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Multi-gradient fluids

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Abstract An internal energy function of the mass density, the volumetric entropy and their gradients at n -order generates the representation of *multi-gradient fluids*. Thanks to Hamilton's principle, we obtain a thermodynamical form of the equation of motion which generalizes the case of perfect compressible fluids. First integrals of flows are extended cases of perfect compressible fluids. The equation of motion and the equation of energy are written for dissipative cases, and are compatible with the second law of thermodynamics.

Keywords Multi-gradient fluids · Equation of motion · Equation of energy · First integrals · Laws of thermodynamics

Mathematics Subject Classification (2000) 76A02 · 76E30 · 76M30

1 Introduction

Phase transitions between liquid and vapour are associated with a bulk internal-energy per unit volume $\varepsilon_0(\rho, \eta)$ which is non-convex function of mass density ρ and volumetric entropy η . Consequently, in continuous theories, the simplest model allowing to study inhomogeneous fluids inside interfacial layers considers an internal energy per unit volume ε in the form

$$\varepsilon = \varepsilon_0(\rho, \eta) + \frac{1}{2} \lambda |\text{grad} \rho|^2, \quad (1)$$

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where the second term is associated with the non-uniformity of mass density ρ and λ is a coefficient independent of η and $\text{grad } \rho$.

The energy form in Eq. (1) has been introduced by van der Waals and is widely used in the literature [1,2]. The model describes interfaces as diffuse layers and has many applications for describing micro-droplets, contact-lines, nanofluidics, thin films, fluid mixtures and vegetal biology [3–6]. Thanks to the second-gradient theory, the model was extended in continuum mechanics, to the behaviour of strongly inhomogeneous media [7–9]. Nonetheless, at equilibrium, expression of energy given by Eq. (1) yields a uniform temperature in all the parts of inhomogeneous fluids [10]. Consequently, the volumetric entropy varies with the mass density in the same way as in the bulks; this fact leads to monotonic variations of densities but they are qualitative features of non-monotonic behaviours in transition layers which require two or more independently varying densities - entropy included - (chapter 8 in [10]); additively, the temperature through liquid-vapour interfaces is not necessarily constant [11]. So, with the form of energy given by Eq. (1), the thermodynamics of inhomogeneous fluids is neglected and it is not possible to model flows with strong variations of temperature such as those across non-isothermal interfaces. In fact, it is difficult to take account of the gradient of temperature in the expression of the internal energy, but we can use the gradient of entropy. The simplest model was called thermocapillary fluids [12]. Such a second-gradient behaviour has also been considered by several authors in physical problems when the temperature is not constant in inhomogeneous parts of complex media [13,14].

We improve the model: when strong variations of mass density and entropy occur, a general case can consider fluids when the volume internal energy depends on mass density, volumetric entropy and their gradients up to a convenient n -order (with $n \in \mathbb{N}$).

The Hamilton principle allows to write the equation of conservative motions in an universal thermodynamic form structurally similar to the equation obtained in the case of conservative perfect fluids [15]. Consequently, all perfect fluid properties, as the Kelvin theorems are preserved and isentropic or isothermal motions can be studied.

The case of dissipative motions is considered. From the equation of motion, the conservation of mass and the balance of entropy, we deduce the balance of equation of energy. The equation of energy is not in the same form as that of classical fluids. A new vector term – in form of flux of energy – appears, which takes the gradients of mass density and entropy into account. The Clausius-Duhem inequality can be deduced from the viscous dissipation and the Fourier inequality: that proves multi-gradient fluids are compatible with the second law of thermodynamics.

Notations: For any vectors \mathbf{a}, \mathbf{b} , term $\mathbf{a}^T \mathbf{b}$ denotes the scalar product (line vector \mathbf{a}^T is multiplied by column vector \mathbf{b}) and tensor $\mathbf{a} \mathbf{b}^T$ (or $\mathbf{a} \otimes \mathbf{b}$) denotes the product of column vector \mathbf{a} by line vector \mathbf{b}^T , where superscript T denotes the transposition. Tensor \mathbf{I} denotes the identity transformation.

Principal nomenclatures:

ρ , η and s denote the mass density, volumetric entropy and specific entropy.

ε denotes the volume internal energy.

$\tilde{\mu}$ and \tilde{T} denote the extended chemical potential and extended temperature.

\mathcal{D}_t denotes the space occupied by the fluid at time t .

\mathcal{D}_0 denotes the reference space.

\mathbf{F} denotes the deformation gradient.

\mathbf{u} and $\boldsymbol{\gamma}$ denote the velocity and acceleration of the fluid.

Ξ , θ and H denote the generalized chemical potential, generalized temperature and generalized enthalpy.

