



**HAL**  
open science

# On moments of exponential functionals of additive processes

Paavo Salminen, Lioudmila Vostrikova

► **To cite this version:**

Paavo Salminen, Lioudmila Vostrikova. On moments of exponential functionals of additive processes. 2018. hal-01730629v1

**HAL Id: hal-01730629**

**<https://hal.science/hal-01730629v1>**

Preprint submitted on 13 Mar 2018 (v1), last revised 15 Oct 2018 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On moments of exponential functionals of additive processes

Paavo Salminen

Åbo Akademi University,  
Faculty of Science and Engineering,  
Fänriksgatan 3 B,  
FIN-20500 Åbo, Finland,

Lioudmila Vostrikova

Université d'Angers,  
Département de Mathématiques,  
2, Bd Lavoisier,  
F-49045 Angers Cedex 01, France,

March 13, 2018

## Abstract

Let  $X = (X_t)_{t \geq 0}$  be a real-valued additive process, i.e., a process with independent increments. In this paper we study the exponential integral functionals of  $X$ , namely, the functionals of the form

$$I_{s,t} = \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty.$$

Our main interest is focused on the moments of  $I_{s,t}$  of order  $\alpha \geq 0$ . In the case when the Laplace exponent of  $X_t$  is explicitly known, we derive a recursive (in  $\alpha$ ) integral equation for the moments. This yields a multiple integral formula for the entire positive moments of  $I_{s,t}$ . From these results emerges an easy-to-apply sufficient condition for the finiteness of all the entire moments of  $I_\infty := I_{0,\infty}$ . The corresponding formulas for Lévy processes are also presented. As examples we discuss the finiteness of the moments of  $I_\infty$  when  $X$  is the first hit process associated with a diffusion. In particular, we discuss the exponential functionals related with Bessel processes and geometric Brownian motions.

Keywords: Additive process, Lévy process, subordinator, first hitting time, diffusion, Bessel process, geometric Brownian motion

AMS Classification: 60J75, 60J60. 60E10

# 1 Introduction

The aim of this paper is to study the exponential integral functionals of an additive process  $X$ , i.e., the functionals of the form

$$I_{s,t} = \int_s^t \exp(-X_u) du, \quad 0 \leq s < t \leq \infty. \quad (1)$$

An interesting special case hereby is  $s = 0$  and  $t = \infty$ , and for this we introduce the notation

$$I_\infty := I_{0,\infty} = \int_0^\infty \exp(-X_s) ds. \quad (2)$$

For convenience of the readers we recall the definition of an additive process (cf. Sato [15], p. 3). A stochastic process  $X = (X_s)_{s \geq 0}$  defined in an appropriate probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  is called an additive process or a process with independent increments if

- (i)  $X_0 = 0$  a.s.,
- (ii)  $X$  is continuous in probability,
- (iii)  $t \mapsto X_t$  is right continuous with left limits a.s.,
- (iv) for all  $n = 1, 2, \dots$  and  $0 \leq t_0 < t_1 < \dots < t_n$ , random variables

$$X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$$

are independent.

A process with independent increments is called a Lévy process if for all  $s \leq t$ , the increment  $X_t - X_s$  is identical in law with  $X_{t-s}$ .

From the independence of the increments it follows that the distribution of  $X_t$  is infinitely divisible. It is known (cf. Sato [15], Theorem 9.8, p.52) that the characteristic function of  $X_t$  has the Lévy-Khintchine representation: for  $\lambda \in \mathbb{R}$  and  $t \geq 0$

$$\mathbf{E} \left( e^{i\lambda X_t} \right) = e^{\Psi(t,\lambda)}, \quad (3)$$

where the characteristic exponent

$$\Psi(t, \lambda) := \exp \left( i\lambda b(t) - \frac{1}{2} \lambda^2 c(t) + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x| < 1\}}) \nu_t(dx) \right)$$

with  $t \mapsto b(t)$ ,  $b(0) = 0$ , continuous real-valued function,  $t \mapsto c(t)$ ,  $c(0) = 0$ , continuous and non-decreasing function, and  $\nu_t$  being a measure on  $\mathbf{R}$  such that

- (a)  $\nu_0(\mathbf{R}) = 0$ ,
- (b) for all  $t$ ,  $\nu_t(\{0\}) = 0$  and  $\int_{\mathbf{R}}(|x|^2 \wedge 1)\nu_t(dx) < \infty$ ,
- (c) for all  $B \in \mathcal{B}(\mathbf{R})$  and  $s \leq t$ ,  $\nu_s(B) \leq \nu_t(B)$ ,
- (d) for all  $B \in \mathcal{B}(\mathbf{R})$  with  $B \subset \{x : |x| > \varepsilon\}$ ,  $\varepsilon > 0$ ,

