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# Phenomenological Determinisms

André Samain and Xavier Garbet

CEA, IRFM, F-13108 Saint-Paul-Lez-Durance, France.

## ABSTRACT

The arrow of time is revisited. It is proposed that it arises from the existence in the cosmos of a phenomenological determinism within the fundamental determinism expressed by the Liouville or Liouville-von Neumann equations. The latter link exactly the fundamental information present in the cosmos at any different times. The proposed phenomenological determinism is expressed by phenomenological laws which allow the observers to deduce, from the phenomenological fraction of the fundamental information accessible to observation at a time  $t$ , the same phenomenological information at a times  $t'$ , but only along the arrow of time  $t' > t$ . It is first shown, in a simplified cosmos, that this phenomenological determinism can exist under two conditions : an initial condition weighing on the fundamental information at a primordial time of the cosmos, which introduces the arrow of time aimed from the primordial time to the observers' times ; a structural condition specifying in fine the nature of the phenomenological information, which imposes that any information which does not become phenomenological very rapidly becomes disconnected forever. It is then shown that these conditions are fulfilled, and the phenomenological determinism exists, thanks to a canonical representation of the phase space in the vicinity of the hamiltonian trajectories, which highlights purely hyperbolic or elliptic motions of the cosmos.

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## I – INTRODUCTION

Physicists since Newton and Maxwell made every possible effort of experimentation and mathematical imagination to discover the laws expressing the fundamental determinism in our cosmos. Admitting that the reality is divided into successive states specified by a time  $t$ ,

the state at time  $t$  expresses a fundamental information assimilated to an appropriate mathematical object  $\mathfrak{I}_t$ . The fundamental determinism produces, from the fundamental information  $\mathfrak{I}_t$  at a time  $t$ , the fundamental information  $\mathfrak{I}_{t'}$  at any time  $t'$  before or after  $t$ . It thus maintains invariant the quantity of information in  $\mathfrak{I}_t$ . It is applicable in the cosmos but also in closed systems quasi isolated from the rest of the cosmos during a finite time interval. In the strictly classical framework, the fundamental information  $\mathfrak{I}_t$  is assimilated to a point  $(x,p)_t$  in a phase space  $(x,p)$  expressing the state of  $2N$  canonically conjugate degrees of freedom  $x^i$  and  $p_i$  of particles and of force fields. An observable  $V$ , assimilated to a function  $V(x,p)$  analytic in  $(x,p)$ , has at each time  $t$  the well-defined value  $V((x,p)_t)$ . The fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$  is expressed by the Hamilton law of motion of the point  $(x,p)_t$ , generated by the hamiltonian  $H(x,p)$ , a real invariant observable representing the cosmos energy  $E$ . The hamiltonian trajectories cover in an ergodic way the closed, differentiable energy surface  $H(x,p)=E$ , the Birkhoff theorem being verified.

The classical fundamental information  $\mathfrak{I}_t$  concentrated within the simple point  $(x,p)_t$  was found too rigid when thermodynamics came in. A radical change, in the Boltzmann-Gibbs framework, has been to identify the fundamental information  $\mathfrak{I}_t$  at time  $t$  in the cosmos with a fundamental distribution function  $\rho_t(x,p)$  in the space phase (Boltzmann 1896 ; Gibbs 1902). The observables  $V$  are still assimilated to analytic functions  $V(x,p)$ . The fundamental distribution function  $\rho_t(x,p)$  is a real, normalized observable, which determines the expectation value  $\langle V \rangle_t = \int d_N x d_N p V(x,p) \rho_t(x,p)$  of each observable. That situation equips naturally the vector space  $\mathfrak{R}$  formed by all observables  $V(x,p)$ ,  $W(x,p)$ , ... with the hermitian product  $V \bullet W = \int d_N x d_N p V(x,p) (W(x,p))^*$ , so that  $\langle V \rangle_t = V \bullet \rho_t$ . The reality at time  $t$  is expressed by the fundamental distribution  $\rho_t(x,p) \Leftrightarrow \mathfrak{I}_t$  or equivalently by the set of all expectation values  $\langle V \rangle_t$ . It remains however recognizable by the observers since many observables are well defined thermodynamic quantities with a small variance around their expectation value. In particular, the cosmos is localized via  $\rho_t(x,p)$  in a thin layer  $H(x,p) \cong E$  and has accordingly a well defined energy close to  $E$ . The fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$  in the Boltzmann-Gibbs framework is expressed by the Liouville equation

generated by the hamiltonian  $H(x,p)$ , namely  $\frac{\partial \rho_t}{\partial t} = \Omega \rho_t = -\{\rho_t, H\}$ , meaning that  $\rho_{t'} = \exp(\Omega(t' - t))\rho_t$  at any time  $t'$  before or after  $t$ . The linear Liouville operator  $\Omega$  acting on the observables  $V \in \mathfrak{R}$  is anti-hermitian for the hermitian product  $V \bullet W$ . Any integral  $I = \int F(\rho_t) \bullet \rho_t$  is then invariant in time, which reflects the invariance of the quantity of information in  $\mathfrak{I}_t \Leftrightarrow \rho_t(x,p)$ . The quantum revolution of first generation assimilates the observables  $V$  to linear operators  $V^\circ$  acting in the vector space of quantum states. A striking point is that it maintains the organization of the fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$  in the Boltzmann- Gibbs framework : the fundamental information  $\mathfrak{I}_t$  is identified with an observable-operator  $\rho_t^\circ$ , the density matrix, which plays the same role as the fundamental distribution function  $\rho_t(x,p)$  by specifying all the expectation values  $\langle V \rangle_t$ , equal to an hermitian product  $V^\circ \bullet \rho_t^\circ$ ; the reality at time  $t$  is expressed by the density matrix or by the set of all expectation values; the fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$  is expressed by the Liouville-von Neumann equation transposing the Liouville equation, which determines  $\rho_{t'}^\circ$  from  $\rho_t^\circ$  whatever  $t$  and  $t'$ . Thermodynamics can be understood via the Boltzmann- Gibbs framework as well as via quantum mechanics if the scales  $(\delta x^i, \delta p_i)$  of the involved observables  $V(x,p)$  are large enough to be not subjected to the Heisenberg uncertainty principle (Landau and Lifshitz 1958 ; Balian 2007). In that case, the Wigner function provides a standard transposition from the Boltzmann- Gibbs form  $V(x,p)$  to the quantum form  $V^\circ$  (Zachos et al. 2005).

The fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$  could be used directly by observers if they knew all the details of the fundamental information  $\mathfrak{I}_t$ . In most practical situations, it is not the case: observers have access only to a fraction  $\mathfrak{I}_{t \ p}$  of the fundamental information  $\mathfrak{I}_t$  at a time  $t$ . A very important fact is that they are able, by using a set of phenomenological laws, to deduce from that accessible fractional information  $\mathfrak{I}_{t \ p}$  at time  $t$ , with accuracy, the accessible fractional information  $\mathfrak{I}_{t' \ p}$  at times  $t' > t$ , along the arrow of time, pointing by convention in the direction  $+$  of the time axis, towards the future and away from the past. The phenomenological laws are not applicable from  $t$  to  $t' < t$  against the arrow: the observers are of course able to deduce from  $\mathfrak{I}_{t \ p}$ , with less accuracy, some elements of  $\mathfrak{I}_{t' \ p}$  at times  $t' < t$ , but by still applying the phenomenological laws along the arrow of time, i.e. by validating an element of  $\mathfrak{I}_{t' \ p}$  for  $t' < t$  if it entails via these laws an element of  $\mathfrak{I}_{t \ p}$ . Those facts may be

interpreted as meaning that the information  $\mathfrak{I}_{t,p}$  at the time  $t$  determines accurately, per se, the information  $\mathfrak{I}_{t',p}$  at the times  $t' > t$  along the arrow of time. In this article we make a working hypothesis in that way, namely, that there exists a phenomenological determinism  $\mathfrak{I}_{t,p} \Rightarrow \mathfrak{I}_{t',p}$  for  $t' > t$ , expressed by specific phenomenological laws, involving at each time  $t$  a phenomenological information  $\mathfrak{I}_{t,p}$  which is a specific fraction of the fundamental information  $\mathfrak{I}_t$ . The following situation is assumed:

1/ the phenomenological information  $\mathfrak{I}_{t,p}$  at time  $t$  determines accurately, via the phenomenological laws, the phenomenological information  $\mathfrak{I}_{t',p}$  at times  $t' > t$  along the arrow of time ; the phenomenological laws are not applicable from  $t$  to  $t' < t$  ;

2/ the phenomenological information  $\mathfrak{I}_{t,p}$  is virtually the only fraction of the fundamental information  $\mathfrak{I}_t$  accessible to observers at time  $t$ .

Our goal is to justify that working hypothesis by showing that the points 1/,2/ may be a consequence of the fundamental determinism  $\mathfrak{I}_t \Rightarrow \mathfrak{I}_{t'} \forall t, t'$ . We admit however two drastic simplifications. First we use the simplest framework allowing one to understand thermodynamics, i.e. the Boltzmann-Gibbs framework. Secondly, we consider a simplified cosmos, with a moderate dimension  $2N$ , where the fundamental distribution  $\rho_t(x,p)$  in the vector space  $\mathfrak{R}$  of all observables exclusively verifies the linear Liouville equation and a linear initial condition at a primordial time, and accordingly verifies the principle of superposition : if  $\rho_t^{\prime}(x,p)$  and  $\rho_t^{\prime\prime}(x,p)$  can occur, any normalized superposition  $\lambda^{\prime}\rho_t^{\prime}(x,p) + \lambda^{\prime\prime}\rho_t^{\prime\prime}(x,p)$  can occur as well. That situation does not apply in the true cosmos where  $\rho_t(x,p)$  is a special solution of the Liouville equation which tends to display factorized structures incompatible with the superposition principle. For instance the existence of the closed systems in their own phase spaces  $(x^{\prime}, p^{\prime})$ ,  $(x^{\prime\prime}, p^{\prime\prime})$ , ... imposes  $\rho_t(x,p) \cong \rho_t(x^{\prime}, p^{\prime}) \rho_t(x^{\prime\prime}, p^{\prime\prime}) \dots$ . Also, in each closed system,  $\rho_t(x,p)$  factually tends to be a ‘‘Boltzmann Gibbs distribution’’  $\exp(\lambda^{\prime}U^{\prime} + \lambda^{\prime\prime}U^{\prime\prime} + \dots)$  (becoming a Maxwell distribution  $\exp(-H/T)$  at thermodynamic equilibrium) which specifies via the expectation values  $U^{\prime} \bullet \rho_t, \dots$  of the observables  $U^{\prime}(x,p), \dots$  well defined thermodynamic quantities, and which minimizes under that constraint the negentropy  $\rho_t \bullet \ln(\rho_t)$  (Balian 1991; Mackey 1991). The principle of

superposition applicable in our simplified cosmos to the fundamental information  $\mathfrak{I}_t \Leftrightarrow \rho_t(x,p)$  must apply also to the phenomenological information  $\mathfrak{I}_{t_p}$ . One is led to assimilate the latter to a phenomenological distribution  $\rho_{t_p}(x,p)$  belonging to a specific sub-space  $\mathfrak{R}_p$  of the vector space  $\mathfrak{R}$  of all observables, and equivalently to the set of expectation values  $\langle V_p \rangle_t$  of all the observables  $V_p(x,p)$  composing  $\mathfrak{R}_p$ . That equivalence implies that the phenomenological distribution  $\rho_{t_p}(x,p) \in \mathfrak{R}_p$  is the projection in the space  $\mathfrak{R}_p$  of the fundamental distribution  $\rho_t(x,p) \in \mathfrak{R}$ . The expectation values  $\langle V_p \rangle_t$  equal to  $V_p \cdot \rho_t$  are thus also equal to  $V_p \cdot \rho_{t_p}$ . In particular,  $\rho_{t_p}(x,p)$  has the same normalization and the same localization in the layer  $H \cong E$  as  $\rho_t(x,p)$ .

With those simplifications, we have to justify a phenomenological determinism  $\mathfrak{I}_{t_p} \Rightarrow \mathfrak{I}_{t'_p}$  for  $t' > t$  meaning that the projection  $\rho_{t_p}$  of  $\rho_t$  in a specific space  $\mathfrak{R}_p$  at time  $t$ , assimilated to the phenomenological information  $\mathfrak{I}_{t_p}$ , determines accurately the same projection  $\rho_{t'_p}$  at times  $t' > t$ . This implies a quasi autonomous evolution of  $\rho_{t'_p}$  when  $t$  varies along the arrow of time expressed by a master equation, which, because of the superposition principle, is linear. A step toward that linear master equation is the Nakajima-Zwanzig equation, which links, under a condition weighing on  $\rho_{t^*}$  at a primordial time  $t^*$ , the projection  $\rho_{t_p}$  of  $\rho_t$  in a vector space  $\mathfrak{R}_p$  at time  $t > t^*$  with the set of values  $\rho_{t''_p}$  at times  $t''$  between  $t^*$  and  $t$  (Zwanzig 1960). Our goal is to go farther by so building up the vector space  $\mathfrak{R}_p$  that the information  $\mathfrak{I}_{t_p} \Leftrightarrow \rho_{t_p}(x,p)$  verifies the above points 1/,2/. We establish in § II that this necessitates the realization of an initial and a structural condition. The initial condition, in its simpler form, imposes that  $\rho_{t^*}(x,p) = \rho_{t^*_p}(x,p) \in \mathfrak{R}_p$  at a primordial time  $t^*$  of the cosmos. It introduces the arrow of time aimed from the time  $t^*$  to the observers' times  $t$ . The structural condition introduces the nature of the observables  $V_p(x,p)$  forming the space  $\mathfrak{R}_p$ . It involves the Liouville propagation operator  $\exp(\Omega\tau)$ : it essentially imposes that the complexity in  $(x,p)$  of the components orthogonal to  $\mathfrak{R}_p$  of the observables  $\exp(\Omega\tau)V_p$  is typically doubled when the time  $\tau$  increases by a small specific time scale  $\Theta$ , which moreover must remain smaller than the time scale of the evolution of the phenomenological information  $\mathfrak{I}_{t_p} \Leftrightarrow \rho_{t_p}(x,p)$ . The realization of that condition for observables  $V_p(x,p) \in \mathfrak{R}_p$  single valued and analytic in  $(X,P)$  is not obvious; It relies on the stochasticity of the hamiltonian

trajectories, i.e. the exponential divergence/convergence of the latter expressed by a set of Lyapounov exponents  $\Lambda > 0$  and  $-\Lambda$ . However it requires more. It is justified in § III by establishing and exploiting a canonical transformation of the phase space  $(x,p)$ , which highlights that the cosmos displays motions in a narrow domain around the hamiltonian trajectories which are purely hyperbolic or elliptic. The time scale  $\Theta$  appears in that context to be the inverse Lyapounov exponents reflecting the hyperbolic motions.

The non-phenomenological component  $\rho_{t\ K}(x,p)$  of the fundamental distribution  $\rho_t(x,p)$  which is orthogonal to the space  $\mathfrak{R}_p$  plays an essential role. If the initial and structural conditions are achieved, that component  $\rho_{t\ K}(x,p)$  at time  $t$  is exclusively produced via the Liouville equation by the phenomenological distribution  $\rho_{t^* P}(x,p)$  at times  $t^*$  between  $t^*$  and  $t$ . The small addition to  $\rho_{t^* K}(x,p)$  created at the time  $t^*$  by  $\rho_{t^* P}(x,p)$  has its complexity in  $(x,p)$  typically doubled during each of the successive time intervals  $\Theta$  following  $t^*$ . The result is that the observable  $\rho_{t\ K}(x,p)$  at a time  $t$  displays a complexity in  $(x,p)$  out of proportion with the complexity of the observable  $\rho_{t^* P}(x,p)$ . This justifies a quasi independence from  $\rho_{t\ K}(x,p)$  of the evolution of  $\rho_{t^* P}(x,p)$  along the arrow of time and therefore the quasi autonomy of that evolution required by the above point 1/. On the other hand the extreme, steadily growing complexity in  $(x,p)$  of  $\rho_{t\ K}(x,p)$  entails that the non-phenomenological information represented by  $\rho_{t\ K}(x,p)$  is not accessible to observers : in practise, the latter have only access to the phenomenological information  $\rho_{t^* P}(x,p) \Leftrightarrow \mathfrak{S}_{t^* P}$ , as required by the point 2/. That functioning of the phenomenological determinism implies a slow gradual disappearance along the arrow of time of the squared norm  $\rho_{t^* P} \bullet \rho_{t^* P}$  of  $\rho_{t^* P} \in \mathfrak{R}_p$  at the benefit of the squared norm  $\rho_{t\ K} \bullet \rho_{t\ K}$  of  $\rho_{t\ K} \perp \mathfrak{R}_p$ , in such a way of course that the squared norm  $\rho_t \bullet \rho_t = \rho_{t^* P} \bullet \rho_{t^* P} + \rho_{t\ K} \bullet \rho_{t\ K}$  of  $\rho_t = \rho_{t^* P} + \rho_{t\ K}$  remains invariant. This leads to consider the quantity of phenomenological information measured by the squared norm  $\rho_{t^* P} \bullet \rho_{t^* P}$  as the decreasing negentropy in our simplified cosmos. One may stress here that simplified versions of the fundamental distribution  $\rho_t(x,p)$  enjoying an autonomous evolution along the arrow of time have been contemplated since the invention of the coarse graining of  $\rho_t(x,p)$  and that such an autonomy has been shown to entail “generalized H theorems” meaning the degradation along the arrow of time of appropriate functionals of the simplified versions (Jancel 1969; Davies 1977). Also, the initial condition considered above imposes that  $\rho_{t^*} \bullet \rho_{t^*} = \rho_{t^* P} \bullet \rho_{t^* P}$  and thereby that the negentropy in the cosmos has at the primordial time  $t^*$  the

unsurpassable level of the quantity of fundamental information. This recovers the consensual view (Davies 1994 ; Halliwell, 1994 ; Lebowitz 1999) that a maximum negentropy in the cosmos at a primordial instant is at the origin of the irreversibility of the evolution of the observable world. One may still note that the gradual degradation along the arrow of time of the accessible information in  $\rho_{t_P} \in \mathfrak{R}_P$ , into an inaccessible information in  $\rho_{t_K} \perp \mathfrak{R}_P$ , gives a clear content to the consensual view that “the irreversible decrease of the negentropy occurs at the benefit of indelible correlations which dissolve in an immense ocean of degrees of freedom” (Prigogine and Stengers 1992) and to the starting point of the quantum decoherence, namely, a transfer of the phenomenological information connected to observers – involving “distinguished” observables (Gell Mann and Hartle 1993) or observables emerging from the “einselection” (Zurek 2003) – into quasi unobservable degrees of freedom of an environment.

One may admit that the phenomenological determinism  $\mathfrak{I}_{t_P} \Rightarrow \mathfrak{I}_{t'_P}$  for  $t' > t$  allows the observers, not only to accurately predict  $\mathfrak{I}_{t'_P}$  from  $\mathfrak{I}_{t_P}$ , but also to influence  $\mathfrak{I}_{t'_P}$  by their own intervention at times  $t < t'$ . The observers may thus hope to exert some control on their future, but none on their irretrievable past. The justification of the phenomenological determinism entails the justification of a causality principle in the following sense: an element of the phenomenological information  $\mathfrak{I}_{t'_P}$  accessible to observers at a time  $t'$  is the effect of causes identifiable with elements of  $\mathfrak{I}_{t_P}$  at a time  $t$  or at various times  $t$  between  $t^*$  and  $t'$ . The phenomenological laws actually consist of these causal relations. When one undertakes their deduction from the laws of the fundamental determinism, the causality principle is an indispensable prior postulate, often implicit, allowing the selection among the possible solutions offered by these laws those which really apply. In that way for example the solutions of the Maxwell equations which physically take place involve electric charge and current densities which precede the electromagnetic field: this leads to the retarded potential and to the Landau damping of electromagnetic waves along the arrow of time of the electromagnetic waves (Landau 1946), which may become an instability in the presence of an inversion of population. One may recall the historical introduction by Boltzmann of his visionary equation at the origin of the H theorem, which relates the distributions in phase space at different times of the particles in a gas. The equation may be selected among the solutions of the Liouville equation by using the above defined causality principle. It is of course justified by the stossahlansatz postulate proposed by Boltzmann (Davies 1977 ;

Cercignani 1988), but the latter could be not strictly imposed by the causality principle and thereby be excessive.

