mathcal{P}$ is K_4.

[29] Modular decomposition can be computed in linear time, see [37].

[30] Not all optimal 1-planar graphs are u-1-planar, cf. [48].

Consider G in $\mathcal{O}1\mathcal{P}$. If it has a vertex x of degree 2 with neighbours y and z, we delete it and we add an edge between y and z. If there are two parallel edges, we remove one. In both cases, we obtain G' in $\mathcal{O}1\mathcal{P}$ with one edge less than G. It has tree-width at most 3 and so has G.

Otherwise, we assume that none of these transformations is applicable. Let $C(x_1, ..., x_n)$ be the Hamiltonian cycle. The internal edges are $x_i x_{i+p}$, such that $1 \leq i$, $2 \leq p$, $i+p \leq n$ (except $x_1 x_n$ because we have eliminated parallel edges).

There exist pairs of internal edges $x_i x_{i+p}$ and $x_j x_{j+q}$ that cross, which means that $i < j < i+p < j+q$. Each of them is not crossed by any other edge. Consider such a pair such that $j+q-i$ is minimal. It is of the form $\{x_i x_{i+2}, x_{i+1} x_{i+3}\}$. We remove x_{i+2}, x_{i+1} and the incident edges, and we add the edge $x_i x_{i+3}$: we get a smaller graph G' in $\mathcal{O}1\mathcal{P}$ having a tree decomposition (T, f) of width at most 3 by the induction hypothesis. Some box $f(u)$ contains x_i and x_{i+3}. By adding to T a box consisting of $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ attached to u, we get a tree decomposition of G of width 3.

From this proof, we can build a graph-grammar of type HR ([20], Chapter 4), and we can express its rules with clique-width operations using 6 labels. Hence, the outer 1-planar graphs have clique-width at most 6. This proof shows that outer planar graphs have clique-width at most[31] 5. □

We conjecture that outer p-planar graphs have bounded tree-width.

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