

STOCHASTIC DEFLATOR FOR AN ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

SUPPLEMENTARY MATERIAL

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1. APPENDIX 1: CONSTRUCT CORRELATED BROWNIAN MOTIONS

In Appendix 1, we provide details of the calculations to construct correlated Brownian motions in our model. We start with the Brownian motion parts of interest rate process and stock price process. Then we derive the Brownian motion parts of default intensity process and convenience yield process step by step. Readers are referred to Shreve (2004) Chapter 3.4 for discussions about quadratic variation.

Step 1

$$\text{Let } dW_r = dW_0, \text{ then } dW_s = \rho_{rs}dW_r + \sqrt{1-\rho_{rs}^2}dW_1.$$

pf.

$$dW_r dW_s = dW_0 \left(\rho_{rs}dW_0 + \sqrt{1-\rho_{rs}^2}dW_1 \right) = \rho_{rs}dt$$

$$\begin{aligned} dW_s dW_s &= \left(\rho_{rs}dW_0 + \sqrt{1-\rho_{rs}^2}dW_1 \right) \left(\rho_{rs}dW_0 + \sqrt{1-\rho_{rs}^2}dW_1 \right) \\ &= \left(\rho_{rs}^2 + 1 - \rho_{rs}^2 \right) dt = dt \end{aligned}$$

Step 2

$$dW_\chi = \rho_{r\chi}dW_r + \rho'_{s\chi}dW_1 + \rho'_{\chi\chi}dW_2,$$

$$\text{where } \rho'_{s\chi} = \frac{\rho_{s\chi} - \rho_{rs}\rho_{r\chi}}{\sqrt{1-\rho_{rs}^2}}, \rho'_{\chi\chi} = \sqrt{\frac{1-\rho_{rs}^2 - \rho_{r\chi}^2 - \rho_{s\chi}^2 + 2\rho_{rs}\rho_{r\chi}\rho_{s\chi}}{1-\rho_{rs}^2}}.$$

pf.

$$dW_r dW_\chi = dW_0 \left(\rho_{r\chi}dW_0 + \rho'_{s\chi}dW_1 + \rho'_{\chi\chi}dW_2 \right) = \rho_{r\chi}dt$$

$$\begin{aligned} dW_s dW_\chi &= \left(\rho_{rs}dW_0 + \sqrt{1-\rho_{rs}^2}dW_1 \right) \left(\rho_{r\chi}dW_0 + \rho'_{s\chi}dW_1 + \rho'_{\chi\chi}dW_2 \right) \\ &= \left(\rho_{rs}\rho_{r\chi} + \rho'_{s\chi}\sqrt{1-\rho_{rs}^2} \right) dt = \rho_{s\chi}dt \end{aligned}$$

$$\rho_{rs}\rho_{r\chi} + \rho'_{s\chi}\sqrt{1-\rho_{rs}^2} = \rho_{s\chi}, \text{ then } \rho'_{s\chi} = \frac{\rho_{s\chi} - \rho_{rs}\rho_{r\chi}}{\sqrt{1-\rho_{rs}^2}}.$$

$$\begin{aligned} dW_\chi dW_\chi &= \left(\rho_{r\chi}dW_0 + \rho'_{s\chi}dW_1 + \rho'_{\chi\chi}dW_2 \right) \left(\rho_{r\chi}dW_0 + \rho'_{s\chi}dW_1 + \rho'_{\chi\chi}dW_2 \right) \\ &= \left(\rho_{r\chi}^2 + \rho_{s\chi}^2 + \rho_{\chi\chi}^2 \right) dt = dt \end{aligned}$$

$$\rho_{r\chi}^2 + \rho_{s\chi}^2 + \rho_{\chi\chi}^2 = 1, \text{ then } \rho_{\chi\chi}^2 = 1 - \rho_{r\chi}^2 - \rho_{s\chi}^2 = 1 - \rho_{r\chi}^2 - \frac{(\rho_{s\chi} - \rho_{rs}\rho_{r\chi})^2}{1-\rho_{rs}^2}.$$

$$\text{Then, } \rho'_{\chi\chi} = \sqrt{\frac{1-\rho_{rs}^2 - \rho_{r\chi}^2 - \rho_{s\chi}^2 + 2\rho_{rs}\rho_{r\chi}\rho_{s\chi}}{1-\rho_{rs}^2}}.$$

Step 3

$$dW_\gamma = \rho_{r\gamma}dW_r + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3,$$

$$\text{where } \rho_{S\gamma}'' = \frac{\rho_{S\gamma} - \rho_{rS}\rho_{r\gamma}}{\sqrt{1 - \rho_{rS}^2}},$$

$$\rho_{XY}'' = \frac{\rho_{XY} - \rho_{rX}\rho_{rY} - \rho_{S_X}\rho_{S_Y} - \rho_{rS}^2\rho_{XY} + \rho_{rS}\rho_{rX}\rho_{S_Y} + \rho_{rS}\rho_{rY}\rho_{S_X}}{\sqrt{1 + \rho_{rS}^4 - 2\rho_{rS}^3\rho_{rX}\rho_{S_X} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{rX}^2 + \rho_{rS}^2\rho_{S_X}^2 - \rho_{rX}^2 - \rho_{S_X}^2 + 2\rho_{rS}\rho_{rX}\rho_{S_X}}},$$

$$\rho_{YY}'' = \sqrt{1 - \rho_{rY}^2 - \rho_{S_Y}''^2 - \rho_{XY}''^2}.$$

pf.

$$dW_r dW_\gamma = dW_0 (\rho_{r\gamma}dW_0 + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3) = \rho_{r\gamma}dt$$

$$\begin{aligned} dW_S dW_\gamma &= (\rho_{rS}dW_0 + \sqrt{1 - \rho_{rS}^2}dW_1)(\rho_{r\gamma}dW_0 + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3) \\ &= (\rho_{rS}\rho_{r\gamma} + \rho_{S\gamma}''\sqrt{1 - \rho_{rS}^2})dt = \rho_{S\gamma}''dt \end{aligned}$$

$$\text{Then, } \rho_{S\gamma}'' = \frac{\rho_{S\gamma} - \rho_{rS}\rho_{r\gamma}}{\sqrt{1 - \rho_{rS}^2}}.$$

$$\begin{aligned} dW_X dW_\gamma &= (\rho_{rX}dW_0 + \rho_{S_X}'dW_1 + \rho_{XX}''dW_2)(\rho_{r\gamma}dW_0 + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3) \\ &= (\rho_{rX}\rho_{r\gamma} + \rho_{S_X}'\rho_{S\gamma}'' + \rho_{XX}''\rho_{XY}'')dt = \rho_{XY}''dt \end{aligned}$$

$$\rho_{rX}\rho_{r\gamma} + \rho_{S_X}'\rho_{S\gamma}'' + \rho_{XX}''\rho_{XY}'' = \rho_{XY}'', \text{ then } \rho_{XY}'' = \frac{\rho_{XY} - \rho_{rX}\rho_{r\gamma} - \rho_{S_X}'\rho_{S\gamma}''}{\rho_{XX}''}.$$