$$\nu_s(B) \rightarrow \nu_t(B) \quad \text{as } s \rightarrow t.$$

Lévy processes constitute a large and important class of additive processes. As discussed below, there exist many results on functionals  $I_\infty$  when  $X$  is a Lévy process. In some important but particular cases the distribution of these functionals are also known. In this case the calculation of the moments can be done directly. For example, the functional  $I_\infty$  with  $X = W^{(\mu)}$ , that is,  $X$  is a Brownian motion with drift  $\mu > 0$ , has been studied by Dufresne [7] and Yor [17] (see also Salminen and Yor [14]), who proved that for  $a > 0$

$$\int_0^\infty \exp(-2aW_s^{(\mu)}) ds \stackrel{(d)}{=} H_0(R^{(\delta)}), \quad (4)$$

where  $R^{(\delta)}$  is a Bessel process of dimension  $\delta = 2(1 - (\mu/a))$  started at  $1/a$ , and  $H_0 := \inf\{t \geq 0 : R_t^{(\delta)} = 0\}$  is the first hitting time of 0, and  $\stackrel{(d)}{=}$  reads "is identical in law with". In particular, for  $a = 1$  it holds

$$\int_0^\infty \exp(-2W_s^{(\mu)}) ds \stackrel{(d)}{=} \frac{1}{2Z_\mu}, \quad (5)$$

where  $Z_\mu$  is a gamma-distributed random variable with rate 1 and shape  $\mu$ . The exponential functionals  $I_\infty$  in the case when  $X$  is a Lévy process have been studied by Carmona et al. in [6] (for English translation, see Yor(2001)). In particular, they consider, e.g., the asymptotic behavior of  $I_\infty$  for  $\alpha$ -stable Lévy processes, give a formula for the positive moments of  $I_\infty$  when  $X$  is a subordinator, and characterize - for a class of Lévy processes - the density of  $I_\infty$  as a solution of an integro-differential equation. The last mentioned problem is also addressed (among other topics) in Bertoin et al. [3]. We

refer to the survey paper [4] by Bertoin and Yor for further results, potential applications, and many references. In [4], some conditions for the finiteness of  $I_\infty$  are also presented.

However, to our best knowledge, the case when  $X$  is a general non-homogeneous additive process has not been sufficiently considered up to now. Nevertheless, these processes arise naturally in particular in the representation of self-decomposable laws on  $\mathbb{R}^+$  (see Sato [15], [16], Jeanblanc, Pitman, Yor [11]). It also occurs in non-parametric Bayesian statistics, where the mean of a random distribution chosen from a neutral-to-the-right prior can be represented as the exponential functional of an increasing additive process (see Epifani, Lijoi, Prnster [8]). In mathematical finance the question is related with the perpetuities containing liabilities, perpetuities under the influence of economical factors (see, for example, Kardaras and Robertson [12]), and also with the price of Asian options and related questions (see, for instance, Dufresne [7], Jeanblanc, Yor, Chesney [10] and references there).

In [13] we studied the Mellin transform of the exponential functionals of the process  $X$  being the process with independent increments and semimartingale with absolutely continuous characteristics. In this paper we study the exponential functionals of the general additive processes. Our results on the moments of the general additive processes will require the following assumption:

**Assumption (A)** *For all  $t \geq 0$ ,  $X_t$  has a finite Laplace exponent for positive values of the parameter, that is, there exists for all  $t \geq 0$  and for all  $\lambda \geq 0$  a function  $t \mapsto \Phi(t, \lambda)$  such that*

$$\mathbf{E}(e^{-\lambda X_t}) = e^{-\Phi(t; \lambda)}. \quad (6)$$

and

$$\Phi(t; 0) = \Phi(0; \lambda) = 0. \quad (7)$$

If  $X$  is a Lévy process we write (with a slight abuse of the notation) formula (6) as

$$\mathbf{E}(e^{-\lambda X_t}) = e^{-t\Phi(\lambda)}. \quad (8)$$

The functions  $\Psi$  and  $\Phi$  are clearly connected as  $\Psi(t, i\lambda) = -\Phi(t, \lambda)$ . We point out that  $t \mapsto \Psi(t, \lambda)$  is continuous and, therefore also  $t \mapsto \Phi(t, \lambda)$  is continuous.

The mentioned Assumption (A) holds for the processes with bounded jumps, for the processes with negative light tails, and in particular, for increasing additive processes. From Sato [15], Theorem 25.3, p. 159, we also deduce that Assumption (A) is equivalent to the condition : for  $t \geq 0$

$$\int_0^t \int_{x < -1} e^{-\lambda x} \nu_s(dx) ds.$$

The additive processes can be constructed in the following ways:

- (1) Deterministic time transformations of Lévy processes, that is, if  $(L_s)_{s \geq 0}$  is a Lévy process and  $s \mapsto g(s)$  is an increasing continuous function such that  $g(0) = 0$  then  $(L_{g(s)})_{s \geq 0}$  is an additive process.
- (2) Integrals of deterministic functions with respect to a Lévy process, that is, if  $(L_s)_{s \geq 0}$  is a Lévy process and  $s \mapsto g(s)$  is a measurable function then

$$Z_t := \int_0^t g(s) dL_s, \quad t \geq 0,$$

is an additive process.

