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SURJECTIONS AND DOUBLE POSETS

LOÏC FOISSY AND FRÉDÉRIC PATRAS

1. INTRODUCTION

The algebraic and Hopf algebraic structures associated to permutations [18, 5, 8] have been intensively studied and applied in various contexts. A reason for their ubiquity is that they occur naturally in geometry, algebraic topology and classical and stochastic integral calculus because their (non-commutative) shuffle products encode cartesian products of simplices [21]. In combinatorics, they appear naturally in the theory of twisted algebras and twisted Hopf algebras (aka Hopf species) [22, Sect. 5] [2]. In the theory of operads, through the associative operad [1] and its Hopf operad structure [17, Sect. 2], and so on...

In another direction, permutations can be encoded by pictures in the sense of Zelevinsky [24]. This encoding is a key ingredient of his proof of the Littlewood-Richardson rule and of the Robinson-Schensted-Knuth correspondence. Recently, the first of us introduced a new approach to Hopf algebras of permutations by means of the notion of double posets [19], closely related to the notion of picture [10, 9, 11]. Besides making a bridge between the combinatorial, picture-theoretic, and the Hopf algebraic approaches to permutations, this new construction is natural in that it originates in the encoding of the statistics of inversions in symmetric groups (recall that, given a permutation σ in the symmetric group S_n , the pair of integers (i, j) in $[n]^2$, with $i < j$, defines an inversion if and only if $\sigma(i) > \sigma(j)$). Inversions are a key notion from the Coxeter group point of view on permutations; among others, their cardinal computes the length of the permutation).

The present article is dedicated to constructing and proving analog objects and results for surjections. Although they are a relatively less familiar object in view of applications of algebra and combinatorics than permutations, there are several reasons to be interested in surjections. For example, through the usual bijection with ordered partitions of finite initial subsets of the integers, they appear naturally in the study of the geometry of Coxeter groups [3] and of twisted Hopf algebras (aka Hopf species) [23, 2]. They have also been involved recently in the modelling of quasi-shuffle products [20, 15, 14] and their applications in stochastics [7, 6].

The theory and structure of the Hopf algebra of surjections (known as WQSym , the Hopf algebra of word quasi-symmetric functions) [16, 4] parallels largely the one of bijections. The study of surjections from a picture

and double poset theoretic point of view, which is the subject of the present article, seems instead new.

The article is organized as follows. We introduce first a family of double posets, weak planar posets, that generalize the planar posets of [10] and are in bijection with surjections or, equivalently, packed words. The following sections investigate their Hopf algebraic properties, which are inherited from the Hopf algebra structure of double posets and their relations with WQSym .

2. WEAK PLANE POSETS

Recall that a double poset is a set equipped with two orders. A quasi-order is a binary relation \leq which is reflexive and transitive but not necessarily antisymmetric (so that one may have $x \leq y$ and $y \leq x$ with $x \neq y$). A quasi-order is total when all elements are comparable (it holds that $x \leq y$ or $y \leq x$ for arbitrary x and y). When $x \leq y$ and $y \leq x$, we write $x \equiv y$ and say that x and y are equivalent (the relation \equiv is an equivalence relation). A quasi-poset is a set equipped with a quasi-order. All the posets, double posets, quasi-posets... we will consider are assumed to be finite (we omit therefore “finite” in our definitions and statements).

Definition 1. *A weak plane poset is a double poset (P, \leq_1, \leq_2) such that:*

- (1) *For all $x, y \in P$, $(x \leq_1 y \text{ and } x \leq_2 y) \implies (x = y)$.*
- (2) *The relation \preceq defined on P by $(x \leq_1 y \text{ or } x \leq_2 y)$ is a total quasi-order.*

In particular, the relation defined by $(x \preceq y \text{ and } y \preceq x)$ is an equivalence relation (we denote it as above by \equiv).

Example. A plane poset is a double poset (P, \leq_1, \leq_2) such that:

- For all $x, y \in P$, if x and y are comparable for both \leq_1 and \leq_2 , then $x = y$.
- For all $x, y \in P$, x and y are comparable for \leq_1 or for \leq_2 .

