



**HAL**  
open science

# On moments of integral exponential functionals of additive processes

Paavo Salminen, Lioudmila Vostrikova

► **To cite this version:**

Paavo Salminen, Lioudmila Vostrikova. On moments of integral exponential functionals of additive processes. 2018. hal-01730629v2

**HAL Id: hal-01730629**

**<https://hal.science/hal-01730629v2>**

Preprint submitted on 29 Mar 2018 (v2), last revised 15 Oct 2018 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On moments of integral exponential functionals of additive processes

Paavo Salminen  
Åbo Akademi University,  
Faculty of Science and Engineering,  
FIN-20500 Åbo, Finland,  
phsalmin@abo.fi

Lioudmila Vostrikova  
Université d'Angers,  
Département de Mathématiques,  
F-49045 Angers Cedex 01, France,  
vostrik@univ-angers.fr

## Abstract

Let  $X = (X_t)_{t \geq 0}$  be a real-valued additive process. In this paper we study the integral exponential functionals of  $X$ , namely, the functionals of the form

$$I_{s,t} = \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty.$$

Our main interest is focused on the moments of  $I_{s,t}$  of order  $\alpha \geq 0$ . In case the Laplace exponent of  $X_t$  exists for positive values of the parameter, we derive a recursive (in  $\alpha$ ) integral equation for the moments. This yields a multiple integral formula for the entire positive moments of  $I_{s,t}$ . From these results emerges an easy-to-apply sufficient condition for the finiteness of all the entire moments of  $I_\infty := I_{0,\infty}$ . The corresponding formulas for Lévy processes are also presented. As examples we discuss the finiteness of the moments of  $I_\infty$  when  $X$  is the first hit process associated with a diffusion. In particular, we discuss Bessel processes and geometric Brownian motions.

Keywords: Additive process, process with independent increments, Lévy process, subordinator, first hitting time, diffusion, Bessel process, geometric Brownian motion

AMS Classification: 60J75, 60J60. 60E10

# 1 Introduction

Let  $X = (X_t)_{t \geq 0}$ ,  $X_0 = 0$ , be a real valued additive process, i.e., a strong Markov process with independent increments having càdlàg sample paths which are continuous in probability (cf. Sato [15] p.3). Important examples of additive processes are:

- (a) Deterministic time transformations of Lévy processes, that is, if  $(L_s)_{s \geq 0}$  is a Lévy process and  $s \mapsto g(s)$  is an increasing continuous function such that  $g(0) = 0$  then  $(L_{g(s)})_{s \geq 0}$  is an additive process.
- (b) Integrals of deterministic functions with respect to a Lévy process, that is, if  $(L_s)_{s \geq 0}$  is a Lévy process and  $s \mapsto g(s)$  is a measurable function then

$$Z_t := \int_0^t g(s) dL_s, \quad t \geq 0,$$

is an additive process.

- (c) First hit processes of one-dimensional diffusions, that is, if  $(Y_s)_{s \geq 0}$  is a diffusion taking values in  $[0, \infty)$ , starting from 0, and drifting to  $+\infty$  then

$$H_a := \inf\{t : Y_t > a\}, \quad a \geq 0,$$

is an additive process.

Of course, Lévy processes themselves constitute a large and important class of additive processes.

The basic tool in our analysis is the Laplace exponent of  $X$ . The structure of this is revealed in the the Lévy-Khintchine representation of the infinitely divisible distribution of  $X_t$ . We recall this briefly from Sato [15] Theorem 9.8 p.52. Therein the characteristic function of  $X_t$  is given as follows

$$\mathbf{E} \left( e^{i\lambda X_t} \right) = e^{\Psi(t, \lambda)}, \quad \lambda \in \mathbb{R}, \quad t \geq 0 \tag{1}$$

where

$$\Psi(t, \lambda) := \exp \left( i\lambda b(t) - \frac{1}{2} \lambda^2 c(t) + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x| < 1\}}) \nu_t(dx) \right),$$

$t \mapsto b(t)$ ,  $b(0) = 0$ , is continuous,  $t \mapsto c(t)$ ,  $c(0) = 0$ , is continuous and non-decreasing, and  $\nu_t$  is a measure on  $\mathbf{R}$  such that

- $\nu_0(\mathbf{R}) = 0$ ,

- for all  $t$ ,  $\nu_t(\{0\}) = 0$  and  $\int_{\mathbf{R}}(|x|^2 \wedge 1)\nu_t(dx) < \infty$ ,
- for all  $B \in \mathcal{B}(\mathbf{R})$  and  $s \leq t$ ,  $\nu_s(B) \leq \nu_t(B)$ ,
- for all  $B \in \mathcal{B}(\mathbf{R})$  with  $B \subset \{x : |x| > \varepsilon\}$ ,  $\varepsilon > 0$ ,

$$\nu_s(B) \rightarrow \nu_t(B) \quad \text{as } s \rightarrow t.$$

Notice that since  $X$  is assumed to be continuous in probability it follows that  $t \mapsto \Psi(t, \lambda)$  is for all  $\lambda \in \mathbb{R}$  continuous.