- (3) First hit processes of one-dimensional diffusions, that is, if  $(Y_s)_{s \geq 0}$  is a diffusion taking values in  $[0, \infty)$ , starting from 0, and drifting to  $+\infty$  then

$$H_a := \inf\{t \geq 0 : Y_t = a\}, \quad a \geq 0,$$

is an additive process.

In this paper we will study the examples related with the construction (3).

The paper is structured as follows. In the next section in Proposition 2.1 the result on the a.s. finiteness of  $I_\infty$  is presented. Section 3 contains the main results of the paper, in particular, the recursive equation for the entire moments  $\mathbf{E}(I_{s,t}^n)$ ,  $n = 1, 2, \dots$  is established (see Theorem 3.2) and moment formula (see Proposition 3.5). We also give formulas for Lévy processes and recover some results in Carmona et al. [6]. The paper is concluded with examples on the first hitting time processes. It is proved that for Bessel processes drifting to  $\infty$  the exponential integral functional of the hit process has all the moments and for geometric Brownian motion the corresponding functional has only some moments.

## 2 Finiteness of $I_\infty$

By the regularity of the sample paths of  $X$  it is clear that  $I_{s,t} < \infty$  a.s. for  $0 \leq s < t < \infty$ , but it may, of course, happen that  $I_{s,\infty} = \infty$  with a positive probability. In the next proposition we give sufficient conditions for the a.s. finiteness of  $I_{s,\infty}$ .

**Proposition 2.1.** *Let  $f : [0, \infty) \mapsto [0, \infty)$  be strictly positive and increasing. Assume that there exist  $l > 0$  and  $l' < l$  such that*

$$\limsup_{t \rightarrow \infty} \frac{X_t}{f(t)} = l \quad \text{a.s.} \quad (9)$$

and

$$\int_0^\infty \exp(-l' f(t)) dt < \infty. \quad (10)$$

Then,

$$\mathbf{P}(I_\infty < \infty) = 1.$$

In particular, if

$$\lim_{t \rightarrow \infty} \frac{X_t}{t} > 0, \quad (11)$$

then  $I_\infty < \infty$  a.s.

*Proof.* From (9) it is seen that for each  $\epsilon > 0$  and almost all  $\omega \in \Omega$  there exists  $t_\epsilon(\omega) > 0$  such that for all  $t > t_\epsilon(\omega)$

$$X_t(\omega) > (l - \epsilon)f(t) \quad (12)$$

Let now  $\epsilon = l - l'$ . Then using (12) and (10) yields for almost all  $\omega$

$$\int_{t_\epsilon(\omega)}^{+\infty} e^{-X_t} dt < \int_{t_\epsilon(\omega)}^{+\infty} e^{-l' f(t)} dt < \infty$$

and  $I_\infty < \infty$   $P$ -a.s. as claimed. The particular case follows immediately, and the proof is complete.  $\square$

**Remark 2.2.** *In the case of Levy processes, the condition (11) is necessary and sufficient for the a.s. finiteness of  $I_\infty$  ( see Theorem 1 in [4] and also for other equivalent conditions).*

### 3 Moments of exponential functionals

Let  $(X_t)_{t \geq 0}$  be an additive process and define for  $0 \leq s \leq t \leq \infty$  and  $\alpha \geq 0$

$$m_{s,t}^{(\alpha)} := \mathbf{E} (I_{s,t}^\alpha) = \mathbf{E} \left( \left( \int_s^t e^{-X_u} du \right)^\alpha \right), \quad \alpha \geq 0, \quad (13)$$

and

$$m_t^{(\alpha)} := m_{0,t}^{(\alpha)}, \quad m_\infty^{(\alpha)} := m_{0,\infty}^{(\alpha)}.$$

In this section we derive a recursive integral equation for  $m_{s,t}^{(\alpha)}$  under the mentioned Assumption (A). We start with the remark about the moments of the order smaller than 1.

**Lemma 3.1.** *Under Assumption (A),  $m_{s,t}^{(\alpha)} < \infty$  for all  $0 \leq s \leq t < \infty$  and  $0 \leq \alpha \leq 1$ .*

*Proof.* For  $0 \leq \alpha \leq 1$  we have

$$\begin{aligned} m_{s,t}^{(\alpha)} &= \mathbf{E}(I_{s,t}^\alpha; I_{s,t} \leq 1) + \mathbf{E}(I_{s,t}^\alpha; I_{s,t} > 1) \\ &\leq 1 + \mathbf{E}(I_{s,t}) = 1 + \int_s^t e^{-\Phi(u;1)} du < \infty. \end{aligned}$$

□

The main result of the paper is given in the next theorem. In the proof we are using similar ideas as in [6] and in [4] Section 3.1 in their study of subordinators. In fact, it is indicated - but not developed further - in [4] that the moment calculations presented therein can be extended for increasing additive processes. However, our case is more general and leads in of we assume that (A) holds for the rest of this section.

