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Mostofa Ameli, Jean Patrick Lebacque, Ludovic Leclercq

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1 **User equilibrium vs. System optimum: an analysis**  
2 **based on demand level breakpoints**

3 **Mostafa Ameli \***

4 University of Paris-Est, IFSTTAR, GRETTIA  
5 Univ. Lyon, IFSTTAR, ENTPE, LICIT  
6 14-20 Boulevard Newton, 77420 Champs-sur-Marne, France  
7 Tel: +33 1 81 66 86 84  
8 Email: [mostafa.ameli@ifsttar.fr](mailto:mostafa.ameli@ifsttar.fr)

9 **Jean-Patrick Lebacque**

10 University of Paris-Est, IFSTTAR, GRETTIA  
11 14-20 Boulevard Newton, 77420 Champs-sur-Marne, France  
12 Tel: +33 1 81 66 86 95  
13 Email: [jean-patrick.lebacque@ifsttar.fr](mailto:jean-patrick.lebacque@ifsttar.fr)

14 **Ludovic Leclercq**

15 Univ. Lyon, IFSTTAR, ENTPE, LICIT  
16 3 Rue Maurice Audin, 69518 Vaulx-en-Velin cedex, France  
17 Tel: +33 4 72 04 77 16  
18 Email: [ludovic.leclercq@ifsttar.fr](mailto:ludovic.leclercq@ifsttar.fr)

19 \* Corresponding Author

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1 **ABSTRACT**

2 The purpose of this study is to compare optimal network loadings related to network equilibria. The  
3 direct comparison of path flows or trajectory patterns is hard to achieve so here we propose a more  
4 aggregate approach based on the comparison of demand level breakpoints. A breakpoint occurs at  
5 a particular demand level when after this level, the pattern of the path flow loading is changed or  
6 the set of active alternatives with at least one assigned user is modified. In the following context,  
7 the structure of the demand is given, but the level of demand increases. We focus on discrete  
8 demand formulation and choices and use a trip-based simulator.

9 This study analyzes the breakpoints for the solution of three popular equilibrium condi-  
10 tions: User equilibrium (UE), System optimum (SO) and Boundary Rational User Equilibrium  
11 (BRUE). First, we investigate breakpoints on a well-known network (Braess) in the static case in  
12 order to better define this concept. Second, breakpoints are investigated on a real network (Lyon,  
13 France) and travel times are calculated by microscopic traffic simulator. Our ultimate goal is to  
14 find, at each level of demand, the critical set of users that can be made to change paths in order  
15 to move the system from demander side (UE and BRUE) to supplier side (SO). The network con-  
16 trollers can guide users to change path, or they can manage the price of paths or can ban some  
17 paths. The numerical results show that our approach can help identify such sets of users.

18 *Keywords:* Trip-based traffic assignment, user equilibrium, system optimum, boundary rational  
19 user equilibrium, variational demand, switch equilibrium

## 1 INTRODUCTION

2 Traffic assignment problem has been studied for more than five decades and a large variety of  
3 analytical and simulation-based models and/or algorithms have been developed to find network  
4 equilibria. Cost can be expressed as a function of time or money or a combination of both. When  
5 all users realize the minimum possible path cost according to the network constraints and other  
6 user path choice, the solution is called User Equilibrium (UE). Transportation engineers aim to  
7 minimize the sum of all user costs. The solution, in this case, is called System Optimum (SO)  
8 (Patriksson, 1). Path costs can be estimated based on models or simulators. Traffic assignment can  
9 be classified into three main groups: Static Traffic Assignment (STA), semi-dynamic and Dynamic  
10 Traffic Assignment (DTA) (Bliemer et al., 2). Since the 1950s there is much research about finding  
11 the assignment solutions for UE and SO. In this paper, based on the review of Szeto and Wong  
12 (3) and following Mahmassani (4), a simulation based approach is preferred in this paper, which is  
13 well-understood, the results of which are easy to interpret and which are relatively close to reality.

14 In the real network, if users know all the information about the network, the network works  
15 in UE condition. If all users follow the command aiming to optimize the system, the network  
16 is loaded by SO path assignment pattern. In practice, both cases are not realistic. The "optimal  
17 command" situation does not exist as we are not able to define the optimal command in practice.  
18 Further users they never fully comply to system guidances even in the hypothetic case where  
19 optimal command could be defined. In uncontrolled networks, users are not perfectly rational even  
20 if perfectly informed, so UE is not necessarily achieved.

21 The main point is that all users try to minimize their own cost. Therefore, the network  
22 moves towards UE (Ehrgott et al., 5). We consider the travel time as the cost function. By looking  
23 at the user behavior, we can find other equilibria which exist in the network. The boundary rational  
24 model is one behavioral model used to relax the perfect rational hypothesis in the definition of UE.  
25 Boundary Rational User Equilibrium (BRUE) can be considered as a relaxation of UE where users  
26 are satisfied with a solution that is close to the optimum. Equilibrium is achieved when all user  
27 costs are within the boundary around the UE solution (Di and Liu, Han et al., 6, 7). In fact, in the  
28 real world, we would have a mixed equilibrium because some users push the network towards UE,  
29 some users push it towards BRUE, and possibly the system controllers will want to meet the SO  
30 conditions and other conditions (Yang et al., 8).

31 In this paper, we consider UE, SO and BRUE independently and try to investigate the  
32 relation between these three equilibria for different demand levels. To the best of our knowledge  
33 the studies in the literature focus mainly on finding the path flow distributions over the global  
34 network related to each equilibrium. Here, we would like to go further and cross-compare the trip  
35 patterns. This can help in the future to design new traffic management methods based on the idea  
36 of incenting users to change paths, so that the network gets closer to SO. The questions are which  
37 users should be switched from one route to another and which routes should be promoted by the  
38 route guidances. We will not directly answer these questions but we aim to move one step in this  
39 direction by studying demand level breakpoints. A breakpoint is characterized by the fact that  
40 one element of the optimal assignment pattern (including path set and path flow structure) changes  
41 when the demand level increases from less to more than the breakpoint level.

42 Generally, the travel demand is not fixed even in the short term. There are few studies that  
43 focus on the impact of different level of demand on the UE (see e.g. Wie et al., Szeto and Lo, Han  
44 et al., Han et al., 9, 10, 11, 12). First we define breakpoints and analyze them. Then, for each  
45 level of demand, and given a fixed active paths set, we try to analyze how we can determine the

1 critical users for rerouting. Critical users are the users who, when they change their path, have  
 2 more impact than others on moving the equilibrium towards SO conditions. There are two main  
 3 points that we aim at investigating:

- 4 • The impact of one origin-destination demand level on the network equilibria.
- 5 • Critical users who can have maximum impact on transforming UE and BRUE to SO by  
 6 change their path.