The aim of this paper is to study integral exponential functionals of  $X$ , i.e., functionals of the form

$$I_{s,t} := \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty, \quad (2)$$

in particular, the moments of  $I_{s,t}$ . We refer also to a companion paper [12] where stochastic calculus is used to study the Mellin transforms of  $I_{s,t}$  when the underlying additive process is a semi martingale with absolutely continuous characteristics.

The main result of the paper is a recursive equation, see (7) in Theorem 2.3, which generalizes the formula for Lévy processes presented in Urbanik [16] Example 3.4 p.309 and Carmona, Petit and Yor [4] Proposition 3.3 p.87, see also Bertoin and Yor [2] Section 3.1 p.195. This formula for Lévy processes is also displayed below in (18). In Epifani, Lijoi and Prünster [7] an extension of the Lévy process formula to integral functionals up to  $t = \infty$  of increasing additive processes is discussed, and their formula (7) in Proposition 5 p.798 can be seen as a special case of our formula (15) – as we found out after finishing our work. Such an extension is also indicated in [2] on p.196. In spite of these closely related results we feel that it is worthwhile to provide a more thorough discussion of this interesting topic. We also give new (to our best knowledge ) applications of the formulas for first hit processes of diffusions and present explicit results for Bessel processes and geometric Brownian motions.

In mathematical finance the interest in integral exponential functionals is coming from the studies of perpetuities containing liabilities, perpetuities under the influence of economical factors (see, for example, Kardaras and Robertson [11]), and also of pricing Asian options and related questions (see, for instance, Dufresne [6], Carr and Wu [5], Jeanblanc, Yor, Chesney [9] and references therein). We refer to the survey paper [2] for further results, potential applications, and many references, in particular, for Lévy

processes but also for additive processes. For applications of increasing additive processes in Bayesian statistics we refer to [7] and references therein. Additive processes can also be found in representations of self-decomposable laws, see Sato [14] and Jeanblanc, Pitman and Yor [10].

The paper is structured as follows. Next section contains the main results of the paper. In particular, the recursive equation for the entire moments  $\mathbf{E}(I_{s,t}^n)$ ,  $n = 1, 2, \dots$  is established. We also give formulas for Lévy processes. The paper is concluded with examples on first hitting time processes. It is proved that for Bessel processes drifting to  $+\infty$  the integral exponential functional of the first hit process has all the moments and for geometric Brownian motion the corresponding functional has only some moments.

## 2 Main results

Let  $(X_t)_{t \geq 0}$  be an additive process and define for  $0 \leq s \leq t \leq \infty$  and  $\alpha \geq 0$

$$m_{s,t}^{(\alpha)} := \mathbf{E}(I_{s,t}^\alpha) = \mathbf{E}\left(\left(\int_s^t e^{-X_u} du\right)^\alpha\right), \quad \alpha \geq 0, \quad (3)$$

and

$$m_t^{(\alpha)} := m_{0,t}^{(\alpha)}, \quad m_\infty^{(\alpha)} := m_{0,\infty}^{(\alpha)}.$$

In this section we derive a recursive integral equation for  $m_{s,t}^{(\alpha)}$  under the following assumption:

- (A)  $X_t$  has for all  $t \geq 0$  a finite Laplace exponent for positive values of the parameter, that is, there exists for all  $t \geq 0$  and for all  $\lambda \geq 0$  a function  $t \mapsto \Phi(t, \lambda)$  such that

$$\mathbf{E}(e^{-\lambda X_t}) = e^{-\Phi(t;\lambda)}. \quad (4)$$

and

$$\Phi(t;0) = \Phi(0;\lambda) = 0. \quad (5)$$

If  $X$  is a Lévy process we write (with a slight abuse of the notation) formula (4) as

$$\mathbf{E}\left(e^{-\lambda X_t}\right) = e^{-t\Phi(\lambda)}. \quad (6)$$

The functions  $\Psi$  and  $\Phi$  are clearly connected as  $\Psi(t, i\lambda) = -\Phi(t, \lambda)$ . As pointed out in Introduction,  $t \mapsto \Psi(t, \lambda)$  is continuous and, therefore also  $t \mapsto \Phi(t, \lambda)$  is continuous. This property has an important – obvious – consequence formulated in the next lemma which is easily proved, e.g., with Jensen's inequality.