## II – HOW MAY WORK A PHENOMENOLOGICAL DETERMINISM?

We study in this section the conditions under which the phenomenological determinism specified in § I can exist in the simplified cosmos that we have elected. These conditions must allow that the projection  $\rho_{t_P}(x,p)$  of the fundamental distribution  $\rho_t(x,p)$  into an appropriate sub- space  $\mathfrak{R}_P$  of the vector space  $\mathfrak{R}$  formed by all observables, determines accurately the same projection  $\rho_{t'_P}(x,p)$  of  $\rho_{t'}(x,p)$  into  $\mathfrak{R}_P$ , for  $t' > t$ . It will appear that they consist of an initial condition on the distribution  $\rho_t(x,p)$  at the primordial time  $t^*$  of the cosmos and of a structural condition involving the vector space  $\mathfrak{R}_P$  and the functioning of the phenomenological determinism via the Liouville propagation operator  $\exp(\Omega\tau)$ .

### II A – CONDITIONS FOR THE ACHIEVEMENT AND ACCURACY OF THE PHENOMENOLOGICAL DETERMINISM

The fundamental distribution function  $\rho_t(x,p) \in \mathfrak{R}$  is decomposed into its projection  $\rho_{t_P}(x,p) \in \mathfrak{R}_P$  and its projection  $\rho_{t_K}(x,p)$  into the sub-vector space  $\mathfrak{R}_K$  of  $\mathfrak{R}$  orthogonal to  $\mathfrak{R}_P$ , assimilated respectively to the phenomenological and to the non-phenomenological information present in the cosmos at time  $t$ . An observable  $V(x,p) \in \mathfrak{R}$  is written  $V_P(x,p) + V_K(x,p)$ , with  $V_P \in \mathfrak{R}_P$  and  $V_K \in \mathfrak{R}_K \perp \mathfrak{R}_P$ . The Liouville equation imposes

$$\frac{\partial}{\partial t} \rho_t = \Omega \rho_t = \frac{\partial}{\partial t} \rho_{t_P} + \frac{\partial}{\partial t} \rho_{t_K} = \Omega \rho_{t_P} + \Omega \rho_{t_K} \quad (\text{II } 1)$$

where the Liouville linear operator  $\Omega$  acting on the observables  $V(x,p) \in \mathfrak{R}$  is such that

$$\Omega V = -\{V, H\} = -\left( \frac{\partial V}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i} \frac{\partial V}{\partial p_i} \right)$$

If the point  $(x,p)[\tau]$  is the hamiltonian image of the point  $(x,p)[0]$ , in the sense that it is the position at the time  $\tau$  on the hamiltonian trajectory passing through  $(x,p)$  at the time 0, one has

$$\rho_{t+\tau}(x,p) = (\exp(\Omega\tau)\rho_t)(x,p) = \rho_t((x,p)[- \tau])$$

We will use the anti-hermitian property of  $\Omega$  with respect to the hermitian product  $V \bullet W$  (the conjugate  $\Omega^*$  such that  $\Omega V \bullet W = V \bullet \Omega^* W \forall V, W \in \mathfrak{R}$  is equal to  $-\Omega$ ). The hermitian multiplication of the equation (II 1) by test observables  $V_P(x,p) \in \mathfrak{R}_P$  produces the derivative

$$\frac{\partial \rho_{t P}}{\partial t} \in \mathfrak{R}_P \text{ via the following principle (Samain and Nguyen 1997)}$$

$$\left(\frac{\partial}{\partial t} \rho_{t P}\right) \bullet V_P + \Sigma_t(V_P) + \mathbf{S}_t(V_P) = 0 \quad \forall V_P \in \mathfrak{R}_P, \quad (\text{II } 2a)$$

where  $\Sigma_t(V_P)$  and  $\mathbf{S}_t(V_P)$ , for given  $\rho_{t P}(x,p)$  and  $\rho_{t K}(x,p)$ , are the linear forms in the complex conjugate  $(V_P(x,p))^*$  of  $V_P(x,p)$

$$\Sigma_t(V_P) = -(\Omega \rho_{t P}) \bullet V_P = \{\rho_{t P}, H\}_P \bullet V_P = -\rho_{t P} \bullet \{V_P, H\}_P = -H \bullet \{\rho_{t P}, V_P\} \quad (\text{II } 2b)$$

$$\mathbf{S}_t(V_P) = -(\Omega \rho_{t K}) \bullet V_P = \{\rho_{t K}, H\}_P \bullet V_P = -\rho_{t K} \bullet \{V_P, H\}_K = -H \bullet \{\rho_{t K}, V_P\} \quad (\text{II } 2c)$$

The dependence of  $\Sigma_t(V_P)$  and  $\mathbf{S}_t(V_P)$  on time  $t$  represents their dependence on  $\rho_{t P}$  and  $\rho_{t K}$ .

As  $\{\rho_{t P}, \rho_{t P}\} = 0$ , the equation (II 2b) and the principle (2a) gives  $\Sigma_t(\rho_{t P}) = 0$  and  $\mathbf{S}_t(\rho_{t P}) =$

$$-\frac{\partial \rho_{t P}}{\partial t} \bullet \rho_{t P} = -\frac{d}{2dt} (\rho_{t P} \bullet \rho_{t P}).$$

The quantity  $\mathbf{S}_t(\rho_{t P})$  thus specifies the decrease time of the squared norm  $\rho_{t P} \bullet \rho_{t P}$  playing in our simplified cosmos the role of negentropy.

The phenomenological determinism along the arrow of time  $\rho_{t P}(x,p) \Rightarrow \rho_{t' P}(x,p)$  for  $t' > t$  is exactly achieved if the principle (II 2a) produces a derivative  $\frac{\partial \rho_{t P}}{\partial t}$  at time  $t$  specified

by  $\rho_{t-\varepsilon P}$  at times  $t-\varepsilon$ ,  $\varepsilon \rightarrow 0^+$ . That is the case with the form  $\Sigma_t(V_P)$  which expresses a direct

path from  $\rho_{t P}$  to  $\frac{\partial \rho_{t P}}{\partial t}$  within the vector space  $\mathfrak{R}_P$ . The linear form  $\Sigma_t(V_P)$  in  $V_P^*$  for given

$\rho_t(x,p)$  is indeed the bilinear form in  $\rho_{t P}, V_P^*$

$$\mathbf{S}_t(\mathbf{V}_P) = -(\Omega_P \rho_{t P}) \bullet \mathbf{V}_P = ((\Omega_P \mathbf{V}_P) \bullet \rho_{t P})^* \quad (\text{II } 3\text{a})$$

generated by the linear operator  $\Omega_P$  acting on the observables  $\mathbf{V}_P \in \mathfrak{R}_P$ , such that

$$\mathbf{V}_P \in \mathfrak{R}_P \Rightarrow \Omega_P \mathbf{V}_P = (\Omega \mathbf{V}_P)_P = -\{\mathbf{V}_P, \mathbf{H}\}_P \in \mathfrak{R}_P, \Omega_P = -\Omega_P^*. \quad (\text{II } 3\text{b})$$

When one applies the principle (II 2a), the form  $\mathbf{S}_t(\mathbf{V}_P) = -(\Omega_P \rho_{t P}) \bullet \mathbf{V}_P$  contributes to the derivative  $\frac{\partial \rho_{t P}}{\partial t}$  at time  $t$  by the observable  $\Omega_P \rho_{t P}$ , which is obviously specified by  $\rho_{t-\varepsilon P}$ ,  $\varepsilon$

$\rightarrow 0^+$ . That situation does not apply at the onset to the form  $\mathbf{S}_t(\mathbf{V}_P)$ , which reflects an indirect

path to  $\frac{\partial \rho_{t P}}{\partial t}$  passing through the vector space  $\mathfrak{R}_K \perp \mathfrak{R}_P$ . It will result in fact,

approximately, from the initial and structural conditions that we will set below for guaranteeing the existence of the phenomenological determinism. These conditions will allow

an approximation of the linear form  $\mathbf{S}_t(\mathbf{V}_P)$  in  $\mathbf{V}_P^*$  by bilinear forms  $\mathbf{S}_t^{(v)}(\mathbf{V}_P) = -(\Omega^{(v)} \rho_{t P}) \bullet$

$\mathbf{V}_P$  in  $\rho_{t P}, \mathbf{V}_P^*$ , generated by specific linear operators  $\Omega^{(v)}$  independent of  $\rho_{t P}$ , acting on the

observables  $\mathbf{V}_P \in \mathfrak{R}_P$ . The contribution of  $\mathbf{S}_t(\mathbf{V}_P)$  to  $\frac{\partial \rho_{t P}}{\partial t}$  via the principle (2a) can then be

approximated by the observable  $\Omega^{(v)} \rho_{t P}$  specified by  $\rho_{t-\varepsilon P}$ ,  $\varepsilon \rightarrow 0^+$ . The construction of the

essential forms  $\mathbf{S}_t^{(v)}(\mathbf{V}_P)$  will be made easier if a limit of the magnitudes of  $\mathbf{S}_t(\mathbf{V}_P)$

corresponding to a given  $\rho_{t P}(x, p)$  is available a priori, for instance if there exists a time scale

$\tau_{\text{phen}}$  depending on  $\rho_{t P}(x, p)$  such that, in magnitude

$$\mathbf{S}_t(\mathbf{V}_P) < \mathcal{O}\left(\frac{1}{\tau_{\text{phen}}} \rho_{t P} \bullet \mathbf{V}_P\right) \quad \forall \mathbf{V}_P \quad (\text{II } 4)$$

The limit (II 4) is an obvious consequence of the principle (II 2a) by taking the time scale  $\tau_{\text{phen}}$

such that  $\frac{1}{\tau_{\text{phen}}} = \text{SUP}\left(\frac{1}{\rho_{t P}} \left| \frac{\partial \rho_{t P}(x, p)}{\partial t} \right| \right)$  at the various points of the phase space. However the

limit (II 4) is of course more useful with a larger  $\tau_{\text{phen}}$ . To move in that direction, one may

admit that the form  $\mathbf{S}_t(\mathbf{V}_P)$  has only to be considered for observables  $\mathbf{V}_P(x, p)$  which cancel

$\Sigma_t(V_P)$ . In view of the equations (II 2b) such observables  $V_P(x,p)$  in each small domain  $(x,p)$  roughly belong to the vector space  $\mathcal{R}$  specified by the equation  $\{\rho_{t P}, V_P\} = 0$ . This suggests to

define the time scale  $\tau_{\text{phen}}$  in the limit (II 4) by  $\frac{1}{\tau_{\text{phen}}} = \text{SUP}\left(\frac{1}{\rho_{t P}} \left| \left( \frac{\partial \rho_{t P}(x, p)}{\partial t} \right)_{\mathcal{R}} \right| \right)$  where  $\left( \frac{\partial \rho_{t P}(x, p)}{\partial t} \right)_{\mathcal{R}}$  is the projection of  $\frac{\partial \rho_{t P}(x, p)}{\partial t}$  in the various spaces  $\mathcal{R}$ .

It is obviously necessary to reach the actual value of  $\mathbf{S}_t(V_P)$  to project the Liouville equation (II 1) into the vector space  $\mathfrak{R}_K \perp \mathfrak{R}_P$

$$\frac{\partial}{\partial t} \rho_{t K} = (\Omega \rho_{t P})_K + (\Omega \rho_{t K})_K = -\{\rho_{t P}, H\}_K + \Omega_K \rho_{t K} \quad (\text{II 5a})$$

where  $\Omega_K$  is the linear operator acting on the observables  $V_K \in \mathfrak{R}_K$ , such that

$$V_K \in \mathfrak{R}_K \Rightarrow \Omega_K V_K = (\Omega V_K)_K = -\{V_K, H\}_K \in \mathfrak{R}_K, \Omega_K = -\Omega_K^* \quad (\text{II 5b})$$

The equation (II 5a) produces the observable  $\rho_{t K}$  at any time  $t > t^*$  after a cosmos primordial time  $t^*$  if one knows the value of  $\rho_{t^* K}$  and the values of  $\rho_{t' P}$  for  $t'$  between  $t^*$  and  $t$

$$\begin{aligned} \rho_{t K} &= \rho_{t^* K}^{\times} + \exp(\Omega_K(t-t^*)) \rho_{t^* K} \\ \rho_{t^* K}^{\times} &= - \int_{t^*}^t dt'' \exp(\Omega_K(t-t'')) \{\rho_{t'' P}, H\}_K \end{aligned}$$

The expression (II 2c) for given  $\rho_t(x,p)$  of the linear form  $\mathbf{S}_t(V_P)$  in  $V_P^*$  then becomes

$$\mathbf{S}_t(V_P) = \mathbf{S}_{t^*}^{\times}(V_P) - \exp(\Omega_K(t-t^*)) \rho_{t^* K} \cdot \{V_P, H\}_K \quad (\text{II 6a})$$

$$\mathbf{S}_{t^*}^{\times}(V_P) = - \rho_{t^* K}^{\times} \cdot \{V_P, H\}_K = \int_{t^*}^t dt'' \exp(\Omega_K(t-t'')) \{\rho_{t'' P}, H\}_K \cdot \{V_P, H\}_K \quad (\text{II 6b})$$

These equations combined with the principle (II 2a) recover the Nakajima-Zwanzig approach evoked in § I. We go farther by imposing the phenomenological determinism  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' > t > t^*$ . The latter is exactly achieved if the derivative  $\frac{\partial}{\partial t} \rho_{t P}$  given by the principle (II 2a) at each time  $t$  between  $t^*$  and  $t'$  is determined by  $\rho_{t-\varepsilon P}$  at the times  $t-\varepsilon$ ,  $\varepsilon \rightarrow 0^+$ . As seen above,

the contribution of the form  $\sum_t(V_P)$  fulfils automatically that demand. On the contrary the expression (II 6a,b) of  $\mathbf{S}_t(V_P)$  leads to an initial and a structural condition. The initial condition imposes that  $\mathbf{S}_t(V_P)$  does not depend on  $\rho_{t^* K}$ . Since  $(\exp(\Omega_K(t-t^*)))^* = \exp(-\Omega_K(t-t^*))$ , it may be written

$$\rho_{t^* K} \cdot \exp(-\Omega_K(t-t^*)) \{V_P, H\}_K = 0 \quad \forall V_P(x,p) \in \mathfrak{R}_p \text{ for } t > t^* \quad (\text{II } 7)$$

It is in particular achieved if  $\rho_{t^* K} = 0$ , i.e.  $\rho_{t^*} = \rho_{t^* P}$ . The structural condition then imposes that the integrand  $\exp(\Omega_K(t-t'')) \{\rho_{t'' P}, H\}_K \cdot \{V_P, H\}_K$  in the integral (II 6b) cancels for  $t''$  within the interval  $[t^*, t-\varepsilon]$ ,  $\varepsilon \rightarrow 0^+$ . This is clearly impossible if  $\{\rho_{t P}, H\}_K \neq 0$  and for instance  $V_P = \rho_{t P}$ . The only way out to impose the phenomenological determinism is to concede that the integrand cancels for  $t''$  within the interval  $[t^*, t-\Theta]$  where  $\Theta$  is a small finite positive interval. Under that concession, the structural condition becomes

$$\exp(\Omega_K(t-t'')) \{\rho_{t'' P}, H\}_K \cdot \{V_P, H\}_K = 0 \quad \forall V_P \in \mathfrak{R}_p \text{ for } t^* < t'' < t - \Theta$$

and is equivalent, since  $\rho_{t'' P}$  may be any element of the vector space  $\mathfrak{R}_p$ , to the condition

$$\exp(\Omega_K \tau) \{V_P, H\}_K \cdot \{W_P, H\}_K = 0 \quad \forall V_P, W_P \in \mathfrak{R}_p \text{ for } \tau > \Theta$$

or, replacing  $\exp(\Omega_K \tau)$  by  $\exp(\Omega_K(\tau-\tau')) = \exp(\Omega_K(\tau))(\exp(\Omega_K(\tau')))^*$ , to the condition

$$\exp(\Omega_K \tau) \{V_P, H\}_K \cdot \exp(\Omega_K \tau') \{W_P, H\}_K = 0 \quad \forall V_P, W_P \in \mathfrak{R}_p \text{ for } |\tau - \tau'| > \Theta \quad (\text{II } 8)$$

The time scale  $\Theta$  becomes of course a basic parameter of the phenomenological determinism. It will be identified in § III as being the inverse of the Lyapounov exponents  $>0$  applying within our cosmos.

Under the conditions (II 7, II 8) the value (II 6a,b) of  $\mathbf{S}_t(V_P)$  at time  $t$  becomes

$$\mathbf{S}_t(V_P) = \int_0^\Theta d\tau \exp(\Omega_K \tau) \{\rho_{t-\tau P}, H\}_K \cdot \{V_P, H\}_K \quad (\text{II } 9)$$

It is not specified by the values of  $\rho_{t-\varepsilon P}$  at the times  $t-\varepsilon$ ,  $\varepsilon \rightarrow 0^+$  and accordingly the phenomenological determinism is not strictly achieved by the conditions (II 7, II 8). However,

if the observable  $\rho_{t-P}(x,p)$  has a small variation during the scale time  $\Theta$ , i.e. if the time  $\tau_{\text{phen}}$  reflecting the fastest evolutions of the phenomenological information is  $\ll \Theta$ , one may expect that the form  $\mathbf{S}_t(\mathbf{V}_P)$  is approximately determined by  $\rho_{t-\varepsilon P}$ ,  $\varepsilon \rightarrow 0^+$  and the phenomenological determinism thus approximately realized. An essential issue is then to evaluate its accuracy under the restriction  $\tau_{\text{phen}} \gg \Theta$ . We will show, by using the limit (II 4), and of course the expression (II 9) resulting from conditions (II 7, II 8), that the form  $\mathbf{S}_t(\mathbf{V}_P)$  at time  $t$  is determined formally by the value of  $\rho_{t-\varepsilon P}$ ,  $\varepsilon \rightarrow 0^+$  “at all orders” in  $\frac{\Theta}{\tau_{\text{phen}}}$ . The phenomenological determinism is thus also achieved “at all orders” in  $\frac{\Theta}{\tau_{\text{phen}}}$ . Its situation with respect to the small ratio  $\frac{\Theta}{\tau_{\text{phen}}}$  appears somewhat similar to the situation of the adiabatic invariance of a nearly periodic fast motion with respect to the small ratio  $\frac{\tau_{\text{fast}}}{\tau_{\text{slow}}}$  of the fast period  $\tau_{\text{fast}}$  to the time scale  $\tau_{\text{slow}}$  of the slow evolution of the trajectory (Kruskal 1962), or to the situation of the WKB approximation with respect to the small ratio  $\frac{\lambda}{L}$  of the short wavelength  $\lambda$  to its long variation scale  $L$ .