2 Model of multi-gradient fluids

Perfect fluids of grade $n + 1$ ($n \in \mathbb{N}$) are continuous media with a volume internal energy ε which is a function of ρ and η and their gradients in form

$$\varepsilon = \varepsilon(\rho, \nabla\rho, \dots, \nabla^p\rho, \dots, \nabla^n\rho, \eta, \nabla\eta, \dots, \nabla^p\eta, \dots, \nabla^n\eta), \quad (2)$$

where ∇^p , $p \in \{1, \dots, n\}$, denotes the successive gradients in space \mathcal{D}_t occupied by the fluid at time t .

$$\text{grad}^p \rho \equiv \nabla^p \rho = \left\{ \rho_{,x_{j_1} \dots x_{j_p}} \right\} \quad \text{and} \quad \text{grad}^p \eta \equiv \nabla^p \eta = \left\{ \eta_{,x_{j_1} \dots x_{j_p}} \right\},$$

where the subscript ‘‘comma’’ indicates the partial derivative and $x_{j_1} \dots x_{j_p}$ belong to components of Euler variables $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathcal{D}_t$, space of the fluid at time t . We indicate indices in subscript position without taking account of the tensor covariance or contravariance. We deduce

$$\begin{aligned} d\varepsilon &= \frac{\partial\varepsilon}{\partial\rho} d\rho + \frac{\partial\varepsilon}{\partial\eta} d\eta + \left(\frac{\partial\varepsilon}{\partial\nabla\rho} \vdots d\nabla\rho \right) + \left(\frac{\partial\varepsilon}{\partial\nabla\eta} \vdots d\nabla\eta \right) \\ &+ \dots + \left(\frac{\partial\varepsilon}{\partial\nabla^n\rho} \vdots d\nabla^n\rho \right) + \left(\frac{\partial\varepsilon}{\partial\nabla^n\eta} \vdots d\nabla^n\eta \right). \end{aligned} \quad (3)$$

Sign \vdots means the complete contraction of tensors such that,

$$\begin{aligned} \left(\frac{\partial\varepsilon}{\partial\nabla^p\rho} \vdots d\nabla^p\rho \right) &\equiv \left(d\nabla^p\rho \vdots \frac{\partial\varepsilon}{\partial\nabla^p\rho} \right) = \varepsilon_{,\rho, x_{j_1} \dots x_{j_p}} d\rho_{,x_{j_1} \dots x_{j_p}}, \\ \left(\frac{\partial\varepsilon}{\partial\nabla^p\eta} \vdots d\nabla^p\eta \right) &\equiv \left(d\nabla^p\eta \vdots \frac{\partial\varepsilon}{\partial\nabla^p\eta} \right) = \varepsilon_{,\eta, x_{j_1} \dots x_{j_p}} d\eta_{,x_{j_1} \dots x_{j_p}}, \end{aligned}$$

where repeated subscripts correspond to the summation ⁽¹⁾. We call the thermodynamical functions,

$$\tilde{\mu} = \frac{\partial\varepsilon}{\partial\rho} \quad \text{and} \quad \tilde{T} = \frac{\partial\varepsilon}{\partial\eta},$$

¹ Due to the fact that $\varepsilon_{,\rho, x_{j_1} \dots x_{j_p}} d\rho_{,x_{j_1} \dots x_{j_p}} = d\rho_{,x_{j_1} \dots x_{j_p}} \varepsilon_{,\rho, x_{j_1} \dots x_{j_p}}$ and $\varepsilon_{,\eta, x_{j_1} \dots x_{j_p}} d\eta_{,x_{j_1} \dots x_{j_p}} = d\eta_{,x_{j_1} \dots x_{j_p}} \varepsilon_{,\eta, x_{j_1} \dots x_{j_p}}$, we indifferently permute the position of the two terms in the summation.

the *extended chemical potential* and the *extended temperature*, respectively. We may also consider the case of an inhomogeneous configuration in the reference space by taking the reference position of the fluid into account; for the sake of simplicity, we will not do it. So, the fluid is supposed to have infinitely short memory and the motion history until an arbitrarily chosen past does not affect the determination of stresses at present time.

3 Equation of motions. Generalized temperature and generalized chemical potential

3.1 Virtual motions

The virtual power principle is a convenient way to obtain the motion equation. For conservative motions, it writes as the Hamilton principle. A particle is identified in Lagrange representation by position $\mathbf{X} = [X_1, X_2, X_3]^T \in \mathcal{D}_0$, reference space of the fluid. The variations of a motion are deduced from the functional family of virtual motions

$$\mathbf{X} = \psi(\mathbf{x}, t; \beta),$$

where $\mathbf{x} \in \mathcal{D}_t$, β is a scalar parameter defined in the vicinity of 0, when the real motion corresponds to $\beta = 0$. The virtual displacement $\delta \mathbf{X}$ associated with any variation of the real motion is written in the form

$$\delta \mathbf{X} = \left. \frac{\partial \psi(\mathbf{x}, t; \beta)}{\partial \beta} \right|_{\beta=0}. \quad (4)$$

The variation is dual and mathematically equivalent to Serrin's one ([15], p. 145). Additively, we only consider variations with compact support in D_0 ; consequently, $\delta \mathbf{X} = 0$ on boundary $\partial \mathcal{D}_0$ of \mathcal{D}_0 .

The mass density satisfies the mass conservation law,

$$\rho \det \mathbf{F} = \rho_0(\mathbf{X}), \quad (5)$$

where ρ_0 is defined on \mathcal{D}_0 and $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$ is the deformation gradient. Equivalently, we can write

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0,$$

where \mathbf{u} is the fluid velocity. The motion is supposed to be conservative and the specific entropy $s = \eta / \rho$ is constant along each trajectory (isentropic motion)

$$s = s_0(\mathbf{X}), \quad (6)$$

or

$$\eta \det \mathbf{F} = \eta_0(\mathbf{X}) \quad \text{with} \quad \eta_0(\mathbf{X}) = \rho_0(\mathbf{X}) s_0(\mathbf{X}).$$

Equivalently to relation (6), we can write

$$\frac{\partial \eta}{\partial t} + \operatorname{div}(\eta \mathbf{u}) = 0.$$

Lemma: *The variations of mass density and volumetric entropy verify*

$$\delta \rho = \frac{\operatorname{div}_0(\rho_0 \delta \mathbf{X})}{\det \mathbf{F}} \quad \text{and} \quad \delta \eta = \frac{\operatorname{div}_0(\eta_0 \delta \mathbf{X})}{\det \mathbf{F}}, \quad (7)$$

where div_0 denotes the divergence in reference space D_0 .