**Lemma 2.1.** *Under assumption (A), for  $0 \leq \alpha \leq 1$  it holds  $m_{s,t}^{(\alpha)} < \infty$  for all  $0 \leq s \leq t < \infty$ .*

Assumption (A) is valid, in particular, for increasing additive processes. Important examples of these are the first hit processes for diffusions (cf. (c) in Introduction). Moments of exponential functionals of some first hit processes are discussed in the next section. Assumption (A) holds also for additive processes of type (a) in Introduction if the underlying Lévy process fullfills Assumption (A).

**Remark 2.2.** *In the case when  $X$  is a semi-martingale with absolutely continuous characteristics, the sufficient condition (see Proposition 1 in [12]) for the existence of the Laplace exponent as in Assumption (A) in terms of the measure  $\nu_t$  is as follows: for  $\lambda \geq 0$  and  $t \geq 0$*

$$\int_{\{|x|>1\}} \left( e^{-\lambda x} + |x| \right) \nu_t(dx) < \infty \quad \Rightarrow \quad \mathbf{E}(e^{-\lambda X_t}) < \infty.$$

The main result of the paper is given in the next theorem. In the proof we are using similar ideas as in [4].

**Theorem 2.3.** *For  $0 \leq s \leq t < \infty$  and  $\alpha \geq 1$  the moments  $m_{s,t}^{(\alpha)}$  are finite and satisfy the recursive equation*

$$m_{s,t}^{(\alpha)} = \alpha \int_s^t m_{u,t}^{(\alpha-1)} e^{-(\Phi(u;\alpha) - \Phi(u;\alpha-1))} du. \quad (7)$$

*Proof.* We start with by introducing the shifted functional  $\widehat{I}_{s,t}$  via

$$\widehat{I}_{s,t} := \int_0^{t-s} e^{-(X_{u+s} - X_s)} du.$$

Clearly,

$$\widehat{I}_{s,t} = e^{X_s} I_{s,t} = e^{X_s} \int_s^t e^{-X_u} du, \quad (8)$$

and we have

$$\frac{d}{ds} I_{s,t}^\alpha = \alpha I_{s,t}^{\alpha-1} \frac{d}{ds} I_{s,t} = -\alpha I_{s,t}^{\alpha-1} e^{-X_s} = -\alpha \widehat{I}_{s,t}^{\alpha-1} e^{-\alpha X_s}.$$

Consequently,

$$I_{s,t}^\alpha - I_{0,t}^\alpha = -\alpha \int_0^s \widehat{I}_{u,t}^{\alpha-1} e^{-\alpha X_u} du$$

The independence of increments implies that  $\widehat{I}_{u,t}^{\alpha-1}$  and  $e^{-\alpha X_u}$  are independent. Hence,

$$\mathbf{E} (I_{s,t}^\alpha - I_{0,t}^\alpha) = -\alpha \int_0^s \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (9)$$

Clearly,  $I_{s,t} \rightarrow 0$  a.s. when  $s \uparrow t$ . Hence, applying monotone convergence in (9) yields

$$\mathbf{E} (I_{0,t}^\alpha) = \alpha \int_0^t \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (10)$$

Putting (9) and (10) together results to the equation

$$\mathbf{E} (I_{s,t}^\alpha) = \alpha \int_s^t \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (11)$$

From (8) evoking the independence of  $\widehat{I}_{u,t}^{\alpha-1}$  and  $e^{-\alpha X_u}$  we have

$$\mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) = \mathbf{E} (I_{u,t}^{\alpha-1}) / \mathbf{E} \left( e^{-(\alpha-1)X_u} \right). \quad (12)$$

Finally, using (12) in (11) and recalling (4) yields (7). The claim that  $m_{s,t}^{(\alpha)}$  is finite follows by induction from the recursive equation (7) and Lemma 2.1.  $\square$

**Remark 2.4.** In [12] the recursive equation (7) is derived via stochastic calculus in case the additive process  $X$  is a semimartingale with absolutely continuous characteristics.