In view of the expression (II 3a,b) of  $\sum_t(\mathbf{V}_P) = -(\Omega_P \rho_{t-P}) \cdot \mathbf{V}_P$  and of the limit (II 4) of the magnitude of  $\mathbf{S}_t(\mathbf{V}_P) < \mathcal{O}(\frac{1}{\tau_{\text{phen}}} \rho_{t-P} \cdot \mathbf{V}_P)$ , the principle (II 2a) produces a derivative  $\frac{\partial}{\partial(-\tau)}$   $\rho_{t-\tau P}$  equal to  $\Omega_P \rho_{t-\tau P} + \mathcal{O}(\frac{1}{\tau_{\text{phen}}} \rho_{t-P})$ . It results that  $\rho_{t-\tau P}$  for  $0 \leq \tau \leq \Theta$  is equal to  $(1 + \mathcal{O}(\frac{\Theta}{\tau_{\text{phen}}})) \exp(-\Omega_P \tau) \rho_{t-P}$ . Substitution of that profile  $\rho_{t-\tau P}$  into the expression (II 9) of  $\mathbf{S}_t(\mathbf{V}_P)$  gives

$$\mathbf{S}_t(\mathbf{V}_P) = \mathbf{S}_t^{(0)}(\mathbf{V}_P) + \mathcal{O}(\frac{\Theta}{\tau_{\text{phen}}} \mathbf{S}_t^{(0)}(\mathbf{V}_P)) = \mathbf{S}_t^{(0)}(\mathbf{V}_P) + \mathcal{O}(\frac{\Theta}{\tau_{\text{phen}}} \frac{1}{\tau_{\text{phen}}} \rho_{t-P} \cdot \mathbf{V}_P) \quad (\text{II } 10a)$$

$$\text{where } \mathbf{S}_t^{(0)}(\mathbf{V}_P) = \int_0^\Theta dt \exp(\Omega_K \tau) \{ \exp(-\Omega_P \tau) \rho_{t-P}, \mathbf{H} \}_K \cdot \{ \mathbf{V}_P, \mathbf{H} \}_K \quad (\text{II } 10b)$$

There exists, independently of  $\rho_{t P}$ , an operator  $\Omega^0$  acting within  $\mathfrak{R}_P$ , such that  $\mathbf{S}^{(0)}_{t(V_P)}$  is equal to the bilinear form  $-(\Omega^0 \rho_{t P}) \bullet V_P$  in  $\rho_{t P}$  and  $V_P^*$ . The form  $\mathbf{S}_{t(V_P)}$  is thus specified by  $\rho_{t-\varepsilon P}(x,p)$ ,  $\varepsilon \rightarrow 0^+$  with a relative error  $\mathcal{O}(\frac{\Theta}{\tau_{\text{phen}}})$ , i.e. at zeroth order in  $\frac{\Theta}{\tau_{\text{phen}}}$ . This expresses the achievement of the phenomenological determinism at the same order.

We now proceed by recurrence. We assume that the situation (II 10a) is achieved at order  $\nu$  in  $\frac{\Theta}{\tau_{\text{phen}}}$ , i.e. that we have

$$\mathbf{S}_{t(V_P)} = \mathbf{S}^{(\nu)}_{t(V_P)} + \mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu+1} \mathbf{S}^{(\nu)}_{t(V_P)}\right) = \mathbf{S}^{(\nu)}_{t(V_P)} + \mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu+1} \frac{1}{\tau_{\text{phen}}} \rho_{t P} \bullet V_P\right) \quad (\text{II 11})$$

where  $\mathbf{S}^{(\nu)}_{t(V_P)}$  is equal to a bilinear form  $-(\Omega^{(\nu)} \rho_{t P}) \bullet V_P$ , being  $\Omega^{(\nu)}$  a linear operator acting in the space  $\mathfrak{R}_P$  independently of  $\rho_{t P}$ . This expresses that  $\mathbf{S}_{t(V_P)}$  is specified by  $\rho_{t-\varepsilon P}(x,p)$ ,  $\varepsilon \rightarrow 0^+$  and that the phenomenological determinism is achieved with a relative error  $\mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu+1}\right)$ , i.e. at order  $\nu$  in  $\frac{\Theta}{\tau_{\text{phen}}}$ . We use the principle (II 2a) with  $\Sigma_{t(V_P)} = -(\Omega_P \rho_{t P}) \bullet V_P$  and

now  $\mathbf{S}_{t(V_P)} = -(\Omega^{(\nu)} \rho_{t P}) \bullet V_P + \mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu+1} \frac{1}{\tau_{\text{phen}}} \rho_{t P} \bullet V_P\right)$ , to deduce profiles  $\rho_{t-\tau P} = (1 + \mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu+2}\right) \exp(-\Omega_P - \Omega^{(\nu)}) \tau) \rho_{t P}$  for  $0 \leq \tau \leq \Theta$ , that we substitute in the expression (II 9) of

$\mathbf{S}_{t(V_P)}$ . This leads to the equations (II 11) with  $\nu$  replaced by  $\nu+1$  and  $\mathbf{S}^{(\nu)}_{t(V_P)} = -(\Omega^{(\nu)} \rho_{t P}) \bullet V_P$  by

$$\mathbf{S}^{(\nu+1)}_{t(V_P)} = \int_0^\Theta dt \exp(\Omega_K \tau) \{ \exp(-(\Omega_P + \Omega^{(\nu)}) \tau) \rho_{t P, H} \}_K \bullet \{V_P, H\}_K, \quad (\text{II 12})$$

There exists again an operator  $\Omega^{(\nu+1)}$ , independent of  $\rho_{t P}$ , acting in the space  $\mathfrak{R}_P$  only, such that  $\mathbf{S}^{(\nu+1)}_{t(V_P)} = -(\Omega^{(\nu+1)} \rho_{t P}) \bullet V_P$ . The situation (II 11) of the form  $\mathbf{S}_{t(V_P)}$  implying the

achievement of the phenomenological determinism at order  $\nu$  is reproduced at order  $\nu+1$ , and thus at all orders in  $\frac{\Theta}{\tau_{\text{phen}}}$ .

If the recurrence formula (II 12) allowing to pass from  $\mathbf{S}^{(\nu)}_{t(V_P)} = -(\Omega^{(\nu)} \rho_{t P}) \cdot V_P$  to  $\mathbf{S}^{(\nu+1)}_{t(V_P)} = -(\Omega^{(\nu+1)} \rho_{t P}) \cdot V_P$  did produce a convergent sequence of forms  $\mathbf{S}^{(\nu)}_{t(V_P)}$ , the form  $\mathbf{S}_{t(V_P)}$  would be exactly equal to the bilinear form  $-\Omega^{(\nu)} \rho_{t P} \cdot V_P$  for  $\nu=\infty$ . Via the principle (II 2a), the phenomenological determinism  $\rho_{t^\circ P} \Rightarrow \rho_{t' P}$  for  $t' > t^\circ > t^*$  would appear achieved exactly under the conditions (II 7, II 8) and the restriction  $\tau_{\text{phen}} \gg \Theta$ . However we know that this is impossible. What necessarily happens is that the convergence mentioned above is broken for  $\nu$  beyond a critical number  $\nu_{\text{cr}}$ . That number then determines the relative error  $\mathcal{O}\left(\left(\frac{\Theta}{\tau_{\text{phen}}}\right)^{\nu_{\text{cr}}}\right)$  with which the phenomenological determinism is achieved. One may

expect that it is reminiscent of the only available dimensionless large ratio  $\left(\frac{\tau_{\text{phen}}}{\Theta}\right)$ . If it is actually the case, the phenomenological determinism  $\rho_{t^\circ P} \Rightarrow \rho_{t' P}$  for  $t' > t^\circ > t^*$ , under the conditions (II 7, II 8), is achieved with a great accuracy as long as the phenomenological information  $\rho_{t P}(x, p)$  is such that the restriction  $\tau_{\text{phen}} \gg \Theta$  applies.

## II B – SIMPLIFIED, SUFFICIENT CONDITIONS – FUNCTIONING OF THE PHENOMENOLOGICAL DETERMINISM

It is possible to give a geometric form to the initial and structural conditions (II 7) and (II 8) allowing the realization of the phenomenological determinism  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' > t > t^*$ . That geometric approach is founded upon sub-spaces  $\tilde{\mathfrak{R}}_n$ ,  $n \in \mathbb{Z}$ , of the vector space  $\mathfrak{R}_K$  orthogonal to the vector space  $\mathfrak{R}_P$ , derived via the time scale  $\Theta$  from  $\mathfrak{R}_P$  as follows

1/ the observables  $\tilde{V}_0(x, p)$  forming the space  $\tilde{\mathfrak{R}}_0$  are the images into  $\mathfrak{R}_K$  via the Liouville operator  $\Omega$  of the phenomenological observables  $V_P(x, p)$  forming  $\mathfrak{R}_P$

$$V_P \in \mathfrak{R}_P \rightarrow \tilde{V}_0 = (\Omega V_P)_K = -\{V_P, H\}_K \in \tilde{\mathfrak{R}}_0 \subset \mathfrak{R}_K \perp \mathfrak{R}_P \quad (\text{II } 13)$$

2/ the observables  $\tilde{V}_n(x,p) \in \tilde{\mathfrak{R}}_n$ , are then the image into  $\mathfrak{R}_K$  via the operator  $\exp(n\Omega_K\Theta)$  of the observables  $\tilde{V}_0(x,p) \in \tilde{\mathfrak{R}}_0 \subset \mathfrak{R}_K$

$$\tilde{V}_0 \in \tilde{\mathfrak{R}}_0 \rightarrow \tilde{V}_n = \exp(n\Omega_K\Theta)\tilde{V}_0 \in \tilde{\mathfrak{R}}_n \subset \mathfrak{R}_K \perp \mathfrak{R}_P$$

$$\Leftrightarrow \tilde{V}_{n\pm 1} = \exp(\pm\Omega_K\Theta)\tilde{V}_n \quad (\text{II } 14)$$

where  $\Omega_K$  is the anti-hermitian operator acting on the observables  $V_K \in \mathfrak{R}_K$  defined by the equation (II 5b). Owing to the relations (II 14), the operator  $\exp(\Omega_K\Theta)$  and its inverse  $\exp(-\Omega_K\Theta)$  transform into itself the vector space  $\sum_n \tilde{\mathfrak{R}}_n \subset \mathfrak{R}_K$ . The operators  $\exp(\Omega_K\Theta)$ ,  $\exp(-\Omega_K\Theta)$ , being functions of  $\Omega_K$ , have the same eigenvectors within  $\sum_n \tilde{\mathfrak{R}}_n$  as  $\Omega_K$ , which therefore transforms also  $\sum_n \tilde{\mathfrak{R}}_n$  into itself. It is also the case with the operator  $\exp(\Omega_K\tau)$  whatever the time  $\tau$ . We give a geometric form to the structural condition (II 8) by taking the times  $\tau$  and  $\tau'$  equal to  $\tau=n\Theta$  and  $\tau'=n'\Theta$ . One then has  $|\tau-\tau'|>\Theta$  for  $|n-n'|>1$ . This leads to the orthogonality relations

$$\tilde{\mathfrak{R}}_{n'} \perp \tilde{\mathfrak{R}}_n \text{ for } n' \neq n \quad (\text{II } 15)$$

As the operators  $\exp(n\Omega_K\Theta)$  are unitary, these orthogonalities can only be due to a change of the complexity in  $(x,p)$  of the observables  $\tilde{V}_n \in \tilde{\mathfrak{R}}_n$  when one goes from a space  $\tilde{\mathfrak{R}}_n$  to another. One may then admit that the observables  $\tilde{V}_n(x,p)$  must display when  $(x,p)$  varies along specific vectors  $(\delta x, \delta p)$  uncorrelated oscillations of wave numbers multiplied by  $O(2)$  if  $n>0$  is increased by 1 and  $n<0$  by  $-1$ . We include that property of the observables  $\tilde{V}_n$  in the structural condition as being the cause of the orthogonalities (II 15).

The initial condition (II 7) may be also given a geometric form, by taking the times  $t > t^*$  of the form  $t=t^*+n\Theta$  with  $n>0$ . It then imposes that the non-phenomenological information  $\rho_{t^* K}$  at the time  $t^*$  is orthogonal to the vector spaces  $\tilde{\mathfrak{R}}_n$  with  $n<0$ . In view of the mutual orthogonality (II 15) of the vector spaces  $\tilde{\mathfrak{R}}_n$ , the condition (II 7) is realized if the observable  $\rho_{t^* K}(x,p)$  belongs to the space  $\sum_{n \geq 0} \tilde{\mathfrak{R}}_n$

$$\rho_{t^* K}(x,p) \in \sum_{n \geq 0} \tilde{\mathfrak{R}}_n \quad (\text{II } 16)$$

We will replace the initial condition (II 7) by the simpler condition (II 16), even if the latter is sufficient but not strictly necessary for entailing the former. The condition (II 16) can be itself replaced by the still simpler condition  $\rho_{t^* K}(x,p)=0$  meaning that at the primordial time  $t^*$  of the cosmos the fundamental distribution  $\rho_{t^*}(x,p) = \rho_{t^* P}(x,p) + \rho_{t^* K}(x,p)$  reduces to the phenomenological distribution  $\rho_{t^* P}(x,p)$  in the space  $\mathfrak{R}_P$ .

In order to obtain a more convenient formulation of the structural condition, we examine the possibility to replace in the equations (II 14), the operator  $\Omega_K$  by the Liouville operator  $\Omega$ . We first show that the non-phenomenological information  $\rho_{t K} \in \sum_{n \geq 0} \tilde{\mathfrak{R}}_n$  influences only via its component in  $\tilde{\mathfrak{R}}_0$  the phenomenological evolution  $\frac{\partial}{\partial t} \rho_{t P}$ . This means that  $\Omega \rho_{t K}$  has a projection  $(\Omega \rho_{t K})_P$  into  $\mathfrak{R}_P$  which is null if  $\rho_{t K} \in \tilde{\mathfrak{R}}_n$  with  $n \neq 0$ . In view of the definition (II 5b) of  $\Omega_K$ , this means that  $\Omega \rho_{t K} = \Omega_K \rho_{t K}$  if  $\rho_{t K} \in \tilde{\mathfrak{R}}_n$  with  $n \neq 0$ , and equivalently that

$$\Omega \tilde{V}_n = \Omega_K \tilde{V}_n \Leftrightarrow (\Omega \tilde{V}_n)_P = 0 \quad \forall \quad \tilde{V}_n \in \tilde{\mathfrak{R}}_n \text{ with } n \neq 0 \quad (\text{II } 17)$$

To prove the statement (II 17), we note that the projection  $(\Omega \tilde{V}_n)_P$  is null if  $(\Omega \tilde{V}_n) \bullet W_P$  is null  $\forall W_P \in \mathfrak{R}_P$  and that it is the case with  $n \neq 0$  since firstly we have  $(\Omega \tilde{V}_n) \bullet W_P = \tilde{V}_n \bullet (-\Omega W_P) = \tilde{V}_n \bullet (-\Omega W_P)_K$ , secondly  $(\Omega W_P)_K$  belongs to  $\tilde{\mathfrak{R}}_0$  in view of the definition (II 13) and thirdly since  $\tilde{V}_n$  with  $n \neq 0$  is orthogonal to  $\tilde{\mathfrak{R}}_0$  in view of the orthogonalities (II 15).

To go further, we associate to an observable  $\tilde{V}_m(x,p)$  belonging to a given vector space  $\tilde{\mathfrak{R}}_m$  the observable  $\tilde{V}_\tau(x,p) = \exp(\Omega_K \tau) \tilde{V}_m(x,p)$  depending on the variable  $\tau$ . That observable  $\tilde{V}_\tau$  belongs to  $\sum_n \tilde{\mathfrak{R}}_n$  and is therefore a superposition of observables  $\tilde{V}_n(x,p) \in \tilde{\mathfrak{R}}_n$ . When  $\tau$  passes from  $-\Theta$  to  $+\Theta$  via 0, it passes from  $\tilde{V}_{m-1}(x,p) \in \tilde{\mathfrak{R}}_{m-1}$  to  $\tilde{V}_{m+1}(x,p) \in \tilde{\mathfrak{R}}_{m+1}$ .

via  $\tilde{V}_m(x,p) \in \tilde{\mathfrak{R}}_m$ . Therefore the wave numbers in  $(x,p)$  of  $\tilde{V}_\tau(x,p)$  for  $|\tau| \leq \Theta$  are close to the wave numbers of the observables that belong to the space  $\tilde{\mathfrak{R}}_{m-1} + \tilde{\mathfrak{R}}_m + \tilde{\mathfrak{R}}_{m+1}$ . This is compatible with the fact that some wave numbers in  $(x,p)$  of the observables  $\tilde{V}_n(x,p) \in \tilde{\mathfrak{R}}_n$  are multiplied by  $O(2)$  when  $|n|$  increases by 1, only if  $\tilde{V}_\tau$  for  $|\tau| \leq \Theta$  belongs to a vector space

$\sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_{m+q}$  where  $q^\circ$  is an integer of at most some units. For  $m \leq -q^\circ$  or  $m \geq q^\circ$ , that space

$\sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_{m+q}$  does not contain the observables  $\tilde{V}_0(x,p) \in \tilde{\mathfrak{R}}_0$ . By virtue of the statement (II 17)

the observables that it contains are then transformed in the same way by the operators  $\Omega$  and

$\Omega_K$ . This means that, for  $|\tau| \leq \Theta$  and  $m \leq -q^\circ$  or  $m \geq q^\circ$ , one has  $\Omega \tilde{V}_\tau = \Omega_K \tilde{V}_\tau$ . As  $\frac{\partial}{\partial \tau} \tilde{V}_\tau = \frac{\partial}{\partial \tau}$

$(\exp(\Omega_K \tau) \tilde{V}_m) = \Omega_K \tilde{V}_\tau$ , this means as well that one has  $\frac{\partial}{\partial \tau} \tilde{V}_\tau = \Omega \tilde{V}_\tau$  and that  $\tilde{V}_\tau$  is equal to

$\exp(\Omega \tau) \tilde{V}_m$ . Taking  $\tau = \pm \Theta$ , and  $m=n$ , it comes

$$\exp(\pm \Omega \Theta) \tilde{V}_n = \exp(\pm \Omega_K \Theta) \tilde{V}_n = \tilde{V}_{n \pm 1} \text{ for } n \leq -q^\circ \text{ or } n \geq q^\circ \quad (\text{II 19a})$$

For  $-q^\circ < n < q^\circ$  the operator  $\exp(\pm \Omega \Theta)$  acting on  $\tilde{V}_n$  produces in addition to  $\exp(\pm \Omega_K \Theta) \tilde{V}_n = \tilde{V}_{n \pm 1}$  a component in the space  $\mathfrak{R}_p$ , and also tends to become a superposition of observables  $\tilde{V}_{n+q}$  with  $-q^\circ < q < q^\circ$ . We accordingly admit

$$\exp(\pm \Omega \Theta) \tilde{V}_n = \tilde{V}_{n \pm 1} + \delta V \text{ where } \delta V \in \sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_q + \mathfrak{R}_p \text{ for } -q^\circ < n < q^\circ \quad (\text{II 19b})$$

It is convenient to reformulate the structural condition on the basis of the equations (II 13), but of the equations (II 19a,b) instead of (II 14). The structural condition then starts with the existence of a time scale  $\Theta \ll \tau_{\text{phen}}$  and of an integer  $q^\circ$  of at most some units, allowing to build up in the space  $\mathfrak{R}_K \perp \mathfrak{R}_p$ , firstly the space  $\tilde{\mathfrak{R}}_0$  defined by the equations (II 13), and secondly the spaces  $\tilde{\mathfrak{R}}_{n \neq 0}$  defined by the equations (II 19a,b). It then imposes that the spaces  $\tilde{\mathfrak{R}}_n$  verify the orthogonalities (II 15), resulting from the fact that the observables  $\tilde{V}_n(x,p) \in \tilde{\mathfrak{R}}_n$

display wave numbers in  $(x,p)$  along specific directions  $(\delta x, \delta p)$  which are multiplied by  $O(2)$  when  $n > 0$  is increased by 1 and  $n < 0$  by  $-1$ .