$$\rho_{XY}'' = \frac{\rho_{XY} - \rho_{rX}\rho_{r\gamma} - \rho_{S_X}\rho_{S_Y} - \rho_{rS}^2\rho_{XY} + \rho_{rS}\rho_{rX}\rho_{S_Y} + \rho_{rS}\rho_{rY}\rho_{S_X}}{\sqrt{1 + \rho_{rS}^4 - 2\rho_{rS}^3\rho_{rX}\rho_{S_X} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{rX}^2 + \rho_{rS}^2\rho_{S_X}^2 - \rho_{rX}^2 - \rho_{S_X}^2 + 2\rho_{rS}\rho_{rX}\rho_{S_X}}}$$

$$\begin{aligned} dW_\gamma dW_\gamma &= (\rho_{r\gamma}dW_0 + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3)(\rho_{r\gamma}dW_0 + \rho_{S\gamma}''dW_1 + \rho_{XY}''dW_2 + \rho_{YY}''dW_3) \\ &= (\rho_{r\gamma}^2 + \rho_{S\gamma}''^2 + \rho_{XY}''^2 + \rho_{YY}''^2)dt = dt \end{aligned}$$

$$\rho_{r\gamma}^2 + \rho_{S\gamma}''^2 + \rho_{XY}''^2 + \rho_{YY}''^2 = 1, \text{ then } \rho_{YY}'' = \sqrt{1 - \rho_{r\gamma}^2 - \rho_{S\gamma}''^2 - \rho_{XY}''^2}.$$

$$\rho_{YY}'' = \sqrt{\frac{1 - \rho_{rS}^2 - \rho_{r\gamma}^2 - \rho_{S\gamma}''^2 + 2\rho_{rS}\rho_{r\gamma}\rho_{S\gamma}''}{1 - \rho_{rS}^2} - \frac{(\rho_{XY} - \rho_{rX}\rho_{r\gamma} - \rho_{S_X}\rho_{S_Y} - \rho_{rS}^2\rho_{XY} + \rho_{rS}\rho_{rX}\rho_{S_Y} + \rho_{rS}\rho_{rY}\rho_{S_X})^2}{1 + \rho_{rS}^4 - 2\rho_{rS}^3\rho_{rX}\rho_{S_X} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{rX}^2 + \rho_{rS}^2\rho_{S_X}^2 - \rho_{rX}^2 - \rho_{S_X}^2 + 2\rho_{rS}\rho_{rX}\rho_{S_X}}}$$

$$= \sqrt{\frac{D}{C}}, \text{ where } C = (1 - \rho_{rS}^2)(1 + \rho_{rS}^4 - 2\rho_{rS}^3\rho_{rX}\rho_{S_X} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{rX}^2 + \rho_{rS}^2\rho_{S_X}^2 - \rho_{rX}^2 - \rho_{S_X}^2 + 2\rho_{rS}\rho_{rX}\rho_{S_X})$$

$$= 1 - \rho_{rS}^6 + 2\rho_{rS}^5\rho_{rX}\rho_{S_X} - \rho_{rS}^4\rho_{rX}^2 - \rho_{rS}^4\rho_{S_X}^2 + 3\rho_{rS}^4 - 4\rho_{rS}^3\rho_{rX}\rho_{S_X} + 2\rho_{rS}^2\rho_{rX}^2 + 2\rho_{rS}^2\rho_{S_X}^2 - 3\rho_{rS}^2 - \rho_{rX}^2 - \rho_{S_X}^2 + 2\rho_{rS}\rho_{rX}\rho_{S_X},$$

$$\begin{aligned} D &= 1 - \rho_{rS}^6 + \rho_{rS}^6\rho_{XY}^2 - 2\rho_{rS}^5\rho_{rX}\rho_{S_Y}\rho_{XY} - 2\rho_{rS}^5\rho_{rY}\rho_{S_X}\rho_{XY} + 2\rho_{rS}^5\rho_{rX}\rho_{S_X} + 2\rho_{rS}^5\rho_{rY}\rho_{S_Y} \\ &\quad - 2\rho_{rS}^4\rho_{rX}\rho_{rY}\rho_{S_X}\rho_{S_Y} + 2\rho_{rS}^4\rho_{rX}\rho_{rY}\rho_{XY} + 2\rho_{rS}^4\rho_{S_X}\rho_{S_Y}\rho_{XY} + \rho_{rS}^4\rho_{rX}^2\rho_{S_Y}^2 + \rho_{rS}^4\rho_{rY}^2\rho_{S_X}^2 \\ &\quad - \rho_{rS}^4\rho_{rX}^2 - \rho_{rS}^4\rho_{rY}^2 - 3\rho_{rS}^4\rho_{XY}^2 - \rho_{rS}^4\rho_{S_X}^2 - \rho_{rS}^4\rho_{S_Y}^2 + 3\rho_{rS}^4 + 4\rho_{rS}^3\rho_{rX}\rho_{S_Y}\rho_{XY} + 4\rho_{rS}^3\rho_{rY}\rho_{S_X}\rho_{XY} \\ &\quad - 4\rho_{rS}^3\rho_{rX}\rho_{S_X} - 4\rho_{rS}^3\rho_{rY}\rho_{S_Y} + 4\rho_{rS}^2\rho_{rX}\rho_{rY}\rho_{S_Y}\rho_{S_X} - 4\rho_{rS}^2\rho_{rX}\rho_{rY}\rho_{XY} - 4\rho_{rS}^2\rho_{S_X}\rho_{S_Y}\rho_{XY} \\ &\quad - 2\rho_{rS}^2\rho_{rX}^2\rho_{S_Y}^2 - 2\rho_{rS}^2\rho_{rY}^2\rho_{S_X}^2 + 2\rho_{rS}^2\rho_{rX}^2 + 2\rho_{rS}^2\rho_{rY}^2 + 2\rho_{rS}^2\rho_{S_X}^2 + 2\rho_{rS}^2\rho_{S_Y}^2 + 3\rho_{rS}^2\rho_{XY}^2 + \rho_{rX}^2\rho_{S_Y}^2 \\ &\quad + \rho_{rY}^2\rho_{S_X}^2 - 3\rho_{rS}^2 - \rho_{rX}^2 - \rho_{rY}^2 - \rho_{S_X}^2 - \rho_{S_Y}^2 - \rho_{XY}^2 - 2\rho_{rS}\rho_{rX}\rho_{S_Y}\rho_{XY} - 2\rho_{rS}\rho_{rY}\rho_{S_X}\rho_{XY} \\ &\quad - 2\rho_{rX}\rho_{rY}\rho_{S_X}\rho_{S_Y} + 2\rho_{rS}\rho_{rX}\rho_{S_X} + 2\rho_{rS}\rho_{rY}\rho_{S_Y} + 2\rho_{rX}\rho_{rY}\rho_{XY} + 2\rho_{S_X}\rho_{S_Y}\rho_{XY} \end{aligned}$$

Following this recursive method, we could construct n correlated Brownian motions. However, the coefficients for each independent Brownian motions would become more and more tedious. Alternatively, if the correlation matrix of n correlated Brownian

motions $\mathbf{C}_{n \times n}$ is symmetric and positive definite, we could decompose \mathbf{C} into $\mathbf{L}\mathbf{L}^T$ by Cholesky decomposition, where $\mathbf{L}_{n \times n}$ is a lower triangular matrix and \mathbf{L}^T is the transpose of \mathbf{L} . Then we could generate n correlated Brownian motions $\mathbf{B}_{n \times 1}$ by $\mathbf{L}\mathbf{W}$, here $\mathbf{W}_{n \times 1}$ is a vector of n independent Brownian motions.