**Corollary 2.5.** Let  $(X_t)_{t \geq 0}$  be a Lévy process with the Laplace exponent as in (6). Then the recursive equation (7) for  $s = 0$  and  $t < \infty$  is equivalent with

$$m_t^{(\alpha)} = \alpha e^{-t\Phi(\alpha)} \int_0^t m_u^{(\alpha-1)} e^{u\Phi(\alpha)} du. \quad (13)$$

*Proof.* Put  $s = 0$  in (7) to obtain

$$m_t^{(\alpha)} = \alpha \int_0^t m_{u,t}^{(\alpha-1)} e^{-u(\Phi(\alpha) - \Phi(\alpha-1))} du. \quad (14)$$

Consider

$$\begin{aligned}
m_{u,t}^{(\alpha-1)} &= \mathbf{E} \left( \left( \int_u^t e^{-X_v} dv \right)^{\alpha-1} \right) \\
&= \mathbf{E} \left( e^{-(\alpha-1)X_u} \left( \int_u^t e^{-(X_v-X_u)} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} \mathbf{E} \left( \left( \int_0^{t-u} e^{-(X_{v+u}-X_u)} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} \mathbf{E} \left( \left( \int_0^{t-u} e^{-X_v} dv \right)^{\alpha-1} \right) \\
&= e^{-u\Phi(\alpha-1)} m_{t-u}^{(\alpha-1)}.
\end{aligned}$$

Substituting this expression in (14) and changing variables yield the claimed equation.  $\square$

For positive integer values on  $\alpha$  the recursive equation (7) can be solved explicitly to obtain the formula (15) in the next proposition. However, we offer another proof highlighting the symmetry properties present in the expressions of the moments of the exponential functional.

**Proposition 2.6.** *For  $0 \leq s \leq t \leq \infty$  and  $n = 1, 2, \dots$  it holds*

$$\begin{aligned}
m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\
&\cdots \int_{t_{n-1}}^t dt_n \exp \left( - \sum_{k=1}^n (\Phi(t_k; n-k+1) - \Phi(t_k; n-k)) \right). \tag{15}
\end{aligned}$$

*In particular,  $m_{s,\infty}^{(n)} < \infty$  if and only if the multiple integral on the right hand side of (15) is finite.*

*Proof.* Let  $t < \infty$  and consider

$$\begin{aligned}
m_{s,t}^{(n)} &= \mathbf{E} \left( \left( \int_s^t e^{-X_u} du \right)^n \right) \\
&= \mathbf{E} \left( \int_s^t \cdots \int_s^t e^{-X_{t_1} - \cdots - X_{t_n}} dt_1 \cdots dt_n \right) \\
&= n! \mathbf{E} \left( \int_s^t dt_1 e^{-X_{t_1}} \int_{t_1}^t dt_2 e^{-X_{t_2}} \cdots \int_{t_{n-1}}^t dt_n e^{-X_{t_n}} \right) \\
&= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \mathbf{E} \left( e^{-(X_{t_1} + \cdots + X_{t_n})} \right),
\end{aligned}$$

where, in the third step, we use that  $(t_1, t_2, \dots, t_n) \mapsto e^{-(X_{t_1} + \dots + X_{t_n})}$  is symmetric. By the independence of the increments

$$\mathbf{E}(e^{-\alpha X_t}) = \mathbf{E}(e^{-\alpha(X_t - X_s) - \alpha X_s}) = \mathbf{E}(e^{-\alpha(X_t - X_s)}) \mathbf{E}(e^{-\alpha X_s}).$$

Consequently,

$$\mathbf{E}(e^{-\alpha(X_t - X_s)}) = \mathbf{E}(e^{-\alpha X_t}) / \mathbf{E}(e^{-\alpha X_s}) = e^{-(\Phi(t; \alpha) - \Phi(s; \alpha))}.$$

Since,

$$X_{t_1} + \dots + X_{t_n} = \sum_{k=1}^n (n - k + 1) (X_{t_k} - X_{t_{k-1}}), \quad t_0 := 0,$$

we have

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \dots \\ &\dots \int_{t_{n-1}}^t dt_n \exp\left(-\sum_{k=1}^n (\Phi(t_k; n - k + 1) - \Phi(t_{k-1}; n - k + 1))\right). \end{aligned}$$

Using here (5) yields the claimed formula (15). The statement concerning the finiteness of  $m_{s,\infty}^{(n)}$  follows by applying the monotone convergence theorem as  $t \rightarrow \infty$  on both sides of (15).  $\square$

In the next corollary we give a sufficient condition for  $m_\infty^{(n)}$  to be finite for all  $n$ .

**Corollary 2.7.** *Variable  $I_\infty$  has all the positive moments if*

$$\int_0^\infty e^{-(\Phi(s;n) - \Phi(s;n-1))} ds < \infty \quad (16)$$

for all  $n = 1, 2, \dots$ .