That formulation gives a clear idea of the functioning of the phenomenological determinism. Let a component of the non-phenomenological information  $\rho_{t,K}$  at time  $t$  belonging to a space  $\tilde{\mathfrak{R}}_n$  to be submitted to the Liouville equation from  $t$  to  $t+r\Theta$ ,  $r = \text{integer} > 0$ , i.e. submitted to the Liouville transport operator  $\exp(\Omega r\Theta)$ . That component has three possible types of evolution, determined by the equations (II 19a,b) :

1/ for  $n \geq q^\circ$ , it is transported in block at the time  $t+r\Theta > t$  into the vector space  $\tilde{\mathfrak{R}}_{n+r}$  ; it thus penetrates, when  $r$  increases, deeper into the vector space  $\sum_{q \geq q^\circ} \tilde{\mathfrak{R}}_q$ , by displaying wave numbers along specific directions multiplied by  $O(2)$  when  $r$  increases by 1 ; it drifts in that way farther and farther from the vector space  $\tilde{\mathfrak{R}}_0$ , the only one according to the statement (II 17) whence it could influence the evolution of the phenomenological distribution in the space  $\mathfrak{R}_p$  ; it is thus disconnected for ever in the future from the phenomenological information ;

2/ for  $n \leq -q^\circ$ , it is also transported in block at a time  $t+r\Theta > t$  into  $\tilde{\mathfrak{R}}_{n+r}$ , without influencing the evolution of the phenomenological information, but only until it enters the vector space  $\sum_{q > -q^\circ} \tilde{\mathfrak{R}}_q$  ; it then has no possibility of returning into the vector space  $\sum_{q \leq -q^\circ} \tilde{\mathfrak{R}}_q$  ;

3/ for  $-q^\circ < n < q^\circ$  the considered component of  $\rho_{t,K}$  in  $\tilde{\mathfrak{R}}_n$  tends to be transported at the time  $t+r\Theta > t$  into  $\tilde{\mathfrak{R}}_{n+r}$  until it enters the vector space  $\sum_{q \geq q^\circ} \tilde{\mathfrak{R}}_q$  where it becomes disconnected for ever in the future from  $\rho_{t,P}$  ; until that entrance, which occurs after a time of the order of  $q^\circ\Theta$ , it occupies the space  $\sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_q$  and influences the evolution of the phenomenological information.

If the non-phenomenological information  $\rho_{t \ K}$  is localized within the vector space  $\sum_{n>-q^\circ} \tilde{\mathfrak{R}}_n$  at a given time  $t$ , that localization remains achieved in the future of that time. Accordingly the initial condition (II 16) imposing at the cosmos primordial time  $t^*$  the localization of  $\rho_{t^\circ \ K}$  within  $\sum_{n \geq 0} \tilde{\mathfrak{R}}_n$  close to  $\sum_{n>-q^\circ} \tilde{\mathfrak{R}}_n$ , combined with the structural condition, entails the same localization at any time  $t > t^*$ . There exists no non-phenomenological information  $\rho_{t^\circ \ K}$  within the vector space  $\sum_{n \leq -q^\circ} \tilde{\mathfrak{R}}_n$ . However, during each interval  $dt''$  around a time  $t''$ , the Liouville equation forces  $\rho_{t'' \ P}$  to produce new components  $(\Omega \rho_{t'' \ P})_K dt''$  of  $\rho_{t'' \ K}$ , located into the space  $\tilde{\mathfrak{R}}_0$  because of the equations (II 13). These components move at a time  $t > t''$  within the space  $\sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_q$  by influencing the derivative  $\frac{\partial}{\partial t} \rho_{t \ P}$ , again through the space  $\tilde{\mathfrak{R}}_0$  because of the statement (II 17). The key point is that they escape at a time  $t'' + O(q^\circ \Theta)$  into the space  $\sum_{n \geq q^\circ} \tilde{\mathfrak{R}}_n$  where they become for ever disconnected from the phenomenological distribution. Accordingly, the part of the derivative  $\frac{\partial}{\partial t} \rho_{t \ P}$  at a time  $t > t^*$  which is determined by  $\rho_{t \ K}$  is created by  $\rho_{t'' \ P}$  at times  $t''$  in the interval  $[t - O(q^\circ \Theta), t]$ . It is specified, along the arrow of time only, with a very small error by  $\rho_{t \ P}$  if  $q^\circ \Theta \approx \Theta$  is much smaller than the time scale  $\tau_{\text{phen}}$  of the evolution of  $\rho_{t \ P}$ . The realization of the phenomenological determinism  $\rho_{t \ P} \Rightarrow \rho_{t' \ P}$  for  $t' > t > t^*$  is thus guaranteed. A crucial consequence of the above functioning of the phenomenological determinism results from the fact that the wave numbers along specific directions  $(\delta x, \delta p)$  of the components of  $\rho_{t \ K}(x, p)$  in the vector spaces  $\tilde{\mathfrak{R}}_n$  for  $n > q^\circ$  are multiplied by  $O(2)$  when  $n$  increases by 1. It is then clear that the greatest share of the non-phenomenological information  $\rho_{t \ K}$  within the space  $\sum_{n \geq 0} \tilde{\mathfrak{R}}_n$  has a much too complex and elusive structure in  $(x, p)$  to be accessible to observers. The latter have practically only access to the phenomenological information  $\rho_{t \ P}$  within  $\mathfrak{R}_P$  and to a small share of the non-phenomenological information  $\rho_{t \ K}$  within spaces  $\tilde{\mathfrak{R}}_n$  with low  $n$  of the order of  $q^\circ$ .

The necessity of the initial condition ((II 16) at the cosmos primordial time  $t^*$  for the achievement of the phenomenological determinism  $\rho_{t \ P} \Rightarrow \rho_{t' \ P}$  for  $t' > t$  is illustrated on the

figure 1. The squares on that figure represent the vector spaces of observables  $\mathfrak{R}_p$  and ...,  $\tilde{\mathfrak{R}}_{-2}$ ,  $\tilde{\mathfrak{R}}_{-1}$ ,  $\tilde{\mathfrak{R}}_0$ ,  $\tilde{\mathfrak{R}}_1$ ,  $\tilde{\mathfrak{R}}_2 \dots \subset \mathfrak{R}_k \perp \mathfrak{R}_p$ . For the sake of clarity the integer  $q^\circ$  is taken equal to 1

and accordingly the space  $\sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_q$  reduces to  $\tilde{\mathfrak{R}}_0$ , the space  $\sum_{n \geq q^\circ} \tilde{\mathfrak{R}}_n$  to  $\sum_{n > 0} \tilde{\mathfrak{R}}_n$ , etc. The

arrows, green if the primordial condition (II 16) is verified, red if it is not the case, indicate

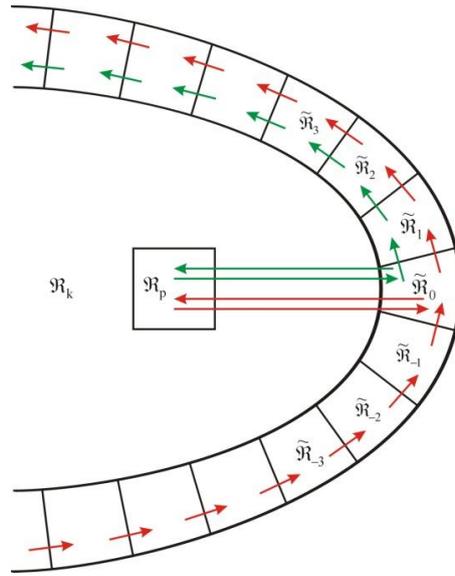


Figure 1. Schematic motion of the phenomenological and non-phenomenological information when the initial condition (II 16) is fulfilled (green arrows) and not fulfilled (red arrows).

1/ the propagation of the non-phenomenological information  $\rho_{t K}$  from  $\tilde{\mathfrak{R}}_n$  to  $\tilde{\mathfrak{R}}_{n+r}$  during  $+\Theta$ , by small arrows ; 2/ the exchanges of information during  $\approx \Theta$  between  $\tilde{\mathfrak{R}}_0$  and  $\mathfrak{R}_p$ , by long arrows. If the condition (II 16) is realized, and only the green arrows are active, the non-phenomenological information  $\rho_{t K}$  is absent from the space  $\sum_{n < 0} \tilde{\mathfrak{R}}_n$ . The phenomenological information  $\rho_{t P}$  within  $\mathfrak{R}_p$  permanently produces via the long green arrow from  $\mathfrak{R}_p$  to  $\tilde{\mathfrak{R}}_0$  new components of  $\rho_{t K}$  into the space  $\tilde{\mathfrak{R}}_0$  which react on the evolution of  $\rho_{t P}$  during  $\approx \Theta$  via the long green arrow from  $\tilde{\mathfrak{R}}_0$  to  $\mathfrak{R}_p$ , before entering the space  $\sum_{n > 0} \tilde{\mathfrak{R}}_n$ , where they no longer influence the evolution of  $\rho_{t P}$ . That process is compatible with the phenomenological

determinism  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' > t$  since the derivative  $\frac{\partial}{\partial t} \rho_{t P}$  at a time  $t$  is influenced by values of  $\rho_{t' P}$  at times  $t''$  in an interval close to  $\Theta$  around  $t$ , and is quasi determined by  $\rho_{t-\varepsilon P}$ ,  $\varepsilon \rightarrow 0^+$  under the restriction  $\tau_{\text{phen}} \gg \Theta$ . If the condition (II 16) is not realized, and the red arrows are active, a non-phenomenological information  $\rho_{t K}$  is present in the space  $\sum_{n < 0} \tilde{\mathfrak{R}}_n$ . It advances when  $t$  increases in that space via the small red arrows, without influencing  $\frac{\partial}{\partial t} \rho_{t P}$  within  $\mathfrak{R}_P$  until it reaches the space  $\tilde{\mathfrak{R}}_0$ , where it does influence the derivative  $\frac{\partial}{\partial t} \rho_{t P}$ , via the long red arrow from  $\tilde{\mathfrak{R}}_0$  to  $\mathfrak{R}_P$ , by components reflecting the state of  $\rho_{t K}$  in the space  $\sum_{n < 0} \tilde{\mathfrak{R}}_n$  without any correlation with  $\rho_{t P}$ . The phenomenological determinism  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' > t$  is then impossible. When the condition (II 16) is fulfilled (and the red arrows inactive) a similar impossibility weighs on an hypothetical phenomenological determinism against the arrow of time,  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' < t$ . Consideration of the latter requires of course that the green arrows on the figure 1 are reversed, the red arrow being inactive. The non-phenomenological information  $\rho_{t K}$  present in the space  $\sum_{n > 0} \tilde{\mathfrak{R}}_n$  moves when  $t$  decreases via the small reverse green arrows, without influencing  $\frac{\partial}{\partial t} \rho_{t P}$  within  $\mathfrak{R}_P$  until it reaches the space  $\tilde{\mathfrak{R}}_0$ , where it influences the derivative  $\frac{\partial}{\partial t} \rho_{t P}$ , via the long red arrow from  $\tilde{\mathfrak{R}}_0$  to  $\mathfrak{R}_P$ , by components reflecting the state of  $\rho_{t K}$  in the space  $\sum_{n > 0} \tilde{\mathfrak{R}}_n$  without any correlation with  $\rho_{t P}$ . Hence it appears that the phenomenological determinism  $\rho_{t P} \Rightarrow \rho_{t' P}$  for  $t' < t$ , against the arrow of time, is made impossible by the initial condition (II 16).

### III– REALIZATION OF A SIMPLIFIED PHENOMENOLOGICAL DETERMINISM

Assuming realized the initial condition (II 16) involving the fundamental distribution  $\rho_{t^*}(x,p)$  at a primordial time  $t^*$  of the cosmos, the existence of the phenomenological determinism depends on the realization of the structural condition specifying in fine the vector space  $\mathfrak{R}_P$  and thereby the phenomenological information  $\rho_{t P}(x,p)$  identified with the

projection of  $\rho_t(x,p)$  into  $\mathfrak{R}_p$ . The key element of the structural condition is the orthogonality (II 15) of the vector spaces  $\tilde{\mathfrak{R}}_n$  generated by the vector space  $\mathfrak{R}_p$  via the equations (II 13, II 19a,b) and a time scale  $\Theta$  much smaller than the time scale  $\tau_{\text{phen}}$  of the evolution of  $\rho_t(x,p)$ . Roughly, the spaces  $\tilde{\mathfrak{R}}_n$  are the hamiltonian images after the time  $n\Theta$  of the space  $\mathfrak{R}_p$ . The cause of their orthogonality is that the observables  $\tilde{V}_n(x,p)$  forming  $\tilde{\mathfrak{R}}_n$  display, when  $(x,p)$  varies along specific vectors  $(\delta x, \delta p)$ , oscillations the wave numbers of which are multiplied by  $O(2)$  if  $n > 0$  is increased by 1 and  $n < 0$  by  $-1$  for  $n > 0$ . It appears that this obliges the vectors  $(\delta x, \delta p)$  to have an hamiltonian image after the time  $\tau$ , the norm of which varies with  $\tau$  as  $\exp(-\frac{\tau}{\Theta})$  for  $n > 0$  and as  $\exp(\frac{\tau}{\Theta})$  for  $n < 0$ . At first sight this seems to merely demand a sufficient stochasticity of the hamiltonian trajectories, expressed by a set of Lyapounov exponents  $\Lambda > 0$ ,  $-\Lambda < 0$  of magnitude  $\sim \frac{1}{\Theta}$ . However the vectors  $(\delta x, \delta p)$  must be real and single valued in  $(x,p)$ , like the observables  $V_p(x,p)$  and  $\tilde{V}_n(x,p)$ . They must then be real eigenvectors, for real eigenvalues  $\exp(\pm \Lambda T) \neq 1$ , of the linear operators transforming the tangent space to the phase space at any point of any closed trajectory of period  $T$  into its hamiltonian image at the same point after the time  $T$ . In fact these operators must have eigenvalues which are real, of the form  $\exp(\pm \Lambda T) \neq 1$ , if they correspond to real eigenvectors  $(\delta x, \delta p)$  carrying the stochasticity. But they may have also eigenvalues of modulus unity corresponding to eigenvectors  $(\delta x', \delta p')$  carrying no stochasticity. That situation of eigenvalues exclusively real or of modulus unity will be shown in § III B to be a reality. It opens the possibility of the justification of the structural condition through a canonical representation of the phase space applicable around the hamiltonian trajectories, which illustrates a motion of the cosmos either of Anosov hyperbolic type via the vectors  $(\delta x, \delta p)$ , or of oscillating elliptic type (via the vectors  $(\delta x', \delta p')$ ). That suited representation is established in a first step in a vicinity of the closed trajectories, which transfer their points into themselves after their immense period  $T$ . In view of the assumed ergodicity of the hamiltonian trajectories in the layer  $H(x,p) \cong E$  containing the cosmos, the closed trajectories run, in infinite number, close to all the points of that layer. It is then validated in a narrow finite domain around any hamiltonian trajectory.

### III A – SUITED REPRESENTATION OF THE PHASE SPACE

We choose a closed trajectory  $\mathbf{T}$  of immense period  $T$  running close to all the points of the layer  $H \cong E$ , and define along  $\mathbf{T}$  a time-like abscissa  $s$ , shown on the figure 2a), starting from a reference origin  $(\mathbf{X}, \mathbf{P})$ . The hamiltonian image  $(\mathbf{X}, \mathbf{P})[s] \in \mathbf{T}$  after the time  $s$  of  $(\mathbf{X}, \mathbf{P}) = (\mathbf{X}, \mathbf{P})[0]$  is an analytical function of  $s$  periodic of period  $T$ . We have in view a representation of the phase space  $(x, p)$  in a vicinity of  $\mathbf{T}$ , realized via  $2N$  canonically conjugate observables, real and analytical in  $(x, p)$ , the first being a time-like observable  $A(x, p)$ , shown on the figure 2b), such that the point  $(\mathbf{X}, \mathbf{P})[A(x, p)] \in \mathbf{T}$  is as close as possible to the point  $(x, p)$

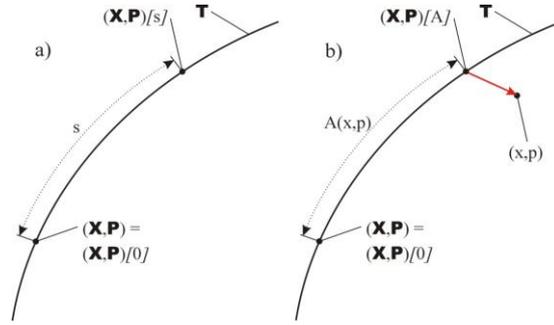


Figure 2 – a) The abscissa  $s$  along a trajectory  $\mathbf{T}$  – b) The canonical observable  $A(x, p)$ .

The real observable  $B(x, p)$  canonically conjugate of  $A(x, p)$  reflects the distance between the points  $(\mathbf{X}, \mathbf{P})[A(x, p)]$  and  $(x, p)$  counted in energy, along an appropriate real vector  $(\delta x, \delta p)_{tr}(\mathbf{X}, \mathbf{P})[A(x, p)]$  originating from the point  $(\mathbf{X}, \mathbf{P})[A(x, p)]$ , transverse to the layer  $H \cong E$ . In view of the quasi localization of the cosmos in that layer, the values taken by the observable  $B(x, p)$  are nearly null. The remaining  $N-1$  pairs of canonically conjugate observables form on one hand  $N'$  pairs of real observables  $Z^\alpha(x, p)$ ,  $Z_\alpha(x, p)$ , labelled by  $\alpha = 1, \dots, N'$ , and on the other hand  $N''$  pairs of complex conjugate observables  $Z^\omega = Z^\omega(x, p)$ ,  $Z_\omega = Z_\omega(x, p) = (Z^\omega(x, p))^*$ , labelled by  $\omega = 1, \dots, N'' = N - N' - 1$ . The latter express in fact for convenience  $N''$  pairs of real canonically conjugate observables  $(Z^\omega(x, p) + Z_\omega(x, p))/2$ ,  $(Z^\omega(x, p) - Z_\omega(x, p))/2i$ . The  $N'$

observables  $Z^\alpha(x,p)$ ,  $Z_\alpha(x,p)$ , and the  $N''$  observables  $Z^\omega, Z_\omega$  reflect the distance between the points  $(\mathbf{X}, \mathbf{P})[A(x,p)] \in \mathbf{T}$  and  $(x,p)$ , along  $N'$  pairs of real vectors  $(\delta x, \delta p)_\alpha(\mathbf{x}, \mathbf{p})[A]$ ,  $(\delta x, \delta p)^\alpha(\mathbf{x}, \mathbf{p})[A]$  and along  $N''$  pairs of complex conjugate vectors  $(\delta x, \delta p)_\omega(\mathbf{x}, \mathbf{p})[A]$ ,  $(\delta x, \delta p)^\omega(\mathbf{x}, \mathbf{p})[A] = ((\delta x, \delta p)_\omega(\mathbf{x}, \mathbf{p})[A])^*$  parallel to the thin layer  $H(x,p)=E$ . These vectors, for each label  $\text{tr}, \alpha, \omega$ , constitute vector functions  $(\delta x, \delta p)_{\text{tr}}(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)_\alpha(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)^\alpha(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)_\omega(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)^\omega(\mathbf{x}, \mathbf{p})[s]$  of the abscissa  $s$  along  $\mathbf{T}$ . With the vector  $(\delta x, \delta p)_{//}(\mathbf{x}, \mathbf{p})[s] = \frac{\partial}{\partial s}(\mathbf{X}, \mathbf{P})[s] = \left( \frac{\partial H(\mathbf{X}, \mathbf{P})[s]}{\partial p}, -\frac{\partial H(\mathbf{X}, \mathbf{P})[s]}{\partial x} \right)$  parallel to the trajectory  $\mathbf{T}$  at the point  $(\mathbf{X}, \mathbf{P})[s]$ , they form a basis of the tangent space to the phase space at  $(\mathbf{X}, \mathbf{P})[s]$ . The standard conjugate observables  $(x,p) = (x^i, p_i)$  and the  $2N$  observables  $A(x,p)$ ,  $B(x,p)$ ,  $Z^\alpha(x,p)$ ,  $Z_\alpha(x,p)$ ,  $Z^\omega(x,p)$ ,  $Z_\omega(x,p)$  are linked by the vector equation

$$(x,p) - (\mathbf{X}, \mathbf{P})[A] = B(\delta x, \delta p)_{\text{tr}}(\mathbf{x}, \mathbf{p})[A] + Z^\alpha(\delta x, \delta p)_\alpha(\mathbf{x}, \mathbf{p})[A] + Z_\alpha(\delta x, \delta p)^\alpha(\mathbf{x}, \mathbf{p})[A] + Z^\omega(\delta x, \delta p)_\omega(\mathbf{x}, \mathbf{p})[A] + Z_\omega(\delta x, \delta p)^\omega(\mathbf{x}, \mathbf{p})[A], \text{ with } A, \dots \text{ standing for } A(x,p), \dots \quad (\text{III } 1)$$

The vector  $(x,p) - (\mathbf{X}, \mathbf{P})[A]$  is illustrated in red on the figure 2b). The vectors  $(\delta x, \delta p)_{//}(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)_{\text{tr}}(\mathbf{x}, \mathbf{p})[s]$ ,  $(\delta x, \delta p)_\alpha(\mathbf{x}, \mathbf{p})[s]$ ,  $\dots$  will be periodic of period  $T$  and analytic in  $s$  to ensure that the observables  $A(x,p)$  (modulo  $T$ ),  $B(x,p)$ ,  $Z^\alpha(x,p)$ ,  $\dots$  are single valued and analytic in  $(x,p)$ . We demand that, in a vicinity of  $\mathbf{T}$ , the observables  $A(x,p)$ ,  $B(x,p)$ ,  $Z^\alpha(x,p)$ ,  $\dots$  are canonically conjugate and in addition that the real observables  $Z^\alpha(x,p)$ ,  $Z_\alpha(x,p)$  express an hyperbolic Anosov motion of the cosmos while the complex conjugate observables  $Z^\omega(x,p)$ ,  $Z_\omega(x,p) = (Z^\omega(x,p))^*$  express an oscillating elliptic motion.