**Theorem 3.2.** *For  $0 \leq s \leq t < \infty$  and  $\alpha \geq 1$  the moments  $m_{s,t}^{(\alpha)}$  are finite and satisfy the recursive equation*

$$m_{s,t}^{(\alpha)} = \alpha \int_s^t m_{u,t}^{(\alpha-1)} e^{-(\Phi(u;\alpha) - \Phi(u;\alpha-1))} du. \quad (14)$$

*Proof.* We start with by introducing the shifted functional  $\widehat{I}_{s,t}$  via

$$\widehat{I}_{s,t} := \int_0^{t-s} e^{-(X_{u+s} - X_s)} du.$$



Clearly,

$$\widehat{I}_{s,t} = e^{X_s} I_{s,t} = e^{X_s} \int_s^t e^{-X_u} du, \quad (15)$$

and we have

$$\frac{d}{ds} I_{s,t}^\alpha = \alpha I_{s,t}^{\alpha-1} \frac{d}{ds} I_{s,t} = -\alpha I_{s,t}^{\alpha-1} e^{-X_s} = -\alpha \widehat{I}_{s,t}^{\alpha-1} e^{-\alpha X_s}.$$

Consequently,

$$I_{s,t}^\alpha - I_{0,t}^\alpha = -\alpha \int_0^s \widehat{I}_{u,t}^{\alpha-1} e^{-\alpha X_u} du$$

The independence of increments implies that  $\widehat{I}_{u,t}^{\alpha-1}$  and  $e^{-\alpha X_u}$  are independent. Hence,

$$\mathbf{E} (I_{s,t}^\alpha - I_{0,t}^\alpha) = -\alpha \int_0^s \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (16)$$

Clearly,  $I_{s,t} \rightarrow 0$  a.s. when  $s \uparrow t$ . Hence, applying monotone convergence in (16) yields

$$\mathbf{E} (I_{0,t}^\alpha) = \alpha \int_0^t \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (17)$$

Putting (16) and (17) together results to the equation

$$\mathbf{E} (I_{s,t}^\alpha) = \alpha \int_s^t \mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) \mathbf{E} (e^{-\alpha X_u}) du. \quad (18)$$

From (15) evoking the independence of  $\widehat{I}_{u,t}^{\alpha-1}$  and  $e^{-\alpha X_u}$  we have

$$\mathbf{E} \left( \widehat{I}_{u,t}^{\alpha-1} \right) = \mathbf{E} (I_{u,t}^{\alpha-1}) / \mathbf{E} (e^{-(\alpha-1)X_u}). \quad (19)$$

Finally, using (19) in (18) and recalling (6) yields (14). The claim that  $m_{s,t}^{(\alpha)}$  is finite follows by induction from the recursive equation (14) and Lemma 3.1.  $\square$

**Remark 3.3.** In [13] the recursive equation (14) is derived via stochastic calculus in case the additive process  $X$  is a semi-martingale with absolutely continuous characteristics.

**Corollary 3.4.** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with the Laplace exponent as in (8). Then the recursive equation (14) for  $s = 0$  and  $t < \infty$  is equivalent with*

$$m_t^{(\alpha)} = \alpha e^{-t\Phi(\alpha)} \int_0^t m_u^{(\alpha-1)} e^{u\Phi(\alpha)} du. \quad (20)$$

*Proof.* Put  $s = 0$  in (14) to obtain

$$m_t^{(\alpha)} = \alpha \int_0^t m_{u,t}^{(\alpha-1)} e^{-u(\Phi(\alpha) - \Phi(\alpha-1))} du. \quad (21)$$

Then consider

$$\begin{aligned} m_{u,t}^{(\alpha-1)} &= \mathbf{E} \left( \left( \int_u^t e^{-X_v} dv \right)^{\alpha-1} \right) \\ &= \mathbf{E} \left( e^{-(\alpha-1)X_u} \left( \int_u^t e^{-(X_v - X_u)} dv \right)^{\alpha-1} \right) \\ &= e^{-u\Phi(\alpha-1)} \mathbf{E} \left( \left( \int_0^{t-u} e^{-(X_{v+u} - X_u)} dv \right)^{\alpha-1} \right) \\ &= e^{-u\Phi(\alpha-1)} \mathbf{E} \left( \left( \int_0^{t-u} e^{-X_v} dv \right)^{\alpha-1} \right) \\ &= e^{-u\Phi(\alpha-1)} m_{t-u}^{(\alpha-1)}. \end{aligned}$$

Substituting this expression in (21) and changing variables yield the claimed equation.  $\square$

For positive integer values on  $\alpha$  the recursive equation (14) can be solved explicitly to obtain the formula (22) in the next proposition. However, we offer another proof highlighting the symmetry properties present in the expressions of the moments of the exponential functional.