7 The next section "Breakpoint definition" presents the definition of the breakpoint and solves  
 8 a simple static traffic model to define the breakpoints in the Braess network. In the section "Break-  
 9 points calculation in the dynamic case", the trip-based simulator is presented first. Second, we  
 10 explain the dynamic equilibrium model and how we calculate the breakpoints for DTA. The net-  
 11 work for numerical experiments and the simulation based optimization process are presented in  
 12 the section "Numerical experiment". The obtained results are discussed in section "Numerical  
 13 results". Finally, we state concluding remarks and introduce future directions of work in the Con-  
 14 clusion section.

## 15 **BREAKPOINT DEFINITION**

16 The assignment pattern determines how many users take each possible path from origins to desti-  
 17 nations. There is a path set which contains all the active paths from origin to destination. The path  
 18 which is chosen by at least one user is an active path. The optimal path set is a component of the  
 19 optimal assignment pattern defined for each equilibrium: UE, SO and BRUE. For a given origin  
 20 and destination pair  $w$ , it is of common knowledge that the active path set for a given equilibrium  
 21 can be the same for different levels of demand. We define the breakpoints as demand levels where  
 22 we observe a modification of the active path set (e.g. one new path in or/and one current path out).  
 23 We will first study the breakpoints in the static case on the classical Braess network (Braess et al.,  
 24 13), because the Braess paradox is induced by different breakpoint values between UE and SO.  
 25 This will help to introduce the concept in a more intuitive way and to show why breakpoints are  
 26 key factors to define management strategies able to get the system closer to SO.

27 The classic Braess network with linear cost functions in the static case is shown in figure 1.  
 28 There are five links with cost functions  $t_{ij}$  for the link  $ij$  connecting node  $i$  to node  $j$ . The flow of  
 29 the link  $ij$  is  $f_{ij}$  and there are 3 alternative paths from origin node 1 to destination node 4: path  
 30 1 (1-3-4), path 2 (1-2-4) and path 3 (1-3-2-4). Therefore the cost functions ( $C_k$ ) of paths  $k$  are as  
 31 follow:

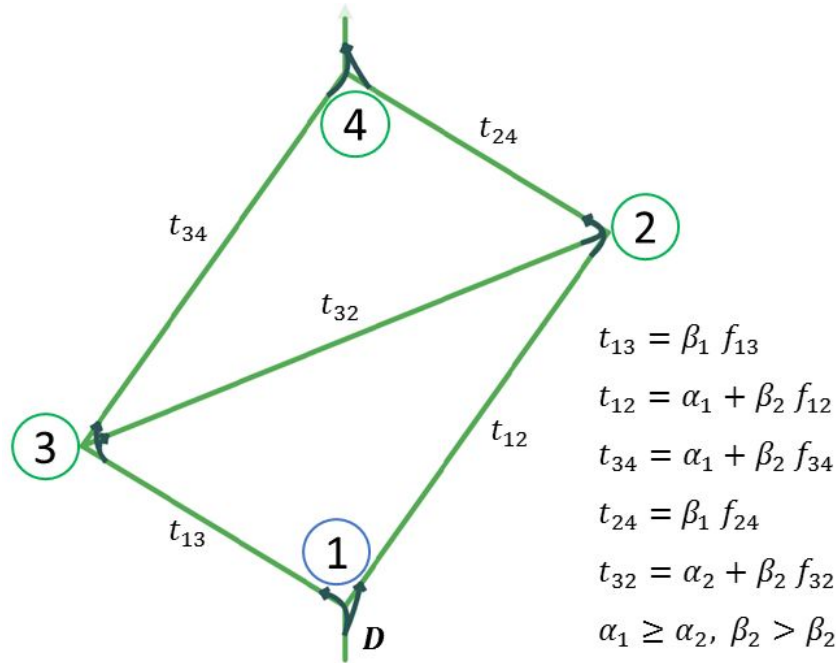
$$C_1(\pi_1) = t_{13} + t_{34} \quad (1)$$

$$C_2(\pi_2) = t_{12} + t_{24} \quad (2)$$

$$C_3(\pi_3) = t_{13} + t_{32} + t_{24} \quad (3)$$

34 Where  $\pi_k$  is the flow of path  $k$ .

35 The two assumptions in figure 1 ( $\alpha_1 \geq \alpha_2$ ,  $\beta_1 > \beta_2$ ) guarantee that by increasing the de-  
 36 mand, there exist three cases of UE (see e.g. Pas and Principio, 14). They also ensure that the  
 37 path 3 is the cheapest free flow path, due to lower coefficients ( $\alpha_2$ ,  $\beta_2$ ) in the path cost function.  
 38 The different demand levels will be defined in relation with the structure of the equilibrium (active



**FIGURE 1 Classic Braess network with the link cost functions and assumptions**

1 paths). Consequently, for UE we will have three scenarios. According to the definition of the UE,  
 2 the conditions of static UE can be stated mathematically as follows:

$$C_p - C_w^* \geq 0 \quad ; \forall p \in P_w \quad (4)$$

3

$$\pi_p (C_p - C_w^*) = 0 \quad ; \forall p \in P_w \quad (5)$$

4

$$\pi_p \geq 0 \quad ; \forall p \in P_w \quad (6)$$

5 Where  $C_w^*$  is the minimum travel cost for origin-destination pair  $w$  and  $P_w$  is the set of possible  
 6 paths for  $w$ . In these conditions, there is an assumption that each traveler is perfectly rational and  
 7 chooses a path that minimizes his travel general cost.

8 The assignment pattern is a  $[1 \times 3]$  vector  $\pi$  that contains the flow value for three paths.  
 9 Note that the feasible path flow vector, which is part of the assignment pattern in this context, is to  
 10 assign the traffic demand onto the possible path for  $w$ :  $\Pi \triangleq \{\pi : \pi \geq 0, \sum_{p \in P_w} \pi_p = D_w\}$ .

11 The assignment pattern for UE is a function  $UE(D)$  of demand level  $D$  and it is defined as follow:

$$UE(D) = \begin{cases} [0, 0, D] & 0 \leq D < \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}, \\ \left[ \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{2(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)D}{\beta_1 + 3\beta_2} \right] & \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} \leq D < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}, \\ \left[ \frac{D}{2}, \frac{D}{2}, 0 \right] & D \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} \end{cases} \quad (7)$$

12 Consequently, there are two breakpoints for UE:  $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$  and  $\frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$ .