2. APPENDIX 2: DERIVE GENERAL FORM OF DEFLATOR $D(t)$

In Appendix 2, we provide details of the calculations to derive the general form of deflator $D(t)$ step by step.

$$dD(t) = \Omega(D(t), t, r(t))dt + \Phi(D(t), t, r(t))dW_r(t) + \Psi(D(t), t, r(t))dW_1(t) + \Gamma(D(t), t, r(t))dW_2(t) + \mathbf{I}(D(t), t, r(t))dW_3(t). \quad (\text{A2.1})$$

Recall that we would like to have $E^Q(\delta(t)X) = E[D(t)X]$ for a nonnegative random variable X . We let $D(t)S(t)$, $D(t)B(t)$, $D(t)P(t, T, r(t))$, $D(t)\chi(t)$, and $D(t)\gamma(t)$ be \mathbf{P} -martingale, then their drift terms are all equal to zero.

A2.1 Short-term saving, $D(t)B(t)$

$$\begin{aligned} d[D(t)B(t)] &= B(t)dD(t) + D(t)dB(t) + dB(t)dD(t) \\ &= [\Omega B(t) + D(t)B(t)r(t)]dt + \Phi B(t)dW_r(t) + \Psi B(t)dW_1(t) + \Gamma B(t)dW_2(t) \\ &\quad + \mathbf{I}B(t)dW_3(t) \end{aligned}$$

Let the drift term of $d[D(t)B(t)]$ equal zero, $\Omega B(t) + D(t)B(t)r(t) = 0$, then $\Omega(D(t), t, r(t))$ is equal to $-D(t)r(t)$.

A2.2 Zero coupon bond of no risk with maturity T , $D(t)P(t, T, r(t))$

$$\begin{aligned} d[D(t)P(t, T, r(t))] &= P(t, T, r(t))dD(t) + D(t)dP(t, T, r(t)) + dP(t, T, r(t))dD(t) \\ &= \left[\Omega P(t, T, r(t)) + \Phi P(t, T, r(t))\tilde{\sigma}(t, r(t)) + D(t)P(t, T, r(t))r(t) \right. \\ &\quad \left. + D(t)P(t, T, r(t))\tilde{\sigma}(t, r(t))\theta(t) \right] dt \\ &\quad + [\Phi P(t, T, r(t)) + D(t)P(t, T, r(t))\tilde{\sigma}(t, r(t))]dW_r(t) \\ &\quad + \Psi P(t, T, r(t))dW_1(t) + \Gamma P(t, T, r(t))dW_2(t) + \mathbf{I}P(t, T, r(t))dW_3(t) \end{aligned}$$

Let the drift term of $d[D(t)P(t, T, r(t))]$ equal zero and plug $\Omega(D(t), t, r(t))$ into the drift term, we have $\Phi(D(t), t, r(t)) = -D(t)\theta(t)$.

A2.3 Stock, $D(t)S(t)$

$$\begin{aligned}
 d[D(t)S(t)] &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
 &= \left[\Omega S(t) + \Phi S(t)\sigma_s(t)\rho_{rs} + \Psi S(t)\sigma_s\sqrt{1-\rho_{rs}^2} + D(t)S(t)\mu_s(t) \right] dt \\
 &\quad + \left[\Phi S(t) + D(t)S(t)\sigma_s(t)\rho_{rs} \right] dW_r(t) + \left[\Psi S(t) + D(t)S(t)\sigma_s(t)\sqrt{1-\rho_{rs}^2} \right] dW_1(t) \\
 &\quad + \Gamma S(t)dW_2(t) + I S(t)dW_3(t)
 \end{aligned}$$

Let the drift term of $d[D(t)S(t)]$ equal zero and plug $\Omega(D(t),t,r(t))$, $\Phi(D(t),t,r(t))$ into the drift term, then $\Psi(D(t),t,r(t))$ is equal to $\frac{D(t)[r(t) + \theta(t)\sigma_s(t)\rho_{rs} - \mu_s(t)]}{\sigma_s(t)\sqrt{1-\rho_{rs}^2}}$.

 A2.4 Default density, $D(t)\chi(t)$

$$\begin{aligned}
 d[D(t)\chi(t)] &= \chi(t)dD(t) + D(t)d\chi(t) + d\chi(t)dD(t) \\
 &= \left[\Omega\chi(t) + \Phi\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} + \Psi\sigma_\chi\rho'_{s\chi}\sqrt{\chi(t)} + \Gamma\sigma_\chi\rho'_{xx}\sqrt{\chi(t)} + D(t)e - D(t)f\chi(t) \right] dt \\
 &\quad + \left[\Phi\chi(t) + D(t)\sigma_\chi\rho_{r\chi}\sqrt{\chi(t)} \right] dW_r(t) + \left[\Psi\chi(t) + D(t)\sigma_\chi\rho'_{s\chi}\sqrt{\chi(t)} \right] dW_1(t) \\
 &\quad + \left[\Gamma\chi(t) + D(t)\sigma_\chi\rho'_{xx}\sqrt{\chi(t)} \right] dW_2(t) + I\chi(t)dW_3(t)
 \end{aligned}$$

Let the drift term of $d[D(t)\chi(t)]$ equal zero and plug $\Omega(D(t),t,r(t))$, $\Phi(D(t),t,r(t))$, $\Psi(D(t),t,r(t))$ into the drift term, we have

$$\Gamma(D(t),t,r(t)) = D(t) \left[\frac{\theta(t)\rho_{r\chi}}{\rho'_{xx}} + \frac{r(t)\chi(t) - e + f\chi(t)}{\sigma_\chi\rho'_{xx}\sqrt{\chi(t)}} + \frac{\rho'_{s\chi}(\mu_s(t) - r(t) - \theta(t)\sigma_s(t)\rho_{rs})}{\rho'_{xx}\sigma_s(t)\sqrt{1-\rho_{rs}^2}} \right]$$

 A2.5 Convenience yield, $D(t)\gamma(t)$

$$\begin{aligned}
 d[D(t)\gamma(t)] &= \gamma(t)dD(t) + D(t)d\gamma(t) + d\gamma(t)dD(t) \\
 &= \left[\Omega\gamma(t) + \Phi\eta\rho_{r\gamma} + \Psi\eta\rho''_{s\gamma} + \Gamma\eta\rho''_{xy} + I\eta\rho''_{yy} \right] dt + \left[\Phi\gamma(t) + D(t)\eta\rho_{r\gamma} \right] dW_r(t) \\
 &\quad + \left[\Psi\gamma(t) + D(t)\eta\rho''_{s\gamma} \right] dW_1(t) + \left[\Gamma\gamma(t) + D(t)\eta\rho''_{xy} \right] dW_2(t) \\
 &\quad + \left(I\gamma(t) + D(t)\eta\rho''_{yy} \right) dW_3(t)
 \end{aligned}$$

