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Abstract

Given a graph G, a *proper k-coloring* of G is a partition $c = (S_i)_{i\in[0,k-1]}$ of $V(G)$ into k stable sets $S_0, \ldots, S_{k-1}$. Given a weight function $w : V(G) \to \mathbb{R}^+$, the *weight of a color* S_i is defined as $w(i) = \max_{v\in S_i} w(v)$ and the *weight of a coloring* c as $w(c) = \sum_{i=0}^{k-1} w(i)$. Guan and Zhu [Inf. Process. Lett., 1997] defined the *weighted chromatic number* of a pair (G, w), denoted by $\sigma(G, w)$, as the minimum weight of a proper coloring of G. For a positive integer r, they also defined $\sigma(G, w; r)$ as the minimum of $w(c)$ among all proper r-colorings c of G.

The complexity of determining $\sigma(G, w)$ when G is a tree was open for almost 20 years, until Araújo et al. [SIAM J. Discrete Math., 2014] recently proved that the problem cannot be solved in time $n^{o(\log n)}$ on n-vertex trees unless the Exponential Time Hypothesis (ETH) fails.

The objective of this article is to provide hardness results for computing $\sigma(G, w)$ and $\sigma(G, w; r)$ when G is a tree or a forest, relying on complexity assumptions weaker than the ETH. Namely, we study the problem from the viewpoint of parameterized complexity, and we assume the weaker hypothesis $\mathsf{FPT} \neq \mathsf{W}[1]$. Building on the techniques of Araújo et al., we prove that when G is a forest, the decision problem of computing $\sigma(G, w)$ is $\mathsf{W}[1]$-hard parameterized by the size of a largest connected component of G, and that computing $\sigma(G, w; r)$ is $\mathsf{W}[2]$-hard parameterized by r. Our results rule out the existence of FPT algorithms for computing these invariants on trees or forests for many natural choices of the parameter.

Keywords: weighted coloring; max-coloring; forests; parameterized complexity; $\mathsf{W}[1]$-hard.

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1 Introduction

A *(vertex) k-coloring* of a graph $G = (V, E)$ is a function $c : V(G) \to \{0, \ldots, k-1\}$. Such coloring c is *proper* if $c(u) \neq c(v)$ for every edge $\{u, v\} \in E(G)$. All the colorings we consider in this paper are proper, hence we may omit the word "proper". The *chromatic number* $\chi(G)$ of G is the minimum integer k such that G admits a k-coloring. Given a graph G, determining $\chi(G)$ is the goal of the classical Vertex Coloring problem. If c is a k-coloring of G, then $S_i = \{u \in V(G) \mid c(u) = i\}$ is a stable (a.k.a. independent) set. Consequently, a k-coloring c can be seen as a partition of $V(G)$ into stable sets $S_0, \ldots, S_{k-1}$. We often see a coloring as a partition in the sequel.

We study a generalization of Vertex Coloring for vertex-weighted graphs that has been defined by Guan and Zhu [11]. Given a graph G and a weight function $w : V(G) \to \mathbb{R}^+$, the *weight of a color* S_i is defined as $w(i) = \max_{v \in S_i} w(v)$. Then, the *weight of a coloring* c is $w(c) = \sum_{i=0}^{k-1} w(i)$. In the Weighted Coloring problem, the goal is to determine the *weighted chromatic number* of a pair (G, w), denoted by $\sigma(G, w)$, which is the minimum weight of a coloring of (G, w). A coloring c of G such that $w(c) = \sigma(G, w)$ is an *optimal weighted coloring*. Guan and Zhu [11] also defined, for a positive integer r, $\sigma(G, w; r)$ as the minimum of $w(c)$ among all r-colorings c of G, or as $+\infty$ if no r-coloring exists. Note that $\sigma(G, w) = \min_{r \geq 1} \sigma(G, w; r)$. It is worth mentioning that the Weighted Coloring problem is also sometimes called Max-Coloring in the literature; see for instance [14, 16].

Guan and Zhu defined this problem in order to study practical applications related to resource allocation, which they describe in detail in [11]. One should observe that if all the vertex weights are equal to one, then $\sigma(G, w) = \chi(G)$, for every graph G. Consequently, determining $\sigma(G, w)$ and $\sigma(G, w; r)$ are NP-hard problems on general graphs [13]. In fact, these problems have been shown to be NP-hard even on split graphs, interval graphs, triangle-free planar graphs with bounded degree, and bipartite graphs [5, 6, 10]. On the other hand, the weighted chromatic number of cographs and of some subclasses of bipartite graphs can be found in polynomial time [5, 6].

In this work we focus on the case where G is a forest, which has attracted considerable attention in the literature. Concerning graphs of bounded treewidth [2], Guan and Zhu [11] showed, by using standard dynamic programming techniques, that on an n-vertex graph of treewidth t the parameter $\sigma(G, w; r)$ can be computed in time

$$n^{O(r)} \cdot r^{O(t)}. \tag{1}$$

Guan and Zhu [11] left as an open problem whether Weighted Coloring is polynomial on trees and, more generally, on graphs of bounded treewidth. Escoffier et al. [10] found a polynomial-time approximation scheme to solve Weighted Coloring on bounded treewidth graphs, and Kavitha and Mestre [14] showed that the problem is in P on the class of trees where vertices with degree at least three induce a stable set.

But the question of Guan and Zhu has been answered only recently, when Araújo et

[2] We will not define treewidth here, just recall that forests are the graphs with treewidth 1; see [4, 7].

al. [1] showed that, unless the Exponential Time Hypothesis (ETH)[3] fails, there is no algorithm computing the weighted chromatic number of n-vertex trees in time $n^{o(\log n)}$.

As discussed in [1], it is worth mentioning that the above lower bound is tight. Indeed, Guan and Zhu [11] showed that the maximum number of colors used by an optimal weighted coloring of any weighted graph (G, w) is at most its so-called *first-fit chromatic number* (see [11] for the definition), denoted by $\chi_{\mathsf{FF}}(G)$. On the other hand, Linhares and Reed [15] proved that for any n-vertex graph G of treewidth at most t, it holds that $\chi_{\mathsf{FF}}(G) = O(t \log n)$. These observations together with Equation (1) imply that the Weighted Coloring problem can be solved on forests in time $n^{O(\log n)}$.