13 There are two main differences in the definition of breakpoints between the BRUE and UE:

- The assignment pattern for BRUE not only depends on the demand level, it also depends on the  $\varepsilon$  value of the indifference band for this origin and destination pair.
- The assignment pattern is not unique in some scenarios, so for the Braess network there are two convex sets (simplices) of flow vectors  $\phi_1$  and  $\phi_2$  which can satisfy the conditions of BRUE:

$$\phi_1 = \{\pi \geq 0 : C_i(\pi) - C_j(\pi) \leq \varepsilon; i, j \in \{1, 2\}, i \neq j\} \quad (8)$$

$$\phi_2 = \{\pi \geq 0 : C_i(\pi) - C_j(\pi) \leq \varepsilon; i, j \in \{1, 2, 3\}, i \neq j\} \quad (9)$$

In equation 9,  $\phi_2$  can be a set that all three paths are active. If the indifference band is equal to zero ( $\varepsilon = 0$ ) the BRUE and UE assignment are the same. Due to the non-uniqueness of the BRUE, for a given  $\varepsilon$ , an active path has a cost lower or equal to the minimum travel cost path plus  $\varepsilon$ . The  $\varepsilon$ -BRUE path flow pattern is defined as:

$$\pi_p > 0 \rightarrow C_p - C_w^* \leq \varepsilon \quad ; \forall p \in P_w \quad (10)$$

The assignment pattern for BRUE is a function of demand level and band value  $BRUE(D, \varepsilon)$  is:

$$BRUE(D, \varepsilon) = \begin{cases} [0, 0, D] & 0 \leq D < \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} - \frac{\varepsilon}{\beta_1 + \beta_2}, \\ \phi_1 \cup \phi_2 & D > \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}, \\ \phi_2 & \text{o.w.} \end{cases} \quad (11)$$

For a detailed calculation of equation 11 readers can refer to [Di et al. \(15\)](#). Therefore, there are two breakpoints but they exist based on the  $\varepsilon$  value:  $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} - \frac{\varepsilon}{\beta_1 + \beta_2}$  and  $\frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}$ .

The third considered equilibria is SO. The SO assignment pattern is based on second principle of [Wardrop \(16\)](#), for minimizing the total travel cost. Therefore, mathematically the goal is to minimize the total travel cost as a function of demand and flow vector:

$$\min T(D, \pi) = \sum_{p \in P_w} \pi_p C_p \quad (12)$$

According to coefficients assumptions for the Braess network (figure 1) and the three scenarios that can happen like the UE case the objective function is:

$$T(D) = \begin{cases} (2\beta_1 + \beta_2)D^2 + \alpha_2 D^2 & \text{if only path 3 used,} \\ (\beta_1 + \beta_2)(\pi_1^2 + \pi_2^2) + \alpha_1 D & \text{if path 1 and 2 used,} \\ (\beta_1 + 2\beta_2)(\pi_1^2 + \pi_2^2) + [\alpha_1 - \alpha_2 - 2(\beta_1 + \beta_2)D](\pi_1 + \pi_2) & \text{if all paths used} \\ + 2\beta_2 \pi_1 \pi_2 + (2\beta_1 + \beta_1)D^2 + \alpha_2 D & \end{cases} \quad (13)$$

Therefore and mathematically, the assignment pattern for SO is defined as follow:

$$SO(D) = \begin{cases} [0, 0, D] & 0 \leq D < \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}, \\ \left[ \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)D}{\beta_1 + 3\beta_2} \right] & \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)} \leq D < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}, \\ \left[ \frac{D}{2}, \frac{D}{2}, 0 \right] & D \geq \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \end{cases} \quad (14)$$

1 There are two breakpoints for SO:  $\frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}$  and  $\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$ .  
 2 As aforementioned, existing studies usually finish the breakpoints analysis here (finding the as-  
 3 signment pattern for equilibria). We are now going to analyze the equilibria based on breakpoints.

#### 4 **Analysis based on breakpoints**

5 The flow diagram for UE, SO and BRUE is shown in figure 2 to present the optimal path set of two  
 6 different types of paths in the Braess network. It is not easy to present the BRUE in flow diagram  
 7 because the  $\varepsilon$  allowance for cost implies that for each path there is a set of possible flow values  
 8 in the BRUE assignment pattern around the UE flow value. Thus in figures 2(a) and 2(b), we  
 9 represent the BRUE solution in terms of the maximum range of flows at equilibrium. Figure 2(a)  
 10 presents the demand-flow diagram for path 1 and 2 in UE, SO and BRUE situations. Breakpoints  
 11 are shown on the demand axis. (a $\leftrightarrow$ b) presents the active path set for UE and SO and also possible  
 12 active path set for BRUE depending on the value of  $\varepsilon$ . The same demand-flow diagram for path 3  
 13 is shown in figure 2(b). Note that the breakpoints of BRUE,  $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} - \frac{\varepsilon}{\beta_1 + \beta_2}$  and  $\frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}$ ,  
 14 are equal to UE breakpoints when  $\varepsilon = 0$ . Figure 2 can help to design control strategies to push  
 15 the network toward SO. First, we discuss the UE-SO breakpoints. Then we explain the BRUE-  
 16 SO breakpoints based on  $\varepsilon$ . Finally, the mathematical formulas for assignment patterns for the  
 17 3 equilibria (UE, SO and BRUE) are shown in table 1. This table is obtained by merging the  
 18 equations 7, 11 and 14 to jointly investigate the breakpoints when demand is increasing.

19 The first breakpoint occurs when  $D = \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}$  for analyzing the UE solution and SO. From  
 20 the first breakpoint to the second one the flow is distributed differently for UE and SO. For UE  
 21 all users are still taking the third path but for SO some of them start to take the paths 1 and 2 to  
 22 minimize the total travel time. It means that for demand level higher than the first breakpoint, Path  
 23 1 and 2 will be added to the active path set in SO solution. Therefore, the network controller can  
 24 promote certain number of users to travel by path 1 or 2. These users are critical for an efficient  
 25 strategy that should guide them to take paths 1 and 2 when the demand level is  $\frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)} \leq D <$   
 26  $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$ .