By proposition 11 of [10], if (P, \leq_1, \leq_2) is a plane poset, then \preceq is a total order, and obviously (1) is also satisfied. So plane posets are weak plane. Moreover, if P is a weak plane poset, then \preceq is an order if, and only if, P is plane: by (2), two distinct elements x, y are always comparable and the fact that \preceq is an order implies that they cannot be comparable for both \leq_1 and \leq_2 .

Lemma 2. *Let (P, \leq_1, \leq_2) be a weak plane poset. Then:*

- (3) *The relation \ll defined on P by $(y \leq_1 x \text{ or } x \leq_2 y)$ is a total order.*

Proof. Let $x, y, z \in P$, such that $x \ll y$ and $y \ll z$. Three cases are possible:

- $(y \leq_1 x \text{ and } z \leq_1 y)$ or $(x \leq_2 y \text{ and } y \leq_2 z)$. Then $(z \leq_1 x \text{ or } x \leq_2 z)$, so $x \ll z$.
- $x \leq_2 y$ and $z \leq_1 y$. As \preceq is a total quasi-order, two subcases are possible.

- $x \preceq z$, then $x \leq_1 z$ or $x \leq_2 z$. If $x \leq_2 z$, then $x \ll z$. If $x \leq_1 z$, then $x \leq_1 y$ and $x \leq_2 y$. By (1), $x = y$, so $x \ll z$.
- $z \preceq x$, then $z \leq_1 x$ or $z \leq_2 x$. If $z \leq_1 x$, then $x \ll z$. If $z \leq_2 x$, then $z \leq_2 y$ and $z \leq_1 y$. By (1), $y = z$, so $x \ll z$.
- $y \leq_1 x$ and $y \leq_2 z$. Similar proof.

Therefore, \ll is transitive.

Let $x, y \in P$, such that $x \ll y$ and $y \ll x$. Two cases are possible:

- (1) $(y \leq_1 x \text{ and } x \leq_1 y)$ or $(x \leq_2 y \text{ and } y \leq_2 x)$: then $x = y$.
- (2) $y \leq_1 x$ and $y \leq_2 x$, or $x \leq_1 y$ and $x \leq_2 y$: by (1), $x = y$.

So \ll is an order.

For all $x, y \in P$, $x \preceq y$ or $y \preceq x$, so $y \leq_1 x$ or $x \leq_2 y$ or $x \leq_1 y$ or $y \leq_2 x$, so $x \ll y$ or $y \ll x$: \ll is total. \square

Remark. (3) implies (1), but not (2).

3. SURJECTIONS AND PACKED WORDS

When a surjection f from $[n]$ to $[k]$ is represented by the sequence of its values, $w_f := f(1) \dots f(n)$, the word w_f is packed: its set of letters identifies with an initial subset of the integers (in that case, $[k]$). Conversely, an arbitrary packed word of length n can always be obtained in that way: packed words (on length n) are in bijection with surjections (with domain $[n]$ and codomain an initial subset of the integers).

Recall also that total quasi-orders \preceq on $[n]$ are in bijection with packed words. An example gives the general rule: assume that $n = 6$ and that the quasi-order is defined by

$$2 \equiv 5 \preceq 1 \equiv 3 \equiv 6 \preceq 4.$$

Then, the corresponding packed word is 212312: the first equivalence class $\{2, 5\}$ gives the position of letter 1, the second, $\{1, 3, 6\}$, the positions of letter 2, and so on.

Proposition 3. *Let w be a packed word of length n . The double poset P_w (also written $Dp(w)$) is defined by $P_w = ([n], \leq_1, \leq_2)$, with:*

$$\begin{aligned} \forall i, j \in [n], i \leq_1 j &\iff (i \geq j \text{ and } w(i) \leq w(j)), \\ i \leq_2 j &\iff (i \leq j \text{ and } w(i) \leq w(j)) \end{aligned}$$

It is a weak plane poset. The total quasi-order \preceq is the one associated bijectively to w .