Therefore, Weighted Coloring on forests is unlikely to be in P, as this would contradict the ETH, and also unlikely to be NP-hard, as in that case all problems in NP could be solved in subexponential time, contradicting again the ETH.

Our results. The objective of this article is to provide hardness results for computing $\sigma(G, w)$ and $\sigma(G, w; r)$ when G is a forest, relying on complexity assumptions weaker than the ETH. Namely, we study the problem from the viewpoint of parameterized complexity (the basic definitions can be found in Section 2), and we assume the weaker hypothesis $\mathsf{FPT} \neq \mathsf{W}[1]$. Indeed, it is well-known [4] that the ETH implies that $\mathsf{FPT} \neq \mathsf{W}[1]$, which in turn implies that $\mathsf{P} \neq \mathsf{NP}$.

Our first result is that when (G, w) is a weighted forest, the decision problem of computing $\sigma(G, w)$ is $\mathsf{W}[1]$-hard parameterized by the size of a largest connected component of G. This is proved by a parameterized reduction from Independent Set that builds on the techniques introduced by Araújo et al. [1]. This result essentially rules out the existence of FPT algorithms for Weighted Coloring on forests for many natural choices of the parameter: cliquewidth, maximum degree, maximum diameter of a connected component, number of colors in an optimal weighted coloring, etc. Indeed, all these parameters are bounded by the size of a largest connected component of G (for the number of colors, this can be proved by using that they are bounded by $\chi_{\mathsf{FF}}(G)$ [11], which is easily seen to be bounded by the size of a largest connected component).

We then move our attention to the parameter $\sigma(G, w; r)$ and we prove, by a parameterized reduction from Dominating Set similar to the first one, that computing $\sigma(G, w; r)$ on forests is $\mathsf{W}[2]$-hard parameterized by r. Interestingly, if we assume the ETH, our reduction together with the results of Chen et al. [3] stating that Dominating Set parameterized by the size of the solution cannot be solved in time $f(k) \cdot n^{o(k)}$ unless the ETH fails, imply that, on graphs of bounded treewidth, the running time given by Equation (1) is asymptotically optimal, that is, there is no algorithm computing $\sigma(G, w; r)$ on n-vertex graphs of bounded treewidth in time $n^{o(r)}$.

We would like to mention that, although our reductions use several key ideas introduced by Araújo et al. [1], our results are incomparable to those of [1].

As further research, it would be interesting to identify "reasonable" parameters yielding FPT algorithms for Weighted Coloring on forests. Probably, it might make sense to consider combined parameters that take into account, on top of the aforementioned

[3] The ETH states that 3-SAT cannot be solved in subexponential time; see [12] for more details.

invariants, the number of distinct weights in the input weighted forest.

Organization of the article. In Section 2 we provide some basic preliminaries about forests, weighted colorings, and parameterized complexity. In Section 3 we introduce some common gadgets that will be used in both reductions. In Section 4 and Section 5 we present the W[1]-hardness and W[2]-hardness reductions, respectively.

2 Preliminaries

Forests and weighted colorings. We use standard graph-theoretic notation, and we consider simple undirected graphs without loops or multiple edges; see [7] for any undefined terminology. Given two integers i and j with $i \leq j$, we denote by $[i, j]$ the set of all integers between i and j, including both i and j.

If T is a rooted tree, we denote by $r(T)$ the root of T. A *weighted graph* is a pair (G, w) where G is a graph and $w : V(G) \to \mathbb{R}^+$ is a weight function. We say that a weighted graph (G, w) is a *weighted forest* if G is a forest and a *weighted rooted tree* if G is a rooted tree. If (G, w) is a weighted rooted tree, we define the *root* of (G, w), denoted by $r((G, w))$, to be the root of G.

When considering a k-coloring c of a graph G, defined in Section 1, for convenience we will usually index its associated stable sets as $c = (S_i)_{i \in [0,k-1]}$. We say that a vertex $v \in V(G)$ is *colored* S_i, for some $i \in [0, k-1]$, if $v \in S_i$.

Parameterized complexity. We refer the reader to [4, 9] for basic background on parameterized complexity, and we recall here only some basic definitions. A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. For an instance $I = (x, k) \in \Sigma^* \times \mathbb{N}$, k is called the *parameter*. A parameterized problem is *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathcal{A}$, a computable function f, and a constant c such that given an instance $I = (x, k)$, $\mathcal{A}$ (called an FPT *algorithm*) correctly decides whether $I \in L$ in time bounded by $f(k) \cdot |I|^c$.

Within parameterized problems, the class W[1] may be seen as the parameterized equivalent to the class NP of classical optimization problems. Without entering into details (see [4, 9] for the formal definitions), a parameterized problem being W[1]-*hard* can be seen as a strong evidence that this problem is *not* FPT. The canonical example of W[1]-hard problem is Independent Set parameterized by the size of the solution [4].

The class W[2] of parameterized problems is a class that contains W[1], and such that the problems that are W[2]-*hard* are even more unlikely to be FPT than those that are W[1]-hard (again, see [4, 9] for the formal definitions). The canonical example of W[2]-hard problem is Dominating Set parameterized by the size of the solution [5].

For $i \in [1, 2]$, to transfer W[i]-hardness from one problem to another, one uses a

[4] Given a graph G and a parameter k, the problem is to decide whether there exists $S \subseteq V(G)$ such that $|S| \geq k$ and $E(G[S]) = \emptyset$.

[5] Given a graph G and a parameter k, the problem is to decide whether there exists $S \subseteq V(G)$ such that $|S| \leq k$ and $N[S] = V(G)$.

parameterized reduction, which given an input $I = (x, k)$ of the source problem, computes in time $f(k) \cdot |I|^c$, for some computable function f and a constant c, an equivalent instance $I' = (x', k')$ of the target problem, such that k' is bounded by a function depending only on k.