The proof of the lemma comes from the velocity definition,

$$\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{u} + \frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial t} = 0,$$

which implies

$$\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \delta \mathbf{u} + \frac{\partial \delta \mathbf{X}}{\partial \mathbf{x}} \mathbf{u} + \frac{\partial \delta \mathbf{X}}{\partial t} = 0 \quad \text{or} \quad \delta \mathbf{u} = -\mathbf{F} \frac{d(\delta \mathbf{X})}{dt}, \quad (8)$$

where d/dt denotes the material derivative. But,

$$\mathbf{F} \mathbf{F}^{-1} = \mathbf{I} \quad \implies \quad \delta \mathbf{F} = -\mathbf{F} \delta \mathbf{F}^{-1} \mathbf{F} = -\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \frac{\partial \delta \mathbf{X}}{\partial \mathbf{X}}.$$

From the Jacobi identity and Eq. (5), we get

$$\delta \det \mathbf{F} = -\det \mathbf{F} \operatorname{tr} \left(\frac{\partial \delta \mathbf{X}}{\partial \mathbf{X}} \right) = -\det \mathbf{F} \operatorname{div}_0 \delta \mathbf{X} \quad \text{and} \quad \rho = \frac{\rho_0(\mathbf{X})}{\det \mathbf{F}},$$

which implies

$$\delta \rho = \frac{1}{\det \mathbf{F}} \left[\rho_0 \operatorname{div}_0(\delta \mathbf{X}) + \frac{\partial \rho_0}{\partial \mathbf{X}} \delta \mathbf{X} \right]$$

and consequently relation (7)¹. The same calculus is suitable to volumetric entropy η and we obtain relation (7)².

Due to definition (4), variation δ and space gradients are independent, so they commute ($\delta \nabla^p \rho = \nabla^p \delta \rho$, $\delta \nabla^p \eta = \nabla^p \delta \eta$). Expression (3) implies,

$$\begin{aligned} \delta \varepsilon &= \frac{\partial \varepsilon}{\partial \rho} \delta \rho + \frac{\partial \varepsilon}{\partial \eta} \delta \eta + \left(\frac{\partial \varepsilon}{\partial \nabla \rho} : \nabla \delta \rho \right) + \left(\frac{\partial \varepsilon}{\partial \nabla \eta} : \nabla \delta \eta \right) + \dots \\ &+ \left(\frac{\partial \varepsilon}{\partial \nabla^n \rho} : \nabla^n \delta \rho \right) + \left(\frac{\partial \varepsilon}{\partial \nabla^n \eta} : \nabla^n \delta \eta \right). \end{aligned} \quad (9)$$

We define operator div_p as follows :

$$\operatorname{div}_p(b_{j_1 \dots j_p}) = (b_{j_1 \dots j_p})_{,x_{j_1} \dots, x_{j_1}}, \quad p \in \mathbb{N} \quad \text{with} \quad x_{j_1}, \dots, x_{j_p} \in \{x_1, x_2, x_3\}.$$

Classically, term $(b_{j_1 \dots j_p})_{,x_{j_1} \dots, x_{j_1}}$ corresponds to the summation on the repeated indices $j_1 \dots j_p$ of the consecutive derivatives of $b_{j_1} \dots j_p$ with respect

to $x_{j_1} \dots x_{j_p}$ ⁽²⁾.

Operator div_p is the *extended divergence at order p*. Operator div_p decreases from p the tensor order and term ∇^p increases from p the tensor order. We deduce,

$$\begin{aligned} \left(\frac{\partial \varepsilon}{\partial \nabla \rho} : \nabla \delta \rho \right) &= \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla \rho} \delta \rho \right) - \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla \rho} \right) \delta \rho \\ \left(\frac{\partial \varepsilon}{\partial \nabla^2 \rho} : \nabla^2 \delta \rho \right) &= \text{div} \left[\left(\frac{\partial \varepsilon}{\partial \nabla^2 \rho} : \nabla \delta \rho \right) - \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla^2 \rho} \right) \delta \rho \right] + \text{div}_2 \left(\frac{\partial \varepsilon}{\partial \nabla^2 \rho} \right) \delta \rho \\ &\vdots \\ \left(\frac{\partial \varepsilon}{\partial \nabla^n \rho} : \nabla^n \delta \rho \right) &= \text{div} A_n + (-1)^n \text{div}_n \left(\frac{\partial \varepsilon}{\partial \nabla^n \rho} \right) \delta \rho \end{aligned}$$

with A_p , $p \in \{1, \dots, n\}$ verifies

$$\begin{aligned} A_p &= \left(\frac{\partial \varepsilon}{\partial \nabla^p \rho} : \nabla^{p-1} \delta \rho \right) - \left(\text{div} \left(\frac{\partial \varepsilon}{\partial \nabla^p \rho} \right) : \nabla^{p-2} \delta \rho \right) + \dots \\ &+ (-1)^{q-1} \left(\text{div}_{q-1} \left(\frac{\partial \varepsilon}{\partial \nabla^p \rho} \right) : \nabla^{p-q} \delta \rho \right) + \dots + (-1)^{p-1} \text{div}_{p-1} \left(\frac{\partial \varepsilon}{\partial \nabla^p \rho} \right) \delta \rho. \end{aligned}$$