**Proposition 3.5.** *For  $0 \leq s \leq t \leq \infty$  and  $n = 1, 2, \dots$  it holds*

$$\begin{aligned} m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \cdots \\ &\cdots \int_{t_{n-1}}^t dt_n \exp \left( - \sum_{k=1}^n (\Phi(t_k; n - k + 1) - \Phi(t_k; n - k)) \right). \end{aligned} \quad (22)$$

In particular,  $m_{s,\infty}^{(n)} < \infty$  if and only if the multiple integral on the right hand side of (22) is finite.

*Proof.* Let  $t < \infty$  and consider

$$\begin{aligned}
m_{s,t}^{(n)} &= \mathbf{E} \left( \left( \int_s^t e^{-X_u} du \right)^n \right) \\
&= \mathbf{E} \left( \int_s^t \dots \int_s^t e^{-X_{t_1} - \dots - X_{t_n}} dt_1 \dots dt_n \right) \\
&= n! \mathbf{E} \left( \int_s^t dt_1 e^{-X_{t_1}} \int_{t_1}^t dt_2 e^{-X_{t_2}} \dots \int_{t_{n-1}}^t dt_n e^{-X_{t_n}} \right) \\
&= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \dots \int_{t_{n-1}}^t dt_n \mathbf{E} (e^{-(X_{t_1} + \dots + X_{t_n})}),
\end{aligned}$$

where, in the third step, we use that

$$(t_1, t_2, \dots, t_n) \mapsto e^{-(X_{t_1} + \dots + X_{t_n})}$$

is symmetric. By the independence of the increments

$$\mathbf{E} (e^{-\alpha X_t}) = \mathbf{E} (e^{-\alpha(X_t - X_s) - \alpha X_s}) = \mathbf{E} (e^{-\alpha(X_t - X_s)}) \mathbf{E} (e^{-\alpha X_s}).$$

Consequently,

$$\mathbf{E} (e^{-\alpha(X_t - X_s)}) = \mathbf{E} (e^{-\alpha X_t}) / \mathbf{E} (e^{-\alpha X_s}) = e^{-(\Phi(t;\alpha) - \Phi(s;\alpha))}.$$

Since,

$$X_{t_1} + \dots + X_{t_n} = \sum_{k=1}^n (n - k + 1) (X_{t_k} - X_{t_{k-1}}), \quad t_0 := 0,$$

we have

$$\begin{aligned}
m_{s,t}^{(n)} &= n! \int_s^t dt_1 \int_{t_1}^t dt_2 \dots \\
&\quad \dots \int_{t_{n-1}}^t dt_n \exp \left( - \sum_{k=1}^n (\Phi(t_k; n - k + 1) - \Phi(t_{k-1}; n - k + 1)) \right).
\end{aligned}$$

Using here (7) yields the claimed formula (22). The statement concerning the finiteness of  $m_{s,\infty}^{(n)}$  follows by applying the monotone convergence theorem as  $t \rightarrow \infty$  on both sides of (22).  $\square$

In the next corollary we give a sufficient condition for  $m_\infty^{(n)}$  to be finite for all  $n$ .

**Corollary 3.6.** *Variable  $I_\infty$  has all the positive moments if for all  $n = 1, 2, \dots$*

$$\int_0^\infty e^{-(\Phi(s;n)-\Phi(s;n-1))} ds < \infty. \quad (23)$$

*Proof.* From (22) we have

$$m_t^{(n)} \leq n! \prod_{k=1}^n \int_0^\infty e^{-(\Phi(s;n)-\Phi(s;n-1))} ds. \quad (24)$$

The right hand side of (24) is finite if (23) holds. Let  $t \rightarrow \infty$  in (24). By monotone convergence,  $m_\infty^{(n)} = \lim_{t \rightarrow \infty} m_t^{(n)}$ , and the claim is proved.  $\square$

Formula (25) below extends the corresponding formula for subordinators found on p. 195 in [4] for Lévy processes satisfying Assumption (A). It is a straightforward implication of Proposition 3.5.

**Corollary 3.7.** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with the Laplace exponent as in (8) and define  $n^* := \min\{n \in \{1, 2, \dots\} : \Phi(n) \leq 0\}$ . Then*

$$m_\infty^{(n)} := \mathbf{E}(I_\infty^n) = \begin{cases} \frac{n!}{\prod_{k=1}^n \Phi(k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*. \end{cases} \quad (25)$$