Hence, an equivalent definition of W[1]-hard (resp. W[2]-hard) problem is any problem that admits a parameterized reduction from Independent Set (resp. Dominating Set) parameterized by the size of the solution.

3 Some useful gadgets

In this section we introduce some gadgets that will be used in the reductions presented in the following sections. As mentioned in the introduction, the first reduction is from Independent Set, and the second one is from Dominating Set. Most of these gadgets are inspired by [1].

Let us first fix (G, k), an instance of either Independent Set or Dominating Set. We denote by (G', w) the instance of Weighted Coloring we are going to construct. We define $n = |V(G)|$ and we fix a bijection $\beta : V(G) \to [0, n-1]$. This bijection will allow us to define our gadgets depending on integers $j \in [0, n-1]$ that correspond, via β, to the vertices of G. We define $M = k(n-1)\varepsilon + \sum_{i \in [0,4k+3]} \frac{1}{2^i}$, where ε is any fixed real number satisfying $0 < \varepsilon < \frac{1}{nk2^{4k+3}}$, which implies that $M < 2$. We define, for each $i \in [0, 4k+3]$ and for each $j \in [0, n]$, $w_i^j = \frac{1}{2^i} + j\varepsilon$. We also define, for each $\ell \in [0, 3]$, $W_\ell = w_{4k+\ell}^0 = \frac{1}{2^{4k+\ell}}$. Note that, in our construction, we will only use the weights $\{w_i^j \mid j \in [0, n], i \in [0, 4k-1]\} \cup \{W_\ell \mid \ell \in [0, 3]\}$.

Following [1], we first define in Definition 3.1 a particular family of *binomial trees* B_i, $i \in [0, 4n+3]$, depicted in Figure 1. They will be crucially used in the construction of (G', w). Their role is to force the color of most of the nodes in any coloring c of G' with $w(c) \leq M$. Note that the notion of binomial trees has also been used, for instance, in [2, 5].

Definition 3.1 For each $i \in [0, 4k+3]$, we define recursively the weighted rooted tree B_i, called *binomial tree*, as follows:

- if $i = 0$, then B_0 has a unique node of weight w_0^0,
- otherwise, B_i has a root r of weight w_i^0 and, for each $p \in [0, i-1]$, we introduce a copy of B_p and we connect its root to r.

Lemma 3.2 (Araújo et al. [1]) *Let $i \in [0, 4k+3]$ and let (T, w) be a weighted forest having B_i as a subtree. If there exists a coloring c of (T, w) with $w(c) \leq M$, then, for any $\ell \in [0, i]$:*

- *all vertices of B_i with weight in w_ℓ^0 receive the same color S_ℓ of c and*
- *there exists a unique color class S_ℓ in c of weight in $\{w_\ell^j \mid j \in [0, n]\}$.*

In our reductions, similarly to Araújo et al. [1], we will extensively use binomial trees

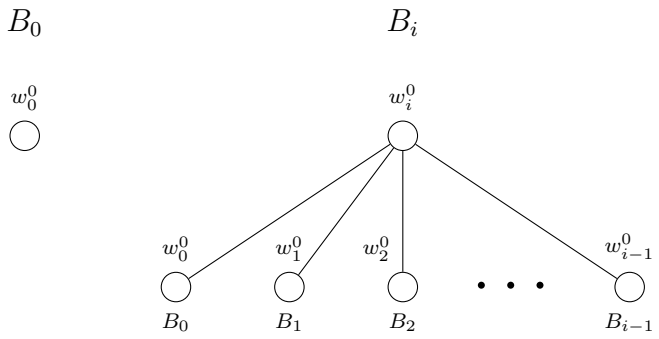

Figure 1. The binomial trees B_0 and B_i, $i > 0$. The vertices labeled B_j are the root of a copy of B_j, for each $j \in [0, i-1]$. The weights are also depicted on top of the vertices.

in order to forbid a subset of the colors at a particular vertex. Namely, assume that c is a coloring of a weighted forest (T, w), with $w(c) \leq M$, such that B_i is a subtree of T attached to a vertex v. Then, by Lemma 3.2, there exists a unique color class S_i in c of weight in $\{w_i^j \mid j \in [0, n]\}$ and, since the root of B_i and v are adjacent, it follows that vertex v cannot be colored S_i in c.

As we shall see later, the choice of the weight of a color class S_ℓ corresponds to choosing (or not) a vertex to be part of the solution of the corresponding problem. Each time that a vertex is chosen, we will have to "pay" an additional weight of $(n-1)\varepsilon$ in the total weight of the coloring. The selected value of M forces that we will be able to choose k vertices.

In every graph we are going to build in the following, we assume that B_{4k+3} is a subtree of our graph. If this is not the case, we introduce a new connected component that contains only B_{4k+3}. This permits to identify a color by its weight. Indeed, in any coloring $c = (S_i)_{i \in [0,\ell]}$, where $\ell \geq 4k+3$, of weight at most M, we have that for each $i \in [0, 4k+3]$, S_i is the only color such that $w(S_i) \in \{w_i^j \mid j \in [0, n]\}$. We define $R_\ell = S_{4k+\ell}$, for each $\ell \in [0, 3]$. As, in our construction, we use only the weights $\{w_i^j \mid j \in [0, n], i \in [0, 4k-1]\} \cup \{W_\ell \mid \ell \in [0, 3]\}$, we have that $w(R_\ell) = W_\ell$, for each $\ell \in [0, 3]$.

We also define the *auxiliary tree* A_i^j for each $i \in [0, 4k-1]$ and each $j \in [0, n]$, as defined in [1]. This auxiliary tree is depicted in Figure 2.

Definition 3.3 For each $i \in [0, 4k-1]$ and each $j \in [0, n]$, we define the weighted rooted tree A_i^j, called *auxiliary tree*, as follows.