We obtain the same relation for the volumetric entropy,

$$\begin{aligned} \left(\frac{\partial \varepsilon}{\partial \nabla \eta} : \nabla \delta \eta \right) &= \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla \eta} \delta \eta \right) - \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla \eta} \right) \delta \eta \\ \left(\frac{\partial \varepsilon}{\partial \nabla^2 \eta} : \nabla^2 \delta \eta \right) &= \text{div} \left[\left(\frac{\partial \varepsilon}{\partial \nabla^2 \eta} : \nabla \delta \eta \right) - \text{div} \left(\frac{\partial \varepsilon}{\partial \nabla^2 \eta} \right) \delta \eta \right] + \text{div}_2 \left(\frac{\partial \varepsilon}{\partial \nabla^2 \eta} \right) \delta \eta \\ &\vdots \\ \left(\frac{\partial \varepsilon}{\partial \nabla^n \eta} : \nabla^n \delta \eta \right) &= \text{div} B_n + (-1)^n \text{div}_n \left(\frac{\partial \varepsilon}{\partial \nabla^n \eta} \right) \delta \eta \end{aligned}$$

with B_p , $p \in \{1, \dots, n\}$ verifies

$$\begin{aligned} B_p &= \left(\frac{\partial \varepsilon}{\partial \nabla^p \eta} : \nabla^{p-1} \delta \eta \right) - \left(\text{div} \left(\frac{\partial \varepsilon}{\partial \nabla^p \eta} \right) : \nabla^{p-2} \delta \eta \right) + \dots \\ &+ (-1)^{q-1} \left(\text{div}_{q-1} \left(\frac{\partial \varepsilon}{\partial \nabla^p \eta} \right) : \nabla^{p-q} \delta \eta \right) + \dots + (-1)^{p-1} \text{div}_{p-1} \left(\frac{\partial \varepsilon}{\partial \nabla^p \eta} \right) \delta \eta. \end{aligned}$$

We denote

$$\begin{cases} \Xi = \tilde{\mu} - \text{div} \Phi_1 + \text{div}_2 \Phi_2 + \dots + (-1)^n \text{div}_n \Phi_n, \\ \theta = \tilde{T} - \text{div} \Psi_1 + \text{div}_2 \Psi_2 + \dots + (-1)^n \text{div}_n \Psi_n, \end{cases} \quad (10)$$

² For example, when $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$\text{div} A = [a_{11,x_1} + a_{21,x_2} + a_{31,x_3}, a_{12,x_1} + a_{22,x_2} + a_{32,x_3}, a_{13,x_1} + a_{23,x_2} + a_{33,x_3}]$, and $\text{div}_2 A = a_{11,x_1,x_1} + a_{21,x_2,x_1} + a_{31,x_3,x_1} + a_{12,x_1,x_2} + a_{22,x_2,x_2} + a_{32,x_3,x_2} + a_{13,x_1,x_3} + a_{23,x_2,x_3} + a_{33,x_3,x_3}$.

with

$$\Phi_p = \frac{\partial \varepsilon}{\partial \nabla p \rho} = \left\{ \varepsilon, \rho, x_{j_1}, \dots, x_{j_p} \right\}, \quad \Psi_p = \frac{\partial \varepsilon}{\partial \nabla p \eta} = \left\{ \varepsilon, \eta, x_{j_1}, \dots, x_{j_p} \right\}, \quad (11)$$

where $x_{j_1}, \dots, x_{j_p} \in \{x_1, x_2, x_3\}$. We call Ξ and θ the *generalized chemical potential* and the *generalized temperature*, respectively.

3.2 The Hamilton action. Generalized enthalpy

Let \mathcal{L} be the Lagrangian of the fluid, i.e.

$$\mathcal{L} = \frac{1}{2} \rho \mathbf{u}^T \mathbf{u} - \varepsilon - \rho \Omega, \quad \text{with} \quad \mathbf{u}^T \mathbf{u} = |\mathbf{u}|^2,$$

where ε is defined by Eq. (2), \mathbf{u} denotes the particle velocity and Ω , function of (\mathbf{x}, t) , denotes the potential of external forces. Between times t_1 and t_2 , the Hamilton action writes

$$a = \int_{t_1}^{t_2} \int_{\mathcal{D}_t} \mathcal{L} \, dx \, dt,$$

where $dx \, dt$ denotes the volume element in time-space $[t_1, t_2] \times D_t$. Classical methods of variation calculus yield the variation of Hamilton's action

$$\delta a = a'(\beta)|_{\beta=0}.$$

From $\delta \Omega = 0$, we deduce

$$\delta a = \int_{t_1}^{t_2} \int_{\mathcal{D}_t} \left[\rho \mathbf{u}^T \delta \mathbf{u} + \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} - \Omega \right) \delta \rho - \delta \varepsilon \right] dx \, dt.$$