The derivation of the equation (III 1) with respect to  $A, B, Z^\alpha, \dots$  allows to obtain the

$$\text{Lagrange brackets } [u, v]_{x,p} = \frac{\partial x^i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial x^i}{\partial v} \frac{\partial p_i}{\partial u} \text{ for any couple of observables } u, v \text{ taken among}$$

$A, B, Z^\alpha, Z_\alpha, Z^\omega, Z_\omega$ . It thus appears that the Lagrange brackets  $[A, B]_{x,p}$ ,  $\dots$  coincide with the

hamiltonian symplectic products  $(\delta x, \delta p)_{//}(\mathbf{x}, \mathbf{p})_{[A]} \wedge (\delta x, \delta p)_{tr}(\mathbf{x}, \mathbf{p})_{[A]}$ ,  $\dots$ , the symplectic product  $(\delta x, \delta p) \wedge (\delta x', \delta p')$  of two vectors  $(\delta x, \delta p)$  and  $(\delta x', \delta p')$  being

$$(\delta x, \delta p) \wedge (\delta x', \delta p') = \delta x^i \delta p'^i - \delta x'^i \delta p_i \quad (\text{III } 2)$$

The canonicity of the observables  $A, B, Z^\alpha, \dots$  in a vicinity of  $\mathbf{T}$  then demands firstly that

$$\begin{aligned} (\delta x, \delta p)_{//}(\mathbf{x}, \mathbf{p})_{[s]} \wedge (\delta x, \delta p)_{tr}(\mathbf{x}, \mathbf{p})_{[s]} &= K, (\delta x, \delta p)_{\alpha}(\mathbf{x}, \mathbf{p})_{[s]} \wedge (\delta x, \delta p)^{\alpha}(\mathbf{x}, \mathbf{p})_{[s]} = K(\alpha), \\ (\delta x, \delta p)_{\omega}(\mathbf{x}, \mathbf{p})_{[s]} \wedge (\delta x, \delta p)^{\omega}(\mathbf{x}, \mathbf{p})_{[s]} &= iK(\omega) \end{aligned} \quad (\text{III } 3a)$$

where  $K, K(\alpha), K(\omega)$  are real constants  $\neq 0$  independent of  $s$ , and secondly that all the other possible symplectic products cancel

$$(\delta x, \delta p)_{//}(\mathbf{x}, \mathbf{p})_{[s]} \wedge (\delta x, \delta p)_{\alpha}(\mathbf{x}, \mathbf{p})_{[s]} = 0, (\delta x, \delta p)_{\alpha}(\mathbf{x}, \mathbf{p})_{[s]} \wedge (\delta x, \delta p)_{\beta}(\mathbf{x}, \mathbf{p})_{[s]} = 0, \dots \quad (\text{III } 3b)$$

The cosmos displays in a vicinity of  $\mathbf{T}$  an hyperbolic Anosov motion in  $Z^\alpha, Z_\alpha$  and an oscillating elliptic motion in  $Z^\omega, Z_\omega$  if the point  $(x, p) \Leftrightarrow (A, B, Z^\alpha, Z_\alpha, Z^\omega, Z_\omega)$  at the time 0 has after the time  $t$  an hamiltonian image  $(x, p)[t] \Leftrightarrow (A[t], B[t], Z^\alpha[t], Z_\alpha[t], Z^\omega[t], Z_\omega[t])$  given at first order in  $B, Z^\alpha, Z_\alpha, Z^\omega, Z_\omega$  by the formulae

$$\begin{aligned} A[t] &= A + t, B[t] = B \\ Z^\alpha[t] &= \exp(-\Lambda(\alpha)t)Z^\alpha, Z_\alpha[t] = \exp(\Lambda(\alpha)t)Z_\alpha \\ Z^\omega[t] &= \exp(-i\Lambda(\omega)t)Z^\omega, Z_\omega[t] = (Z^\omega[t])^* = \exp(i\Lambda(\omega)t)Z_\omega \end{aligned} \quad (\text{III } 4)$$

The  $N'$  pairs of real coefficients  $-\Lambda(\alpha) < 0, \Lambda(\alpha) > 0$  coincide with  $N'$  pairs  $-\Lambda < 0, \Lambda > 0$  of Lyapounov exponents. According to the Oseledec theorem (Oseledec 1968; Raghunathan 1979) these Lyapounov pairs are recovered without change everywhere in the layer  $H \cong E$ . The  $N''$  pairs of exponents  $-i\Lambda(\omega), i\Lambda(\omega)$  giving the evolution of the  $N''$  pairs of complex conjugate  $Z^\omega, Z_\omega$  must be purely imaginary  $\neq 0$ . They thus entail a purely oscillatory behaviour of  $Z^\omega$  and  $Z_\omega$  of period  $\frac{2\pi}{\Lambda(\omega)}$ . Accordingly they do not contribute to the

stochasticity : in the context of the Oseledec theorem, they reflect the existence of  $N''$  pairs of Lyapounov exponents  $\Lambda$  having exactly a null value. By analogy, we state that the pairs of exponents  $-i\Lambda(\omega), i\Lambda(\omega)$ , like the Lyapounov pairs  $-\Lambda(\alpha), \Lambda(\alpha)$ , are recovered without

change everywhere in the cosmos. To simplify we will assume in what follows that the  $N'$  real pairs  $-\Lambda(\alpha) < 0, \Lambda(\alpha) > 0$  and the  $N''$  purely imaginary pairs  $-i\Lambda(\omega), i\Lambda(\omega) \neq 0$  are different.

We express the constraints (III 4) in a vicinity of  $\mathbf{T}$  by introducing at each point  $(X,P)$  of the phase space, and for each time interval  $\tau$ , the differential operator  $C_\tau (X,P)$ , which transforms the small vectors  $(\delta x, \delta p)$  of the tangent space to the phase space at the point  $(X,P)$  into their hamiltonian image  $C_\tau (X,P)(\delta x, \delta p)$  after the time  $\tau$  in the tangent space at the hamiltonian image  $(X,P)/[\tau]$  of  $(X,P)$  (Lichtenberg and Lieberman 1983 ; McDuff and Salamon 2005). That operator is specified by the hamiltonian  $H(x,p)$  and its derivatives as an analytic function of the time  $\tau$  and the position  $(X,P)$ . It basically verifies the equations  $C_{\tau=0} (X,P) = I$  and  $C_{\tau+\tau'} (X,P) = C_{\tau'} (X,P)/[\tau'] C_\tau (X,P)$ . It may be expressed in the form

$$C_\tau (X,P) = C_\varepsilon (X,P)/[\tau] \dots C_\varepsilon (X,P)/[3\varepsilon] C_\varepsilon (X,P)/[2\varepsilon] C_\varepsilon (X,P)/[\varepsilon] C_\varepsilon (X,P), \varepsilon \rightarrow 0$$

$$C_\varepsilon (X,P) = I + \varepsilon D_{(X,P)} \text{ for } \varepsilon \rightarrow 0 \quad (\text{III 5a})$$

The operator  $D_{(X,P)}$  transforms a vector  $(\delta x, \delta p)$  of the tangent space to the phase space at the point  $(X,P)$  into  $D_{(X,P)}(\delta x, \delta p) = (\delta \xi, \delta \pi)$  in the same tangent space. The components  $\delta \xi^i$  and  $\delta \pi_i$  are classically related to  $\delta x^i$  and  $\delta p_i$  through the second derivatives of the hamiltonian

$H(x,p)$ . In the case where  $H(x,p) = \mathcal{O}(\bar{x}) + \frac{p^2}{2}$ , one has

$$\delta \xi^i = \delta p_i, \delta \pi^i = -\left(\frac{\partial^2 \mathcal{O}(\bar{X})}{\partial x^i \partial x^j}\right) \delta x^j \quad (\text{III 5b})$$

If the point  $(x,p)$  is in a vicinity of the trajectory  $\mathbf{T}$ , the vector equation (III 1) imposes that the hamiltonian image after the time  $\tau$  of the small vector  $(x,p) - (\mathbf{X}, \mathbf{P})/s]$  is the sum  $B C_\tau (\mathbf{x}, \mathbf{p})/s](\delta x, \delta p)_{tr} (\mathbf{x}, \mathbf{p})/s] + Z^\alpha C_\tau (\mathbf{x}, \mathbf{p})/s](\delta x, \delta p)_{\alpha} (\mathbf{x}, \mathbf{p})/s] + \dots$ . The equations (III 4) imposes that the same hamiltonian image equals  $B(\delta x, \delta p)_{tr} (\mathbf{x}, \mathbf{p})/s+\tau] + \exp(-\Lambda(\alpha)\tau) Z^\alpha (\delta x, \delta p)_{\alpha} (\mathbf{x}, \mathbf{p})/s+\tau] + \dots$ . It results that the hyperbolic / elliptic motions (III 4) of the cosmos in  $Z^\alpha, Z_\alpha / Z^0, Z_\omega$  is ensured if

$$C_\tau (\mathbf{x}, \mathbf{p})/s](\delta x, \delta p)_{\alpha} (\mathbf{x}, \mathbf{p})/s] = \exp(-\Lambda(\alpha)\tau) (\delta x, \delta p)_{\alpha} (\mathbf{x}, \mathbf{p})/s+\tau],$$

$$\begin{aligned}
C_{\tau}(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)^{\alpha}(\mathbf{x},\mathbf{p})/s/ &= \exp(\Lambda(\alpha)\tau)(\delta x,\delta p)^{\alpha}(\mathbf{x},\mathbf{p})/s+\tau/, \\
C_{\tau}(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)_{\omega}(\mathbf{x},\mathbf{p})/s/ &= \exp(-i\Lambda(\omega)\tau)(\delta x,\delta p)_{\omega}(\mathbf{x},\mathbf{p})/s+\tau/, \\
C_{\tau}(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)^{\omega}(\mathbf{x},\mathbf{p})/s/ &= \exp(i\Lambda(\omega)\tau)(\delta x,\delta p)^{\omega}(\mathbf{x},\mathbf{p})/s+\tau/
\end{aligned} \tag{III 6}$$

These equations, combined with the periodicity of period T in s of the vector  $(\delta x,\delta p)_{\alpha}(\mathbf{x},\mathbf{p})/s/$ ,  
 $\dots$ , impose that the vectors  $(\delta x,\delta p)_{\alpha}(\mathbf{x},\mathbf{p})/s/,\dots$  are eigenvectors of the operator  $C_T(\mathbf{x},\mathbf{p})/s/$

$$\begin{aligned}
C_T(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)_{\alpha}(\mathbf{x},\mathbf{p})/s/ &= \exp(-\Lambda(\alpha)T)(\delta x,\delta p)_{\alpha}(\mathbf{x},\mathbf{p})/s/, \\
C_T(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)^{\alpha}(\mathbf{x},\mathbf{p})/s/ &= \exp(\Lambda(\alpha)T)(\delta x,\delta p)^{\alpha}(\mathbf{x},\mathbf{p})/s/, \\
C_T(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)_{\omega}(\mathbf{x},\mathbf{p})/s/ &= \exp(-i\Lambda(\omega)T)(\delta x,\delta p)_{\omega}(\mathbf{x},\mathbf{p})/s/, \\
C_T(\mathbf{x},\mathbf{p})/s/(\delta x,\delta p)^{\omega}(\mathbf{x},\mathbf{p})/s/ &= \exp(i\Lambda(\omega)T)(\delta x,\delta p)^{\omega}(\mathbf{x},\mathbf{p})/s/
\end{aligned} \tag{III 7}$$

The operators  $C_T(\mathbf{x},\mathbf{p})/s/$  must accordingly admit  $N'$  pairs of eigenvalues of the form  $\exp(-\Lambda(\alpha)T)$ ,  $\exp(\Lambda(\alpha)T)$  located on the real axis, real, inverse to each other, and  $N''$  pairs of eigenvalues of the form  $\exp(-i\Lambda(\omega)T)$ ,  $\exp(i\Lambda(\omega)T)$ , located on the circle of radius 1, complex conjugate, inverse to each other. We have already assumed that these eigenvalues are different, i.e. non degenerate. They are in number  $2N'+2N'' = 2N-2$  supplemented up to  $2N$  by the degenerate value 1 for the eigenvectors  $(\delta x,\delta p)_{//}(\mathbf{x},\mathbf{p})/s/$  and  $(\delta x,\delta p)_{tr}(\mathbf{x},\mathbf{p})/s/$ .

The constraints (III 6,III 7) are better understood by taking into account the property of the operators  $C_{\tau}(\mathbf{x},\mathbf{p})$  to be symplectic (McDuff and Salamon 2005), namely, to maintain invariant the symplectic product (III 2): whatever the vectors  $(\delta x,\delta p)$  and  $(\delta x',\delta p')$  originating from  $(\mathbf{X},\mathbf{P})$ , their images  $C_{\tau}(\mathbf{x},\mathbf{p})(\delta x,\delta p)$  and  $C_{\tau}(\mathbf{x},\mathbf{p})(\delta x',\delta p')$  originating from  $(\mathbf{X},\mathbf{P})/\tau/$  verify

$$C_{\tau}(\mathbf{x},\mathbf{p})(\delta x,\delta p) \wedge C_{\tau}(\mathbf{x},\mathbf{p})(\delta x',\delta p') = (\delta x,\delta p) \wedge (\delta x',\delta p') \tag{III 8a}$$

If  $(\delta x,\delta p)_c$  and  $(\delta x,\delta p)_{c'}$  are eigenvectors of  $C_{\tau}(\mathbf{x},\mathbf{p})$  for the eigenvalues c and c' one has therefore

$$(\delta x,\delta p)_c \wedge (\delta x,\delta p)_{c'} = 0 \text{ if } 1-cc' \text{ is } \neq 0 \tag{III 8b}$$

and as the vector  $((\delta x,\delta p)_c)^*$  complex conjugate of  $(\delta x,\delta p)_c$  is eigenvector for the eigenvalue  $c^*$

$$(\delta x,\delta p)_c \wedge ((\delta x,\delta p)_c)^* = 0 \text{ if } 1-cc^* \text{ is } \neq 0 \tag{III 8c}$$

One may deduce from the equation (III 8b) that if the operator  $C_{\tau (X,P)}$  admits the eigenvalue  $c$ , it admits also  $\frac{1}{c}$ . As it admits  $c^*$ , its eigenvalues run in principle by groups of four:  $c, c^*, \frac{1}{c}, \frac{1}{c^*}$  (Lichtenberg and Lieberman 1983). In fact, concerning a non-degenerate eigenvalue  $c$  of  $C_{\tau (X,P)}$  considered per se, there are two distinct possibilities :

1/  $c=c^*$  is located on the real axis, or  $c=\frac{1}{c^*}$  is located on the circle of radius 1, and is then only accompanied by the eigenvalue  $\frac{1}{c} \neq c$  on that axis or on that circle ;

2/  $c \neq c^*$  and  $\frac{1}{c} \neq \frac{1}{c^*}$  is located neither on the real axis nor on the circle of radius 1, and is then accompanied by the eigenvalues  $\frac{1}{c}, c^*, \frac{1}{c^*}$  different and different of  $c$ .

The symplectic equation (III 8c) is involved non trivially in the possibility 2/ alone. One may show that the latter is equivalent to a unique real, symmetric relation  $A(c,c^*) = 0$  between  $c$  and  $c^*$ , specific of the operator  $C_{\tau (X,P)}$ . This imposes to  $c$ , when the possibility 2/ occurs, a localization in the complex plane on a specific line  $L$ . The possibilities 1/, 2/ therefore correspond to distinct locations of  $c$  in the complex plane, equally plausible in principle : 1/ on the real axis or on the circle of radius 1, 2/ on the line  $L$ . The constraints (III 6, III 7) oblige of course the operators  $C_{\tau (X,P)/s}$ , for  $\tau = T$  at the points  $(X,P)/s$  of  $T$ , to fall in the possibilities 1/, never in the possibility 2/.

This is a priori not obvious. One may expect from numerical calculations with randomly chosen operators  $C_{\tau (X,P)}$  that the operators  $C_{\tau (X,P)/s}$  fall erratically in the possibilities 1/ or 2/. In that context, the operators  $C_T (X,P)/s$  must be exceptional with respect to the operators  $C_{\tau (X,P)/s}$  with  $\tau \neq T$ . That exceptional character is shown for instance by the fact that the operators  $C_T (X,P)/s$  transform the tangent space at the point  $(X,P)/s$  into itself at the point  $(X,P)/s+T = (X,P)/s$  implies that their eigenvalues have an hamiltonian meaning, i.e. are invariant when one changes the standard canonical representation  $(x,p) = (x^i, p_i)$ , which

is not the case in general with the operators  $C_{\tau}(\mathbf{X},\mathbf{P})/s]$  with  $\tau \neq T$ . More important, the operators  $C_T(\mathbf{X},\mathbf{P})/s]$  verify the equation

$$C_T(\mathbf{X},\mathbf{P})/s+\tau] = C_{\tau}(\mathbf{X},\mathbf{P})/s]C_T(\mathbf{X},\mathbf{P})/s]C_{-\tau}(\mathbf{X},\mathbf{P})/s+\tau] = C_{\tau}(\mathbf{X},\mathbf{P})/s]C_T(\mathbf{X},\mathbf{P})/s](C_{\tau}(\mathbf{X},\mathbf{P})/s])^{-1} \quad \forall \tau \quad (\text{III } 9)$$

This is a consequence of the equation  $C_{\tau+\tau'}(\mathbf{X},\mathbf{P}) = C_{\tau'}(\mathbf{X},\mathbf{P})/s]C_{\tau}(\mathbf{X},\mathbf{P})$  and of the periodicity  $(\mathbf{X},\mathbf{P})/s] = (\mathbf{X},\mathbf{P})/s+T]$ . The equation (III 9) obviously implies that the eigenvalues of  $C_T(\mathbf{X},\mathbf{P})/s]$  at the point  $(\mathbf{X},\mathbf{P})/s]$  are recovered without change at all the points  $(\mathbf{X},\mathbf{P})/s+\tau]$  of  $\mathbf{T}$ . It furthermore implies that, for each eigenvalue, an eigenvector of  $C_T(\mathbf{X},\mathbf{P})/s+\tau]$  is obtained within a scalar factor by applying the operator  $C_{\tau}(\mathbf{X},\mathbf{P})/s]$  to an eigenvector of  $C_T(\mathbf{X},\mathbf{P})/s]$ . A remarkable situation then results from the statement justified in § III B, namely that the operator  $C_T(\mathbf{X},\mathbf{P})/s]$  admits, for one value of  $s$  (for instance for  $(\mathbf{X},\mathbf{P})/s=0]=(\mathbf{X},\mathbf{P})$ ), eigenvalues exclusively on the real axis or on the circle of radius 1, which then go by pairs inverse  $\exp(-\Lambda(\alpha)T)$ ,  $\exp(\Lambda(\alpha)T)$  or  $\exp(-i\Lambda(\omega)T)$ ,  $\exp(i\Lambda(\omega)T)$ . Firstly the same eigenvalues occur at all values of  $s$ . Secondly, it readily appears that the corresponding eigenvectors  $(\delta x, \delta p)_{\alpha}(\mathbf{X},\mathbf{P})/s]$ ,  $(\delta x, \delta p)_{\alpha}^{\alpha}(\mathbf{X},\mathbf{P})/s]$ ,  $(\delta x, \delta p)_{\omega}(\mathbf{X},\mathbf{P})/s]$ ,  $(\delta x, \delta p)_{\omega}^{\omega}(\mathbf{X},\mathbf{P})/s]$  for each abscissa  $s$  may be so adjusted in amplitude that they verify the constraints (III 6). They thus constitute, for each exponent  $-\Lambda(\alpha) < 0$ ,  $\Lambda(\alpha) > 0$  and  $-i\Lambda(\omega)$ ,  $i\Lambda(\omega)$  purely imaginary  $\neq 0$ , vector functions of  $s$  periodic of period  $T$  and analytic, satisfying the equations (III 6). The observables  $A(x,p)$ ,  $B(xp)$ ,  $Z^{\alpha}(x,p)$ , ... defined by the equation (III 1) then verify in a vicinity of  $\mathbf{T}$  the equations (III 4) illustrating an hyperbolic / elliptic motion of the cosmos in  $Z^{\alpha}, Z_{\alpha}$ , /  $Z^{\omega}, Z_{\omega}$ . It also appears, in view of the equations (II 9a,b), that the vectors  $(\delta x, \delta p)_{\alpha}(\mathbf{X},\mathbf{P})/s]$ , ... verify the equations (III 3a,b) guaranteeing in a vicinity of  $\mathbf{T}$  the canonicity of the representation  $A(x,p)$ ,  $B(xp)$ ,  $Z^{\alpha}(x,p)$ , ...