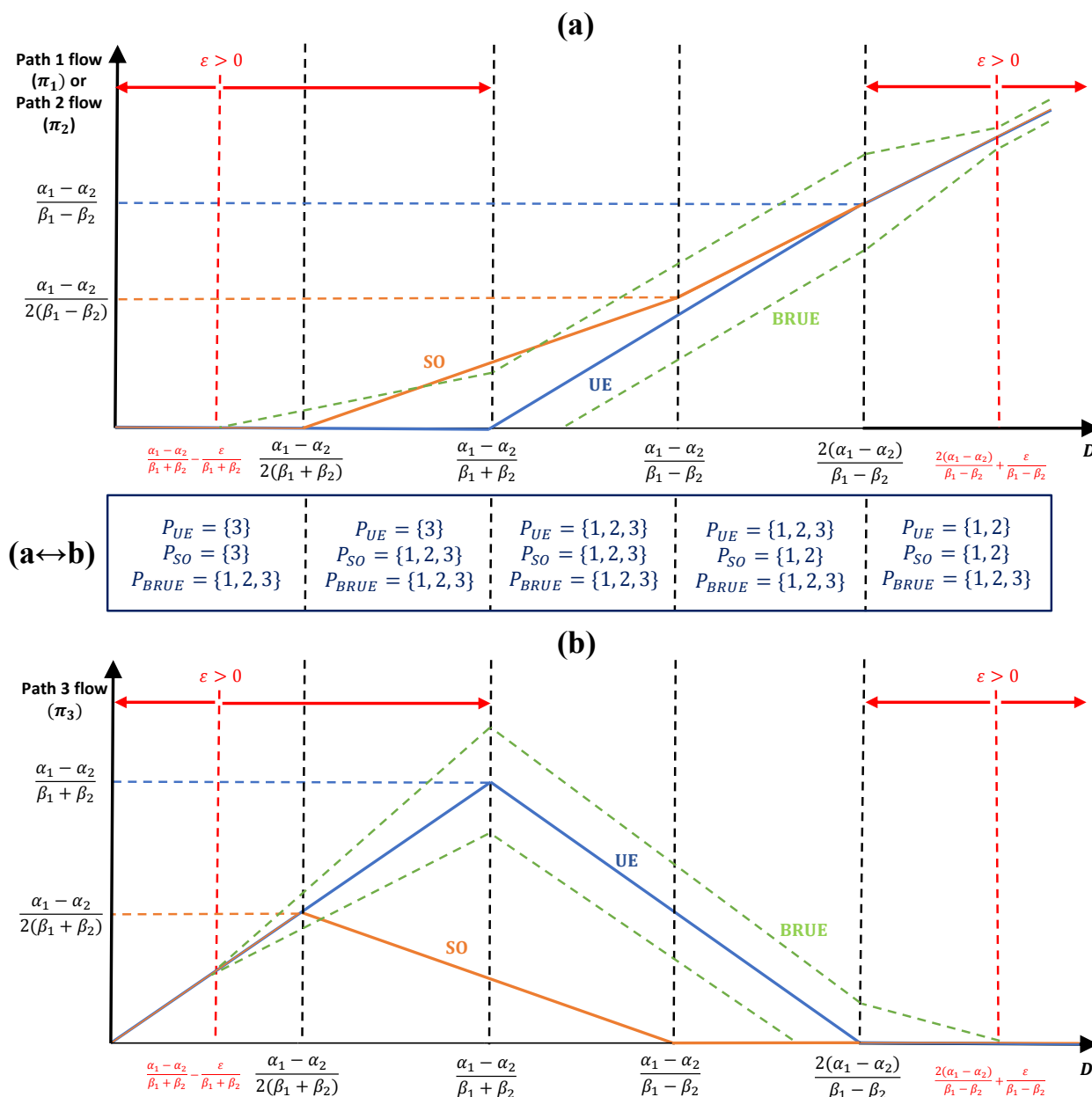
27 The active path set is the same for UE and SO when the demand level is  $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} \leq D < \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$ .  
 28 So the system controllers only need to promote the critical users. The interval between the third and  
 29 fourth breakpoints has different active path sets for equilibria (UE) and (SO). Path 3 is not in the  
 30 active path set for SO when  $\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \leq D < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$ . In this demand interval the system controller  
 31 can ban path 3 or design a control strategy to prevent users to travel on path 3 (e.g. increasing the  
 32 path toll with pricing strategies). In Braess network, after the last breakpoint,  $D \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$ , the  
 33 UE and SO assignment pattern will be the same.

34 In general, the BRUE assignment pattern is close to UE but the solution is not symmetric  
 35 with respect to paths 1 and 2 as in SO and UE. In BRUE-SO analysis,  $D = \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}$  is the first  
 36 breakpoint like UE-SO only if  $\varepsilon \leq \frac{\alpha_1 - \alpha_2}{2}$ . Because if  $\varepsilon > \frac{\alpha_1 - \alpha_2}{2}$ , the first breakpoint is located in  
 37 the lower demand level of the first SO breakpoint. Path 1 and(or) path 2 are added to the path set  
 38 after the first breakpoint. For higher demand level, by taking to account  $\varepsilon$ , the control strategies  
 39 is same as UE-SO breakpoints until the last breakpoint, when path 3 exits the active path set. If  
 40  $D \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$ , the BRUE and SO assignment pattern will be the same only if  $\varepsilon = 0$ . It means, for  
 41  $\varepsilon > 0$ , the last breakpoint will be  $\frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}$ . All paths are active for BRUE and path 3 is



1 not active for SO while  $\frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} \leq D < \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}$ . Consequently, the assignment pattern is  
 2 the same for BRUE and SO when  $D \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2} + \frac{\varepsilon}{\beta_1 - \beta_2}$ . Note that BRUE accepts 3 non zero path-  
 3 flows for values of  $D < \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} - \frac{\varepsilon}{\beta_1 + \beta_2}$  when  $\varepsilon > \alpha_1 - \alpha_2$ . Moreover, if  $\varepsilon \leq \frac{\alpha_1 - \alpha_2}{2}$  and  $D < \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}$ ,  
 4 the solution is the same for all 3 equilibria.

5         In urban networks, transportation system controllers can guide (e.g. by Advanced Traveler  
 6 Information System (ATIS)) the limited number of users that are equipped. Generally, it will be  
 7 useful to know how many users need to change their path to push the network from one equilibrium  
 8 in the user side (UE or BRUE) to one equilibrium in the system side (SO). By the development of  
 9 the controlling facilities in the network, this idea can improve the system objectives. This part of  
 10 the study has attempted to explain the assignment pattern breakpoints concept and to define them  
 11 in a simple manner using the static assignment on the Braess network. In the following parts of  
 12 the paper, we present the dynamic trip-based framework to find the breakpoints and identify the  
 13 critical user(s) for rerouting and critical path(s) for banning.



**FIGURE 2 (a): Path 1 or Path 2 flow-demand diagram for UE, SO and BRUE. (a↔b): Possible path set of optimal solution [ $P_x$  = Possible path set of optimal solution for  $x$ ]. Note that for BRUE, it depends on the  $\varepsilon$ . (b): Path 3 flow-demand diagram for UE, SO and BRUE. Red dash lines in figures (a) and (b) presents the value of breakpoints in BRUE assignment pattern based on the given  $\varepsilon$ . It can change in ranges that are specified by the red arrows.**

**TABLE 1 Assignment pattern in Braess network for UE, SO and BRUE**

| Demand level (D)   | User Equilibrium (UE)  | System optimum (SO)   | Band Value ( $\epsilon$ )  | Boundary rational UE            |
|--|--|---|--|---------------------------------|
| $0 \leq D \leq \frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)}$   | $[0, 0, D]$  | $[0, 0, D]$   | $\epsilon < \alpha_1 - \alpha_2 - (\beta_1 + \beta_2)D$  | $[0, 0, D]$                     |
|  |  |   | $\epsilon > \alpha_1 - \alpha_2 - (\beta_1 + \beta_2)D$  | $\varphi_2$                     |
| $\frac{\alpha_1 - \alpha_2}{2(\beta_1 + \beta_2)} \leq D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2}$ | $[0, 0, D]$  | $[\frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)D}{\beta_1 + 3\beta_2}]$ | $\epsilon < \alpha_1 - \alpha_2 - (\beta_1 + \beta_2)D$  | $[0, 0, D]$                     |
|  |  |   | $\epsilon > \alpha_1 - \alpha_2 - (\beta_1 + \beta_2)D$  | $\varphi_2$                     |
| $\frac{\alpha_1 - \alpha_2}{\beta_1 + \beta_2} \leq D \leq \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}$    | $[\frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{2(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)D}{\beta_1 + 3\beta_2}]$ | $[\frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{-\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)D}{\beta_1 + 3\beta_2}, \frac{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)D}{\beta_1 + 3\beta_2}]$ | All values   | $\varphi_2$                     |
|  |  |   | $\frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2} \leq D \leq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$ | $[\frac{D}{2}, \frac{D}{2}, 0]$ |
| $D \geq \frac{2(\alpha_1 - \alpha_2)}{\beta_1 - \beta_2}$  | $[\frac{D}{2}, \frac{D}{2}, 0]$  | $[\frac{D}{2}, \frac{D}{2}, 0]$   | $\epsilon < -2(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)D$  | $\varphi_1 \cup \varphi_2$      |
|  |  |   | $\epsilon > -2(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)D$  | $\varphi_2$                     |