- We first introduce two vertices u and v such that u is the root of A_i^j, v is connected to u, $w(u) = W_0$, and $w(v) = w_i^j$.
- for each $\ell \in [0, i-2]$, we introduce a copy of B_ℓ and we connect the root of this copy to v.
- for each $\ell \in [0, 4k-1] \setminus \{i-1\}$, we introduce a copy of B_ℓ and we connect the root of this copy to u.

The vertex v is called the *subroot* of A_i^j. Note that A_i^j consists of 2^{4k} nodes.

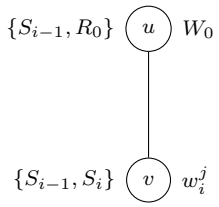

Figure 2. The auxiliary tree A_i^j, $i \in [0, 4k-1]$ and $j \in [0, n]$. The binomial trees are not depicted. Next to each vertex, its weight and the set of colors this vertex can receive (see Lemma 3.4) are depicted.

Lemma 3.4 (Araújo et al. [1]) *Let $i \in [0, 4k-1]$, let $j \in [0, n]$, and let (T, w) be any weighted forest having B_{4k+3} and A_i^j as subtrees. Let u and v be the root and the subroot of A_i^j, respectively. For any coloring c of (T, w) with weight $w(c) \leq M$, it holds that:*

- *either v is colored S_{i-1} and u must be colored with the color R_0,*
- *or v is colored S_i (therefore, $w(S_i) \geq w_i^j$) and u is colored either with S_{i-1} or with the color R_0.*

We also need the R_i-AND gadget, $i \in [0, 1]$, depicted in Figure 3, and which is strongly inspired by a similar gadget presented in [1] (called *clause tree*) corresponding to the logical 'OR'.

Definition 3.5 Let $i \in [0, 1]$. Given two vertices I_1, I_2, we define the R_i-AND *gadget* between the *input vertices* I_1 and I_2 as follows:

- We add four new vertices v_1, v_2, v_3, and O and the edges $\{v_1, I_1\}$, $\{v_2, I_2\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_3, O\}$.
- For each $j \in [1, 3]$ and each $\ell \in [0, 4k-1]$, we introduce a copy of B_ℓ and we connect its root to v_j.
- For each $\ell \in [0, 4k-1]$, we introduce a copy of B_ℓ and we connect its root to O.
- For each $j \in [1, 2]$ we introduce a copy of B_{4k+1-i} and we connect its root to v_j.
- We introduce a copy of B_{4k+i} and a copy of B_{4k+2} and we connect their roots to v_3.
- We set $w(v_1) = W_2$, $w(v_2) = W_3$, $w(v_3) = W_3$, and $w(O) = W_1$.

The vertex O is called the *output vertex* of the R_i-AND gadget. Note that in this gadget, the binomial trees and the weight assignments are used to forbid an appropriately chosen set of colors at a particular vertex. This results in the set of allowed colors depicted in Figure 3.

We naturally extend the definition of the R_i-AND gadget to ℓ input vertices with $\ell \geq 2$ by introducing $\ell - 1$ R_i-AND gadgets in a sequential way as follows: given ℓ input vertices $I_1, \ldots, I_\ell$, let $O_1 = I_1$ and, for $i \in [2, \ell]$, we let O_i be the output vertex of an R_i-AND gadget having O_{i-1} and I_i as input vertices. We define the output vertex of the whole gadget to be O_ℓ.

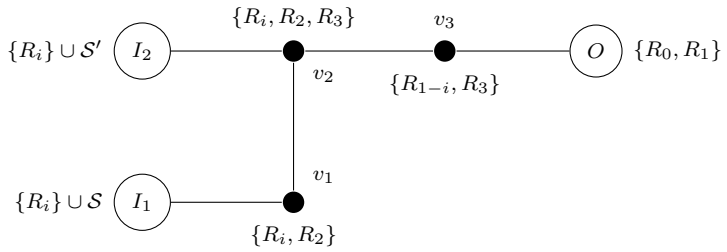

Figure 3. The R_i-AND gadget, for some $i \in [0, 1]$, where I_1 and I_2 are the input vertices and O is the output vertex, and where $\mathcal{S}$ and $\mathcal{S}'$ are subsets of $\{S_\ell \mid \ell \in [0, 4k-1]\} \cup \{R_0, R_1\}$. For each vertex, the associated set is the set of colors that the vertex can receive. Again, the binomial trees are not depicted.

Lemma 3.6 *Let $i \in [0, 1]$, let I_1 and I_2 be the two input vertices of an R_i-AND gadget, and let O be its output vertex. If I_1 and I_2 are colored R_i, then O must be colored R_i. Moreover, if either I_1 or I_2 is not colored R_i, then O can be colored either R_0 or R_1.*

PROOF: First, assume that I_1 and I_2 are colored R_i. This sequentially implies that v_1 must be colored R_2, v_2 must be colored R_3, v_3 must be colored R_{1-i}, and O must be colored R_i. Secondly, assume that I_1 is not colored R_i. This sequentially implies that v_1 can be colored R_i, v_2 can be colored R_2, v_3 can be colored R_3, and therefore O can be colored either R_0 or R_1. Finally, assume that I_2 is not colored R_i. This sequentially implies that v_2 can be colored R_i, v_3 can be colored R_3, and so O can be colored either R_0 or R_1. □

Finally, we define, for each $i \in [0, k-1]$ and $j \in [0, n-1]$, the *vertex tree* T_i^j, depicted in Figure 4, which is also inspired by a similar construction given in [1], called *variable tree*. The main difference with respect to [1] is that, in our case, the color given to the root of a vertex tree codifies a binary value corresponding to picking or not a vertex in the solution, whereas the gadget of [1] codifies an integer corresponding to the assignment of a group of variables in the *integral* version of 3-SAT that they consider.

Definition 3.7 For each $i \in [0, k-1]$ and for each $j \in [0, n-1]$, we define the *vertex tree* T_i^j to be the weighted rooted tree, representing the vertex $\beta^{-1}(j)$, defined as follows.