Let the drift term of $d[D(t)\gamma(t)]$ equal zero and plug $\Omega(D(t),t,r(t))$, $\Phi(D(t),t,r(t))$, $\Psi(D(t),t,r(t))$, $\Gamma(D(t),t,r(t))$ into the drift term, then

$$I(D(t),t,r(t)) = D(t) \left\{ \frac{\rho_{r\gamma}\theta(t)}{\rho''_{yy}} + \frac{r(t)\gamma(t)}{\eta\rho''_{yy}} - \frac{\rho''_{xy}\rho_{r\chi}\theta(t)}{\rho''_{yy}\rho'_{xx}} + \frac{\rho''_{xy}[e - r(t)\chi(t) - f\chi(t)]}{\rho''_{yy}\rho'_{xx}\sigma_\chi\sqrt{\chi(t)}} \right\} + \frac{(\rho''_{s\gamma}\rho'_{xx} - \rho''_{xy}\rho'_{s\chi})[\mu_s(t) - r(t) - \rho_{rs}\theta(t)\sigma_s(t)]}{\rho''_{yy}\rho'_{xx}\sigma_s(t)\sqrt{1-\rho_{rs}^2}}$$

A2.6 Deflator, $D(t)$

We could derive the general form of deflator $D(t)$ now. Let $y(t)$ equal $\ln(D(t))$, then the first partial derivative $(\frac{\partial y}{\partial D})$ and the second partial derivative $(\frac{\partial^2 y}{\partial D^2})$ of $y(t)$ with respect to $D(t)$ are $\frac{1}{D(t)}$ and $-\frac{1}{D^2(t)}$ respectively. By Itô formula,

$$dy(t) = \frac{\partial y}{\partial D} dD(t) + \frac{1}{2} \frac{\partial^2 y}{\partial D^2} dD(t)dD(t) \quad .$$

$$dy(t) = \left[\begin{array}{c} \frac{\Omega(D(t),t,r(t))}{D(t)} - \frac{\Phi^2(D(t),t,r(t))}{2D^2(t)} - \frac{\Psi^2(D(t),t,r(t))}{2D^2(t)} \\ - \frac{\Gamma^2(D(t),t,r(t))}{2D^2(t)} - \frac{I^2(D(t),t,r(t))}{2D^2(t)} \end{array} \right] dt + \frac{\Phi(D(t),t,r(t))}{D(t)} dW_r(t) \\ + \frac{\Psi(D(t),t,r(t))}{D(t)} dW_1(t) + \frac{\Gamma(D(t),t,r(t))}{D(t)} dW_2(t) + \frac{I(D(t),t,r(t))}{D(t)} dW_3(t)$$

(10)

Let

$$K_\Psi(t,r(t)) = \frac{\Psi(D(t),t,r(t))}{D(t)} = \frac{r(t) + \theta(t)\sigma_s(t)\rho_{rs} - \mu_s(t)}{\sigma_s(t)\sqrt{1-\rho_{rs}^2}} \quad ,$$

$$K_\Gamma(t,r(t)) = \frac{\Gamma(D(t),t,r(t))}{D(t)} = \frac{\theta(t)\rho_{r\chi}}{\rho'_{\chi\chi}} + \frac{r(t)\chi(t) - e + f\chi(t)}{\sigma_\chi\rho'_{\chi\chi}\sqrt{\chi(t)}} + \frac{\rho'_{s\chi}[\mu_s(t) - r(t) - \theta(t)\sigma_s(t)\rho_{rs}]}{\rho'_{\chi\chi}\sigma_s(t)\sqrt{1-\rho_{rs}^2}}$$

$$K_I(t,r(t)) = \frac{I(D(t),t,r(t))}{D(t)} = \frac{\rho_{rr}\theta(t)}{\rho''_{rr}} + \frac{r(t)\gamma(t)}{\eta\rho''_{rr}} - \frac{\rho''_{\chi\gamma}\rho_{r\chi}\theta(t)}{\rho''_{rr}\rho'_{\chi\chi}} + \frac{\rho''_{\chi\gamma}[e - r(t)\chi(t) - f\chi(t)]}{\rho''_{rr}\rho'_{\chi\chi}\sigma_\chi\sqrt{\chi(t)}} \quad , \text{ and}$$

$$+ \frac{(\rho''_{s\gamma}\rho'_{\chi\chi} - \rho''_{\chi\gamma}\rho'_{s\chi})[\mu_s(t) - r(t) - \rho_{rs}\theta(t)\sigma_s(t)]}{\rho''_{rr}\rho'_{\chi\chi}\sigma_s(t)\sqrt{1-\rho_{rs}^2}}$$

integrate both sides of $dy(t)$, plugging $\frac{\Omega(D(t),t,r(t))}{D(t)} = -r(t)$ and

$\frac{\Phi(D(t),t,r(t))}{D(t)} = -\theta(t)$ into $y(t)$. We have $y(t)$ as follows.

$$y(t) = y(0) + \int_0^t \left[-r(s) - \frac{1}{2}\theta^2(s) - \frac{1}{2}K_\Psi^2(t,r(t)) - \frac{1}{2}K_\Gamma^2(t,r(t)) - \frac{1}{2}K_I^2(t,r(t)) \right] ds \\ - \int_0^t \theta(s) dW_r(t) + \int_0^t K_\Psi(t,r(t)) dW_1(t) + \int_0^t K_\Gamma(t,r(t)) dW_2(t) + \int_0^t K_I(t,r(t)) dW_3(t)$$

(11)

Take exponential both sides of $y(t)$, we have the general form of deflator.

$$D(t) = D(0) \exp \left\{ -\int_0^t r(s) ds - \int_0^t \frac{1}{2} \left[\theta^2(s) + K_\Psi^2(t, r(t)) + K_\Gamma^2(t, r(t)) + K_I^2(t, r(t)) \right] ds \right\} \\ \times \exp \left[-\int_0^t \theta(s) dW_r(t) + \int_0^t K_\Psi(t, r(t)) dW_1(t) + \int_0^t K_\Gamma(t, r(t)) dW_2(t) + \int_0^t K_I(t, r(t)) dW_3(t) \right] \quad (12)$$

3. APPENDIX 3: IMPLEMENTATION OF SIMPLIFIED SECOND MILSTEIN METHOD

In Appendix 3, we present the implementation of simplified Second Milstein method in our example. $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$ where X_t , W_t , $a(t, X_t)$, and $b(t, X_t)$ are as follows.

$$X_t = \begin{bmatrix} r(t) \\ \theta(t) \\ B(t) \\ P(t, T, r(t)) \\ S(t) \\ \chi(t) \\ \gamma(t) \\ D(t) \end{bmatrix}, W_t = \begin{bmatrix} W_r(t) \\ W_1(t) \\ W_2(t) \\ W_3(t) \\ W_\theta(t) \end{bmatrix} \quad (A3.1)$$

$$a(t, X_t) = \begin{bmatrix} a_r - b_r r(t) + \theta(t) \sigma_r \sqrt{r(t)} \\ a_\theta - b_\theta \theta(t) \\ B(t) r(t) \\ P(t, T, r(t)) r(t) + P_r \sigma_r \sqrt{r(t)} \theta(t) \\ S(t) \mu_s \\ e - f \chi(t) \\ 0 \\ -r(t) D(t) \end{bmatrix} \quad (A3.2)$$

$$b(t, X_t) = \begin{bmatrix} \sigma_r \sqrt{r(t)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta \sqrt{\theta(t)} \\ 0 & 0 & 0 & 0 & 0 \\ P_r \sigma_r \sqrt{r(t)} & 0 & 0 & 0 & 0 \\ S(t) \sigma_S \rho_{rS} & S(t) \sigma_S \sqrt{1 - \rho_{rS}^2} & 0 & 0 & 0 \\ \sigma_\chi \rho_{r\chi} \sqrt{\chi(t)} & \sigma_\chi \rho'_{S\chi} \sqrt{\chi(t)} & \sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)} & 0 & 0 \\ \eta \rho_{r\gamma} & \eta \rho''_{S\gamma} & \eta \rho''_{\chi\gamma} & \eta \rho''_{r\gamma} & 0 \\ -\theta(t) D(t) & \Psi(D(t), t, r(t)) & \Gamma(D(t), t, r(t)) & I(D(t), t, r(t)) & 0 \end{bmatrix}$$