## 1 BREAKPOINTS CALCULATION IN THE DYNAMIC CASE

2 In this work, we use Symuvia as a trip-based simulator for calculating the needed variables in the  
 3 network. Symuvia has been developed by the LICIT laboratory in IFSTTAR. It is a microscopic  
 4 simulator based on the Lagrangian resolution of the LWR model (Leclercq et al. (17)). It has a  
 5 simulation time-step equal to 1 second and obtains the information at the link level every 1 minute.  
 6 [Note that Symuvia is an open source simulator that will be available starting winter 2018.]

7 Consider a network  $G(N, A)$  with a finite set of nodes  $N$  and a finite set of directed links  
 8  $A$ . The demand is time-dependent. The period of interest (planning horizon) of duration  $H$  is  
 9 discretized into a set of small time intervals indexed by  $\tau$  ( $\tau \in T = \{\tau_0, \tau_0 + \sigma, \tau_0 + 2\sigma, \dots, \tau_0 +$   
 10  $M\sigma\}$  and  $\tau_0 + M\sigma = H$ ).  $\sigma$  is the duration of the time intervals. In an interval  $\tau$ , travel time and  
 11 traffic conditions do not change. The important notations to introduce the dynamic equilibrium  
 12 model are as follows:

13  $W$ : OD pairs, subset of origin  $\times$  destination nodes,  $W \subset N \times N$ .

14  $a$ : index of link,  $a \in A$ .

15  $w$ : index of origin and destination pair,  $w \in W$ .

16  $P_{w, \tau}$ : set of paths for  $w$  in departure time  $\tau$ .

17  $P_{w, \tau}^*$ : set of shortest paths for  $w$  in departure time interval  $\tau$ .

18  $p$ : index of path,  $p \in P_{w, \tau}$ .

19  $p^*$ : index of shortest path,  $p^* \in P_{w, \tau}^*$ .

20  $D_w$ : total demand for  $w$  pair.

21  $C_{p, \tau}$ : travel cost of path  $p$  in departure time  $\tau$ .

22  $C_{w, \tau}^*$ : minimum travel cost of OD pair  $w$  in departure time  $\tau$ .

23  $\hat{C}_{p, \tau}$ : marginal travel cost of path  $p$  in departure time  $\tau$ .

24  $\hat{C}_{w, \tau}^*$ : minimum marginal travel cost of OD  $w$  in departure time  $\tau$ .

25 According to the definition of the time interval in DTA and based on the study of (Sbayti et al.,  
 26 18), the conditions of dynamic UE can be mathematically restated from equations 4, 5 and 6:

$$C_{p, \tau} - C_{w, \tau}^* \geq 0 \quad ; \forall w \in W, p \in P_{w, \tau}, \tau \in T \quad (15)$$

27

$$\pi_{p, \tau} (C_{p, \tau} - C_{w, \tau}^*) = 0 \quad ; \forall w \in W, p \in P_{w, \tau}, \tau \in T \quad (16)$$

28

$$\pi_{p, \tau} \geq 0 \quad ; \forall p \in P_{w, \tau}, \tau \in T \quad (17)$$

29 If we violate the assumption that each traveler is perfectly rational, we can rewrite the equation 10  
 30 for BRUE:

$$\pi_{p, \tau} > 0 \rightarrow C_{p, \tau} - C_{w, \tau}^* \leq \varepsilon \quad ; \forall p \in P_{w, \tau} \quad (18)$$

31 The SO conditions in the dynamic case mathematically stated in equations 19, 20 and 21. These  
 32 conditions are presented based on marginal travel time and they state that if the path flow is positive,  
 33 then the experienced path marginal travel time should be equal to the minimum path marginal travel  
 34 times.

$$\hat{C}_{p, \tau} - \hat{C}_{w, \tau}^* \geq 0 \quad ; \forall w \in W, p \in P_{w, \tau}, \tau \in T \quad (19)$$

35

$$\pi_{p, \tau} (\hat{C}_{p, \tau} - \hat{C}_{w, \tau}^*) = 0 \quad ; \forall w \in W, p \in P_{w, \tau}, \tau \in T \quad (20)$$

$$\pi_{p,\tau} \geq 0 \quad ; \forall p \in P_{w,\tau}, \tau \in T \quad (21)$$

Computing the path marginal travel time analytically is very costly. In [Leclercq et al. \(19\)](#) it has been shown that using the simulation-based approach to compute the path marginals is also very costly, even in a simple grid network. Therefore, we use the simulator to compute a surrogate model for the marginal travel time. The simulator updates the marginal of a link every time a vehicle is exiting the link by calculating the marginal variation since the last exit the link. Finally, link marginal are averaged using the 1 minute window. Consequently, the path marginal travel time can be obtained by:

$$\hat{C}_{p,\tau} = \sum_{a \in A_p} \left[ \frac{\sum_{t=\tau}^{\tau+\sigma} \hat{C}_{a,t}}{\sigma} \right] \quad (22)$$

Where  $\hat{C}_{a,\tau}$  is the marginal travel time of link  $a$  at second  $t$ ,  $\sigma$  is the length of each time interval and  $A_p$  is the subset of link set  $A$  which defines path  $p$ . In fact, the surrogate function by [Peeta and Mahmassani \(20\)](#) consider the sum of links marginal travel time as the actual function. In the SO problem, we are looking for minimizing equation 22 for all users.