*Proof.* From (15) we have

$$m_t^{(n)} \leq n! \prod_{k=1}^n \int_0^\infty e^{-(\Phi(s;n) - \Phi(s;n-1))} ds. \quad (17)$$

The right hand side of (17) is finite if (16) holds. Let  $t \rightarrow \infty$  in (17). By monotone convergence,  $m_\infty^{(n)} = \lim_{t \rightarrow \infty} m_t^{(n)}$ , and the claim is proved.  $\square$

Formula (18) below extends the corresponding formula for subordinators found [16], see also [2] p.195, for Lévy processes satisfying Assumption (A). It is a straightforward implication of Proposition 2.6.

**Corollary 2.8.** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with the Laplace exponent as in (6) and define  $n^* := \min\{n \in \{1, 2, \dots\} : \Phi(n) \leq 0\}$ . Then*

$$m_\infty^{(n)} := \mathbf{E}(I_\infty^n) = \begin{cases} \frac{n!}{\prod_{k=1}^n \Phi(k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*. \end{cases} \quad (18)$$

**Example 2.9.** A much studied functional is obtained when taking  $X = (X_t)_{t \geq 0}$  with  $X_t = \sigma W_t + \mu t$ ,  $\sigma > 0, \mu > 0$ , where  $(W_t)_{t \geq 0}$  is a standard Brownian motion. In the papers by Dufresne [6] and Yor [17] (see also Salminen and Yor [13]) it is proved that

$$I_\infty := \int_0^\infty e^{-(\sigma W_s + \mu s)} ds \sim H_0^{(\delta)}, \quad (19)$$

where  $H_0^\delta$  is the first hitting time of 0 for a Bessel process of dimension  $\delta = 2(1 - (\mu/\sigma^2))$  starting from  $\sigma/2$ , and  $\sim$  means "is identical in law with". In particular, it holds

$$\int_0^\infty \exp(-(2W_s + \mu s)) ds \sim \frac{1}{2Z_\mu}, \quad (20)$$

where  $Z_\mu$  is a gamma-distributed random variable with rate 1 and shape  $\mu/2$ . We refer to [6] for a discussion showing how the functional on the left hand side of (19) arises as the present value of a perpetuity in a discrete model after a limiting procedure. Since the Lévy exponent in this case is

$$\Phi(\lambda) = \lambda\mu - \frac{1}{2}\lambda^2\sigma^2,$$

the criterium in Corollary 2.8 yields

$$\mathbf{E}(I_\infty^n) < \infty \Leftrightarrow n < 2\mu/\sigma^2,$$

which readily can also be checked from (20).

### 3 First hit processes of one-dimensional diffusions

We recall first some facts concerning the first hitting times of one-dimensional (or linear) diffusions. Let now  $Y = (Y_s)_{s \geq 0}$  be a linear diffusion taking values in an interval  $I$ . To fix ideas assume that  $I$  equals  $\mathbf{R}$  or  $(0, \infty)$  or  $[0, \infty)$  and that

$$\limsup_{s \rightarrow \infty} Y_s = +\infty \quad \text{a.s.} \quad (21)$$

Assume  $Y_0 = v$  and consider for  $a \geq v$  the first hitting time

$$H_a := \inf\{s : Y_s > a\}.$$

Defining  $X_t := H_{t+v}$ ,  $t \geq 0$ , it is easily seen – since  $Y$  is a strong Markov process – that  $X = (X_t)_{t \geq 0}$  is an increasing purely discontinuous additive process starting from 0. Moreover, from assumption (21) it follows that  $X_t < \infty$  a.s. for all  $t$ . The process  $X$  satisfies Assumption (A) in Section 2. Indeed, using the well known characterization of the Laplace transform of  $H_a$  we have

$$\mathbf{E}_v(e^{-\beta X_t}) = \mathbf{E}_v(e^{-\beta H_{t+v}}) = \frac{\psi_\beta(v)}{\psi_\beta(t+v)}, \quad t \geq 0, \quad (22)$$

where  $\beta \geq 0$ ,  $\mathbf{E}_v$  is the expectation associated with  $Y$  starting from  $v$ , and  $\psi_\beta$  is a unique (up to a multiple) positive and increasing solution of the ODE

$$Gf(x) = \beta f(x) \quad (23)$$

satisfying the appropriate boundary condition at 0 in case  $I = [0, \infty)$  and 0 is reflecting. In (23)  $G$  denotes the differential operator associated with  $Y$ . In the absolutely continuous case  $G$  is of the form