One must stress that the exponents  $-\Lambda(\alpha), \dots$  defined through the statement in § III B on the eigenvalues  $\exp(-\Lambda(\alpha)T), \dots$  of the operators  $C_T(\mathbf{X},\mathbf{P})/s]$  on any closed trajectory  $\mathbf{T}$  of immense period  $T$  are recovered by virtue of the Oseledec theorem without change on all  $\mathbf{T}s$ . The point  $(\mathbf{X},\mathbf{P})/s] \in \mathbf{T}$ , when the abscissa  $s$  varies within the period  $T$  runs by virtue of the ergodicity of  $\mathbf{T}$  close to all the points  $(X,P)$  of the layer  $H \cong E$  containing the cosmos. This

associates, via that ergodicity, to a vector function  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})/[s], \dots$  of the abscissa  $s$  along  $\mathbf{T}$  a vector field  $(\delta x, \delta p)_\alpha (X, P), \dots$  which interpolates to the best at each point  $(X, P)$  the values of  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})/[s], \dots$  at the points  $(\mathbf{X}, \mathbf{P})/[s]$  neighbouring  $(X, P)$ . The vector field  $(\delta x, \delta p)_\alpha (X, P), \dots$  is single valued in  $(X, P)$  in view of the periodicity of period  $T$  in  $s$  of the vector function  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})/[s], \dots$ . An important thing is that it is also analytic in  $(X, P)$  by displaying finite variation scales in the directions transverse to  $\mathbf{T}$ . It means that  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})/[T'], \dots$  is neighbouring  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p}), \dots$  if the time  $T' < T$  is such that  $(\mathbf{X}, \mathbf{P})/[T']$  is neighbouring  $(\mathbf{X}, \mathbf{P}) = (\mathbf{X}, \mathbf{P})/[T]$ . A reason for this is that the statement in § III B implies that the operators  $C_{T' \cdot (\mathbf{x}, \mathbf{p})}$  and  $C_{T-T' \cdot (\mathbf{x}, \mathbf{p})/[T']}$  have eigenvalues which are quasi equal to  $\exp(-\Lambda(\alpha)T'), \dots$  and  $\exp(-\Lambda(\alpha)(T-T')), \dots$ , thus forming a product quasi equal to the eigenvalues  $\exp(-\Lambda(\alpha)T), \dots$  of  $C_T (\mathbf{x}, \mathbf{p}) = C_{T-T' \cdot (\mathbf{x}, \mathbf{p})/[T']} C_{T' \cdot (\mathbf{x}, \mathbf{p})}$ . The corresponding eigenvectors of  $C_{T' \cdot (\mathbf{x}, \mathbf{p})}$  and  $C_{T-T' \cdot (\mathbf{x}, \mathbf{p})/[T']}$  are then necessarily quasi equal to the eigenvectors  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p}), \dots$  of  $C_T (\mathbf{x}, \mathbf{p})$ : : we have  $C_{T' \cdot (\mathbf{x}, \mathbf{p})}(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p}), \dots$  quasi equal to  $\exp(-\Lambda(\alpha)T')(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p}), \dots$ , i.e., in view of the relations (III 6),  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})/[T'], \dots$  quasi equal to  $(\delta x, \delta p)_\alpha (\mathbf{x}, \mathbf{p})$ .

A further step is to admit that the various  $\mathbf{T}$ s produce for a given exponent  $-\Lambda(\alpha), \dots$  an unique field  $(\delta x, \delta p)_\alpha (X, P), \dots$  displaying finite variation scales in  $(X, P)$ . There then exists around any closed trajectory  $\mathbf{T}$  a narrow but finite domain  $\mathbf{D}$ , shown on the figure (3a), such that each vector field  $(\delta x, \delta p)_\alpha (X, P), \dots$  is quasi uniform over each section of  $\mathbf{D}$  transverse to  $\mathbf{T}$ . The observables  $Z^\alpha(x, p), \dots$  established around  $\mathbf{T}$  inside  $\mathbf{D}$  by the equation (III i), taken at two points  $(x, p)$  and  $(x', p') = (x, p) + (d\xi, d\pi)$  very close to each other produce differences  $Z^\alpha(x', p') - Z^\alpha(x, p), \dots$  quasi equal to the observables  $Z^\alpha(d\xi, d\pi), \dots$  establishes by the equation (III 1) around a closed trajectory passing very near  $(x, p)$ . This generalizes to the domain  $\mathbf{D}$  the canonicity of the observables  $A, B, Z^\alpha, Z_\alpha, Z^0, Z_\omega$  and the hyperbolic / elliptic

motion (III 4) of the cosmos in  $Z^\alpha, Z_\alpha / Z^\omega, Z_\omega$ . Via an appropriate normalization of  $(\delta x, \delta p)_\alpha (X,P), \dots$ , the sections of **D** transverse to **T** are delimited via the equations (III 1) by squares  $|Z^\alpha| < 1$  along  $(\delta x, \delta p)_\alpha (X,P)$  and  $|Z_\alpha| < 1$  along  $(\delta x, \delta p)^\alpha (X,P)$ , or by circles  $|Z^\omega| = |Z_\omega| < 1$ . The domain **D** around a trajectory **T** should cover many times the thin layer  $H(x,p)=E$  by overlapping effect. The orientations of the vector fields  $(\delta x, \delta p)_\alpha (X,P), \dots$  in overlapping zones are of course similar, but on the contrary, as shown on figure 3b) become quite uncorrelated in non-overlapping zones.

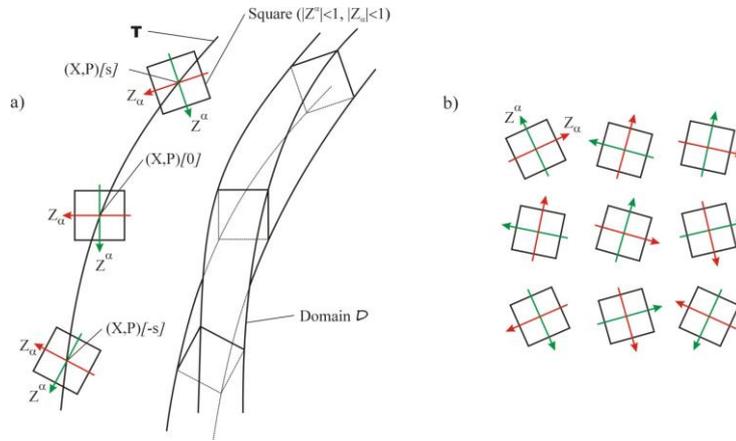


Figure 3 – a) The domain **D** around a closed trajectory **T**, which may be any Hamiltonian trajectory, delimited along the vector  $(\delta x, \delta p)_\alpha (X,P)$ ,  $(\delta x, \delta p)^\alpha (X,P)$  by squares  $|Z^\alpha| < 1$ ,  $|Z_\alpha| < 1$  – b) Absence of correlation of the orientations of the vector fields  $(\delta x, \delta p)_\alpha (X,P)$ , ... in non overlapping transverse sections of **D**.

The vector fields  $(\delta x, \delta p)_\alpha (X,P)$ ,  $(\delta x, \delta p)^\alpha (X,P)$  and  $(\delta x, \delta p)_\omega (X,P)$ ,  $(\delta x, \delta p)^\omega (X,P)$ , for given exponents  $-\Lambda(\alpha) < 0$ ,  $\Lambda(\alpha) > 0$  and  $-i\Lambda(\omega)$ ,  $i\Lambda(\omega)$  purely imaginary  $\neq 0$ , may be better understood by noting, in the context of the Oseledec theorem, that the first one for instance belong to the set of real vectors  $(\delta x, \delta p)_{(X,P)}$  originating from  $(X,P)$  such that the norm  $\|C_\tau (X,P)(\delta x, \delta p)_{(X,P)}\|$  multiplied by  $\exp(\Lambda\tau)$  is null or finite for  $\tau \rightarrow \infty$  if  $\Lambda$  is  $\leq \Lambda(\alpha)$  and for  $\tau \rightarrow -\infty$  if  $\Lambda$  is  $\geq \Lambda(\alpha)$ . Their analytical character in  $(X,P)$  is then suggested by a theorem (Ruelle, 1979) which implies that the vectors  $(\delta x, \delta p)_{(X,P)}$  originating from  $(X,P)$  such that the norm  $\|C_\tau (X,P)(\delta x, \delta p)_{(X,P)}\|$  is null or finite for  $\tau \rightarrow \pm\infty$  are tangent to a differentiable manifold. Another way relies on equations specifying on a trajectory **T** the vector functions  $(\delta x, \delta p)_\alpha (X,P)/s, \dots$  in  $s$ , obtained by differentiating with respect to  $\tau$  the relations (III 6). One

finds by using the formulae (III 5a,b) that the components  $(\delta x)_{\alpha (\mathbf{x},\mathbf{p})/s}$  and  $(\delta p)_{\alpha (\mathbf{x},\mathbf{p})/s}$  of the vector function  $(\delta x, \delta p)_{\alpha (\mathbf{x},\mathbf{p})/s}$  for given  $-\Lambda(\alpha)$  are specified by the equations

$$\begin{aligned} \mathbf{L}(i, -\Lambda(\alpha), (\delta x)_{\alpha (\mathbf{x},\mathbf{p})/s}) &\equiv \left( \frac{\partial^2}{\partial x^i \partial x^j} V((\mathbf{X})/s) \right) (\delta x^j)_{\alpha (\mathbf{x},\mathbf{p})/s} + \frac{\partial^2}{\partial s^2} (\delta x^i)_{\alpha (\mathbf{x},\mathbf{p})/s} + \\ &(-\Lambda(\alpha))^2 (\delta x^i)_{\alpha (\mathbf{x},\mathbf{p})/s} + 2(-\Lambda(\alpha)) \frac{\partial}{\partial s} (\delta x^i)_{\alpha (\mathbf{x},\mathbf{p})/s} = 0 \quad \forall i, \text{ with } (\mathbf{X}, \mathbf{P})/s = ((\mathbf{X})/s, (\mathbf{P})/s) \\ (\delta p_i)_{\alpha (\mathbf{x},\mathbf{p})/s} &= -\Lambda(\alpha) (\delta x^i)_{\alpha (\mathbf{x},\mathbf{p})/s} + \frac{\partial}{\partial s} (\delta x^i)_{\alpha (\mathbf{x},\mathbf{p})/s} \end{aligned}$$

One may expect from these equations that the variation scale in  $s$  of  $(\delta x, \delta p)_{\alpha (\mathbf{x},\mathbf{p})/s}$  is of the order of  $1/\Lambda(\alpha)$ . By changing  $-\Lambda(\alpha)$  into  $\Lambda(\alpha)$ ,  $-i\Lambda(\omega)$ ,  $i\Lambda(\omega)$ , one specifies  $(\delta x, \delta p)_{\alpha (\mathbf{x},\mathbf{p})/s}$ ,  $(\delta x, \delta p)_{\omega (\mathbf{x},\mathbf{p})/s}$ ,  $(\delta x, \delta p)_{\omega (\mathbf{x},\mathbf{p})/s}$ . It is convenient to introduce a test vector function  $(\delta \xi)_{(\mathbf{x},\mathbf{p})/s}$  and to specify  $(\delta x, \delta p)_{\alpha (\mathbf{x},\mathbf{p})/s}$  by the principle that the functional

$$\mathbf{I}(-\Lambda(\alpha), (\delta x)_{\alpha (\mathbf{x},\mathbf{p})/s}, (\delta \xi)_{(\mathbf{x},\mathbf{p})/s}) \equiv \int_0^T ds \sum_i \mathbf{L}(i, -\Lambda(\alpha), (\delta x)_{\alpha (\mathbf{x},\mathbf{p})/s}) (\delta \xi^i)^*_{(\mathbf{x},\mathbf{p})/s}$$

cancels whatever  $(\delta \xi)_{(\mathbf{x},\mathbf{p})/s}$ . That functional becomes via the Birkhoff theorem a functional  $\mathbf{J}(-\Lambda(\alpha), (\delta x)_{\alpha (\mathbf{x},\mathbf{p})}, (\delta \xi)_{(\mathbf{x},\mathbf{p})})$  of same value involving the vector fields  $(\delta x)_{\alpha (\mathbf{x},\mathbf{p})}$  and  $(\delta \xi)_{(\mathbf{x},\mathbf{p})}$  in the layer  $H \cong E$  associated with the vector functions  $(\delta x)_{\alpha (\mathbf{x},\mathbf{p})/s}$ , and  $(\delta \xi)_{(\mathbf{x},\mathbf{p})/s}$  along the trajectory  $\mathbf{T}$  via the ergodicity of the latter. Derivatives  $\frac{\partial \bullet}{\partial \mathbf{X}} \frac{\partial H(\mathbf{X}, \mathbf{P})}{\partial \mathbf{P}} - \frac{\partial \bullet}{\partial \mathbf{P}} \frac{\partial H(\mathbf{X}, \mathbf{P})}{\partial \mathbf{X}}$  replace of course in the new functional  $\mathbf{J}$  the derivatives  $\frac{\partial \bullet}{\partial s}$

present in  $\mathbf{I}$  and integration  $\int_{\text{layer } H \cong E} d_N \mathbf{X} d_N \mathbf{P}$  in  $\mathbf{J}$  replaces the integration  $\int_0^T ds$  in  $\mathbf{I}$ . The vector

field  $(\delta x)_{\alpha (\mathbf{x},\mathbf{p})}$  is specified by the principle that the functional  $\mathbf{J}(-\Lambda(\alpha), (\delta x)_{\alpha (\mathbf{x},\mathbf{p})}, (\delta \xi)_{(\mathbf{x},\mathbf{p})})$  cancels whatever the vector field  $(\delta \xi)_{(\mathbf{x},\mathbf{p})}$ . It is then obviously independent of the trajectory  $\mathbf{T}$ , and therefore unique around any hamiltonian trajectory. One may expect that its variation scales along the latter, are of the order of  $1/\Lambda(\alpha)$ , but related to the variation scales of the hamiltonian  $H(\mathbf{X}, \mathbf{P})$  in the transverse directions.

### III B – EIGENVALUES OF THE OPERATORS $C_T(\mathbf{x},\mathbf{p})_{[s]}$

We consider a closed hamiltonian trajectory  $\mathbf{T}$  of period  $T$  formed of points  $(\mathbf{X},\mathbf{P})_{[s]}$  hamiltonian images after the time  $s$  of a reference point  $(\mathbf{X},\mathbf{P}) = (\mathbf{X},\mathbf{P})_{[s=0]}$  (see figure 2a), and the linear operator  $C_\tau(\mathbf{x},\mathbf{p})_{[s]}$  transforming the small vectors  $(\delta x, \delta p)$  in the tangent space to the phase space at the point  $(\mathbf{X},\mathbf{P})_{[s]}$  into their hamiltonian image after the time  $\tau$  in the tangent space at the point  $(\mathbf{X},\mathbf{P})_{[s+\tau]}$ . We will justify the following statement : the eigenvalues of the operators  $C_\tau(\mathbf{x},\mathbf{p})_{[s]}$  obtained by taking  $\tau$  equal to the period  $T$ , transporting therefore  $(\delta x, \delta p)$  from the tangent space at  $(\mathbf{X},\mathbf{P})_{[s]}$  into the same tangent space at the point  $(\mathbf{X},\mathbf{P})_{[s+T]} = (\mathbf{X},\mathbf{P})_{[s]}$ , have their eigenvalues located either on the real axis or on the circle of radius 1. It results from the equation (III 9) that the operators  $C_T(\mathbf{x},\mathbf{p})_{[s]}$  for the various  $s$  have the same eigenvalues  $c$ . We just need accordingly to justify our statement for the operator  $C_T(\mathbf{x},\mathbf{p})$  starting from the reference point  $(\mathbf{X},\mathbf{P})$ . A possible way exploits that the symplectic constraint (III 9c), which applies if  $c$  is outside the real axis and the circle of radius 1, cannot be verified by an eigenvector of  $C_T(\mathbf{x},\mathbf{p})$ . We present here a narrower perspective, relying directly on the expression (III 5a,b) of the operators  $C_\tau(\mathbf{x},\mathbf{p})$  applicable for  $H(\mathbf{x},\mathbf{p}) = \mathcal{O}(\bar{x}) + \frac{\mathbf{p}^2}{2}$ . We will show that the eigenvalues  $c + \frac{1}{c}$  of the sum  $C_T(\mathbf{x},\mathbf{p}) + C_T'(\mathbf{x},\mathbf{p})$  of  $C_T(\mathbf{x},\mathbf{p})$  and of its inverse  $C_T'(\mathbf{x},\mathbf{p}) = (C_T(\mathbf{x},\mathbf{p}))^{-1}$  are real, which entails that  $\text{Im}(c + \frac{1}{c}) = \text{Im}(c) - \frac{\text{Im}(c)}{|c|^2}$  is null , i.e. that  $c$  is either real or of modulus 1.

We introduce, alongside the labels  $i, j, k, \dots$  running from 1 to  $N$ , the labels  $u, v, w, \dots$  running from 1 to the dimension  $2N$  of the phase space. We identify a vector  $(\delta x, \delta p) = (\delta x^i, \delta p_i)$  with a vector  $(\Xi) = (\Xi^u)$  such that  $\Xi^{u=i} = \delta x^i$  and  $\Xi^{u=i+N} = \delta p_i$ . A linear operator  $L$  transforming a vector  $(\delta x, \delta p) = (\Xi)$  into  $L(\delta x, \delta p) = (\delta x', \delta p') = L(\Xi) = (\Xi')$  is represented by a

$2N \times 2N$  matrix  $L^u_v$  such that  $\Xi^u = L^u_v \Xi^v$ . The matrix  $(LM)^u_v$  is equal to  $L^u_w M^w_v$ . We

introduce a small fraction  $\frac{T}{Q} = \varepsilon \rightarrow 0$  of the period  $T$ . We obtain via the formulae (III 5a,b)

$$C_T(\mathbf{x}, \mathbf{p}) = C_{Q\varepsilon}(\mathbf{x}, \mathbf{p}) = (I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/Q\varepsilon}) \dots (I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon}) \dots (I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/0\varepsilon}) \quad (\text{III } 10a)$$

with  $I^u_v = \delta(u, v)$ ,  $D_{(\mathbf{x}, \mathbf{p})/q\varepsilon} = \Delta + \mathcal{G}(q\varepsilon)$

$$\Delta^{u=i}_{v=j+N} = \delta(i, j) \text{ and } \Delta^u_v = 0 \text{ otherwise} \quad (\text{III } 10b)$$

$$(\mathcal{G}(s))^{u=i+N}_{v=j} = -\frac{\partial^2 \mathcal{V}(\mathbf{X}[s])}{\partial x^i \partial x^j} \text{ and } (\mathcal{G}(s))^u_v = 0 \text{ otherwise} \quad (\text{III } 10c)$$

The operator  $C_Q(\mathbf{x}, \mathbf{p})$  with  $Q\varepsilon \neq T$  is simulated by replacing  $Q$  by  $\underline{Q}$  in the expression (III 10a).