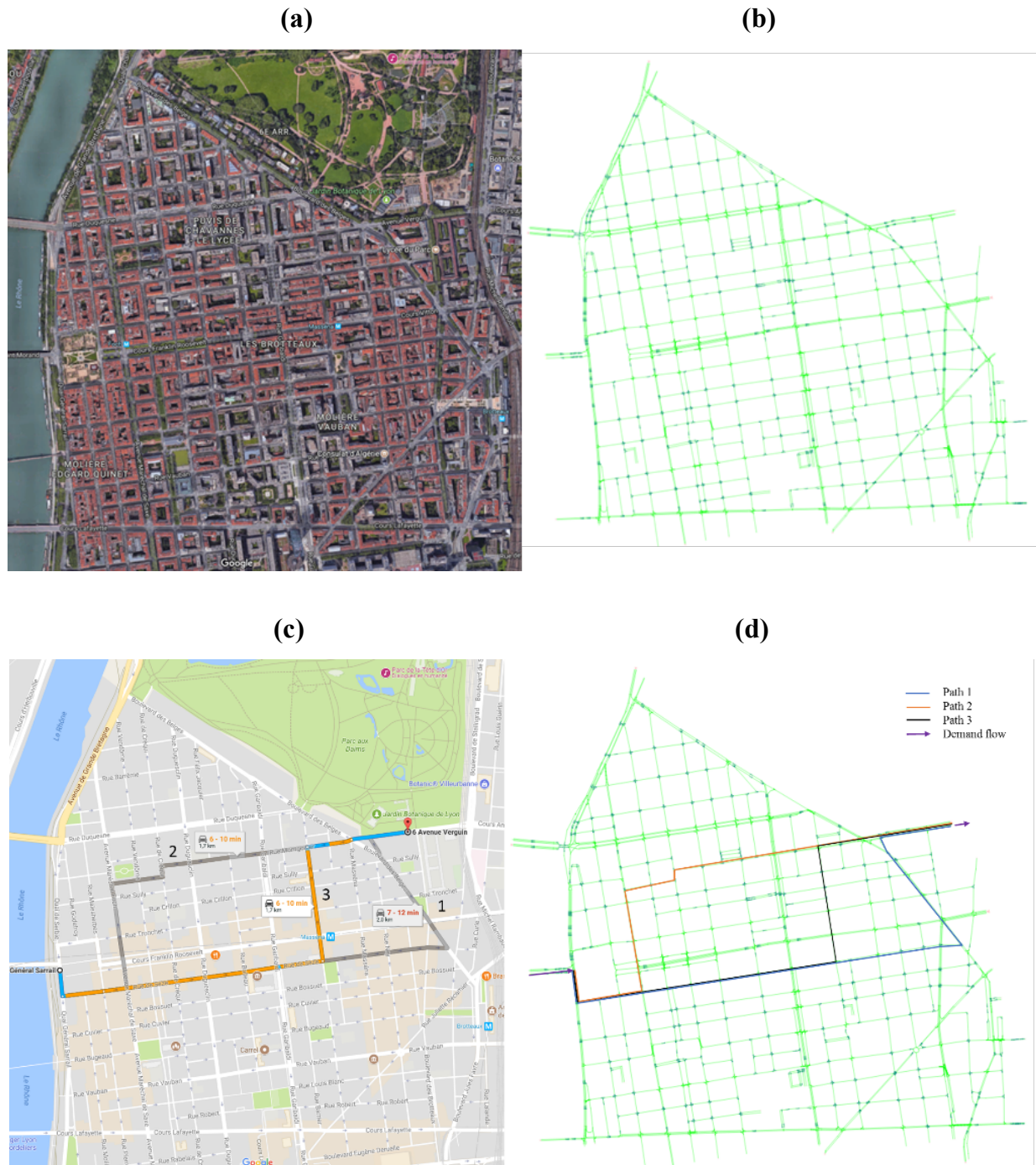
## NUMERICAL EXPERIMENTS

In this section, we could consider the Braess network with a flow-based dynamic model and draw the flow demand diagram with breakpoints as a function of the demand level. But the goal of this paper is to consider the trip-based DTA on an urban network and to find the breakpoints for the three equilibrium conditions as a function of the demand level. As mentioned earlier, we use a sub-area of the Lyon full regional network. It is a network of the Lyon 6 district (figures 3(a) and 3(b)). We are looking for the breakpoints of different demand level and a specific OD pair that correspond to travel from the west (Quai de Serbie) to the east of the network (6 Avenue Verguin) when the network is loaded by the travelers of other ODs to represent the peak half an hour of the network based on study of [Krug et al. \(21\)](#). We consider various levels of demand. The simulation and optimization are carried out for each level of demand with a 30 minutes horizon. The demand pattern has been set up to adjust the regular level observed in this area during the peak hour. We select the 3 most likely routes from the origin to destination to define the set of paths candidates. They are shown in figures 3(c) and 3(d). The network of Lyon 6 has:

- 430 nodes
- 786 links
- 26 origins and 24 destinations
- 6405 trips for ODs without the demand of Quai de Serbie to 6 Avenue Verguin

We search for the three equilibria in the complete network. It means that the optimization process is executed for all ODs and users in the network and we try to analyze the breakpoints by increasing the demand level on one specific OD. A simulation-based DTA is used to find the UE, SO and BRUE at every level of the demand on the three predefined paths. The departure time for the test users is spread uniformly in the 30 minutes duration of the horizon. Moreover, the users with other ODs travel with fixed departure time. According to the scale of the network, for calculating

- 1 the BRUE assignment, we consider  $\varepsilon = 0.1C_{p,\tau}$ . It means users will be satisfied if they perceive
- 2 maximum ten percent more than shortest path cost.



**FIGURE 3** (a) Network of Lyon 6: Satellite view of Lyon 6, France (Mapping data ©Google 2017) (b) Lyon 6 network consider by this study (c) Origin, destination and paths (Mapping data ©Google 2017) (d) The paths on the network of this study.

## 1 Optimization process

2 The Method of Successive Average (MSA) has been presented for the first time by Robbins and  
 3 Monro (22). In this study, we use the modified MSA in simulation-based DTA. Consider  $i$  as an  
 4 index of iteration in the optimization process and  $\pi_{p,\tau}^i$  is the number of users on path  $p$  in departure  
 5 interval  $\tau$  and iteration  $i$  of the optimization. In iteration  $i$ , MSA swaps a fraction  $\frac{1}{i+1}$  of users on  
 6 each path with higher cost value than the lowest cost path, to the lowest cost path(s). The direction  
 7 of the optimization process is defined by an all-or-nothing assignment vector ( $y_{p,\tau}^i$ ) for each OD at  
 8 departure interval  $\tau$ . With this step size, path assignments  $\pi_{p,\tau}^i$  at iteration  $i$  are updated in order to  
 9 obtain the path assignments  $\pi_{p,\tau}^{i+1}$  for iteration  $(i+1)$  as follows:

$$\pi_{p,\tau}^{i+1} = \frac{i}{i+1} \pi_{p,\tau}^i + \frac{1}{i+1} y_{p,\tau}^i \quad ; \forall p \in P_{w,\tau} \quad (23)$$

10 The main drawback of MSA is that it swaps users from all non-lowest cost paths without consid-  
 11 ering the gap between path costs. In fact, there is no priority for expensive paths to swap more  
 12 users. In this study, we use MSA ranking by (18) to remove this drawback for trip-based DTA. The  
 13 idea of MSA ranking is first, rank the users by the experienced travel time then swap a maximum  
 14 number of users based on MSA method. Maximum of number of swaps  $NS^i$  is observed when we  
 15 have an empty lowest cost path for swapping. It means:

$$NS^i = \frac{1}{i+1} D_w \quad (24)$$

16 This method swaps the user from most expensive paths so it accelerates convergence and provides  
 17 the best quality solution for the trip-based problem. With a large number of users traveling between  
 18 many ODs and with many paths per OD, ranking the users is very costly and has no justification  
 19 in a large-scale problem. But here with the sub-area network it is efficient to find the UE, SO and  
 20 BRUE with this ranking method.