## 4 First hit processes of one-dimensional diffusions

We recall first some facts concerning the first hitting times of one-dimensional (or linear) diffusions. Let now  $Y = (Y_s)_{s \geq 0}$  be a linear diffusion taking values in an interval  $I$ . To fix ideas assume that  $I$  equals  $\mathbf{R}$  or  $(0, \infty)$  or  $[0, \infty)$  and that

$$\limsup_{s \rightarrow \infty} Y_s = +\infty \quad \text{a.s.} \quad (26)$$

Assume  $Y_0 = v$  and consider for  $a \geq v$  the first hitting time

$$H_a := \inf\{s \geq 0 : Y_s = a\}.$$

Defining  $X_t := H_{t+v}$ ,  $t \geq 0$ , it is easily seen – since  $Y$  is a strong Markov process – that  $X = (X_t)_{t \geq 0}$  is an increasing purely discontinuous additive process starting from 0. Moreover, from assumption (26) it follows that  $X_t < \infty$  a.s. for all  $t$ . The process  $X$  satisfies Assumption (A) given in Introduction. Indeed, using the well known characterization of the Laplace transform of  $H_a$  we have

$$\mathbf{E}_v(e^{-\beta X_t}) = \mathbf{E}_v(e^{-\beta H_{t+v}}) = \frac{\psi_\beta(v)}{\psi_\beta(t+v)}, \quad t \geq 0, \quad (27)$$

where  $\beta \geq 0$ ,  $\mathbf{E}_v$  is the expectation associated with  $Y$  starting from  $v$ , and  $\psi_\beta$  is a unique (up to a multiple) positive and increasing solution of the ODE

$$(Gf)(x) = \beta f(x) \quad (28)$$

satisfying the appropriate boundary condition at 0 in case  $I = [0, \infty)$  and 0 is reflecting. In (28)  $G$  denotes the differential operator associated with  $Y$ . In the absolutely continuous case  $G$  is of the form

$$(Gf)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \quad f \in C^2(I), x \in I,$$

where  $\sigma$  and  $\mu$  are continuous functions. For details about diffusions and further references, see [9] and [5]. The Laplace transform of  $X_t$  can also be represented as follows

$$\mathbf{E}_v(e^{-\beta X_t}) = \exp\left(-\int_v^{t+v} S(du) \int_0^\infty (1 - e^{-\beta x})n(u, dx)\right), \quad (29)$$

where  $S$  is the scale function, and  $n$  is a kernel such that for all  $v \in I$  and  $t \geq 0$

$$\int_v^{t+v} \int_0^\infty (1 \wedge x)n(u, dx)S(du) < \infty.$$

Representation (29) clearly reveals the structure of  $X$  as a process with independent increments. Comparing with the notation in Introduction, we have

$$\nu_t(dx) = \int_v^{t+v} n(u, dx)S(du).$$

From (27) and (29) we may conclude that

$$\int_0^\infty (1 - e^{-\beta x})n(u, dx) = \lim_{w \rightarrow u^-} \frac{1 - \mathbf{E}_w(e^{-\beta X_u})}{S(u) - S(w)}. \quad (30)$$

We now pass to present examples of exponential functionals of the first hit processes. First we study Bessel processes satisfying (26) and show, in particular, that the exponential functional of the first hit process has all the moments. In our second example it is seen that the exponential functional of the first hit process of geometric Brownian motion has only finitely many moments depending on the values of the parameters.

**Example 4.1. Bessel processes.** Let  $Y$  be a Bessel process starting from  $v > 0$ . The differential operator associated with  $Y$  is given by

$$Gf(x) = \frac{1}{2}f''(x) + \frac{\delta - 1}{2x}f'(x),$$

where  $\delta \in \mathbf{R}$  is called the dimension parameter. From [5] we extract the following information

- a) For  $\delta \geq 2$  the boundary point 0 is entrance-not-exit and (26) holds.
- b) For  $0 < \delta < 2$  the boundary point 0 is non-singular and (26) holds when the boundary condition at 0 is reflection.
- c) For  $\delta \leq 0$  (26) does not hold.

In case when (26) is valid the Laplace exponent for the first hit process  $X = (X_t)_{t \geq 0}$  is given for  $v > 0$  and  $t \geq 0$  by

$$\mathbf{E}_v(e^{-\beta X_t}) = \frac{\psi_\beta(v)}{\psi_\beta(t)} = \frac{v^{1-\frac{\delta}{2}} I_{\frac{\delta}{2}-1}(v\sqrt{2\beta})}{t^{1-\frac{\delta}{2}} I_{\frac{\delta}{2}-1}((t+v)\sqrt{2\beta})}, \quad (31)$$

where  $I$  denotes the modified Bessel function of the first kind. For simplicity, we wish to study the exponential functional of  $X$  when  $v = 0$ . To find the Laplace exponent when  $v = 0$  we let  $v \rightarrow 0$  in (31). For this, recall that for  $p \neq -1, -2, \dots$

$$I_p(v) \simeq \frac{1}{\Gamma(p+1)} \left(\frac{v}{2}\right)^p \quad \text{as } v \rightarrow 0. \quad (32)$$

Consequently,

$$\begin{aligned}
\mathbf{E}_0(e^{-\beta X_t}) &= \lim_{\nu \rightarrow 0} \mathbf{E}_\nu(e^{-\beta X_t}) \\
&= \frac{1}{\Gamma(\nu + 1)} \left( \frac{\sqrt{2\beta}}{2} \right)^{\frac{\delta}{2}-1} \frac{t^{\frac{\delta}{2}-1}}{I_{\frac{\delta}{2}-1}(t\sqrt{2\beta})} \\
&=: e^{-\Phi(t;\beta)}.
\end{aligned}$$