$$Gf(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \quad f \in C^2(I), x \in I,$$

where  $\sigma$  and  $\mu$  are continuous functions. For details about diffusions (and further references), see Itô and McKean [8], and [3]. The Laplace transform of  $X_t$  can also be represented as follows

$$\mathbf{E}_v(e^{-\beta X_t}) = \exp\left(-\int_v^{t+v} S(du) \int_0^\infty (1 - e^{-\beta x})n(u, dx)\right), \quad (24)$$

where  $S$  is the scale function, and  $n$  is a kernel such that for all  $v \in I$  and  $t \geq 0$

$$\int_v^{t+v} \int_0^\infty (1 \wedge x)n(u, dx)S(du) < \infty.$$

Representation (24) clearly reveals the structure of  $X$  as a process with independent increments. Comparing with the notation in Introduction, we have

$$\nu_t(dx) = \int_v^{t+v} n(u, dx)S(du).$$

From (22) and (24) we may conclude that

$$\int_0^\infty (1 - e^{-\beta x})n(u, dx) = \lim_{w \rightarrow u-} \frac{1 - \mathbf{E}_w(e^{-\beta X_u})}{S(u) - S(w)}. \quad (25)$$

We now pass to present examples of exponential functionals of first hit processes. Firstly, we study Bessel processes satisfying (21) and show, in particular, that the exponential functional of the first hit process has all the moments. In our second example it is seen that the exponential functional of the first hit process of geometric Brownian motion has only finitely many moments depending on the values of the parameters.

**Example 3.1. Bessel processes.** Let  $Y$  be a Bessel process starting from  $v > 0$ . The differential operator associated with  $Y$  is given by

$$Gf(x) = \frac{1}{2}f''(x) + \frac{\delta - 1}{2x}f'(x), \quad x > 0,$$

where  $\delta \in \mathbf{R}$  is called the dimension parameter. From [3] we extract the following information

- for  $\delta \geq 2$  the boundary point 0 is entrance-not-exit and (21) holds,
- for  $0 < \delta < 2$  the boundary point 0 is non-singular and (21) holds when the boundary condition at 0 is reflection,
- for  $\delta \leq 0$  (21) does not hold.

In case when (21) is valid the Laplace exponent for the first hit process  $X = (X_t)_{t \geq 0}$  is given for  $v > 0$  and  $t \geq 0$  by

$$\mathbf{E}_v(e^{-\beta X_t}) = \frac{\psi_\beta(v)}{\psi_\beta(t)} = \frac{v^{1-\delta/2} I_{\delta/2-1}(v\sqrt{2\beta})}{t^{1-\delta/2} I_{\delta/2-1}((t+v)\sqrt{2\beta})}, \quad (26)$$

where  $\mathbf{E}_v$  is the expectation associated with  $Y$  when started from  $v$  and  $I$  denotes the modified Bessel function of the first kind. For simplicity, we wish to study the exponential functional of  $X$  when  $v = 0$ . To find the

Laplace exponent when  $v = 0$  we let  $v \rightarrow 0$  in (26). For this, recall that for  $p \neq -1, -2, \dots$

$$I_p(v) \simeq \frac{1}{\Gamma(p+1)} \left(\frac{v}{2}\right)^p \quad \text{as } v \rightarrow 0. \quad (27)$$

Consequently,

$$\begin{aligned} \mathbf{E}_0(e^{-\beta X_t}) &= \lim_{v \rightarrow 0} \mathbf{E}_v(e^{-\beta X_t}) \\ &= \frac{1}{\Gamma(\nu+1)} \left(\frac{\sqrt{2\beta}}{2}\right)^{\delta/2-1} \frac{t^{\delta/2-1}}{I_{\delta/2-1}(t\sqrt{2\beta})} \\ &=: e^{-\Phi(t;\beta)}. \end{aligned}$$

The validity of (16), that is, the finiteness of the positive moments, can now be checked by exploiting the asymptotic behaviour of  $I_p$  saying that for all  $p \in \mathbf{R}$  (see Abramowitz and Stegun [1], 9.7.1 p.377)

$$I_p(t) \simeq e^t / \sqrt{2\pi t} \quad \text{as } t \rightarrow \infty. \quad (28)$$

Indeed, for  $n = 1, 2, \dots$

$$\begin{aligned} e^{-(\Phi(t;n)-\Phi(t;n-1))} &= \frac{n^{\delta/2-1}}{I_{\delta/2-1}(t\sqrt{2n})} \frac{I_{\delta/2-1}(t\sqrt{2(n-1)})}{(n-1)^{\delta/2-1}} \\ &\simeq \left(\frac{n}{n-1}\right)^{\delta/2-1} \left(\frac{n}{n-1}\right)^{1/4} e^{-t(\sqrt{2n}-\sqrt{2(n-1)})}, \end{aligned}$$

which clearly is integrable at  $+\infty$ . Consequently, by Corollary 2.7, the integral functional

$$\int_0^\infty e^{-X_t} dt$$

has all the (positive) moments.