Proof. The fact that P_w is a double poset is obvious. For all $i, j \in [n]$, if $i \leq_1 j$ and $i \leq_2 j$, then $i \geq j$ and $i \leq j$, so $i = j$: (1) is verified. Moreover:

$$i \preceq j \iff w(i) \leq w(j),$$

so \preceq is indeed a total quasi-order. Finally, remark that the total order \ll agrees with the natural order:

$$i \ll j \iff i \leq j.$$

□

Theorem 4. *The set of packed words of length n and of isomorphism classes of weak plane posets are in bijection through Dp . The inverse map $pack$ is given as follows. Let P a weak double poset. By Lemma 2 we can assume that $P = [n]$ with \ll the natural order. Then, $Dp^{-1}(P) =: pack(P)$ is the packed word associated to the total quasi-order \preceq .*

Proof. Let us show first that for any packed words w, w' , the double posets P_w and $P_{w'}$ are isomorphic if, and only if, $w = w'$. Let $f : P_w \rightarrow P_{w'}$ be an isomorphism. Then f is increasing from $([n], \ll)$ to $([n'], \ll')$. As \ll and \ll' are the usual total orders of $[n]$ and $[n']$, $n = n'$ and $f = Id_{[n]}$. Consequently, for all $i, j \in [n]$, assuming that $i \leq j$:

$$w(i) \leq w(j) \iff i \leq_2 j \iff i \leq'_2 j \iff w'(i) \leq w'(j).$$

As w and w' are packed words, $w = w'$.

Let P now be a weak plane poset and let us show that $Dp(pack(P)) = P$. We can assume that $(P, \ll) = ([n], \leq)$. The packed word $w = pack(P)$ is such that for all $i, j \in [n]$, $i \preceq j \iff w(i) \leq w(j)$. We denote by \leq'_1 and \leq'_2 the orders of P_w . Then, for all $i, j \in [n]$:

$$\begin{aligned} i \leq'_1 j &\iff (j \leq i) \text{ and } (w(i) \leq w(j)) \\ &\iff (j \ll i) \text{ and } (i \preceq j) \\ &\iff (i \leq_1 j \text{ or } j \leq_2 i) \text{ and } (i \leq_1 j \text{ or } i \leq_2 j) \\ &\iff (i \leq_1 j) \text{ or } (j \leq_2 i \text{ and } i \leq_2 j) \\ &\iff (i \leq_1 j) \text{ or } (i = j) \\ &\iff i \leq_1 j. \end{aligned}$$

So $\leq'_1 = \leq_1$. Similarly, $\leq'_2 = \leq_2$. □

4. THE SELF-DUAL HOPF ALGEBRA STRUCTURE

We denote by \mathcal{H}_{WPP} the vector space generated by isomorphism classes of weak plane posets and show below how definitions and results in [19, 13] apply in this context (definitions and results relative to double posets are taken from [19]).

Let P, Q be two double posets. Two preorders are defined on $P \sqcup Q$ by:

$$\begin{aligned} \forall i, j \in P \sqcup Q, i \leq_1 j \text{ if } & (i, j \in P \text{ and } i \leq_1 j) \\ & \text{or } (i, j \in Q \text{ and } i \leq_1 j); \\ i \leq_2 j \text{ if } & (i, j \in P \text{ and } i \leq_2 j) \\ & \text{or } (i, j \in Q \text{ and } i \leq_2 j) \\ & \text{or } (i \in P \text{ and } j \in Q). \end{aligned}$$

This defines a double poset denoted by PQ . Extending this product by bilinearity makes the linear span \mathcal{H}_{DP} of double posets an associative algebra, whose unit is the empty double poset denoted 1. If P and Q are weak plane posets, then so is PQ : \mathcal{H}_{WPP} is a subalgebra of \mathcal{H}_{DP} .

Definition 5. *Let P be a double poset (resp. weak double poset) and let $X \subseteq P$.*

- X is also a double poset (resp. weak) by restriction of \leq_1 and \leq_2 : we denote this double poset (resp. weak) by $P|_X$.
- We shall say that X is an open set of P if:

$$\forall i, j \in P, i \leq_1 j \text{ and } i \in X \implies j \in X.$$

The set of open sets of P is denoted by $Top(P)$.

- A coproduct is defined on \mathcal{H}_{DP} (resp. \mathcal{H}_{WPP}) by

$$\Delta(P) = \sum_{O \in Top(P)} P|_{P \setminus O} \otimes P|_O.$$

Theorem 6. *The product and the coproduct equip \mathcal{H}_{DP} and therefore its subspace \mathcal{H}_{WPP} with the structure of a graded, connected Hopf algebra.*

See [19] for a proof of the compatibility properties between the product and the coproduct characterizing a Hopf algebra.