(A3.3)

For each $i = 1, \dots, d$,

$$Y_{n+1,i} = Y_{n,i} + a_i(n, Y_n) \Delta t + \sum_{k=1}^m b_{ik}(n, Y_n) \Delta W_{n,k} + \frac{1}{2} L^0 a_i(n, Y_n) (\Delta t)^2 + \frac{1}{2} \sum_{k=1}^m [L^k a_i(n, Y_n) + L^0 b_{ik}(n, Y_n)] \Delta W_{n,k} \Delta t + \frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m L^j b_{ik}(n, Y_n) (\Delta W_{n,j} \Delta W_{n,k} - V_{jk}).$$

(A3.4)

In order to calculate Y_{n+1} at each time step $n+1$, we have to calculate operators L^0 and L^k for each $a_i(n, X_n)$ and $b_{ik}(n, X_n)$. We would skip the cases where $a_i(n, X_n)$ and $b_{ik}(n, X_n)$ equal zero or a constant since their operators L^0 and L^k are equal to zero. We show the calculations for the rest as follows.

A3.1 $a_1(t, X_t)$

$$\begin{aligned} a_1(t, X_t) &= a_r - b_r r(t) + \theta(t) \sigma_r \sqrt{r(t)}, \quad \frac{\partial a_1(t, X_t)}{\partial x_1} = -b_r + \frac{1}{2} \theta(t) \sigma_r \frac{1}{\sqrt{r(t)}}, \quad \frac{\partial a_1(t, X_t)}{\partial x_2} = \sigma_r \sqrt{r(t)} \\ \frac{\partial^2 a_1(t, X_t)}{\partial x_1 \partial x_1} &= -\frac{1}{4} \theta(t) \sigma_r \frac{1}{\sqrt{r^3(t)}}, \quad \frac{\partial^2 a_1(t, X_t)}{\partial x_1 \partial x_2} = \frac{\partial^2 a_1(t, X_t)}{\partial x_2 \partial x_1} = \frac{1}{2} \sigma_r \frac{1}{\sqrt{r(t)}} \\ L^0 a_1(t, X_t) &= \frac{\partial a_1(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_1(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_1(t, X_t)}{\partial x_i \partial x_j} \\ &= a_1(t, X_t) \left[-b_r + \frac{1}{2} \theta(t) \sigma_r \frac{1}{\sqrt{r(t)}} \right] + a_2(t, X_t) \sigma_r \sqrt{r(t)} - \frac{1}{8} \Sigma_{t,11} \sigma_r \theta(t) \frac{1}{\sqrt{r^3(t)}} \\ L^k a_1(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_1(t, X_t)}{\partial x_i} \\ &= b_{1k}(t, X_t) \left[-b_r + \frac{1}{2} \theta(t) \sigma_r \frac{1}{\sqrt{r(t)}} \right] + b_{2k}(t, X_t) \sigma_r \sqrt{r(t)} \end{aligned}$$

A3.2 $a_2(t, X_t)$

$$a_2(t, X_t) = a_\theta - b_\theta \theta(t), \frac{\partial a_2(t, X_t)}{\partial x_2} = -b_\theta$$

$$\begin{aligned} L^0 a_2(t, X_t) &= \frac{\partial a_2(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_2(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_2(t, X_t)}{\partial x_i \partial x_j} \\ &= -a_2(t, X_t) b_\theta \end{aligned}$$

$$\begin{aligned} L^k a_2(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_2(t, X_t)}{\partial x_i} \\ &= -b_{2k}(t, X_t) b_\theta \end{aligned}$$

A3.3 $a_3(t, X_t)$

$$a_3(t, X_t) = B(t)r(t), \frac{\partial a_3(t, X_t)}{\partial x_1} = B(t), \frac{\partial a_3(t, X_t)}{\partial x_3} = r(t)$$

$$\frac{\partial^2 a_3(t, X_t)}{\partial x_1 \partial x_3} = \frac{\partial^2 a_3(t, X_t)}{\partial x_3 \partial x_1} = 1$$

$$\begin{aligned} L^0 a_3(t, X_t) &= \frac{\partial a_3(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_3(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_3(t, X_t)}{\partial x_i \partial x_j} \\ &= a_1(t, X_t) B(t) + a_3(t, X_t) r(t) \end{aligned}$$

$$\begin{aligned} L^k a_3(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_3(t, X_t)}{\partial x_i} \\ &= b_{1k}(t, X_t) B(t) + b_{3k}(t, X_t) r(t) \end{aligned}$$

A3.4 $a_4(t, X_t)$

$$a_4(t, X_t) = P(t, T, r(t))r(t) + \sigma_r P_r \sqrt{r(t)}\theta(t), \quad \frac{\partial a_4(t, X_t)}{\partial x_1} = P(t, T, r(t)) + \frac{1}{2} \sigma_r P_r \theta(t) \frac{1}{\sqrt{r(t)}},$$

$$\frac{\partial a_4(t, X_t)}{\partial x_2} = \sigma_r P_r \sqrt{r(t)}, \quad \frac{\partial a_4(t, X_t)}{\partial x_4} = r(t)$$

$$\frac{\partial^2 a_1(t, X_t)}{\partial x_1 \partial x_1} = -\frac{1}{4} \sigma_r P_r \theta(t) \frac{1}{\sqrt{r^3(t)}}, \quad \frac{\partial^2 a_1(t, X_t)}{\partial x_1 \partial x_2} = \frac{\partial^2 a_1(t, X_t)}{\partial x_2 \partial x_1} = \frac{1}{2} \sigma_r P_r \frac{1}{\sqrt{r(t)}},$$

$$\frac{\partial^2 a_1(t, X_t)}{\partial x_1 \partial x_4} = \frac{\partial^2 a_1(t, X_t)}{\partial x_4 \partial x_1} = 1$$

$$\begin{aligned} L^0 a_4(t, X_t) &= \frac{\partial a_4(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_4(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_4(t, X_t)}{\partial x_i \partial x_j} \\ &= a_1(t, X_t) \left[P(t, T, r(t)) + \frac{1}{2} \sigma_r P_r \theta(t) \frac{1}{\sqrt{r(t)}} \right] + a_2(t, X_t) \sigma_r P_r \sqrt{r(t)} + a_4(t, X_t) r(t) \\ &\quad - \frac{1}{8} \Sigma_{t,11} \sigma_r P_r \theta(t) \frac{1}{\sqrt{r^3(t)}} + \Sigma_{t,14} \end{aligned}$$

$$\begin{aligned} L^k a_4(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_4(t, X_t)}{\partial x_i} \\ &= b_{1k}(t, X_t) \left[P(t, T, r(t)) + \frac{1}{2} \sigma_r P_r \theta(t) \frac{1}{\sqrt{r(t)}} \right] + b_{2k}(t, X_t) \sigma_r P_r \sqrt{r(t)} + b_{4k}(t, X_t) r(t) \end{aligned}$$