**Example 3.2. Geometric Brownian motion.** Let  $Y = (Y_s)_{s \geq 0}$  be a geometric Brownian motion with parameters  $\sigma^2 > 0$  and  $\mu \in \mathbf{R}$ , i.e.,

$$Y_s = \exp\left(\sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s\right)$$

where  $W = (W_s)_{s \geq 0}$  is a standard Brownian motion initiated at 0. Since  $W_s/s \rightarrow 0$  a.s. when  $s \rightarrow \infty$  it follows

- $\lim_{s \rightarrow \infty} Y_s = +\infty$  a.s if  $\mu > \frac{1}{2}\sigma^2$ ,

- $\lim_{s \rightarrow \infty} Y_s = 0$  a.s if  $\mu < \frac{1}{2}\sigma^2$ .
- $\limsup_{s \rightarrow \infty} Y_s = +\infty$  and  $\liminf_{s \rightarrow \infty} Y_s = 0$  a.s. if  $\mu = \frac{1}{2}\sigma^2$ .

Consequently, condition (21) is valid if and only if  $\mu \geq \frac{1}{2}\sigma^2$ . Since  $Y_0 = 1$  we consider the first hitting times of the points  $a \geq 1$ . Consider

$$\begin{aligned}
H_a &:= \inf\{s : Y_s = a\} \\
&= \inf\left\{s : \exp\left(\sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s\right) = a\right\} \\
&= \inf\left\{s : \sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s = \log a\right\} \\
&= \inf\left\{s : W_s + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} s = \frac{1}{\sigma} \log a\right\}.
\end{aligned}$$

We assume now that  $\sigma > 0$  and  $\mu \geq \frac{1}{2}\sigma^2$ . Let  $\nu := (\mu - \frac{1}{2}\sigma^2)\sigma$ . Then  $H_a$  is identical in law with the first hitting time of  $(\log a)/\sigma$  for Brownian motion with drift  $\nu \geq 0$  starting from 0. Consequently, letting  $X_t := H_{1+t}$  we have for  $t \geq 0$

$$\begin{aligned}
\mathbf{E}_1(e^{-\beta X_t}) &= \frac{\psi_\beta(0)}{\psi_\beta(\log(1+t)/\sigma)} \\
&= \exp\left(-\left(\sqrt{2\beta + \nu^2} - \nu\right) \frac{\log(1+t)}{\sigma}\right) \\
&= (1+t)^{-\left(\sqrt{2\beta + \nu^2} - \nu\right)/\sigma}, \\
&=: \exp(-\Phi(t; \beta)).
\end{aligned} \tag{29}$$

where  $\mathbf{E}_1$  is the expectation associated with  $Y$  when started from 1 and

$$\psi_\beta(x) = \exp\left(\left(\sqrt{2\beta + \nu^2} - \nu\right)x\right)$$

is the increasing fundamental solution for Brownian motion with drift (see [3] p.132). Notice that the additive process  $X$  is a deterministic time change of the first hit process of Brownian motion with drift, which is a subordinator. We use now Proposition 2.6 to study the moments of the perpetual integral functional

$$I_\infty = \int_0^\infty e^{-X_s} ds.$$

To simplify the notation (cf. (29)) introduce

$$\rho(\beta) := \left(\sqrt{2\beta + \nu^2} - \nu\right) / \sigma.$$

By formula (15) the  $n$ th moment is given by

$$\begin{aligned} \mathbf{E}_1(I_\infty^n) &= n! \int_0^\infty dt_1 (1+t_1)^{-(\rho(n)-\rho(n-1))} \int_{t_1}^\infty dt_2 (1+t_2)^{-(\rho(n-1)-\rho(n-2))} \\ &\quad \times \int_{t_2}^\infty dt_3 \cdots \int_{t_{n-1}}^\infty dt_n (1+t_n)^{-\rho(1)} \\ &= \begin{cases} \frac{n!}{\prod_{k=1}^n (\rho(k) - k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*, \end{cases} \end{aligned}$$