From eqs. (9) to (11), we deduce the value of $\delta \varepsilon$ and

$$\begin{aligned} \delta a = \int_{t_1}^{t_2} \int_{\mathcal{D}_t} \left\{ \left[\rho \mathbf{u}^T \delta \mathbf{u} + \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} - \Xi - \Omega \right) \delta \rho - \theta \delta \eta \right] \right. \\ \left. + \operatorname{div} (\mathbf{A}_1 + \dots + \mathbf{A}_n + \mathbf{B}_1 + \dots + \mathbf{B}_n) \right\} dx \, dt, \end{aligned}$$

where term $\operatorname{div} (\mathbf{A}_1 + \dots + \mathbf{A}_n + \mathbf{B}_1 + \dots + \mathbf{B}_n)$ can be integrated on ∂D_t and consequently has a zero contribution. Then,

$$\delta a = \int_{t_1}^{t_2} \int_{\mathcal{D}_0} \left[\rho_0 \mathbf{u}^T \delta \mathbf{u} + \det \mathbf{F} \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} - \Xi - \Omega \right) \delta \rho - (\det \mathbf{F}) \theta \delta \eta \right] dX \, dt,$$

where dX denotes the volume element in D_0 . Let us denote $m = \frac{1}{2} \mathbf{u}^T \mathbf{u} - \Xi - \Omega$; taking account of Eqs. (7) and (8), we obtain

$$\begin{aligned} \delta a = & \int_{t_1}^{t_2} \int_{\mathcal{D}_0} \left[-\rho_0 \mathbf{u}^T \mathbf{F} \frac{d\delta \mathbf{X}}{dt} + (\det \mathbf{F}) m \delta \rho - (\det \mathbf{F}) \theta \delta \eta \right] dX dt = \\ & \int_{t_1}^{t_2} \int_{\mathcal{D}_0} \left\{ \left[\rho_0 \frac{d(\mathbf{u}^T \mathbf{F})}{dt} - \rho_0 \nabla_0^T m + \eta_0 \nabla_0^T \theta \right] \delta \mathbf{X} \right\} dX dt + \\ & \int_{t_1}^{t_2} \int_{\mathcal{D}_0} \left[-\frac{d(\rho_0 \mathbf{u}^T \mathbf{F} \delta \mathbf{X})}{dt} + \operatorname{div}_0 (m \rho_0 \delta \mathbf{X}) - \operatorname{div}_0 (\theta \eta_0 \delta \mathbf{X}) \right] dX dt, \end{aligned}$$

where ∇_0 and div_0 denotes the gradient and divergence operators in \mathcal{D}_0 , respectively. The last integral can be integrated on the boundary of $[t_1, t_2] \times D_0$ and consequently is null. Then,

$$\delta a = \int_{t_1}^{t_2} \int_{\mathcal{D}_0} \left\{ \left[\rho_0 \frac{d(\mathbf{u}^T \mathbf{F})}{dt} - \rho_0 \nabla_0^T m + \eta_0 \nabla_0^T \theta \right] \delta \mathbf{X} \right\} dX dt.$$

The fundamental lemma of variation calculus and $\eta_0 = \rho_0 s_0$ yield

$$\frac{d(\mathbf{u}^T \mathbf{F})}{dt} - \nabla_0^T m + s_0 \nabla_0^T \theta = 0.$$

Due to

$$\frac{d(\mathbf{u}^T \mathbf{F})}{dt} = \left(\boldsymbol{\gamma}^T + \mathbf{u}^T \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \mathbf{F},$$

where $\boldsymbol{\gamma}$ denotes the acceleration vector, we obtain the equation of conservative motions,

$$\boldsymbol{\gamma} + \operatorname{grad}(\Xi + \Omega) + s \operatorname{grad} \theta = 0 \quad \text{or} \quad \boldsymbol{\gamma} + \operatorname{grad}(H + \Omega) - \theta \operatorname{grad} s = 0, \quad (12)$$

where $H = \Xi + s \theta$ denotes the *generalized enthalpy*. Relation (12) is a thermodynamic form of the equation of isentropic motions for perfect fluids which generalizes relation (29.8) in [15].

3.3 Isothermal motions

We assume that

$$\int_{t_1}^{t_2} \int_{\mathcal{D}_t} \eta \, dx \, dt = S_0, \quad (13)$$

where S_0 is a constant. Only mass conservation (5) is considered. The volumetric entropy becomes an independent variable which is subject to constraint (13) and action a can be replaced by

$$b = \int_{t_1}^{t_2} \int_{\mathcal{D}_t} (\mathcal{L} + \theta_0 \eta) \, dx \, dt,$$

where scalar θ_0 is a constant Lagrange multiplier associated with integral constraint (13).

For any variation κ of η , with $\delta\mathbf{X} = 0$, we immediately deduce

$$\int_{t_1}^{t_2} \int_{\mathcal{D}_0} [\det \mathbf{F} (\theta_0 - \theta) \kappa] dX dt = 0$$

and consequently

$$\theta = \theta_0,$$

that corresponds to isothermal motions in the sense of *generalized temperature* θ . When $\kappa = 0$, for any variation of $\delta\mathbf{X}$, we get the equation of motion in the form

$$\boldsymbol{\gamma} + \text{grad}(\Xi + \Omega) = 0.$$

At equilibrium, without body forces,

$$\Xi = \mu_0 \quad \text{and} \quad \theta = \theta_0,$$

where μ_0 denotes the chemical-potential value in fluid bulks.

3.4 Conservative properties of perfect multi-gradient fluids

The circulation of velocity vector \mathbf{u} on a closed fluid-curve \mathcal{C} is $\mathcal{J} = \oint_{\mathcal{C}} \mathbf{u}^T d\mathbf{x}$.