- We introduce one copy of A_{4i+1}^{j+1} and A_{4i+3}^{n-j} and an R_0-AND gadget whose inputs are the two roots of A_{4i+1}^{j+1} and A_{4i+3}^{n-j}. We call u the output of the R_0-AND gadget and we set u to be the root of T_i^j.
- We introduce one copy of A_{4i+1}^{j}, A_{4i+1}^{j+1}, A_{4i+3}^{n-j}, and A_{4i+3}^{n-j-1},
  - we connect $r(A_{4i+1}^{j})$ to $r(A_{4i+3}^{n-j})$ and $r(A_{4i+1}^{j+1})$ to $r(A_{4i+3}^{n-j-1})$, and
  - we connect u to $r(A_{4i+1}^{j})$ and to $r(A_{4i+3}^{n-j-1})$.

The usefulness of a vertex tree T_i^j associated with a vertex v of the instance of INDEPENDENT SET or DOMINATING SET corresponding to the integer j is the following. The color of the root u codifies whether vertex v has been chosen in the solution or not.

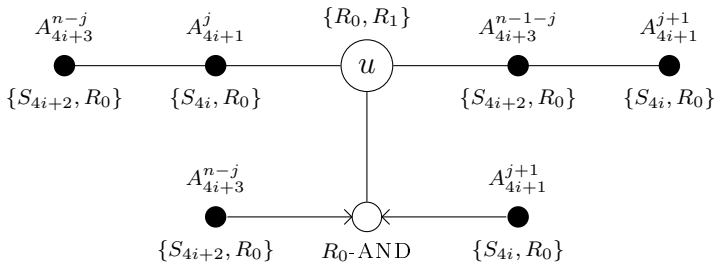

Figure 4. The vertex tree T_i^j, $i \in [0, k-1]$ and $j \in [0, n-1]$. The vertices labeled A_p^q are the roots of a copy of A_p^q. For each vertex, the associated set is the set of colors that the vertex can receive. The R_0-AND circle corresponds to the vertices of the R_0-AND gadget without the inputs and the output. The two input vertices are connected to it with an arrow and the output is u.

Namely, if u gets color R_0 (resp. R_1), this means that vertex v is (resp. is not) part of the solution. The following lemma formalizes this idea and guarantees that the choices are *consistent*, in the sense that the choices made in all vertex trees corresponding to the same vertex are the same. It is also important to note that, as we will see in the proof of Lemma 3.8, because of the definition of the weights w_i^j, each time we choose to color a root of a vertex tree with R_0, we have to "pay" $(n-1)\varepsilon$ in the total weight. Making k such choices is forced by the properties of the gadgets and the value of M.

Lemma 3.8 *Let (T, w) be any weighted forest having B_{4k+3} as a subtree and containing, for each $(i, j) \in [0, k-1] \times [0, n-1]$, T_i^j as a subtree. Let c be a coloring of (T, w) with $w(c) \leq M$. Then, there exist $(j_i)_{i \in [0,k-1]} \in [0, n-1]^k$ such that each root u of each subtree T_i^j, $(i, j) \in [0, k-1] \times [0, n-1]$, satisfies:*

- *if $j = j_i$ for some $i \in [0, k-1]$, then the color of u in c must be R_0, and*
- *otherwise, the color of u in c must be R_1.*

Proof: By Lemma 3.2 and since we assume that $w(c) \leq M$ and that B_{4k+3} occurs in (T, w) as a subtree, it follows that we can write $c = (S_i)_{i \in [0,\ell]}$ with $\ell \geq 4k+3$, so that for each $i \in [0, 4k+3]$, $w(S_i) \in \{w_i^j \mid j \in [0, n]\}$.

Let $i \in [0, k-1]$. Given $j \in [0, n]$, as T_i^j or T_i^{j-1} is a subgraph of T (in fact, if $j \notin \{0, n\}$, both are), we know that there exist a copy of A_{4i+1}^j with root r_{4i+1}^j and a copy of A_{4i+3}^{n-j} with root r_{4i+3}^{n-j} such that r_{4i+1}^j and r_{4i+3}^{n-j} are adjacent. This implies that, for each $j \in [0, n]$,

$$c(r_{4i+1}^j) \neq R_0 \text{ or} \tag{1_j}$$
$$c(r_{4i+3}^{n-j}) \neq R_0. \tag{2_j}$$

Note that, by Lemma 3.4, for each $j \in [0, n]$, (1_j) implies that $w(S_{4i+1}) \geq w_{4i+1}^j$ and (2_j) implies that $w(S_{4i+3}) \geq w_{4i+3}^{n-j}$. Therefore, one of the following cases necessarily occurs:

- (1_n) is satisfied and so $w(S_{4i+1}) \geq w_{4i+1}^n$,
- (2_0) is satisfied and so $w(S_{4i+3}) \geq w_{4i+3}^{n-0}$, or

- (1_0) and (2_n) are satisfied and, since for each $j \in [0,n]$ at least one of (1_j) and (2_j) holds, the integer $j^* = \min\{j \mid 0 \leq j \leq n-1$ and property (2_{j+1}) is satisfied$\}$ is well-defined. It follows that both (1_{j^*}) and (2_{j^*+1}) are satisfied, which implies that $w(S_{4i+1}) \geq w^{j^*}_{4i+1}$ and $w(S_{4i+3}) \geq w^{n-(j^*+1)}_{4i+3}$.

In the first two cases, using that $w(S_{4i+1}) \geq w^0_{4i+1}$ and $w(S_{4i+3}) \geq w^0_{4i+3}$, we obtain $w(S_{4i+1}) + w(S_{4i+3}) \geq w^0_{4i+1} + w^0_{4i+3} + n\varepsilon$. In the third case, we obtain $w(S_{4i+1}) + w(S_{4i+3}) \geq (w^0_{4i+1} + j^*\varepsilon) + (w^0_{4i+3} + (n-(j^*+1))\varepsilon) = w^0_{4i+1} + w^0_{4i+3} + (n-1)\varepsilon$.

Thus, it always holds that $w(S_{4i+1}) + w(S_{4i+3}) \geq w^0_{4i+1} + w^0_{4i+3} + (n-1)\varepsilon$.