The inverse  $C'_T(\mathbf{x}, \mathbf{p}) = (C_T(\mathbf{x}, \mathbf{p}))^{-1}$  of  $C_T(\mathbf{x}, \mathbf{p})$  is simulated by

$$C'_T(\mathbf{x}, \mathbf{p}) = (I - \varepsilon D_{(\mathbf{x}, \mathbf{p})/0\varepsilon}) \dots (I - \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon}) \dots (I - \varepsilon D_{(\mathbf{x}, \mathbf{p})/Q\varepsilon}) \quad (\text{III } 10d)$$

The simulations (III 10a,d) basically require that the factors  $I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon}$  are very close to the identity  $I$ . One notes from the equations (III b,c) that the operators  $\Delta$  and  $\mathcal{G}(q\varepsilon)$  which compose  $D_{(\mathbf{x}, \mathbf{p})/q\varepsilon}$  verify  $\Delta\Delta = 0$ ,  $\mathcal{G}(q\varepsilon)\mathcal{G}(q\varepsilon) = 0$ , and that only the matrix elements

$$(\Delta\mathcal{G}(q\varepsilon))^{u=i}_{v=j} = (\mathcal{G}(q\varepsilon)\Delta)^{u=i+N}_{v=j+N} = -\frac{\partial^2 \mathcal{V}(\mathbf{X}[q\varepsilon])}{\partial x^i \partial x^j} \text{ are non-null. The eigenvalues of}$$

$I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon}$  then appear equal to  $1 +$  the square roots of the eigenvalues of the  $N \times N$  matrix  $\varepsilon^2$

$$\left(-\frac{\partial^2 \mathcal{V}(\mathbf{X}[q\varepsilon])}{\partial x^i \partial x^j}\right). \text{ The basic constraint } I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon} \cong I \text{ therefore means that the}$$

dimensionless numbers  $\varepsilon \left| \frac{\partial^2 \mathcal{V}(\mathbf{X}[q\varepsilon])}{\partial x^i \partial x^i} \right|^{1/2}$  must be  $\ll 1$ . On the other hand, it appears that the

numbers  $Q\varepsilon \left| \frac{\partial^2 \mathcal{V}(\mathbf{X}[q\varepsilon])}{\partial x^i \partial x^i} \right|^{1/2}$  reflect the numbers  $\Lambda(\alpha)T$ ,  $\Lambda(\omega)T$  which are physically  $\gg 1$ .

The matrix  $\left(-\frac{\partial^2 \mathcal{V}(\mathbf{X}[q\varepsilon])}{\partial x^i \partial x^j}\right)$  is self adjoint and the square roots of its eigenvalues are real or purely imaginary. Thus our statement holds for a single factor  $(I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/q\varepsilon})$ . It is then

possible that it holds automatically for operators  $C_{Q\varepsilon}(\mathbf{x}, \mathbf{p})$  containing a small number  $Q$  of factors, corresponding to numbers  $|Q| \varepsilon \left| \frac{\partial^2 \mathcal{O}(\mathbf{X} | q\varepsilon J)}{\partial x^i \partial x^i} \right|^{1/2}$  under a threshold value.

The operator  $\mathcal{G}(s)$  being an analytical function of  $s$  of period  $T$  may be simulated as

$$\mathcal{G}(s) = \sum_{r=0}^R (\underline{\mathcal{G}}(r)) \cos(\Omega(r)s + \phi(r)) \quad (\text{III 11a})$$

where for each label  $r$  the frequency  $\Omega(r)$  is a multiple of  $\frac{2\pi}{T}$  (with, e.g.,  $\Omega(0)=0$ ), and the phase  $\phi(r)$  is a priori arbitrary. In view of the equation (III 10c), the matrices  $(\underline{\mathcal{G}}(r))_v^u$  must verify

$$(\underline{\mathcal{G}}(r))^{u=i+N}_{v=j} = (S(r))(i,j) \text{ and } (\underline{\mathcal{G}}(r))_v^u = 0 \text{ otherwise} \quad (\text{III 11b})$$

We therefore simulate the operators  $C_T(\mathbf{x}, \mathbf{p})$  and  $C_T'(\mathbf{x}, \mathbf{p})$  by the product (III 10a,d), with

$$(I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/s})_v^u = \begin{pmatrix} (\delta(i,j)) & (\varepsilon \delta(i,j)) \\ \left( \varepsilon \sum_{r=1}^R S(r)(i,j) \cos(\Omega(r)s + \phi(r)) \right) & (\delta(i,j)) \end{pmatrix} \quad (\text{III 11c})$$

The  $N \times N$  matrices  $(S(r))(i,j)$  must reflect in structure and in magnitude the matrices

$-\frac{\partial^2 \mathcal{O}(\mathbf{X} | q\varepsilon J)}{\partial x^i \partial x^i}$ . This imposes that they are self adjoint

$$(S(r))(i,j) = (S(r))(j,i) = ((S(r))(i,j))^* \quad (\text{III 11d})$$

and satisfy the basic constraint

$$\varepsilon |S(r)(i,i)|^{1/2} \sim \varepsilon \left| \frac{\partial^2 \mathcal{O}(\mathbf{X} | q\varepsilon J)}{\partial x^i \partial x^i} \right|^{1/2} \ll 1 \quad (\text{III 11e})$$

The numbers  $Q\varepsilon |S(r)(i,i)|^{1/2}$  are physically  $\gg 1$ . It is possible that our statement holds automatically for  $C_{Q\varepsilon}(\mathbf{x}, \mathbf{p})$  if the numbers  $|Q| \varepsilon |S(r)(i,i)|^{1/2}$  are under a threshold value.

We may use for an evaluation of the eigenvalues of the operator  $C_T(\mathbf{x}, \mathbf{p})$  any location of the reference point  $(\mathbf{X}, \mathbf{P})$  on the trajectory  $\mathbf{T}$ . We choose a convenient location by exploiting that the differentiable layer  $H(x,p) = \mathcal{O}(\bar{x}) + \frac{p^2}{2} \cong E$  containing the cosmos,

symmetric with respect to the plane  $p_i=0$ , contains an element  $\mathbf{U}$  of that plane, and that the closed hamiltonian trajectory  $\mathbf{T}$  containing the points  $(\mathbf{X},\mathbf{P})/s$  is accompanied by a similar one  $\mathbf{T}'$  containing the points  $(\mathbf{X}',\mathbf{P}')/s$  such that  $\mathbf{X}'/s = \mathbf{X}/[-s]$ ,  $\mathbf{P}'/s = -\mathbf{P}/[-s]$ . Both  $\mathbf{T}$  and  $\mathbf{T}'$  run close to all the points of the thin layer  $H \cong E$  and in particular of the element  $\mathbf{U}$ . Without inconvenience for the construction of our suited representation of the phase space, we may assume that the trajectories  $\mathbf{T}$  at the basis of that construction are not only closed but also coincide with  $\mathbf{T}'$ , by containing a same point of  $\mathbf{U}$ . Taking that point as reference point  $(\mathbf{X},\mathbf{P}) = (\mathbf{X},\mathbf{P})/s=0$  on  $\mathbf{T}$ , it appears that the functions  $\mathbf{X}/s$ ,  $\mathbf{P}/s$  periodic of period  $T$  in  $s$  are also respectively even and odd in  $s$ . This choice of the reference point  $(\mathbf{X},\mathbf{P})$  entails accordingly cancelling phases  $\phi(r)$  in the formulae (III 11a,c). We will justify below our statement with such cancelling phases. However the statement will apply as well if  $(\mathbf{X},\mathbf{P})$  is changed into any other point  $(\mathbf{X},\mathbf{P})/h$  of  $\mathbf{T}$ , which introduces in the formulae (III 11a,c) phases  $\phi(r)$  equal to  $\Omega(r)h$ . The arbitrariness of the time  $h$  means a partial arbitrariness of these new phases  $\phi(r)$ . One may conjecture that our statement justified below with cancelling  $\phi(r)$  applies with arbitrary  $\phi(r)$ . A numerical test of that conjecture is attempted in fine.

We analyse the operator  $C_T(\mathbf{x},\mathbf{p})$  and its inverse  $C'_T(\mathbf{x},\mathbf{p})$ , by considering the components  $C_A$  of  $C_T(\mathbf{x},\mathbf{p})$  and  $C'_A$  of  $C'_T(\mathbf{x},\mathbf{p})$  which are of order  $A$  in  $\varepsilon$ , the integer  $A$  being of course between 0 and  $Q = \frac{T}{\varepsilon}$ . We introduce for each value of  $A$  the combinations of times

$\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}$  such that

$$0 \leq q_1\varepsilon < q_2\varepsilon < \dots < q_{A-1}\varepsilon < q_A\varepsilon \leq Q\varepsilon = T \quad (\text{III } 12)$$

It comes from the formulae (III 10a,d)

$$C_T(\mathbf{x},\mathbf{p}) = \sum_{A=0}^Q C_A, \quad C_A = \varepsilon^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} D_{(\mathbf{x},\mathbf{p})/q_A\varepsilon} D_{(\mathbf{x},\mathbf{p})/q_{A-1}\varepsilon} \dots D_{(\mathbf{x},\mathbf{p})/q_2\varepsilon} D_{(\mathbf{x},\mathbf{p})/q_1\varepsilon}$$

$$C'_T(\mathbf{x},\mathbf{p}) = \sum_{A=0}^Q C'_A, \quad C'_A = (-\varepsilon)^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} D_{(\mathbf{x},\mathbf{p})/q_1\varepsilon} D_{(\mathbf{x},\mathbf{p})/q_2\varepsilon} \dots D_{(\mathbf{x},\mathbf{p})/q_A\varepsilon}$$

The operator  $C_A$  is then a sum  $|C_A + ||C_A$  where, in view of the equations (III 10b,c), the operators  $|C_A$  and  $||C_A$  have the following structure :

– for uneven A

$$|C_A = \varepsilon^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} \Delta G(q_2\varepsilon) \Delta G(q_4\varepsilon) \dots \Delta G(q_{A-1}\varepsilon) \Delta,$$

$$\text{such that } (|C_A)_v^u \neq 0 \text{ only if } u = i \text{ and } v = j+N \quad (\text{III 13a})$$

$$\|C_A = \varepsilon^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} G(q_1\varepsilon) \Delta G(q_3\varepsilon) \Delta \dots \Delta G(q_A\varepsilon),$$

$$\text{such that } (\|C_A)_v^u \neq 0 \text{ only if } u = i+N \text{ and } v = j \quad (\text{III 13b})$$

– for even A

$$|C_A = \varepsilon^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} \Delta G(q_2\varepsilon) \Delta G(q_4\varepsilon) \dots \Delta G(q_A\varepsilon),$$

$$\text{such that } (|C_A)_v^u \neq 0 \text{ only if } u = i \text{ and } v = j \quad (\text{III 13c})$$

$$\|C_A = \varepsilon^A \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} G(q_1\varepsilon) \Delta G(q_3\varepsilon) \Delta \dots G(q_{A-1}\varepsilon) \Delta,$$

$$\text{such that } (\|C_A)_v^u \neq 0 \text{ only if } u = i+N \text{ and } v = j+ N \quad (\text{III 13d})$$

The operator  $C'_A$  is a sum  $|C'_A + \|C'_A$  where  $|C'_A$  and  $\|C'_A$  verify the formulae (III 13a,b,c,d) with the following changes: the factor  $\varepsilon^A$  is replaced by  $(-\varepsilon)^A$  and the times  $q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon$  in  $\Delta G(q_2\varepsilon) \dots$  or  $G(q_1\varepsilon) \Delta \dots$  are replaced by  $q_A\varepsilon, q_{A-1}\varepsilon, \dots, q_2\varepsilon, q_1\varepsilon$ .

Let us consider in greater detail the operators  $C_A = |C_A + \|C_A$  and  $C'_A = |C'_A + \|C'_A$  for uneven A. It comes from the formulae (III 13a) and (III 11a,b) with the phases  $\phi(r)$  taken null  $(|C_A)_{v=j+N}^{u=i} = |\Gamma_A)_j^i$ ,  $(|C_A)_v^u = 0$  otherwise

$$(|\Gamma_A)_j^i = \varepsilon^A \sum_{r_1=0, r_2=0, \dots, r_{(A-1)/2}=0}^R (S(r_1))(i,k)(S(r_2))(k,l) \dots (S(r_{(A-1)/2}))(l,j) \Pi(r_1, r_2, \dots, r_{(A-1)/2})$$

$$\Pi(r_1, r_2, \dots, r_{(A-1)/2}) = \sum_{\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}} \cos(\Omega(r_1)q_2\varepsilon) \cos(\Omega(r_2)q_4\varepsilon) \dots \cos(\Omega(r_{(A-1)/2})q_{A-1}\varepsilon)$$

The same formula applies to  $|C'_A$  with  $\varepsilon^A$  replaced by  $(-\varepsilon)^A$  and  $\Pi(r_1, r_2, \dots, r_{(A-1)/2})$  replaced by  $\Pi(r_{(A-1)/2}, \dots, r_2, r_1)$ . The key point of our demonstration is that a combination of times  $\{q_1\varepsilon, q_2\varepsilon, \dots, q_{A-1}\varepsilon, q_A\varepsilon\}$  verifying the conditions (III 12) may be replaced by the combination

$\{T-q_A\varepsilon, T-q_{A-1}\varepsilon, \dots, T-q_2\varepsilon, T-q_1\varepsilon\}$  which verifies the same conditions. Taking into account that the numbers  $\Omega(r_1)T, \Omega(r_2)T, \dots$  are multiple of  $2\pi$ , and the parity  $\cos(x)=\cos(-x)$ , that point entails that  $\Pi(r_1, r_2, \dots, r_{(A-1)/2})$  equals  $\Pi(r_{(A-1)/2}, \dots, r_2, r_1)$ . Of course that equation involving explicitly the periodicity of period  $T$  of the closed trajectory  $\mathbf{T}$  would not be systematically applicable if we were considering operators  $C_\tau(\mathbf{x}, \mathbf{p})$  with  $\tau \neq T$ . Since for uneven  $A$  one has  $(-\varepsilon)^A = -\varepsilon^A$ , it means that  $|C_A = -|C'_A$ . Similarly it comes via the formulae (III 13b) and (III 11a,b) that  $\|C_A = -\|C'_A$ . Finally it appears that the components  $C_A + C'_A$  of the operator  $C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p})$  of order  $A$  in  $\varepsilon$  are null for uneven  $A$ . That operator consists only of components  $C_A + C'_A$  of even order  $A$ . Using the formulae (III 13c,d), it comes

$$(C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p}))^{u-i}_{v=j} = (C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p}))^{u=i+N}_{v=j+N} = \sum_{\text{even } A} G_A(i, j),$$

$(C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p}))^u_v = 0$  otherwise, and

$$G_A(i, j) = \sum_{r_1=0, r_2=0, \dots, r_{A/2}=0}^R (S(r_1))(i, k)(S(r_2))(k, l) \dots (S(r_{A/2}))(l, j) P(r_1, r_2, \dots, r_{A/2})$$

where  $P(r_1, r_2, \dots, r_{A/2}) = P(r_{A/2}, \dots, r_2, r_1)$ . In view of the properties (III 11d) of  $S(r)(i, j)$  the matrix  $(C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p}))^u_v$  appears self adjoint. The eigenvalues of the operator  $C_T(\mathbf{x}, \mathbf{p}) + C'_T(\mathbf{x}, \mathbf{p})$  are accordingly real as demanded by our statement.

It is natural to try to recover our statement through a numerical computation of the eigenvalues of the operator  $C_T(\mathbf{x}, \mathbf{p})$  as it is simulated by the formulae (III 10a, III 11c). Such a computation via the PYTHON software appears accurate enough in the following situation : a dimension  $2N=200$  of the phase space, a number  $Q=1000$  of factors  $(I + \varepsilon D_{(\mathbf{x}, \mathbf{p})/Q\varepsilon})$  and

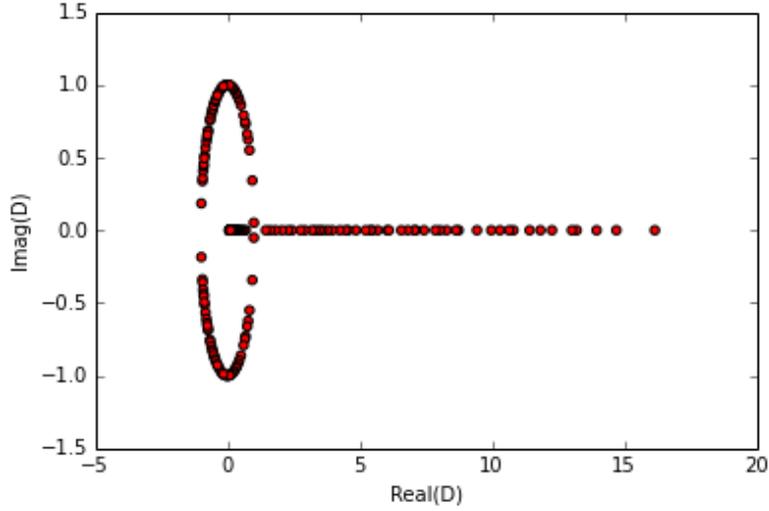


Figure 4.– Set of eigenvalues of the operator  $C_T(\mathbf{x},\mathbf{p})$  in the conditions indicated in the text. All these eigenvalues are real or on the circle of radius 1

a number  $R=10$  of frequencies  $\Omega(r)$  multiple of  $\frac{2\pi}{T}$ . Taking the period  $T$  as time unit, and

accordingly the frequencies  $\Omega(r)$  multiple of  $2\pi$ , the time  $\varepsilon = \frac{T}{Q}$  is equal to  $10^{-3}$ . The matrix

$(S(r))(i,j)$  is randomly chosen under the symmetries (III 11d), the limit (III 11e) imposing

$|S(r)(i,i)|^{1/2} \ll 1/\varepsilon = 1000$ . The results, illustrated by an example on the figure (4), confirm the

statement in those conditions. This occurs with cancelling and not cancelling phases  $\phi(r)$  as

well. However, it appears that the statement applies automatically to operators  $C_{Q\varepsilon}(\mathbf{x},\mathbf{p})$  if the

dimensionless numbers  $|Q|\varepsilon|S(r)(i,i)|^{1/2}$  are under a threshold of the order of 10. For instance it

ceases to apply to the operator  $C_{T/2}(\mathbf{x},\mathbf{p})$ , i.e. for  $Q=500$  and  $Q\varepsilon=0.5$ , while applying to

$C_T(\mathbf{x},\mathbf{p})$ , for values  $|S(r)(i,i)|^{1/2}$  beyond roughly 10, which still verify the limit  $|S(r)(i,i)|^{1/2} \ll$

1000.

### III C– REALISATION OF THE STRUCTURAL CONDITION

We finally examine whether the canonical representation  $A(x,p)$ ,  $B(x,p)$ ,  $Z^\alpha(x,p)$ , ... of the phase space in the domain  $\mathbf{D}$  around the hamiltonian trajectories  $\mathbf{T}$ , where the cosmos

displays the laws of motion (III 4), allows the realisation of the structural condition. The formulation of the latter given in § II B is based upon a small time scale  $\Theta$  and an integer  $q^\circ$  of the order of some units, which allow to build up spaces  $\tilde{\mathfrak{R}}_n$  from the space  $\mathfrak{R}_p$  via the equations (II 13) and (II 19a,b). It imposes that  $\mathfrak{R}_p$ ,  $\Theta$  and  $q^\circ$  are such that the spaces  $\tilde{\mathfrak{R}}_n$ , orthogonal to  $\mathfrak{R}_p$ , are mutually orthogonal, because of an increase of the wave numbers of the observables  $\tilde{V}_n(x,p) \in \tilde{\mathfrak{R}}_n$  in specific directions of the phase space, by a factor  $O(2)$  when  $n>0$  is increased by 1 and  $n<0$  by  $-1$ .