A3.5 $a_5(t, X_t)$

$$a_5(t, X_t) = S(t) \mu_S, \quad \frac{\partial a_5(t, X_t)}{\partial x_5} = \mu_S$$

$$\begin{aligned} L^0 a_5(t, X_t) &= \frac{\partial a_5(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_5(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_5(t, X_t)}{\partial x_i \partial x_j} \\ &= a_5(t, X_t) \mu_S \end{aligned}$$

$$\begin{aligned} L^k a_5(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_5(t, X_t)}{\partial x_i} \\ &= b_{5k}(t, X_t) \mu_S \end{aligned}$$

A3.6 $a_6(t, X_t)$

$$a_6(t, X_t) = e - f\chi(t), \frac{\partial a_6(t, X_t)}{\partial x_6} = -f$$

$$\begin{aligned} L^0 a_6(t, X_t) &= \frac{\partial a_6(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_6(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_6(t, X_t)}{\partial x_i \partial x_j} \\ &= -a_6(t, X_t) f \end{aligned}$$

$$\begin{aligned} L^k a_6(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_6(t, X_t)}{\partial x_i} \\ &= -b_{6k}(t, X_t) f \end{aligned}$$

A3.7 $a_8(t, X_t)$

$$a_8(t, X_t) = -r(t)D(t), \frac{\partial a_8(t, X_t)}{\partial x_1} = -D(t), \frac{\partial a_8(t, X_t)}{\partial x_8} = -r(t)$$

$$\frac{\partial^2 a_8(t, X_t)}{\partial x_1 \partial x_8} = \frac{\partial^2 a_8(t, X_t)}{\partial x_8 \partial x_1} = -1$$

$$\begin{aligned} L^0 a_8(t, X_t) &= \frac{\partial a_8(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial a_8(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 a_8(t, X_t)}{\partial x_i \partial x_j} \\ &= -a_1(t, X_t)D(t) - a_8(t, X_t)r(t) - \Sigma_{t,18} \end{aligned}$$

$$\begin{aligned} L^k a_8(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial a_8(t, X_t)}{\partial x_i} \\ &= -b_{1k}(t, X_t)D(t) - b_{8k}(t, X_t)r(t) \end{aligned}$$

A3.8 $b_{11}(t, X_t)$

$$b_{11}(t, X_t) = \sigma_r \sqrt{r(t)}, \frac{\partial b_{11}(t, X_t)}{\partial x_1} = \frac{1}{2} \sigma_r \frac{1}{\sqrt{r(t)}}$$

$$\frac{\partial^2 b_{11}(t, X_t)}{\partial x_1 \partial x_1} = -\frac{1}{4} \sigma_r \frac{1}{\sqrt{r^3(t)}}$$

$$\begin{aligned} L^0 b_{11}(t, X_t) &= \frac{\partial b_{11}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{11}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{11}(t, X_t)}{\partial x_i \partial x_j} \\ &= \frac{1}{2} a_1(t, X_t) \sigma_r \frac{1}{\sqrt{r(t)}} - \frac{1}{8} \Sigma_{t,11} \sigma_r \frac{1}{\sqrt{r^3(t)}} \end{aligned}$$

$$\begin{aligned} L^k b_{11}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{11}(t, X_t)}{\partial x_i} \\ &= \frac{1}{2} b_{1k}(t, X_t) \sigma_r \frac{1}{\sqrt{r(t)}} \end{aligned}$$

A3.9 $b_{25}(t, X_t)$

$$b_{25}(t, X_t) = \sigma_\theta \sqrt{\theta(t)}, \frac{\partial b_{25}(t, X_t)}{\partial x_2} = \frac{1}{2} \sigma_\theta \frac{1}{\sqrt{\theta(t)}}$$

$$\frac{\partial^2 b_{25}(t, X_t)}{\partial x_2 \partial x_2} = -\frac{1}{4} \sigma_\theta \frac{1}{\sqrt{\theta^3(t)}}$$

$$\begin{aligned} L^0 b_{25}(t, X_t) &= \frac{\partial b_{25}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{25}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{25}(t, X_t)}{\partial x_i \partial x_j} \\ &= \frac{1}{2} a_2(t, X_t) \sigma_\theta \frac{1}{\sqrt{\theta(t)}} - \frac{1}{8} \Sigma_{t,22} \sigma_\theta \frac{1}{\sqrt{\theta^3(t)}} \end{aligned}$$

$$\begin{aligned} L^k b_{25}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{25}(t, X_t)}{\partial x_i} \\ &= \frac{1}{2} b_{2k}(t, X_t) \sigma_\theta \frac{1}{\sqrt{\theta(t)}} \end{aligned}$$

A3.10 $b_{41}(t, X_t)$

$$b_{41}(t, X_t) = \sigma_r P_r \sqrt{r(t)}, \frac{\partial b_{41}(t, X_t)}{\partial x_1} = \frac{1}{2} \sigma_r P_r \frac{1}{\sqrt{r(t)}}$$

$$\frac{\partial^2 b_{41}(t, X_t)}{\partial x_1 \partial x_1} = -\frac{1}{4} \sigma_r P_r \frac{1}{\sqrt{r^3(t)}}$$

$$\begin{aligned} L^0 b_{41}(t, X_t) &= \frac{\partial b_{41}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{41}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{41}(t, X_t)}{\partial x_i \partial x_j} \\ &= \frac{1}{2} a_1(t, X_t) \sigma_r P_r \frac{1}{\sqrt{r(t)}} - \frac{1}{8} \Sigma_{t,11} \sigma_r P_r \frac{1}{\sqrt{r^3(t)}} \end{aligned}$$

$$\begin{aligned} L^k b_{41}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{41}(t, X_t)}{\partial x_i} \\ &= \frac{1}{2} b_{1k}(t, X_t) \sigma_r P_r \frac{1}{\sqrt{r(t)}} \end{aligned}$$

A3.11 $b_{51}(t, X_t)$

$$b_{51}(t, X_t) = S(t) \sigma_s \rho_{rs}, \frac{\partial b_{51}(t, X_t)}{\partial x_5} = \sigma_s \rho_{rs}$$

$$\begin{aligned} L^0 b_{51}(t, X_t) &= \frac{\partial b_{51}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{51}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{51}(t, X_t)}{\partial x_i \partial x_j} \\ &= a_5(t, X_t) \sigma_s \rho_{rs} \end{aligned}$$

$$\begin{aligned} L^k b_{51}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{51}(t, X_t)}{\partial x_i} \\ &= b_{5k}(t, X_t) \sigma_s \rho_{rs} \end{aligned}$$