Therefore,

$$\begin{aligned} w(c) &\geq \sum_{i\in[0,k-1]} (w(S_{4i}) + w(S_{4i+1}) + w(S_{4i+2}) + w(S_{4i+3})) + \sum_{i\in[0,3]} w(R_i) \\ &\geq \sum_{i\in[0,k-1]} (w^0_{4i} + w^0_{4i+1} + w^0_{4i+2} + w^0_{4i+3} + (n-1)\varepsilon) + \sum_{i\in[0,3]} W_i \\ &= M. \end{aligned}$$

By definition of c, we have $w(c) = M$, for each $i \in [0,3]$, $w(R_i) = W_i$, and for each $i \in [0,k-1]$, $w(S_{4i}) = w^0_{4i}$, $w(S_{4i+2}) = w^0_{4i+2}$, and $w(S_{4i+1}) + w(S_{4i+3}) = w^0_{4i+1} + w^0_{4i+3} + (n-1)\varepsilon$. Moreover, for each $4k+3 < i \leq \ell$, $w(S_i) = 0$.

Let us fix $i^* \in [0,k-1]$. The equation $w(S_{4i^*+1}) + w(S_{4i^*+3}) = w^0_{4i^*+1} + w^0_{4i^*+3} + (n-1)\varepsilon$ implies the existence of $j^* \in [0,n-1]$ such that $w(S_{4i^*+1}) = w^{j^*}_{4i^*+1}$ and $w(S_{4i^*+3}) = w^{n-1-j^*}_{4i^*+3}$. Thus, for each $j > j^*$, the root of any copy of $A^j_{4i^*+1}$ must be colored R_0 and for each $j < j^*$, the root of any copy of $A^{n-1-j}_{4i^*+3}$ must be colored R_0. This implies that for each $j \in [0,n-1] \setminus \{j^*\}$, the root of $T^j_{i^*}$ must be colored R_1. Moreover, as in $T^{j^*}_{i^*}$ the roots of the copy of $A^{j^*+1}_{4i^*+1}$ and the copy of $A^{n-j^*}_{4i^*+3}$ must be colored R_0 (otherwise, $w(S_{4i^*+1}) \geq w^{j^*+1}_{4i^*+1} > w^{j^*}_{4i^*+1}$ or $w(S_{4i^*+3}) \geq w^{n-j^*}_{4i^*+3} > w^{n-1-j^*}_{4i^*+3}$), the R_0-AND gadget ensures that the root of $T^{j^*}_{i^*}$ is colored R_0. □

4 W[1]-hardness reduction

In this section we present a parameterized reduction from Independent Set to Weighted Coloring on forests.

Theorem 4.1 *Given a weighted forest (G,w), the decision problem of computing $\sigma(G,w)$ is* W[1]*-hard when parameterized by the size of a largest connected component of G.*

Proof: We reduce from Independent Set parameterized by the size of the solution, which is well-known to be W[1]-hard [8]. Let (G,k) be an instance of Independent Set, and let $n = |V(G)|$. Recall that $M = k(n-1)\varepsilon + \sum_{i\in[0,4k+3]} \frac{1}{2^i}$ where ε is any real number satisfying $0 < \varepsilon < \frac{1}{nk2^{4k+3}}$, which implies that $M < 2$. Let $\beta : V(G) \to [0,n-1]$ be a bijection. For each edge $\{v_1,v_2\} \in E(G)$ and each $i_1, i_2 \in [0,k-1]$, we define the weighted rooted tree $H_{\{v_1,v_2\},i_1,i_2}$ as follows.

- We introduce a copy of $T_{i_1}^{\beta(v_1)}$ and a copy of $T_{i_2}^{\beta(v_2)}$, and call the roots r_1 and r_2, respectively.
- We introduce an R_0-AND gadget where the input vertices are r_1 and r_2 and the output is a new vertex r.
- We introduce a copy of B_{4k} and we connect its root to r.
- We set r to be the root of $H_{\{v_1,v_2\},i_1,i_2}$.

An illustration of $H_{\{v_1,v_2\},i_1,i_2}$ is shown in Figure 5.

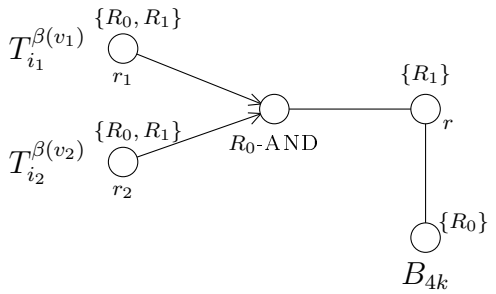

Figure 5. The weighted rooted tree $H_{\{v_1,v_2\},i_1,i_2}$ defined in the proof of Theorem 4.1. For each vertex, the associated set is the set of colors that the vertex can receive. The R_0-AND circle corresponds to the vertices of the R_0-AND gadget without the inputs and the output. The two input vertices are r_1 and r_2 and the output is r.

Note that, since a copy of B_{4k} is attached to the root r and the output vertex of an R_0-AND gadget can only be colored R_0 or R_1, it follows that r has to be colored R_1. We also define, for each vertex v in $V(G)$ and each i_1, i_2 in $[0, k-1]$ with $i_1 \neq i_2$, the weighted rooted tree H_{v,i_1,i_2} to be the tree $H_{\{v_1,v_2\},i_1,i_2}$ defined above with $v_1 = v_2 = v$.

We define (G', w) as the disjoint union of the weighted tree B_{4k+3}, of each weighted tree of $\{H_{e,i_1,i_2} \mid e \in E(G), i_1, i_2 \in [0, k-1]\}$, of each weighted tree of $\{H_{v,i_1,i_2} \mid v \in V(G), i_1, i_2 \in [0, k-1], i_1 \neq i_2\}$, and of each weighted tree of $\{T_i^j \mid i \in [0, k-1], j \in [0, n-1]\}$. Note that the size of each connected component of G' is bounded by a function depending only on k. Indeed, the size of any connected component is bounded by the size of those of type H_{e,i_1,i_2}, which can be easily checked to be at most $2 \cdot (6 \cdot 2^{4k} + 4) + 4 + 2^{4k} = 13 \cdot 2^{4k} + 12$. Note that the construction of (G', w) can be performed in time $f(k) \cdot n^{O(1)}$, as required.