The validity of (23), that is, the finiteness of the positive moments, can now be checked by exploiting the asymptotic behaviour of  $I_p$  saying that for all  $p \in \mathbf{R}$  (see [1], 9.7.1 p. 377)

$$I_p(t) \simeq e^t / \sqrt{2\pi t} \quad \text{as } t \rightarrow \infty. \quad (33)$$

Indeed, for  $n = 1, 2, \dots$

$$\begin{aligned}
e^{-(\Phi(t;n)-\Phi(t;n-1))} &= \frac{n^{\frac{\delta}{2}-1} I_{\frac{\delta}{2}-1}(t\sqrt{2(n-1)})}{I_{\frac{\delta}{2}-1}(t\sqrt{2n}) (n-1)^{\frac{\delta}{2}-1}} \\
&\simeq \left( \frac{n}{n-1} \right)^{\frac{\delta}{2}-1} \left( \frac{n}{n-1} \right)^{1/4} e^{-t(\sqrt{2n}-\sqrt{2(n-1)})},
\end{aligned}$$

which clearly is integrable at  $+\infty$ . Consequently, by Corollary 3.6, the integral functional

$$\int_0^\infty e^{-X_t} dt$$

has all the (positive) moments.

**Example 4.2. Geometric Brownian motion.** Let  $Y = (Y_s)_{s \geq 0}$  be a geometric Brownian motion with parameters  $\sigma^2 > 0$  and  $\mu \in \mathbf{R}$ , i.e.,

$$Y_s = \exp \left( \sigma W_s + \left( \mu - \frac{1}{2} \sigma^2 \right) s \right)$$

where  $W = (W_s)_{s \geq 0}$  is a standard Brownian motion initiated at 0. Since  $W_s/s \rightarrow 0$  a.s. when  $s \rightarrow \infty$  it follows

- a)  $Y_s \rightarrow +\infty$  a.s as  $s \rightarrow \infty$  if  $\mu > \frac{1}{2}\sigma^2$ ,
- b)  $Y_s \rightarrow 0$  a.s as  $s \rightarrow \infty$  if  $\mu < \frac{1}{2}\sigma^2$ .

c)  $\limsup_{s \rightarrow \infty} Y_s = +\infty$  and  $\liminf_{s \rightarrow \infty} Y_s = 0$  a.s. if  $\mu = \frac{1}{2}\sigma^2$ .

Consequently, condition (26) is valid if and only if  $\mu \geq \frac{1}{2}\sigma^2$ . Since  $Y_0 = 1$  we consider the first hitting times of the points  $a \geq 1$ . Consider

$$\begin{aligned} H_a &:= \inf\{s \geq 0 : Y_s = a\} \\ &= \inf\left\{s \geq 0 : \exp\left(\sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s\right) = a\right\} \\ &= \inf\left\{s \geq 0 : \sigma W_s + \left(\mu - \frac{1}{2}\sigma^2\right)s = \log a\right\} \\ &= \inf\left\{s \geq 0 : W_s + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} s = \frac{1}{\sigma} \log a\right\}. \end{aligned}$$

We assume now that  $\sigma > 0$  and  $\mu \geq \frac{1}{2}\sigma^2$ . Let  $\nu := \frac{1}{\sigma}(\mu - \frac{1}{2}\sigma^2)$ . Then  $H_a$  is identical in law with the first hitting time of  $\frac{1}{\sigma} \log a$  for Brownian motion with drift  $\nu \geq 0$  starting from 0. Consequently, letting  $X_t := H_{1+t}$  we have for  $t \geq 0$

$$\begin{aligned} \mathbf{E}_1(e^{-\beta X_t}) &= \frac{\psi_\beta(0)}{\psi_\beta(\log(1+t)/\sigma)} \\ &= \exp\left(-\left(\sqrt{2\beta + \nu^2} - \nu\right) \frac{\log(1+t)}{\sigma}\right) \\ &= (1+t)^{-\left(\sqrt{2\beta + \nu^2} - \nu\right)/\sigma}, \\ &=: \exp(-\Phi(t; \beta)). \end{aligned} \tag{34}$$

where  $\mathbf{E}_1$  is the expectation associated with  $Y$  when started from 1 and

$$\psi_\beta(x) = \exp\left(\left(\sqrt{2\beta + \nu^2} - \nu\right)x\right)$$

is the increasing fundamental solution for Brownian motion with drift (see [5] p. 132). Notice that the additive process  $X$  is a deterministic time change of the first hit process of Brownian motion with drift, which is a subordinator. We use now Proposition 3.5 to study the moments of the perpetual integral functional