A3.12 $b_{52}(t, X_t)$

$$b_{52}(t, X_t) = S(t) \sigma_s \sqrt{1 - \rho_{rs}^2}, \frac{\partial b_{52}(t, X_t)}{\partial x_5} = \sigma_s \sqrt{1 - \rho_{rs}^2}$$

$$L^0 b_{52}(t, X_t) = \frac{\partial b_{52}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{52}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{52}(t, X_t)}{\partial x_i \partial x_j}$$

$$= a_5(t, X_t) \sigma_s \sqrt{1 - \rho_{rs}^2}$$

$$L^k b_{52}(t, X_t) = \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{52}(t, X_t)}{\partial x_i}$$

$$= b_{5k}(t, X_t) \sigma_s \sqrt{1 - \rho_{rs}^2}$$

 A3.13 $b_{61}(t, X_t)$

$$b_{61}(t, X_t) = \sigma_\chi \rho_{r\chi} \sqrt{\chi(t)}, \frac{\partial b_{61}(t, X_t)}{\partial x_6} = \frac{1}{2} \sigma_\chi \rho_{r\chi} \frac{1}{\sqrt{\chi(t)}}$$

$$\frac{\partial^2 b_{61}(t, X_t)}{\partial x_6 \partial x_6} = -\frac{1}{4} \sigma_\chi \rho_{r\chi} \frac{1}{\sqrt{\chi^3(t)}}$$

$$L^0 b_{61}(t, X_t) = \frac{\partial b_{61}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{61}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{61}(t, X_t)}{\partial x_i \partial x_j}$$

$$= \frac{1}{2} a_6(t, X_t) \sigma_\chi \rho_{r\chi} \frac{1}{\sqrt{\chi(t)}} - \frac{1}{8} \Sigma_{t,66} \sigma_\chi \rho_{r\chi} \frac{1}{\sqrt{\chi^3(t)}}$$

$$L^k b_{61}(t, X_t) = \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{61}(t, X_t)}{\partial x_i}$$

$$= \frac{1}{2} b_{6k}(t, X_t) \sigma_\chi \rho_{r\chi} \frac{1}{\sqrt{\chi(t)}}$$

 A3.14 $b_{62}(t, X_t)$

$$b_{62}(t, X_t) = \sigma_\chi \rho'_{s\chi} \sqrt{\chi(t)}, \frac{\partial b_{62}(t, X_t)}{\partial x_6} = \frac{1}{2} \sigma_\chi \rho'_{s\chi} \frac{1}{\sqrt{\chi(t)}}$$

$$\frac{\partial^2 b_{62}(t, X_t)}{\partial x_6 \partial x_6} = -\frac{1}{4} \sigma_\chi \rho'_{s\chi} \frac{1}{\sqrt{\chi^3(t)}}$$

$$\begin{aligned} L^0 b_{62}(t, X_t) &= \frac{\partial b_{62}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{62}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{62}(t, X_t)}{\partial x_i \partial x_j} \\ &= \frac{1}{2} a_6(t, X_t) \sigma_{\lambda} \rho'_{S\lambda} \frac{1}{\sqrt{\chi(t)}} - \frac{1}{8} \Sigma_{t,66} \sigma_{\lambda} \rho'_{S\lambda} \frac{1}{\sqrt{\chi^3(t)}} \end{aligned}$$

$$\begin{aligned} L^k b_{62}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{62}(t, X_t)}{\partial x_i} \\ &= \frac{1}{2} b_{6k}(t, X_t) \sigma_{\lambda} \rho'_{S\lambda} \frac{1}{\sqrt{\chi(t)}} \end{aligned}$$

A3.15 $b_{63}(t, X_t)$

$$b_{63}(t, X_t) = \sigma_{\lambda} \rho'_{\lambda\lambda} \sqrt{\chi(t)}, \quad \frac{\partial b_{63}(t, X_t)}{\partial x_6} = \frac{1}{2} \sigma_{\lambda} \rho'_{\lambda\lambda} \frac{1}{\sqrt{\chi(t)}}$$

$$\frac{\partial^2 b_{63}(t, X_t)}{\partial x_6 \partial x_6} = -\frac{1}{4} \sigma_{\lambda} \rho'_{\lambda\lambda} \frac{1}{\sqrt{\chi^3(t)}}$$

$$\begin{aligned} L^0 b_{63}(t, X_t) &= \frac{\partial b_{63}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{63}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{63}(t, X_t)}{\partial x_i \partial x_j} \\ &= \frac{1}{2} a_6(t, X_t) \sigma_{\lambda} \rho'_{\lambda\lambda} \frac{1}{\sqrt{\chi(t)}} - \frac{1}{8} \Sigma_{t,66} \sigma_{\lambda} \rho'_{\lambda\lambda} \frac{1}{\sqrt{\chi^3(t)}} \end{aligned}$$

$$\begin{aligned} L^k b_{63}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{63}(t, X_t)}{\partial x_i} \\ &= \frac{1}{2} b_{6k}(t, X_t) \sigma_{\lambda} \rho'_{\lambda\lambda} \frac{1}{\sqrt{\chi(t)}} \end{aligned}$$

A3.16 $b_{81}(t, X_t)$

$$b_{81}(t, X_t) = -\theta(t) D(t), \quad \frac{\partial b_{81}(t, X_t)}{\partial x_2} = -D(t), \quad \frac{\partial b_{81}(t, X_t)}{\partial x_8} = -\theta(t),$$

$$\frac{\partial^2 b_{81}(t, X_t)}{\partial x_2 \partial x_8} = \frac{\partial^2 b_{81}(t, X_t)}{\partial x_8 \partial x_2} = -1$$

$$\begin{aligned} L^0 b_{81}(t, X_t) &= \frac{\partial b_{81}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{81}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{81}(t, X_t)}{\partial x_i \partial x_j} \\ &= -a_2(t, X_t) D(t) - a_8(t, X_t) \theta(t) - \Sigma_{t,28} \end{aligned}$$

$$\begin{aligned} L^k b_{81}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{81}(t, X_t)}{\partial x_i} \\ &= -b_{2k}(t, X_t) D(t) - b_{8k}(t, X_t) \theta(t) \end{aligned}$$

A3.17 $b_{82}(t, X_t)$

$$\begin{aligned}
 b_{82}(t, X_t) &= \Psi(D(t), t, r(t)) = \frac{D(t)[r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s]}{\sigma_s\sqrt{1-\rho_{rs}^2}}, \\
 \frac{\partial b_{82}(t, X_t)}{\partial x_1} &= \frac{D(t)}{\sigma_s\sqrt{1-\rho_{rs}^2}}, \quad \frac{\partial b_{82}(t, X_t)}{\partial x_2} = \frac{D(t)\rho_{rs}}{\sqrt{1-\rho_{rs}^2}}, \quad \frac{\partial b_{82}(t, X_t)}{\partial x_8} = \frac{r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s}{\sigma_s\sqrt{1-\rho_{rs}^2}}, \\
 \frac{\partial^2 b_{82}(t, X_t)}{\partial x_1\partial x_8} &= \frac{\partial^2 b_{82}(t, X_t)}{\partial x_8\partial x_1} = \frac{1}{\sigma_s\sqrt{1-\rho_{rs}^2}}, \quad \frac{\partial^2 b_{82}(t, X_t)}{\partial x_2\partial x_8} = \frac{\partial^2 b_{82}(t, X_t)}{\partial x_8\partial x_2} = \frac{\rho_{rs}}{\sqrt{1-\rho_{rs}^2}}, \\
 L^0 b_{82}(t, X_t) &= \frac{\partial b_{82}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{82}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{i,j} \frac{\partial^2 b_{82}(t, X_t)}{\partial x_i\partial x_j} \\
 &= a_1(t, X_t) \frac{D(t)}{\sigma_s\sqrt{1-\rho_{rs}^2}} + a_2(t, X_t) \frac{D(t)\rho_{rs}}{\sqrt{1-\rho_{rs}^2}} + a_8(t, X_t) \left[\frac{r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s}{\sigma_s\sqrt{1-\rho_{rs}^2}} \right] \\
 &\quad + \Sigma_{1,18} \frac{1}{\sigma_s\sqrt{1-\rho_{rs}^2}} + \Sigma_{1,28} \frac{\rho_{rs}}{\sqrt{1-\rho_{rs}^2}} \\
 L^k b_{82}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{82}(t, X_t)}{\partial x_i} \\
 &= b_{1k}(t, X_t) \left[\frac{D(t)}{\sigma_s\sqrt{1-\rho_{rs}^2}} \right] + b_{2k}(t, X_t) \left[\frac{D(t)\rho_{rs}}{\sqrt{1-\rho_{rs}^2}} \right] + b_{8k}(t, X_t) \left[\frac{r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s}{\sigma_s\sqrt{1-\rho_{rs}^2}} \right]
 \end{aligned}$$