The idea of the construction is that the trees $H_{\{v_1,v_2\},i_1,i_2}$ defined above guarantee that, for each edge $\{v_1, v_2\}$ of G, at most one of v_1 and v_2 belongs to the independent set. More formally, as the root r of such tree has to be colored R_1, by the R_0-AND gadget at least one of r_1 and r_2 has to be colored R_1, which translates to the fact that at least one of v_1 and v_2 does *not* belong to the independent set (recall the paragraph after Definition 3.7). Similarly, by construction, the trees H_{v,i_1,i_2} guarantee that the same vertex is not picked more than once in the solution.

More formally, we now prove that there exists a solution of Independent Set on (G, k) if and only if $\sigma(G', w) \leq M$.

Assume first that Z is a solution of Independent Set on (G,k). We may assume that Z is of size exactly k. Let $\delta : Z \to [0,k-1]$ be a bijection. For each $i \in [0,k-1]$, we define $v_i = \delta^{-1}(i)$. We are going to define a coloring $c = (S_i)_{i\in[0,4k+3]}$ of weight at most M such that for each $i \in [0,4k+3]$, $w(S_i) \in \{w_i^j \mid j \in [0,n]\}$. By Lemma 3.2, we can (and we must) color every tree B_i in that way, for each $i \in [0,4k+3]$. Then for each $j \leq \beta(v_i)$ and each $j' \geq \beta(v_i)$, we set the color of the subroot of each A_{4i+1}^j and each $A_{4i+3}^{m-j'-1}$ to be to be color S_{4i+1} and S_{4i+3}, respectively, and their root to be colored S_{4i} and S_{4i+2}, respectively. For each $j > \beta(v_i)$ and each $j' < \beta(v_i)$, we set the color of the roots of each A_{4i+1}^j and each $A_{4i+3}^{m-j'-1}$ to be R_0 and the color of their subroots to be S_{4i+1} and S_{4i+3}, respectively. This coloring is possible by Lemma 3.4. Note also that for each $i \in [0,k-1]$, if $j_i = \beta(v_i)$, then we have $w(S_{4i}) = w_i^0$, $w(S_{4i+1}) = w_i^{j_i}$, $w(S_{4i+2}) = w_i^0$, and $w(S_{4i+3}) = w_i^{m-j_i-1}$. We set the color of the root of each T_i^j such that $j = \beta(\delta^{-1}(i))$ to R_0, and we set the color of the root of each T_i^j such that $j \neq \beta(\delta^{-1}(i))$ to R_1. The colors of the other vertices are forced by the R_0-AND gadgets.

As Z is an independent set, for each edge $\{v_1,v_2\}$ of G, at least one of the extremities, say v_1, is not in Z. Thus, for each i_1, i_2 in $[0,k-1]$, the root of $T_{i_1}^{\beta(v_i)}$ is colored R_1 and therefore the root of $H_{\{v_1,v_2\},i_1,i_2}$ can be colored R_1, which is the only color available for this vertex. As in this coloring, for each $\ell \in [0,3]$, $w(R_\ell) = W_\ell$, we obtain that $\sigma(G',w) \leq M$.

Conversely, assume that there is an integer ℓ and a coloring $c = (S_i)_{i\in[0,\ell]}$ of G' such that $w(c) \leq M$. As there is no weight below W_3, from Lemma 3.2 it follows that $\ell = 4k+3$ and for each $i \in [0,4k+3]$, $w(S_i) \in \{w_i^j \mid j \in [0,n]\}$. By Lemma 3.8, for each $i \in [0,k-1]$, there exists an index j_i such that the root of each $T_i^{j_i}$ is colored R_0. Let us define $Z = \{\beta^{-1}(j_i) \mid i \in [0,k-1]\}$. Given i_1 and i_2 in $[0,k-1]$, we claim that there is no edge in G between $\beta^{-1}(j_{i_1})$ and $\beta^{-1}(j_{i_2})$. Indeed, if the root of $T_{i_1}^{j_{i_1}}$ and the root of $T_{i_2}^{j_{i_2}}$ are colored R_0, then the root of $H_{\{\beta^{-1}(j_{i_1}),\beta^{-1}(j_{i_2})\},i_1,i_2}$ should also be colored R_0 because of the R_0-AND gadget, but this is not possible because of the tree B_{4k} that is connected to it. A similar argument shows that, because of the trees H_{v,i_1,i_2}, for any i_1, i_2 in $[0,k-1]$ with $i_1 \neq i_2$, it holds that $\beta^{-1}(j_{i_1}) \neq \beta^{-1}(j_{i_2})$, that is, the same vertex does not occur more than once in Z. This implies that Z is an independent set in G of size exactly k, concluding the proof. □

5 W[2]-hardness reduction

In this section we present a reduction from Dominating Set to Weighted Coloring on forests when the number of colors is prescribed. The reduction is similar to the one presented in Theorem 4.1, but it is somehow simpler and uses the R_1-AND gadget instead of the R_0-AND gadget.

Theorem 5.1 *Given a weighted forest (G,w) and a positive integer r, the problem of computing $\sigma(G,w;r)$ is* W[2]*-hard when parameterized by r.*

Proof: We reduce from Dominating Set parameterized by the size of the solution,

which is well-known to be W[2]-hard (see [4, 9]). Let (G, k) be an instance of Dominating Set, and let $n = |V(G)|$. Recall again that $M = k(n-1)\varepsilon + \sum_{i\in[0,4k+3]} \frac{1}{2^i}$ where ε is any real number satisfying $0 < \varepsilon < \frac{1}{nk2^{4k+3}}$, which implies that $M < 2$. Let $\beta : V(G) \to [0, n-1]$ be a bijection. For each vertex $v \in V(G)$, we define the weighted rooted tree H_v as follows.