From [15] p. 162,

$$\frac{d}{dt} \oint_{\mathcal{C}} \mathbf{u}^T d\mathbf{x} = \oint_{\mathcal{C}} \boldsymbol{\gamma}^T d\mathbf{x}$$

Thanks to Eq. (12), we deduce:

- The velocity circulation on a closed, isentropic fluid-curve is constant.
- In a homentropic motion (the specific entropy is uniform in the fluid), the velocity circulation on a fluid-curve is constant.
- The velocity circulation on a closed fluid-curve such that $\theta = \theta_0$ is constant.

From

$$\boldsymbol{\gamma} - \frac{1}{2} \text{grad}(\mathbf{u}^T \mathbf{u}) = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{u} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^T \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + \text{rot} \mathbf{u} \times \mathbf{u}.$$

and

$$\boldsymbol{\gamma} + \text{grad}(H + \Omega) - \theta \text{grad} s = 0,$$

in the case of stationary motions, we obtain

$$\text{rot} \mathbf{u} \times \mathbf{u} = \theta \text{grad} s - \text{grad} \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + H + \Omega \right). \quad (14)$$

Equation (14) is the *generalized Crocco-Vazsonyi* relation for multi-gradient fluids.

The Noether theorem proves that any law of conservation can be represented by an invariance group. It has been proved that the conservation laws expressed by Kelvin's theorems correspond to group of permutations consisting of particles of equal specific entropy [16]. It is clear that this group keeps the equation of motions invariant both for the classical perfect fluids and also for multi-gradient perfect fluids. Consequently, it is natural to surmise that the most general perfect fluids are continuous media whose motions verify Kelvin's theorems.

4 Equation of energy and thermodynamical relations

4.1 Equation of balance of energy

Let us consider a dissipative fluid. The equation of motion written in the conservative case can be extended to viscous fluids in the form

$$\rho \gamma + \rho \operatorname{grad} \Xi + \eta \operatorname{grad} \theta - \operatorname{div} \boldsymbol{\sigma}_v + \rho \operatorname{grad} \Omega = 0,$$

where $\boldsymbol{\sigma}_v$ denotes the viscous stress tensor of the fluid. Due to the relaxation time of viscous stresses, the viscosity does not take into account of any gradient terms. We denote \mathbf{M} , B , N and F as,

$$\left\{ \begin{array}{l} \mathbf{M} = \rho \gamma + \rho \operatorname{grad} \Xi + \eta \operatorname{grad} \theta - \operatorname{div} \boldsymbol{\sigma}_v + \rho \operatorname{grad} \Omega, \\ B = \frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{u}), \\ N = \rho \theta \frac{ds}{dt} + \operatorname{div} \mathbf{Q} - r - \operatorname{tr} (\boldsymbol{\sigma}_v D), \\ F = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta - \Pi + \rho \Omega \right) + \\ \operatorname{div} \left\{ \left[\left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta + \rho \Omega \right) \mathbf{I} + \boldsymbol{\sigma}_v \right] \mathbf{u} + \boldsymbol{\chi} \right\} + \operatorname{div} \mathbf{Q} - r - \rho \frac{\partial \Omega}{\partial t}. \end{array} \right.$$

Terms \mathbf{Q} and r represent the heat flux vector and the heat supply, respectively. The Legendre transformation of volume internal energy ε with respect to variables $\rho, \eta, \nabla \rho, \nabla \eta, \dots, \nabla^n \rho, \nabla^n \eta$ is denoted Π ,

$$\Pi = \rho \mu + \eta T + \left(\boldsymbol{\Phi}_1 \dot{\cdot} \nabla \rho \right) + \left(\boldsymbol{\Psi}_1 \dot{\cdot} \nabla \eta \right) + \dots + \left(\boldsymbol{\Phi}_n \dot{\cdot} \nabla^n \rho \right) + \left(\boldsymbol{\Psi}_n \dot{\cdot} \nabla^n \eta \right) - \varepsilon. \quad (15)$$

Then, Π is a function of $\mu, T, \Phi_1, \Psi_1, \dots, \Phi_n, \Psi_n$.

Term $D = \frac{1}{2} \left(\partial \mathbf{u} / \partial \mathbf{x} + (\partial \mathbf{u} / \partial \mathbf{x})^T \right)$ is the velocity deformation tensor and

$$\begin{aligned} \boldsymbol{\chi} &= \rho \frac{\partial \Phi_1}{\partial t} + \eta \frac{\partial \Psi_1}{\partial t} \\ &+ \left(\nabla \rho : \frac{\partial \Phi_2}{\partial t} \right) - \rho \operatorname{div} \frac{\partial \Phi_2}{\partial t} + \left(\nabla \eta : \frac{\partial \Psi_2}{\partial t} \right) - \eta \operatorname{div} \frac{\partial \Psi_2}{\partial t} + \dots \\ &\vdots \\ &+ \left(\nabla^{n-1} \rho : \frac{\partial \Phi_n}{\partial t} \right) - \left(\nabla^{n-2} \rho : \operatorname{div} \frac{\partial \Phi_n}{\partial t} \right) + \dots \\ &+ (-1)^{p-1} \left(\nabla^{n-p} \rho : \operatorname{div}_{p-1} \frac{\partial \Phi_n}{\partial t} \right) + \dots + (-1)^{n-1} \rho \operatorname{div}_{n-1} \frac{\partial \Phi_n}{\partial t} \\ &+ \left(\nabla^{n-1} \eta : \frac{\partial \Psi_n}{\partial t} \right) - \left(\nabla^{n-2} \eta : \operatorname{div} \frac{\partial \Psi_n}{\partial t} \right) + \dots \\ &+ (-1)^{p-1} \left(\nabla^{n-p} \eta : \operatorname{div}_{p-1} \frac{\partial \Psi_n}{\partial t} \right) + \dots + (-1)^{n-1} \eta \operatorname{div}_{n-1} \frac{\partial \Psi_n}{\partial t}. \end{aligned}$$

Let us note that partial derivative $\partial/\partial t$ and div_p commute. Vector $\boldsymbol{\chi}$ is similar to the flux of energy vector obtained in [17].