$$I_\infty = \int_0^\infty e^{-X_s} ds.$$

To simplify the notation (cf. (34)) introduce

$$\rho(\beta) := \frac{1}{\sigma} \left(\sqrt{2\beta + \nu^2} - \nu\right).$$



and let

$$n^* := \min\{n \in \{1, 2, \dots\} : \rho(n) - n \leq 0\}.$$

By formula (22) the  $n$ th moment is given by

$$\begin{aligned} \mathbf{E}_1(I_\infty^n) &= n! \int_0^\infty dt_1 (1+t_1)^{-(\rho(n)-\rho(n-1))} \int_{t_1}^\infty dt_2 (1+t_2)^{-(\rho(n-1)-\rho(n-2))} \\ &\quad \times \int_{t_2}^\infty dt_3 \cdots \int_{t_{n-1}}^\infty dt_n (1+t_n)^{-\rho(1)} \\ &= \begin{cases} \frac{n!}{\prod_{k=1}^n (\rho(k) - k)}, & \text{if } n < n^*, \\ +\infty, & \text{if } n \geq n^*. \end{cases} \end{aligned}$$

Condition (23) in Corollary 3.6 takes in this case the form

$$\rho(n) - \rho(n-1) > 1. \quad (35)$$

This being a sufficient condition for the finiteness of  $m_\infty^{(n)}$  we have

$$\rho(n) - \rho(n-1) > 1 \quad \Rightarrow \quad \rho(n) - n > 0. \quad (36)$$

Consider now the case  $\nu = 0$ . Then

$$\rho(n) > n \quad \Leftrightarrow \quad n < \frac{2}{\sigma^2}, \quad (37)$$

i.e., smaller the volatility (i.e.  $\sigma$ ) more moments of  $I_\infty$  exist, as expected. Moreover, in this case

$$\begin{aligned} \rho(n) - \rho(n-1) > 1 &\quad \Leftrightarrow \quad \sqrt{2n} + \sqrt{2(n-1)} < \frac{2}{\sigma} \\ &\quad \Leftrightarrow \quad 2n - 1 + \sqrt{4n(n-1)} < \frac{2}{\sigma^2} \end{aligned} \quad (38)$$

showing, in particular, that the condition (38) is stronger than (37).

## 5 Acknowledgements

This research was partially supported by Defimath project of the Research Federation of "Mathématiques des Pays de la Loire" and by PANORisk project "Pays de la Loire" region.

## References

- [1] M. Abramowitz, I. Stegun 1970. *Mathematical functions, 9th printing*, Dover publications, Inc., New York.
- [2] J. Bertoin. (1996) *Lévy processes*, Cambridge University Press.
- [3] J. Bertoin, A. Lindler, R. Maller (2008.) *On continuity properties of the law of integrals of Lévy processes*, Séminaire de probabilités XLI, 1934, 137–159.
- [4] J. Bertoin, M. Yor (2005). *Exponential functionals of Lévy processes*, Probability Surveys, 191-212.
- [5] A. Borodin, P. Salminen (2002). *Handbook of Brownian motion - Facts and Formulae, 2nd ed.*, Birkhäuser Verlag, Basel-Boston-Berlin.
- [6] P. Carmona, F. Petit, M. Yor (2004). *On the distribution and asymptotic results for exponential functionals of Levy processes*, in "Exponential functionals and principal values related to Brownian motion", Biblioteca de la Revista Matematica IberoAmericana (1997) ISSN 0207-2230.
- [7] D. Dufresne (1990). *The distribution of a perpetuity, with applications to risk theory and pension funding*. Scand. Actuarial J., 1-2, 39-79.
- [8] I. Epifani, A. Lijoi and I. Prnster (2003) *Exponential functionals and means of neutral-to-the-right priors*, Biometrika 90, 791-808.
- [9] K. Itô, H.P. McKean (1974). *Diffusion Processes and Their Sample Paths*, Springer Verlag, Berlin, Heidelber.
- [10] M. Jeanblanc, M. Yor, M. Chesnay. (2009 ) *Mathematical Methods for Financial Markets*, Springer Finance Textbook.
- [11] M. Jeanblanc, J. Pitman, and M. Yor (2002) *Self-similar processes with independent increments associated with Lévy and Bessel Stochastic Process*, Stochastic processes Appl. 100, 223-231.
- [12] C. Kardaras, S. Robertson (2016). *Continuous time perpetuities and time reversal of diffusions*, Finance and Stochastics. ISSN 0949-2984.

- [13] P. Salminen, L. Vostrikova (2018). *On exponential functionals of processes with independent increments*, Theory of probab. appl., to appear.
- [14] P. Salminen, M. Yor (2005). *Perpetual Integral Functionals as Hitting and Occupation Times*, Electronic Journal of Probability, Vol. 10, Issue 11, 371-419.
- [15] K. Sato (2013). *Lévy Processes and Infinitely Divisible Distributions*, 2nd ed., Cambridge University Press.
- [16] K. Sato (1991) *Self-similar processes with independent increments*, Probab. Theory Related Fields 89, 285-300.
- [17] M. Yor (1992). *On some exponential functionals of Brownian motion*, Adv. Appl. Probab., Vol. 24, 509-531.