The space  $\mathfrak{R}_p$  is at the basis of the phenomenological determinism : it contains the phenomenological information that the observers can predict along the arrow of time, assimilated in our simplified cosmos to the projection  $\rho_{t p}(x,p)$  in the space  $\mathfrak{R}_p$  of the fundamental distribution function  $\rho_t(x,p)$ . Our first step is obviously to propose a definition of the observables  $V_P(x,p)$  forming  $\mathfrak{R}_p$  as functions of  $A(x,p)$ ,  $B(x,p)$ ,  $Z^\alpha(x,p)$ ,  $\dots$  for  $(x,p)$  in the domain  $\mathbf{D}$ . We find that the simplest possible observables  $V_P(x,p)$  are weakly dependent of  $Z^\alpha, Z_\alpha$ , when they are expressed as functions of  $A, B, Z^\alpha, \dots$ . Assuming, as on the figure 3, that the domain  $\mathbf{D}$  is located in  $Z^\alpha, Z_\alpha$  within the squares  $|Z^\alpha|<1, |Z_\alpha|<1$ , an observable  $V_P(x,p)$  is then a sum of terms proportional to  $\exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha))$ , the numbers  $(h_\alpha, h^\alpha)$  having a magnitude  $\ll 1$

$$V_P(x,p) = \sum_{(h_\alpha, h^\alpha) \in S_P} V_P^\diamond(A, B, Z^\alpha, Z_\alpha, h_\alpha, h^\alpha) \exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha)) \text{ for } (x,p) \in \mathbf{D}$$

$$\text{where } (h_\alpha, h^\alpha) \in S_P \Rightarrow |h_\alpha| \ll 1 \text{ and } |h^\alpha| \ll 1 \quad (\text{III 14})$$

The observables  $\tilde{V}_0(x,p) \in \tilde{\mathfrak{R}}_0$ , derived from the observables  $V_P \in \mathfrak{R}_p$ , via the equation (II 13), are the projection  $(\Omega V_P)_K = -\{V_P, H\}_K$  of the observables  $\Omega V_P = -\{V_P, H\}$  in the space  $\mathfrak{R}_K \perp \mathfrak{R}_p$ . One obtains the structure of the Poisson bracket  $\{V_P, H\}$  in the representation

$$A(x,p), B(x,p), Z^\alpha(x,p), \dots \text{ by writing } \{V_P, H\} = \frac{\partial V_P(A, B, Z^\alpha, \dots)}{\partial A} \{A, H\} + \dots \text{ and}$$

$$\text{deriving the expressions (III 4) of } A[t], \dots \text{ with respect to } t \text{ to obtain } \{A, H\} = \frac{d}{dt} A[t], \dots$$

for  $t = 0$ . One thus recovers for  $\{V_P, H\}$  the structure  $\sum_{(h_\alpha, h^\alpha)} \{V_P, H\}^\diamond(A, B, Z^\omega, Z_\omega, h_\alpha, h^\alpha)$   $\exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha))$ , where of course the numbers  $(h_\alpha, h^\alpha)$  may be different from the numbers  $(h_\alpha, h^\alpha) \in S_P$  applicable to the observables  $V_P$ . One then notes that the numbers  $(h_\alpha, h^\alpha)$  which apply to the projection  $\{V_P, H\}_K = \tilde{V}_0(x, p)$  of  $\{V_P, H\}$  into  $\mathfrak{R}_K \perp \mathfrak{R}_P$  have a magnitude  $< 1$  but well above the magnitude of  $(h_\alpha, h^\alpha) \in S_P$  so as not to belong to  $\mathfrak{R}_P$ . We accordingly write

$$\tilde{V}_0(x, p) = \sum_{(h_\alpha, h^\alpha) \in \tilde{S}_0} \tilde{V}_0^\diamond(A, B, Z^\omega, Z_\omega, h_\alpha, h^\alpha) \exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha)) \text{ for } (x, p) \in \mathbf{D} \quad (\text{III } 15)$$

$(h_\alpha, h^\alpha) \in \tilde{S}_0$  being  $< 1$ , well above  $(h_\alpha, h^\alpha) \in S_P$ .

The structural condition essentially requires for  $|n| \geq q^\circ$  that the spaces  $\tilde{\mathfrak{R}}_n$  are orthogonal, and that their orthogonalities reflect that the observables  $\tilde{V}_n(x, p) \in \tilde{\mathfrak{R}}_n$  display wave numbers along specific directions of the phase space which increase by a factor  $O(2)$  when  $|n|$  increases by 1. That increase by a factor  $O(2)$  is achieved if one has

$$\tilde{V}_n(x, p) = \sum_{(h_\alpha, h^\alpha) \in \tilde{S}_n} \tilde{V}_n^\diamond(A, B, Z^\omega, Z_\omega, h_\alpha, h^\alpha) \exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha)) \text{ for } (x, p) \in \mathbf{D} \quad (\text{III } 16)$$

and the numbers  $(h_\alpha, h^\alpha)$  in the set  $\tilde{S}_n$  for  $|n| \geq q^\circ$  verify

$$|h_\alpha| = O(2^{|n|}), \quad |h^\alpha| \ll |h_\alpha| \text{ for } n \geq q^\circ \quad (\text{III } 17a)$$

$$|h^\alpha| = O(2^{|n|}), \quad |h_\alpha| \ll |h^\alpha| \text{ for } n \leq -q^\circ \quad (\text{III } 17b)$$

The canonicity of the representation  $A(x, p)$ ,  $B(x, p)$ ,  $Z^\alpha(x, p)$ ,  $\dots$  within the domains  $\mathbf{D}$ , i.e. within the squares  $Z^\alpha < 1$ ,  $|Z_\alpha| < 1$ , then entails the orthogonalities of the spaces  $\tilde{\mathfrak{R}}_n$ . It indeed allows to replace the volume element  $d_N x d_N p$  by  $dA dB dZ^\alpha dZ_\alpha \text{Im}(dZ^\omega dZ_\omega^*)$ , so that two observables displaying numbers  $h_\alpha$  and  $h_\alpha'$  with differences  $h_\alpha - h_\alpha'$  or  $h^\alpha - h^{\alpha'}$  of magnitude beyond or of order 1 are orthogonal. The problem is to justify that the structure (III 17a,b) is a consequence of the equations of the motion of the cosmos (III 4) in the domain  $\mathbf{D}$ , and of the equations (II 19a) which applies for  $|n| \geq q^\circ$ . We use the facts that these equations impose  $\tilde{V}_{n \pm 1} = \exp(\pm \Omega \Theta) \tilde{V}_n$  and that the value of the observable  $\exp(\pm \Omega \Theta) \tilde{V}_n$  at a point  $(x, p)$  is the

value of the observable  $\tilde{V}_n$  at the hamiltonian image  $(x,p)/[\mu \Theta]$  of  $(x,p)$  after the time  $\mu \Theta$ .

Expressing those facts together with the equations (III 4), it comes

$$\begin{aligned} \tilde{V}_{n\pm 1}(x,p) &= \sum_{(h_\alpha, h^\alpha) \in \tilde{S}_{n\pm 1}} \tilde{V}_{n\pm 1}^\diamond(A, B, Z^\omega, Z_\omega, h_\alpha, h^\alpha) \exp(i(h_\alpha Z^\alpha + h^\alpha Z_\alpha)) = \\ & \sum_{(h_\alpha, h^\alpha) \in \tilde{S}_n} \tilde{V}_n^\diamond(A \mu \Theta, B, \exp(\pm i\Lambda(\omega)\Theta)Z^\omega, \exp(\mu i\Lambda(\omega)\Theta)Z_\omega, h_\alpha, h^\alpha) \\ & \exp(i(h_\alpha \exp(\pm \Lambda(\alpha)\Theta)Z^\alpha + h^\alpha \exp(\mu \Lambda(\alpha)\Theta)Z_\alpha)) \end{aligned} \quad (\text{III } 18)$$

The numbers  $h_\alpha$  and  $h^\alpha$  in the set  $\tilde{S}_{n\pm 1}$  applicable to the observables  $\tilde{V}_{n\pm 1}$  are therefore the numbers  $h_\alpha$  and  $h^\alpha$  in the set  $\tilde{S}_n$  applicable to the observables  $\tilde{V}_n$  multiplied by  $\exp(\pm \Lambda(\alpha)\Theta)$  and  $\exp(\mu \Lambda(\alpha)\Theta)$ , respectively. This justifies the structure (III 17a,b) if firstly the time scale

$\Theta$  is identified with  $\mathcal{O}\left(\frac{1}{\text{MAX}(\Lambda(\alpha))}\right)$  where  $\text{MAX}(\Lambda(\alpha))$  is the maximum Lyapounov

exponent in the cosmos. Secondly the structure (III 17a) and (III 17b) must be realized for  $n=q^\circ$  and  $n=-q^\circ$ . That realization is justified by the behaviour of the observables  $\tilde{V}_n \in \tilde{\mathfrak{R}}_n$  for

$|n| < q^\circ$  under the equations (II 19b) :  $\exp(\Omega\Theta) \tilde{V}_n = \tilde{V}_{n+1} + \delta V$  where  $\delta V \in \sum_{-q^\circ < q < q^\circ} \tilde{\mathfrak{R}}_q + \mathfrak{R}_p$ . One

may show that the latter, starting from  $\tilde{V}_0(x,p) \in \tilde{\mathfrak{R}}_0$  of the form (III 15), allow the orthogonalities  $\tilde{\mathfrak{R}}_n \perp \tilde{\mathfrak{R}}_n$  and  $\tilde{\mathfrak{R}}_n \perp \mathfrak{R}_p$ , for  $|n| < q^\circ$  without the requirement that the numbers  $h_\alpha$  or  $h^\alpha$  in the expression (III 16) of  $\tilde{V}_n(x,p)$  have a magnitude beyond or of order 1. On the other hand the equation (III 17) based on the strict relation  $\tilde{V}_{n\pm 1} = \exp(\pm \Omega\Theta) \tilde{V}_n$  is still approximately applicable. The magnitude of the numbers  $h_\alpha$  or  $h^\alpha$  in the structure (III 20a) for  $|n| < q^\circ$  therefore increases from their value  $< 1$  for  $n=0$ , up to a value above 1 when  $n$  reaches a level  $\pm q^\circ$  of the order of some units.

The phenomenological determinism essentially relies on the hyperbolic motion of the cosmos expressed by the coordinates  $Z^\alpha, Z_\alpha$ . That motion implies that the hamiltonian trajectories tend, during each time interval  $+\Theta$ , to contract any half square  $|Z^\alpha| \sim 1, |Z_\alpha| \sim \frac{1}{2}$  by a factor  $\exp(-\Lambda(\alpha)\Theta) \sim \frac{1}{2}$  in the direction  $Z^\alpha$  and to dilate it by a factor  $\exp(\Lambda(\alpha)\Theta) \sim 2$  in the direction  $Z_\alpha$ . That transformation of half squares  $|Z^\alpha| \sim 1, |Z_\alpha| \sim \frac{1}{2}$  into half squares  $|Z^\alpha| \sim \frac{1}{2}$ ,

$|Z_\alpha| \sim 1$  recalls and justifies the baker transformation, intuitively employed sometimes to simulate the complexification of the distribution function  $\rho_t(x,p)$  along the arrow of time (I. Prigogine and I. Stengers, 1992, H. H. Hasegawa and W. C. Saphir, 1992). However the baker transformation is a sequence of contraction/dilatation of squares along two directions  $Z'/Z''$  followed by a reset via a non-Hamiltonian, non-analytic displacement. That reset is not present in the transformation of the phase space via the hyperbolic motion in  $Z^\alpha, Z_\alpha$ .

#### IV – CONCLUSION

We have tried to better understand the arrow of time which permeates the cosmos through the existence and the functioning of a phenomenological determinism: the latter allows the observers to apply specific phenomenological laws to a specific fraction of the fundamental information expressing the reality at a time  $t$ , and thus accurately anticipate the same fraction at times  $t' > t$ , along the arrow of time. Unlike the fundamental determinism applicable to the fundamental information in both time directions, the phenomenological determinism is not applicable against the arrow of time. We have admitted two drastic simplifications : first the fundamental determinism is expressed by the Boltzmann–Gibbs framework, meaning that the fundamental information at time  $t$  is assimilated to a fundamental distribution function  $\rho_t(x,p)$  in a phase space  $(x,p)$ , governed by the Liouville equation ; secondly the principle of superposition applies to the evolution of  $\rho_t(x,p)$ . It then applies as well to the evolution of the phenomenological fraction of  $\rho_t(x,p)$ , which must be identified with the projection  $\rho_{t P}(x,p)$  of  $\rho_t(x,p)$  in a specific vector space  $\mathfrak{R}_p$  of observables. We have shown in § II that the existence of the phenomenological determinism  $\rho_t(x,p) \Rightarrow \rho_{t'}(x,p)$  for  $t' > t$  requires the combined realization of two conditions: an initial condition, which in its simplest form imposes that  $\rho_{t^*}(x,p)$  coincides with  $\rho_{t^* P}(x,p)$  at a primordial time  $t^*$  of the cosmos, and thereby imposes the arrow of time aimed from  $t^*$  to the observers times  $t > t^*$ ; a structural condition which in fine imposes the nature of the phenomenological space  $\mathfrak{R}_p$  : essentially, there must exist a small time scale  $\Theta$  such that the non-phenomenological component  $\rho_{t'' K}$  of  $\rho_{t''}(x,p)$  orthogonal to  $\mathfrak{R}_p$  created at a time  $t''$  between  $t^*$  and  $t$  by the phenomenological component  $\rho_{t'' P}(x,p)$ , displays after successive time intervals  $\Theta$  along the arrow of time following  $t''$  exponentially growing wavenumbers along specific directions of the space phase  $(x,p)$ . The greatest share of the non-phenomenological component  $\rho_{t K}$  of

$\rho_t(x,p)$  at time  $t$  is then thoroughly disconnected from the phenomenological component  $\rho_{t P}(x,p)$  and is thereby unable to influence the evolution of the latter. The phenomenological determinism then occurs if the time scale  $\Theta$  is smaller than the time scale  $\tau_{\text{phen}}$  of the evolution of the phenomenological distribution  $\rho_t(x,p)$ . A further consequence of the exponentially increasing complexity of the non-phenomenological components of  $\rho_t(x,p)$  is that most of the latter are not accessible to observers, who have accordingly only access in practice to the phenomenological information  $\rho_{t P}(x,p)$ .

The realization of the structural condition relies in its principle on the Lyapounov stochasticity of the hamiltonian trajectories. However, as explained in § III, a mere exponential divergence/convergence of the latter does not ensure that  $\rho_{t P}(x,p)$  is duly real, single valued and analytic in  $(x,p)$ . The realization of the structural condition is in fact only possible thanks to the existence of a canonical representation of the phase space in a narrow domain **D** around the trajectories, which highlights in that domain either a purely Anosov hyperbolic motion or a purely elliptic motion of the cosmos. The hyperbolic motion introduces via  $N'$  pairs of conjugate observables  $Z^\alpha(x,p)$ ,  $Z_\alpha(x,p)$  the stochasticity in the cosmos, expressed by  $N'$  pairs of non-null Lyapounov exponents  $\Lambda(\alpha) > 0, -\Lambda(\alpha) < 0$ . The elliptic motion is expressed by  $N''$  pairs of conjugate observables  $Z^\omega(x,p), Z_\omega(x,p) = (Z^\omega(x,p))^*$  oscillating at real frequencies  $\Lambda(\omega), -\Lambda(\omega)$ . It produces no stochasticity (and implies in fact the existence of  $N''$  pairs of exactly cancelling Lyapounov exponents). The canonical representation including  $Z^\alpha$ ,  $Z_\alpha$  and  $Z^\omega, Z_\omega$  has been schematically established in § III, but much remains to be done to understand the structure of the observables  $Z^\alpha(x,p), \dots$  in terms of the degrees of liberty  $x$ ,  $p$ , and of the attractive and repulsive effects introduced by the hamiltonian  $H(x,p)$ .

The canonical representation of the phase space in the domain **D** around the hamiltonian trajectories allow to specify the projections  $\rho_{t P}(x,p)$  of  $\rho_t(x,p)$ , which enters a phenomenological determinisms. The simplest  $\rho_{t P}(x,p)$  is the component of  $\rho_t(x,p)$  depending slowly on  $Z^\alpha(x,p)$  and  $Z_\alpha(x,p)$ . The phenomenological information  $\rho_{t P}(x,p)$  may retain a complex structure accessible to observers and submitted to the phenomenological determinism in particular through its dependence on  $Z^\omega(x,p)$  and  $Z_\omega(x,p)$ . A key point is that the non-phenomenological information expressed by the component  $\rho_{t K}(x,p)$  orthogonal to

the space  $\mathfrak{R}_p$  appears as an observable depending at a given time  $t$  slowly on  $Z_\alpha$  but more and more rapidly on  $Z^\alpha$  when  $t$  increases : namely  $\rho_{t\kappa}(x,p)$  appears as a sum of terms depending on  $Z^\alpha$  and  $Z_\alpha$  as  $\text{FONC}(Z_\alpha(x,p))\exp(ih_\alpha Z^\alpha(x,p))$ , the numbers  $h_\alpha$  being multiplied by  $\exp(\Lambda(\alpha)\Theta)$  during each time interval  $\Theta$ . This realizes the deconnection of  $\rho_{t\kappa}(x,p)$  from  $\rho_t(x,p)$  required by the structural condition, with  $\Theta$  taken of the order of the inverse of the Lyapounov exponents  $\Lambda(\alpha) > 0$ , and therefore  $\exp(\Lambda(\alpha)\Theta)$  of the order of 2.

The observable  $\rho_t(x,p)$  depending slowly on  $Z_\alpha(x,p)$  corresponds to a quantum operator  $\rho_{t\text{P}}^\circ$  represented by a quasi-diagonal matrix  $\langle v | \rho_{t\text{P}}^\circ | v' \rangle$  in the orthonormal basis of quantum eigenstates  $\langle v |$  of the quantum operator  $Z_\alpha^\circ$ . The Wigner formalism associates to an observable  $\mathcal{X}(x,p) = \text{FONC}(Z_\alpha(x,p))\exp(ih_\alpha Z^\alpha(x,p))$  a quantum operator  $\mathcal{X}^\circ$  represented by a matrix  $\langle v | \mathcal{X}^\circ | v' \rangle$  involving differences  $v' - v$  proportional to the numbers  $h_\alpha$ . The exponential increase of these numbers  $h_\alpha$  responsible for the deconnection of the observable  $\rho_{t\kappa}(x,p)$  from  $\rho_{t\text{P}}(x,p)$  entails an exponential increase of the differences  $v - v'$  producing a similar deconnection of the quantum operator  $\rho_{t\kappa}^\circ$  from  $\rho_{t\text{P}}^\circ$ . It actually appears that the Wigner formalism, when it is applicable, allows a direct transposition of the phenomenological determinism from the Boltzmann-Gibbs framework to the quantum framework. However the full extension of the phenomenological determinism considered in this article is of course problematic. This is also the case when one drops the simplification that the principle of superposition applies to the evolution of  $\rho_t(x,p)$  and therefore to the phenomenological fraction of  $\rho_t(x,p)$ . The latter is no longer a simple projection of  $\rho_t(x,p)$  in a vector space  $\mathfrak{R}_p$  enjoying a linear autonomous equation of evolution along the arrow of time. One must take into account the factorizations incompatible with the principle of superposition which reflect, in the true cosmos, in particular the existence of the closed systems and of the Boltzmann Gibbs distributions evoked in § I satisfying a minimum negentropy principle. The difficulty is that the initial and structural conditions at the basis of the simplified phenomenological determinism must explain at the same time the factorizations.

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