Theorem: *relation*

$$F - \mathbf{M}^T \mathbf{u} - \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + \Xi + s \theta + \Omega \right) B - N \equiv 0$$

is an algebraic identity.

Equation $\mathbf{M} = 0$ represents the equation of motion, $B = 0$ is the mass conservation and $N = 0$ is the classical entropy relation, then for dissipative multi-gradient fluids, $F = 0$ is the equation of energy,

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta - \Pi + \rho \Omega \right) + \\ \operatorname{div} \left\{ \left[\left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta + \rho \Omega \right) \mathbf{I} + \boldsymbol{\sigma}_v \right] \mathbf{u} + \boldsymbol{\chi} \right\} + \operatorname{div} \mathbf{Q} - r - \rho \frac{\partial \Omega}{\partial t} &= 0. \end{aligned}$$

Proof : Firstly, the proof comes from the identities in Ω terms, $(\operatorname{div} \mathbf{Q} - r)$ terms and $\boldsymbol{\sigma}_v$ terms. Secondly, we denote

$$\left\{ \begin{array}{l} \mathbf{M}_0 = \rho \boldsymbol{\gamma} + \rho \operatorname{grad} \Xi + \eta \operatorname{grad} \theta, \\ B = \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}), \\ N_0 = \rho \theta \frac{ds}{dt}, \\ F_0 = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta - \Pi \right) + \operatorname{div} \left\{ \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta \right) \mathbf{u} + \boldsymbol{\chi} \right\}, \end{array} \right.$$

then,

$$\begin{aligned} \rho \gamma^T \mathbf{u} + \rho \frac{\partial \Xi}{\partial \mathbf{x}} \mathbf{u} + \eta \frac{\partial \theta}{\partial \mathbf{x}} \mathbf{u} &= \rho \frac{d}{dt} \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + \Xi \right) + \eta \frac{d\theta}{dt} - \rho \frac{\partial \Xi}{\partial t} - \eta \frac{\partial \theta}{\partial t} \\ &= \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta \right) \mathbf{u} \right] - \rho \theta \frac{ds}{dt} \\ &\quad - \rho \frac{\partial \Xi}{\partial t} - \eta \frac{\partial \theta}{\partial t} - \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + \Xi + s \theta \right) \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \right). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left(\nabla \rho : \frac{\partial \Phi_1}{\partial t} \right) &= \operatorname{div} \left(\rho \frac{\partial \Phi_1}{\partial t} \right) - \rho \frac{\partial \operatorname{div} \Phi_1}{\partial t} \\ \left(\nabla^2 \rho : \frac{\partial \Phi_2}{\partial t} \right) &= \operatorname{div} \left[\left(\nabla \rho : \frac{\partial \Phi_2}{\partial t} \right) - \rho \frac{\partial \operatorname{div} \Phi_2}{\partial t} \right] + \rho \frac{\partial \operatorname{div}_2 \Phi_2}{\partial t} \\ &\quad \vdots \\ \left(\nabla^n \rho : \frac{\partial \Phi_n}{\partial t} \right) &= \operatorname{div} \left[\left(\nabla^{n-1} \rho : \frac{\partial \Phi_n}{\partial t} \right) - \left(\nabla^{n-2} \rho : \frac{\partial \operatorname{div} \Phi_n}{\partial t} \right) + \dots \right. \\ &\quad \left. + (-1)^{p-1} \left(\nabla^{n-p} \rho : \frac{\partial \operatorname{div}_{p-1} \Phi_n}{\partial t} \right) + \dots + (-1)^{n-1} \rho \frac{\partial \operatorname{div}_{n-1} \Phi_n}{\partial t} \right] \\ &\quad + (-1)^n \rho \frac{\partial \operatorname{div}_n \Phi_n}{\partial t}, \end{aligned}$$

and a similar relation for η ,

$$\begin{aligned} \left(\nabla \eta : \frac{\partial \Psi_1}{\partial t} \right) &= \operatorname{div} \left(\eta \frac{\partial \Psi_1}{\partial t} \right) - \eta \frac{\partial \operatorname{div} \Psi_1}{\partial t} \\ \left(\nabla^2 \eta : \frac{\partial \Psi_2}{\partial t} \right) &= \operatorname{div} \left[\left(\nabla \eta : \frac{\partial \Psi_2}{\partial t} \right) - \eta \frac{\partial \operatorname{div} \Psi_2}{\partial t} \right] + \eta \frac{\partial \operatorname{div}_2 \Psi_2}{\partial t} \\ &\quad \vdots \\ \left(\nabla^n \eta : \frac{\partial \Psi_n}{\partial t} \right) &= \operatorname{div} \left[\left(\nabla^{n-1} \eta : \frac{\partial \Psi_n}{\partial t} \right) - \left(\nabla^{n-2} \eta : \frac{\partial \operatorname{div} \Psi_n}{\partial t} \right) + \dots \right. \\ &\quad \left. + (-1)^{p-1} \left(\nabla^{n-p} \eta : \frac{\partial \operatorname{div}_{p-1} \Psi_n}{\partial t} \right) + \dots + (-1)^{n-1} \eta \frac{\partial \operatorname{div}_{n-1} \Psi_n}{\partial t} \right] \\ &\quad + (-1)^n \eta \frac{\partial \operatorname{div}_n \Psi_n}{\partial t}. \end{aligned}$$