Recall that, for $P = (P, \leq_1, \leq_2)$, $\iota(P) := (P, \leq_2, \leq_1)$. If P is a weak plane poset, then so is $\iota(P)$. Recall also that there exists a pairing on \mathcal{H}_{DP} defined, for two double posets P, Q by

$$\langle P, Q \rangle := \sharp Pic(P, Q),$$

where $Pic(P, Q)$ stands for the number of pictures between P and Q (a picture between P and Q is a bijection f such that

$$i \leq_1 j \implies f(i) \leq_2 f(j), \quad f(i) \leq_1 f(j) \implies i \leq_2 j.)$$

The Hopf algebra of double poset \mathcal{H}_{DP} is self-dual for this pairing. By Proposition 24 of [13]:

Proposition 7. *The Hopf algebra \mathcal{H}_{WPP} is a self-dual Hopf subalgebra of the Hopf algebra of double poset \mathcal{H}_{DP} .*

5. LINEAR EXTENSIONS AND **WQSym**.

Definition 8. Let $P = (P, \leq_1, \leq_2)$ be a weak plane poset. We assume that $(P, \ll) = ([n], \leq)$. A linear extension of P is a surjective map $f : [n] \rightarrow [k]$ such that:

- (1) For all $i, j \in [n]$, $i \leq_1 j \implies f(i) \leq f(j)$.
- (2) For all $i, j \in [n]$, $f(i) = f(j) \implies i \equiv j$.

The set of linear extensions of P is denoted by $\text{Lin}(P)$.

If f is a linear extension of a weak plane poset P , we see it as a packed word $f(1) \dots f(n)$.

Let us denote by $PW(n)$ the set of packed words of length n . Recall that **WQSym** is given two products, its usual one, denoted by \cdot , and the shifted shuffle product \sqcup [12]. Denoting by Δ the usual coproduct of **WQSym**, both $(\mathbf{WQSym}, \sqcup, \Delta)$ and $(\mathbf{WQSym}, \cdot, \Delta)$ are Hopf algebras.

For example:

$$\begin{aligned} (1) \sqcup (1) &= (12) + (21), \\ (1) \cdot (2) &= (12) + (21) + (11). \end{aligned}$$

Proposition 9. The following map is a Hopf algebra morphism:

$$\phi : \begin{cases} \mathcal{H}_{WPP} & \longrightarrow (\mathbf{WQSym}, \sqcup, \Delta) \\ P & \longrightarrow \sum_{f \in \text{Lin}(P)} f. \end{cases}$$

Proof. We omit it since it is similar to the proof of Theorem 18 of [10]. \square

Proposition 10. For all $f, g \in PW(n)$, we shall say that $f \leq g$ if:

- (1) For all $i, j \in [n]$, $i \geq j$ and $f(i) \leq f(j) \implies g(i) \leq g(j)$.
- (2) For all $i, j \in [n]$, $g(i) = g(j) \implies f(i) = f(j)$.

Then \leq is an order on $PW(n)$. Moreover, for all $f \in PW(n)$:

$$\phi(P_f) = \sum_{f \leq g} g.$$

Proof. The relation \leq is clearly transitive and reflexive. Let us assume that $f \leq g$ and $g \leq f$. By (2), for all $i, j \in [n]$, $f(i) = f(j)$ if, and only if, $g(i) = g(j)$. Hence, putting $k = \max(f) = \max(g)$, there exists a unique permutation $\sigma \in \mathfrak{S}_k$ such that for all $i \in [k]$, $f^{-1}(i) = g^{-1}(\sigma(i))$. By (1), if $i \geq j$, then $g(i) \leq g(j) \iff f(i) \leq f(j)$. Hence:

$$\begin{aligned} \max f^{-1}(1) &= \max\{i \in [n] \mid \forall j \leq i, f(i) \leq f(j)\} \\ &= \max\{i \in [n] \mid \forall j \leq i, g(i) \leq g(j)\} \\ &= \max g^{-1}(1) \\ &= \max(g^{-1}(\sigma(1))), \end{aligned}$$