- For each $i \in [0, k-1]$ and each $j \in \beta(N[v])$, we introduce a copy of T_i^j and call its root r_i^j.
- We introduce an R_1-AND gadget where the input vertices are the vertices of $\{r_i^j \mid i \in [0, k-1], j \in \beta(N[v])\}$, and let r be the output.
- We introduce a copy of B_{4k+1} and we connect its root to r.
- We set r to be the root of H_v.

We then define (G', w) as the disjoint union of the weighted tree B_{4k+3}, of each weighted tree of $\{H_v \mid v \in V(G)\}$, and of each weighted tree of $\{T_i^j \mid i \in [0, k-1], j \in [0, n-1]\}$. Finally, we set $r = 4k + 4$. Note that r depends only on k and that the construction of (G', w) can be performed in time $f(k) \cdot n^{O(1)}$, as required.

The idea of this construction is to guarantee that a dominating set in G must contain, for each $v \in V(G)$, at least one vertex in $N[v]$. In the tree H_v, this is captured by forbidding its root r to be colored R_1, which by the R_1-AND gadget implies that at least one of the roots of the trees T_i^j must be colored R_0, meaning that at least one vertex in $N[v]$ belongs to the solution.

Formally, we now prove that there exists a solution of Dominating Set on (G, k) if and only if $\sigma(G', w; r) \leq M$.

First assume that Z is a solution of Dominating Set on (G, k). We may assume that Z is of size exactly k. Let $\delta : Z \to [0, k-1]$ be a bijection. For each $i \in [0, k-1]$, we define $v_i = \delta^{-1}(i)$. We are going to define a coloring $c = (S_i)_{i\in[0,4k+3]}$ of weight at most M such that for each $i \in [0, 4k+3]$, $w(S_i) \in \{w_i^j \mid j \in [0, n]\}$, in the same way we did for Theorem 4.1. By Lemma 3.2, we can (and we must) color every tree B_i in that way, for $i \in [0, 4k+3]$. Then for each $j \leq \beta(v_i)$ and each $j' \geq \beta(v_i)$, we set the color of the subroot of each A_{4i+1}^j and each $A_{4i+3}^{m-j'-1}$ to be to be color S_{4i+1} and S_{4i+3}, respectively, and their root to be colored S_{4i} and S_{4i+2}, respectively. For each $j > \beta(v_i)$ and each $j' < \beta(v_i)$, we set the color of the roots of each A_{4i+1}^j and each $A_{4i+3}^{m-j'-1}$ to be R_0 and the color of their subroot to be S_{4i+1} and S_{4i+3}, respectively. Again, this coloring is possible by Lemma 3.4. Note also that for each $i \in [0, k-1]$, if $j_i = \beta(v_i)$, then we have $w(S_{4i}) = w_i^0$, $w(S_{4i+1}) = w_i^{j_i}$, $w(S_{4i+2}) = w_i^0$, and $w(S_{4i+3}) = w_i^{m-j_i-1}$. We set the color of the root of each T_i^j such that $j = \beta(\delta^{-1}(i))$ to R_0, and we set the color of the root of each T_i^j such that $j \neq \beta(\delta^{-1}(i))$ to R_1. The colors of the other vertices are forced by the R_1-AND gadgets.

As Z is a dominating set of G, for each $v \in V(G)$, at least one of the vertices r_i^j, $i \in [0, k-1]$, $j \in \beta(N[v])$, of H_v is colored R_0. So we can affect the color R_0 to the root of H_v, which is, by construction, the only available color for this vertex. As in this coloring, for each $\ell \in [0, 3]$, $w(R_\ell) = W_\ell$, we obtain that $\sigma(G', w; r) \leq M$.

Conversely, assume that there is an integer ℓ and a coloring $c = (S_i)_{i\in[0,\ell-1]}$ of G' certifying that $\sigma(G', w; \ell) \leq M$. As there is no weight below W_3, from Lemma 3.2 it follows that $\ell = 4k + 4$ and for each $i \in [0, 4k + 3]$, $w(S_i) \in \{w_i^j \mid j \in [0, n]\}$. By Lemma 3.8, for each $i \in [0, k - 1]$, there exists an index j_i such that the root of each $T_i^{j_i}$ is colored R_0. Let us define $Z = \{\beta^{-1}(j_i) \mid i \in [0, k - 1]\}$, where the same vertex may have been chosen for different indices in $[0, k - 1]$. Let v be a vertex of G. As, by construction, the root of H_v can only receive the color R_0 in any coloring of weight at most M, this implies that at least one vertex $r_{i^*}^{j^*}$, $i^* \in [0, k - 1]$, $j^* \in \beta(N[v])$, of H_v is colored R_0. This implies, by Lemma 3.8, that $\beta^{-1}(j^*) \in Z$. Moreover, $\beta^{-1}(j^*) \in N[v]$. It follows that Z is a dominating set in G of size at most k. □

Note that the proof of Theorem 5.1 shows that, if (G, k) in an instance of Dominating Set, then the number of colors of the constructed instance satisfies $r = 4k + 4 = O(k)$. Note also that it is easy to strengthen the lower bound given by Theorem 5.1 to apply to *trees* instead of forests. Indeed, we can just add a new vertex v, attach it to every connected component of the forest G' built in the reduction, and give to v a weight that does not conflict with any of the weights of its neighbors. By possibly using a new color containing only v, it still holds that $r = O(k)$.

The above paragraph together with the fact that, assuming the ETH, Dominating Set parameterized by the size of the solution cannot be solved in time $f(k) \cdot n^{o(k)}$ for any computable function f [3] imply the following corollary.

Corollary 5.2 *Assuming the ETH, there is no algorithm that, given a weighted tree (G, w) and a positive integer r, computes $\sigma(G, w; r)$ in time $f(r)\cdot n^{o(r)}$ for any computable function f.*

In particular, Corollary 5.2 implies that on forests, and more generally on graphs of bounded treewidth, the running time stated in Equation (1), which in this case is equal to $n^{O(r)}$, is asymptotically optimal under the ETH.

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