## 21 NUMERICAL RESULTS

22 As mentioned earlier, we focus on travels from one specific OD pair. We solve the problem with  
 23 different demand levels. Demand is increased from one user to 1400 users over the simulation time  
 24 horizon = 30 min. The departure time is fixed for all users who do not travel between this OD pair.  
 25 The distribution of departure time for this OD is uniform. The optimization process has been done  
 26 for all three equilibria and the full network. All experiments for UE and SO are reproducible and  
 27 lead to a unique solution but for BRUE the solution depends on the first network initialization and  
 28 the value of  $\varepsilon$ . In each experiment, each equilibrium is measured and the breakpoint occurs when  
 29 new active path(s) appear or when the assignment pattern is changed. Changes in slope (derivative  
 30 of path flow with respect of demand) are also indicators of breakpoints. These changes in slope  
 31 are connected to changes in active path sets. The results of breakpoint detection for UE and SO  
 32 are presented in table 2. Also, we present the BRUE assignment plan of breakpoints with all-or-  
 33 nothing initialization and  $\varepsilon = 0.1C_{p,\tau}$  in this figure. The solution space is not continuous because  
 34 of the trip-based simulation (flows have integer values). Therefore, for each experiment, we have  
 35 three integer numbers as the flow on each path in the vector of assignment.

36 To represent and analyze the breakpoints in continuous space we draw the flow diagram by  
 37 the integer data with linear regression method with  $R^2 < 0.95$  and the breakpoints are the boundary

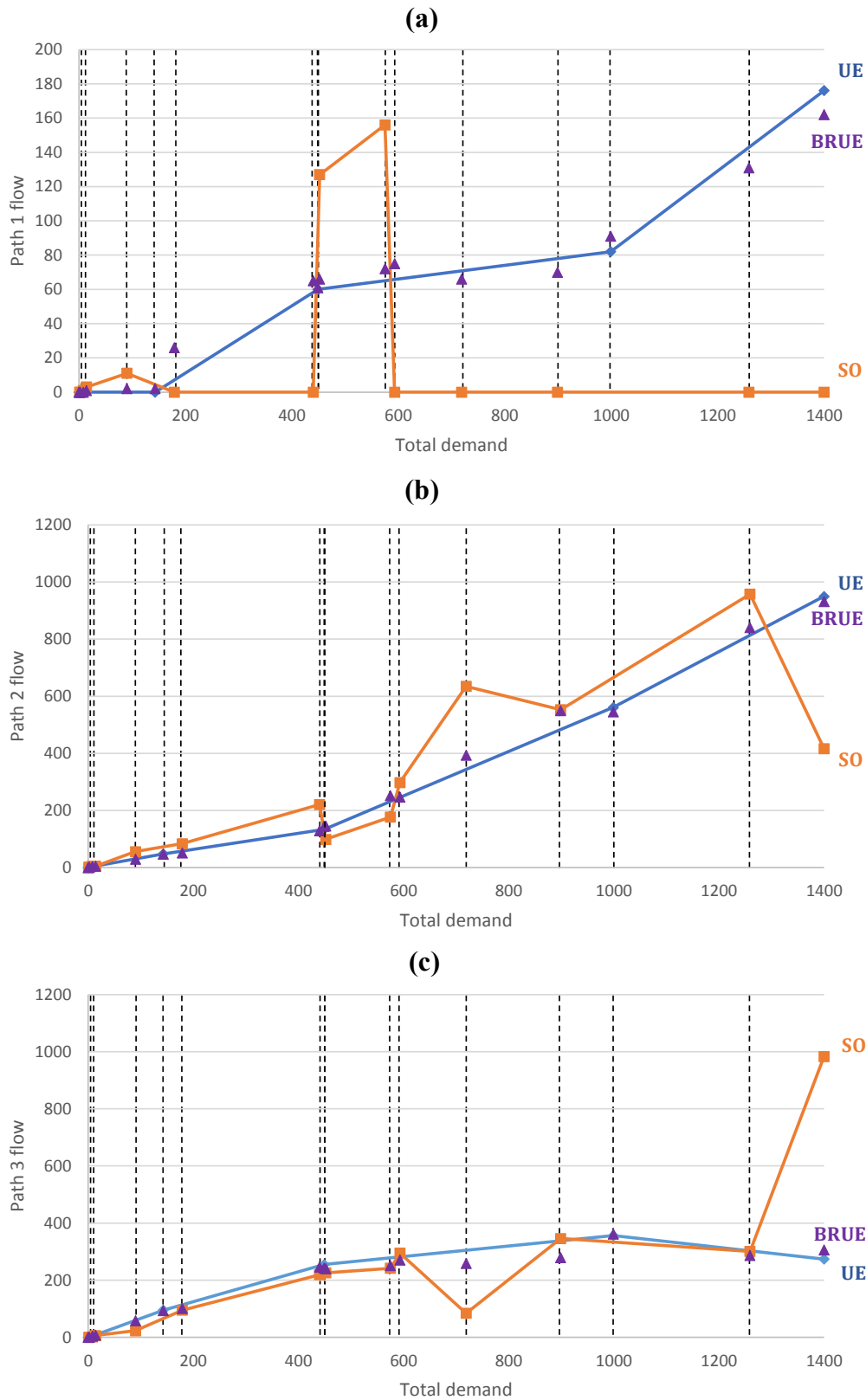
1 of the regression lines. The demand-flow diagram for each path is presented in figure 4. The idea  
2 of moving from one equilibrium to another is applied between 2 breakpoints when the assignment  
3 patterns are different. When both equilibria have the same path set but different flows, we need  
4 to design a strategy to reroute the critical users. The number of critical users, in this case, is the  
5 difference between the two flow values of the paths. We present the swap matrix to determine the  
6 number of users needed to swap from one path to another. It is a  $n \times n$  matrix where  $n$  is the number  
7 of the paths (in this case,  $n = 3$ ). Element  $ij$  of this matrix demonstrates how many users should  
8 swap from path  $i$  to path  $j$  in order to move the system from UE to SO. This matrix is presented  
9 in table 2 for these experiments. For instance, in this network, the pattern of each equilibrium is  
10 constant between two breakpoints. The traffic management system can induce a change of path of  
11 the critical users. Also, when the difference in flow is high for two equilibria, then promoting users  
12 to use one specific path may also be efficient.

13 On the other hand, if the path set is not the same for the two equilibria (paths used in UE  
14 which are not used in SO), the system can ban some paths (unused in SO) through routing advises.  
15 In this experiments, when the demand level exceeds 593, the SO flow for the first path is zero  
16 (figure 4(a)) so the system can ban some links in order to prevent users to take the first path to  
17 reach the SO.