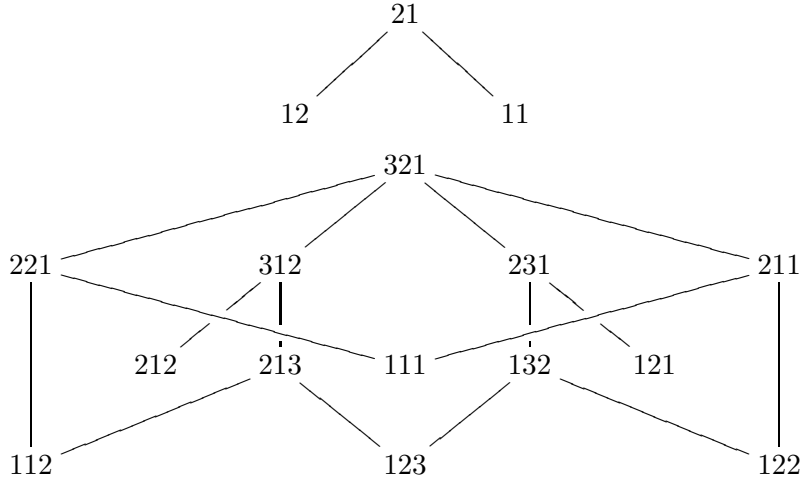
so $\sigma(1) = 1$. Iterating this, one shows that $\sigma = Id_k$, so $f = g$: the relation \leq is an order. Let $f, g \in PW(n)$. Then:

$$g \in Lin(f) \iff \begin{cases} \forall i, j \in [n], i \geq j \text{ and } f(i) \leq f(j) \implies g(i) \leq g(j), \\ \forall i, j \in [n], g(i) = g(j) \implies f(i) = f(j) \end{cases} \\ \iff f \leq g.$$

So $Lin(P_f) = \{g \in PW(n), f \leq g\}$. □

Corollary 11. ϕ is a Hopf algebra isomorphism.

Here are the Hasse graphs of $(PW(2), \leq)$ and $(PW(3), \leq)$.



Remark. If f and g are two permutations, then:

$$f \leq g \iff \forall i, j \in [n], i > j \text{ and } f(i) < f(j) \implies g(i) < g(j) \\ \iff Desc(f) \subseteq Desc(g).$$

So the restriction of \leq to \mathfrak{S}_n is the right weak Bruhat order.

Definition 12. Let $P = (P, \leq_1, \leq_2)$ be a weak plane poset. We assume that $(P, \ll) = ([n], \leq)$. A weak linear extension of P is a surjective map $f : [n] \rightarrow [k]$ such that:

- (1) For all $i, j \in [n]$, $i \leq_1 j \implies f(i) \leq f(j)$.
- (2) For all $i, j \in [n]$, if $i \leq_1 j$ and $f(i) = f(j) \implies i \equiv j$.

The set of weak linear extensions of P is denoted by $WLin(P)$.

In [12], an order is defined on $PW(n)$: for all $f, g \in PW(n)$, $f \prec g$ if

- (1) For all $i, j \in [n]$, $g(i) \leq g(j) \implies f(i) \leq f(j)$.
- (2) For all $i, j \in [n]$, $i < j$ and $g(i) > g(j) \implies f(i) > f(j)$.

It is proved that the following map is a Hopf algebra isomorphism:

$$\psi : \begin{cases} (\mathbf{WQSym}, \sqcup, \Delta) & \longrightarrow & (\mathbf{WQSym}, \cdot, \Delta) \\ f & \longrightarrow & \sum_{g \preceq f} g. \end{cases}$$

Lemma 13. *Let P be a weak plane poset. Then:*

$$WLin(P) = \bigsqcup_{f \in Lin(P)} \{g \in PW(n), g \preceq f\}.$$

Proof. \subseteq . Let $g \in WLin(P)$. For any $p \in [\max(g)]$, we put $P_p = P_{|g^{-1}(p)}$. Then P_p is a weak plane poset, so there exists a unique packed word f_p such that P_p is isomorphic to P_{f_p} . Let us define g' by $g'(i) = f_p(i) + \max(f_1) + \dots + \max(f_{p-1})$ for any $i \in P_p$; g' is a packed word.