From Eqs. (3) and (15), we deduce

$$dH = \rho d\mu + \eta dT + \left(\nabla \rho : d\Phi_1 \right) + \left(\nabla \eta : d\Psi_1 \right) + \dots + \left(\nabla^n \rho : d\Phi_n \right) + \left(\nabla^n \eta : d\Psi_n \right)$$

which implies

$$\frac{\partial H}{\partial t} = \rho \frac{\partial \mu}{\partial t} + \eta \frac{\partial T}{\partial t} + \left(\nabla \rho : \frac{\partial \Phi_1}{\partial t} \right) + \left(\nabla \eta : \frac{\partial \Psi_1}{\partial t} \right) + \dots + \left(\nabla^n \rho : \frac{\partial \Phi_n}{\partial t} \right) + \left(\nabla^n \eta : \frac{\partial \Psi_n}{\partial t} \right).$$

Partial derivative $\partial/\partial t$ and div_p commute and consequently,

$$\begin{aligned} \frac{\partial \Pi}{\partial t} &= \rho \frac{\partial \mu}{\partial t} - \rho \frac{\partial \text{div} \Phi_1}{\partial x} + \dots + (-1)^n \rho \frac{\partial \text{div}_n \Phi_n}{\partial t} \\ &+ \eta \frac{\partial T}{\partial t} - \eta \frac{\partial \text{div} \Psi_1}{\partial t} + \dots + (-1)^n \eta \frac{\partial \text{div}_n \Psi_n}{\partial t} \\ &+ \text{div} \left[\rho \frac{\partial \Phi_1}{\partial t} + \dots + \left(\nabla^{n-1} \rho : \frac{\partial \Phi_n}{\partial t} \right) - \left(\nabla^{n-2} \rho : \text{div} \frac{\partial \Phi_n}{\partial t} \right) + \dots \right. \\ &+ (-1)^{p-1} \left(\nabla^{n-p} \rho : \text{div}_{p-1} \frac{\partial \Phi_n}{\partial t} \right) + \dots + (-1)^{n-1} \rho \text{div}_{n-1} \frac{\partial \Phi_n}{\partial t} \\ &+ \eta \frac{\partial \Psi_1}{\partial t} + \dots + \left(\nabla^{n-1} \eta : \frac{\partial \Psi_n}{\partial t} \right) - \left(\nabla^{n-2} \eta : \text{div} \frac{\partial \Psi_n}{\partial t} \right) + \dots \\ &\left. + (-1)^{p-1} \left(\nabla^{n-p} \eta : \text{div}_{p-1} \frac{\partial \Psi_n}{\partial t} \right) + \dots + (-1)^{n-1} \eta \text{div}_{n-1} \frac{\partial \Psi_n}{\partial t} \right], \end{aligned}$$

and from (10), we get

$$\frac{\partial \Pi}{\partial t} = \rho \frac{\partial \Xi}{\partial t} + \eta \frac{\partial \theta}{\partial t} + \text{div} \chi.$$

Then,

$$\begin{aligned} \mathbf{M}_0^T \mathbf{u} &= \frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta - \Pi \right) + \text{div} \left[\left(\frac{1}{2} \rho \mathbf{u}^T \mathbf{u} + \rho \Xi + \eta \theta \right) \mathbf{u} + \chi \right] \\ &- \rho \theta \frac{ds}{dt} - \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + \Xi + s \theta \right) \left(\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) \right), \end{aligned}$$

which implies

$$F_0 - \mathbf{M}_0^T \mathbf{u} - \left(\frac{1}{2} \mathbf{u}^T \mathbf{u} + \Xi + s \theta \right) B - N_0 \equiv 0$$

and proves the theorem.

4.2 The Planck and the Clausius-Duhem inequalities

For all dissipative fluid motions,

$$\text{tr}(\boldsymbol{\sigma}_v D) \geq 0.$$

Then, from relation $N = 0$, we deduce the Planck inequality as in [18],

$$\rho \theta \frac{ds}{dt} + \text{div} \mathbf{Q} - r \geq 0.$$

We consider the Fourier inequality in the general form

$$\mathbf{Q}^T \text{grad} \theta \leq 0$$

and we obtain

$$\rho \frac{ds}{dt} + \operatorname{div} \frac{\mathbf{Q}}{\theta} - \frac{\mathbf{r}}{\theta} \geq 0,$$

which is the extended form of the Clausius-Duhem inequality.

Consequently, multi-gradient fluids are compatible with the first and second laws of thermodynamics.

Final remark: In a forthcoming article, we will prove that system of equations of multi-gradient fluids is a quasi-linear hyperbolic-parabolic system of conservation laws which can be written in Hermitian symmetric-form implying the stability of constant solutions.

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