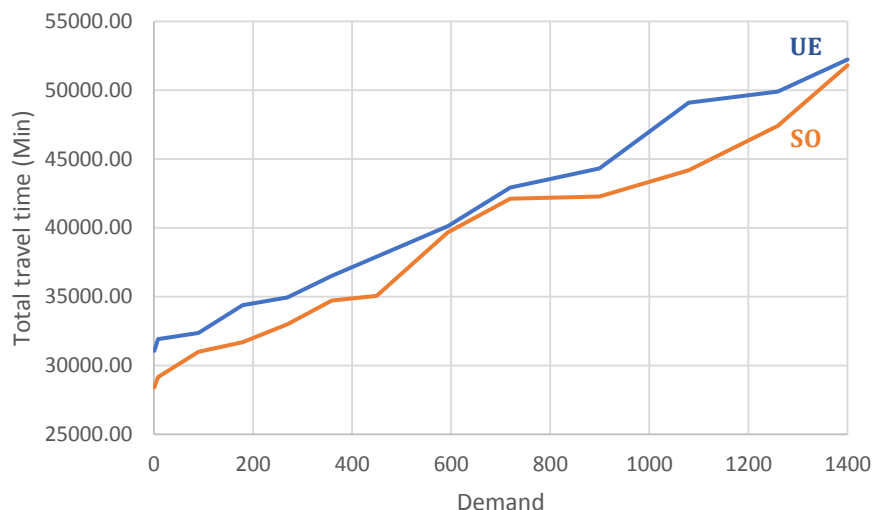
**TABLE 2 Assignment pattern of three paths for UE, SO on Lyon 6 traffic network**

| Demand level( $D$ ) | User Equilibrium (UE) | System Optimum (SO) | Critical user swap matrix   |
|---------------------|-----------------------|---------------------|---|
| 1                   | [0, 1, 0]             | [0, 1, 0]           | –   |
| 8                   | [0, 5, 3]             | [1, 5, 2]           | $\begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 1 & 0 & - \end{bmatrix}$     |
| 14                  | [0, 5, 9]             | [3, 5, 6]           | $\begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 3 & 0 & - \end{bmatrix}$     |
| 90                  | [0, 30, 60]           | [11, 56, 23]        | $\begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 11 & 26 & - \end{bmatrix}$   |
| 143                 | [0, 48, 95]           | [4, 63, 76]         | $\begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 4 & 15 & - \end{bmatrix}$    |
| 179                 | [24, 53, 102]         | [0, 84, 95]         | $\begin{bmatrix} - & 24 & 0 \\ 0 & - & 0 \\ 0 & 7 & - \end{bmatrix}$    |
| 440                 | [59, 131, 250]        | [0, 221, 219]       | $\begin{bmatrix} - & 59 & 0 \\ 0 & - & 0 \\ 0 & 31 & - \end{bmatrix}$   |
| 449                 | [60, 134, 255]        | [84, 134, 231]      | $\begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 24 & 0 & - \end{bmatrix}$    |
| 452                 | [60, 137, 255]        | [127, 99, 226]      | $\begin{bmatrix} - & 0 & 0 \\ 38 & - & 0 \\ 29 & 0 & - \end{bmatrix}$   |
| 575                 | [66, 253, 256]        | [156, 177, 242]     | $\begin{bmatrix} - & 0 & 0 \\ 76 & - & 0 \\ 14 & 0 & - \end{bmatrix}$   |
| 593                 | [67, 256, 270]        | [0, 298, 295]       | $\begin{bmatrix} - & 42 & 25 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix}$   |
| 719                 | [73, 396, 250]        | [0, 635, 84]        | $\begin{bmatrix} - & 73 & 0 \\ 0 & - & 0 \\ 0 & 166 & - \end{bmatrix}$  |
| 899                 | [83, 565, 251]        | [0, 553, 346]       | $\begin{bmatrix} - & 0 & 83 \\ 0 & - & 12 \\ 0 & 0 & - \end{bmatrix}$   |
| 999                 | [82, 561, 356]        | [0, 665, 334]       | $\begin{bmatrix} - & 82 & 0 \\ 0 & - & 0 \\ 0 & 22 & - \end{bmatrix}$   |
| 1259                | [146, 813, 300]       | [0, 958, 301]       | $\begin{bmatrix} - & 145 & 1 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix}$   |
| 1400                | [176, 950, 274]       | [0, 417, 983]       | $\begin{bmatrix} - & 0 & 176 \\ 0 & - & 533 \\ 0 & 0 & - \end{bmatrix}$ |



**FIGURE 4 (a) Path 1 flow-demand diagram for UE and SO. (b) Path 2 flow-demand diagram for UE and SO. (c) Path 3 flow-demand diagram for UE and SO. Breakpoints are presented by black vertical dash lines on total demand axis.**

- 1 Figure 5 presents the total travel time of the breakpoints in UE and SO. We present it here in order  
 2 to show that optimizing the surrogate model (equation 22) instead of the actual value of the path  
 3 marginal yields a better total travel time than UE and BRUE.



**FIGURE 5 Total travel time for each level of demand for UE and SO**

#### 4 CONCLUSION

5 This paper considers static and dynamic traffic assignment in order to study the impact of different  
 6 demand levels on three equilibria (User Equilibrium, System Optimum and Boundary Rational  
 7 User Equilibrium). It attempts to find the breakpoints and to investigate the possibility to move  
 8 from one equilibrium to another. At any level of demand we define a critical user set as a sub-set  
 9 of all users. The critical user has more impact than others on the assignment solution. In fact, the  
 10 system controller can design a strategy to reroute the critical users in order to move the network  
 11 from a UE or BRUE situation towards a SO flow pattern. Also at some level of demand when  
 12 the active path sets are different for UE/BRUE and SO, we can ban/promote some paths in order  
 13 to achieve the SO. This paper proposes a novel approach to analyze network DTA equilibrium  
 14 as a function of demand level. The potential implications of this approach for network suppliers  
 15 concern the analysis of the network status and the design of rerouting strategies (e.g. with the  
 16 communication devices) in order to move from an initial UE or BRUE situation towards SO.

17 The numerical experiments were conducted on a static classic Braess network and on a  
 18 dynamic real sub network (Lyon 6) to examine the equilibrium patterns at different demand levels.  
 19 In the dynamic network, we consider the full network equilibrium and analyze the pattern when  
 20 the demand of one origin-destination pair is changed. Note that the experiments of this study  
 21 have been carried out with homogeneous users. The approach in this paper can be carried over to  
 22 heterogeneous users and can consider the profile of each user, which allows the study to consider  
 23 the BRUE in the stable form. It is also found that using the surrogate model instead of analytical  
 24 path marginal cost is feasible, improves equilibrium but it can lead to sub-optimal SO equilibria.  
 25 This may be the cause of the shape and change of pattern of the path flow between breakpoints  
 26 143 to 440 in figure 4. Therefore, the authors plan to improve the calculation method for SO  
 27 to investigate if what we observe here is caused by a non-optimal solution or related to network

1 effects (e.g. correlations of the effects between multiple ODs). Finally, the results of this study  
2 shows the existence of the breakpoints, which is to be expected from the static case but has not  
3 been analyzed in the dynamic case before. For future work, the authors are looking for the impact  
4 of initialization on the equilibrium in different demand level and also apply the breakpoint analysis  
5 process to more than one ODs with considering of correlation between paths.

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9 nUM project)

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