Let us show first that $g' \in Lin(p)$. Assume that $i \leq_1 j$. Then $g(i) \leq g(j)$. Let us show that we also have $g'(i) \leq g'(j)$:

- If we don't have $i \equiv j$, then, as g is a weak linear extension of P , $g(i) < g(j)$, which implies $g'(i) < g'(j)$.
- If $g(i) = g(j) = p$, then $i \equiv j$ in P_p , so $f_p(i) = f_p(j)$ and finally $g'(i) = g'(j)$.

Now, if $g'(i) = g'(j)$, then $g(i) = g(j) = p$ and $f_p(i) = f_p(j)$, so $i \equiv j$ in P_p and finally $i \equiv j$: $g' \in Lin(P)$.

Let us show finally that $g \preceq g'$. If $g'(i) \leq g'(j)$, then necessarily $g(i) \leq g(j)$. Let us assume $i < j$ and $g'(i) > g'(j)$. Then $g(i) \geq g(j)$. If $g(i) = g(j) = p$, then $f_p(i) > f_p(j)$, so $j \leq_i 1$ and we do not have $i \equiv j$: this contradicts the fact that g is a weak linear extension. So $g(i) > g(j)$, and finally $g \preceq g'$.

\supseteq . Let $f \in Lin(P)$ and $g \preceq f$. If $i \leq_1 j$, then $f(i) \leq f(j)$, so $g(i) \leq g(j)$. If moreover $g(i) = g(j)$, as $i \geq j$ (because $i \leq_1 j$), we can not have $f(i) < f(j)$ as $g \preceq f$, so $f(i) = f(j)$ and $i \equiv j$. So $g \in WLin(P)$.

Disjoint union. Let $f, f' \in Lin(P)$, such that there exists $g \in PW(n)$, $g \preceq f, f'$. Let us consider $i < j$. If $f(i) > f(j)$, then $g(i) > g(j)$. If $f'(i) \leq f'(j)$, we would have $g(i) \leq g(j)$, contradiction. Hence, by symmetry:

$$\begin{aligned} \forall i, j \in [n] \text{ such that } i < j, f(i) > f(j) &\iff f'(i) > f'(j), \\ f(i) \leq f(j) &\iff f'(i) \leq f'(j). \end{aligned}$$

Let us assume that $i < j$ and $f(i) = f(j)$. Then $i \equiv j$, and $f'(i) \leq f'(j)$. As P is isomorphic to P_h for a certain packed word h , $i < j$ and $h(i) = h(j)$, so $j \leq_1 i$ in P ; consequently, $f'(j) \leq f'(i)$ and finally $f'(i) = f'(j)$. As a conclusion:

$$\begin{aligned} \forall i, j \in [n] \text{ such that } i < j, f(i) > f(j) &\iff f'(i) > f'(j), \\ f(i) = f(j) &\iff f'(i) = f'(j), \\ f(i) < f(j) &\iff f'(i) < f'(j). \end{aligned}$$

So $P_f = P_{f'}$, which implies $f = f'$. □

Proposition 14. *The following map is a Hopf algebra isomorphism:*

$$\phi' = \psi \circ \phi : \begin{cases} \mathcal{H}_{WPP} & \longrightarrow (\mathbf{WQSym}, \cdot, \Delta) \\ P_f & \longrightarrow \sum_{f \in WLin(P)} f. \end{cases}$$

Proof. Indeed, for any packed word f , by the preceding lemma:

$$\psi \circ \phi(P_f) = \sum_{f \in Lin(P)} \sum_{g \preceq f} g = \sum_{f \in WLin(P)} f.$$

By composition, ϕ' is an isomorphism. \square

Examples. We order the packed words of degree 2 in the following way: (11,12,21).

- (1) The matrices of ϕ and ϕ' from the basis $(P_f)_{f \in PW(2)}$ to the basis $PW(2)$ are respectively given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

- (2) The matrix of the pairing of \mathcal{H}_{WPP} in the basis $(P_f)_{f \in PW(2)}$ is given by:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- (3) Via ϕ and ϕ' , $(\mathbf{WQSym}, \sqcup, \Delta)$ and $(\mathbf{WQSym}, \cdot, \Delta)$ inherit nondegenerate Hopf pairings. The matrices of these pairings in the basis $PW(2)$ are respectively given by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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