where

$$n^* := \min\{n \in \{1, 2, \dots\} : \rho(n) - n \leq 0\}.$$

Condition (16) in Corollary 2.7 takes in this case the form

$$\rho(n) - \rho(n-1) > 1. \quad (30)$$

This being a sufficient condition for the finiteness of  $m_\infty^{(n)}$  we have

$$\rho(n) - \rho(n-1) > 1 \quad \Rightarrow \quad \rho(n) - n > 0. \quad (31)$$

Consider now the case  $\nu = 0$ . Then

$$\rho(n) > n \quad \Leftrightarrow \quad n < \frac{2}{\sigma^2}, \quad (32)$$

i.e., smaller the volatility (i.e.  $\sigma$ ) more moments of  $I_\infty$  exist, as expected. Moreover, in this case

$$\begin{aligned} \rho(n) - \rho(n-1) > 1 &\quad \Leftrightarrow \quad \sqrt{2n} + \sqrt{2(n-1)} < \frac{2}{\sigma} \\ &\quad \Leftrightarrow \quad 2n - 1 + \sqrt{4n(n-1)} < \frac{2}{\sigma^2} \end{aligned} \quad (33)$$

showing, in particular, that when  $\sigma$  is “small” there exist “many”  $n$  satisfying (32) but not (33).

**Acknowledgement.** This research was partially supported by Defimath project of the Research Federation of “Mathématiques des Pays de la Loire”, by PANORisk project “Pays de la Loire” region, and by the Magnus Ehrnrooth Foundation, Finland.

## References

- [1] Abramowitz, M., and Stegun, I. : *Mathematical Functions, 9th printing*, Dover publications, Inc., New York, 1970.
- [2] Bertoin, J. and Yor, M: *Exponential functionals of Lévy processes*, Probability Surveys, Vol. 2, 191–212 (2005).
- [3] Borodin, A. and Salminen, P.: *Handbook of Brownian motion - Facts and Formulae, 2nd ed., Corrected printing*, Birkhäuser Verlag, Basel-Boston-Berlin, 2015.
- [4] Carmona, P., Petit, F. and Yor, M.: *On the distribution and asymptotic results for exponential functionals of Levy processes*, in "Exponential functionals and principal values related to Brownian motion; a collection of research papers", ed. M. Yor, Biblioteca de la Revista Matematica IberoAmericana (1997).
- [5] Carr, P. and Wu, L.: *Time-changed Lévy processes and option pricing*, Journal of Financial Economics, Vol. 71,113–141 (2004).
- [6] Dufresne, D.: *The distribution of a perpetuity, with applications to risk theory and pension funding*. Scand. Actuarial J., Vol. 1-2, 39–79, (1990).
- [7] Epifani, I., Lijoi, A. and Prünster, I.: *Exponential functionals and means of neutral-to-right priors*, Biometrika, Vol. 90(4), 791–808 (2003).
- [8] Itô, K. and McKean, H.P.: *Diffusion Processes and Their Sample Paths*, Springer Verlag, Berlin, Heidelberg, 1974.
- [9] Jeanblanc, M., Yor, M. and Chesnay, M.: *Mathematical Methods for Financial Markets*, Springer Finance Textbook, 2009.
- [10] Jeanblanc, M. , Pitman, J. and Yor, M.: *Self-similar processes with independent increments associated with Lévy and Bessel*, Stochastic Process. Appl., Vol. 100, 223–231 (2002).
- [11] Kardaras, C. and Robertson, S. : *Continuous time perpetuities and time reversal of diffusions*, Preprint: arXiv: 1411.7551v1, (2014).
- [12] Salminen, P. and Vostrikova, L.: *On exponential functionals of processes with independent increments*, Probab. Theory Appl. (to appear), Preprint: arXiv:1610.08732, (2016).

- [13] Salminen, P. and Yor, M.: *Perpetual integral functionals as hitting and occupation times*, Electronic Journal of Probability, Vol. 10, Issue 11, 371-419, (2005).
- [14] Sato, K.: *Self-similar processes with independent increments*, Probab. Theory Related Fields, Vol. 89, 285-300 (1991).
- [15] Sato, K.: *Lévy Processes and Infinitely Divisible Distributions, 2nd ed.*, Cambridge University Press., 2013.
- [16] K. Urbanik: *Functionals of transient stochastic processes with independent increments*, Studia Math., Vol. 103(3), 299-315, (1992).
- [17] Yor, M.: *On some exponential functionals of Brownian motion*, Adv. Appl. Probab., Vol. 24, 509-531 (1992).