 A3.18 $b_{83}(t, X_t)$

$$\begin{aligned}
 b_{83}(t, X_t) &= \Gamma(D(t), t, r(t)) = D(t) \left[\frac{\theta(t)\rho_{rz} + \frac{r(t)\chi(t) + \chi(t)f - e}{\sigma_z\rho'_{zz}\sqrt{\chi(t)}} - \frac{\rho'_{sz}(r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s)}{\sigma_s\rho'_{zz}\sqrt{1-\rho_{rs}^2}} \right], \\
 \frac{\partial b_{83}(t, X_t)}{\partial x_1} &= D(t) \left[\frac{\sqrt{\chi(t)}}{\sigma_z\rho'_{zz}} - \frac{\rho'_{sz}}{\sigma_s\rho'_{zz}\sqrt{1-\rho_{rs}^2}} \right], \quad \frac{\partial b_{83}(t, X_t)}{\partial x_2} = D(t) \left[\frac{\rho_{rz}}{\rho'_{zz}} - \frac{\rho_{rs}\rho'_{sz}}{\rho'_{zz}\sqrt{1-\rho_{rs}^2}} \right], \\
 \frac{\partial b_{83}(t, X_t)}{\partial x_6} &= D(t) \left[\frac{r(t) + f}{2\sigma_z\rho'_{zz}\sqrt{\chi(t)}} + \frac{e}{2\sigma_z\rho'_{zz}\sqrt{\chi^3(t)}} \right], \\
 \frac{\partial b_{83}(t, X_t)}{\partial x_8} &= \frac{\theta(t)\rho_{rz}}{\rho'_{zz}} + \frac{r(t)\chi(t) + \chi(t)f - e}{\sigma_z\rho'_{zz}\sqrt{\chi(t)}} - \frac{\rho'_{sz}(r(t) + \theta(t)\sigma_s\rho_{rs} - \mu_s)}{\sigma_s\rho'_{zz}\sqrt{1-\rho_{rs}^2}}, \\
 \frac{\partial^2 b_{83}(t, X_t)}{\partial x_1\partial x_6} &= \frac{\partial^2 b_{83}(t, X_t)}{\partial x_6\partial x_1} = \frac{D(t)}{2\sigma_z\rho'_{zz}\sqrt{\chi(t)}}, \\
 \frac{\partial^2 b_{83}(t, X_t)}{\partial x_1\partial x_8} &= \frac{\partial^2 b_{83}(t, X_t)}{\partial x_8\partial x_1} = \frac{\sqrt{\chi(t)}}{\sigma_z\rho'_{zz}} - \frac{\rho'_{sz}}{\sigma_s\rho'_{zz}\sqrt{1-\rho_{rs}^2}}, \\
 \frac{\partial^2 b_{83}(t, X_t)}{\partial x_2\partial x_8} &= \frac{\partial^2 b_{83}(t, X_t)}{\partial x_8\partial x_2} = \frac{\rho_{rz}}{\rho'_{zz}} - \frac{\rho_{rs}\rho'_{sz}}{\rho'_{zz}\sqrt{1-\rho_{rs}^2}}, \\
 \frac{\partial^2 b_{83}(t, X_t)}{\partial x_6\partial x_6} &= -D(t) \left[\frac{r(t) + f}{4\sigma_z\rho'_{zz}\sqrt{\chi^3(t)}} + \frac{3e}{4\sigma_z\rho'_{zz}\sqrt{\chi^5(t)}} \right], \\
 \frac{\partial^2 b_{83}(t, X_t)}{\partial x_6\partial x_8} &= \frac{\partial^2 b_{83}(t, X_t)}{\partial x_8\partial x_6} = \frac{r(t) + f}{2\sigma_z\rho'_{zz}\sqrt{\chi(t)}} + \frac{e}{2\sigma_z\rho'_{zz}\sqrt{\chi^3(t)}},
 \end{aligned}$$

$$\begin{aligned}
L^0 b_{83}(t, X_t) &= \frac{\partial b_{83}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{83}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{83}(t, X_t)}{\partial x_i \partial x_j} \\
&= a_1(t, X_t) D(t) \left[\frac{\sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi}} - \frac{\rho'_{S\chi}}{\sigma_S \rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] + a_2(t, X_t) D(t) \left[\frac{\rho_{r\chi}}{\rho'_{\chi\chi}} - \frac{\rho_{rS} \rho'_{S\chi}}{\rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] \\
&\quad + a_6(t, X_t) D(t) \left[\frac{r(t)+f}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} + \frac{e}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi^3(t)}} \right] \\
&\quad + a_8(t, X_t) \left[\frac{\theta(t) \rho_{r\chi}}{\rho'_{\chi\chi}} + \frac{r(t) \chi(t) + \chi(t) f - e}{\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} - \frac{\rho'_{S\chi} (r(t) + \theta(t) \sigma_S \rho_{rS} - \mu_S)}{\sigma_S \rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] \\
&\quad + \Sigma_{t,16} \frac{D(t)}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} + \Sigma_{t,18} \left[\frac{\sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi}} - \frac{\rho'_{S\chi}}{\sigma_S \rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] \\
&\quad + \Sigma_{t,28} \left[\frac{\rho_{r\chi}}{\rho'_{\chi\chi}} - \frac{\rho_{rS} \rho'_{S\chi}}{\rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] - \frac{1}{2} \Sigma_{t,66} D(t) \left[\frac{r(t)+f}{4\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi^3(t)}} + \frac{3e}{4\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi^5(t)}} \right] \\
&\quad + \Sigma_{t,68} \left[\frac{r(t)+f}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} + \frac{e}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi^3(t)}} \right] \\
L^k b_{83}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{83}(t, X_t)}{\partial x_i} \\
&= b_{1k}(t, X_t) D(t) \left[\frac{\sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi}} - \frac{\rho'_{S\chi}}{\sigma_S \rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] + b_{2k}(t, X_t) D(t) \left[\frac{\rho_{r\chi}}{\rho'_{\chi\chi}} - \frac{\rho_{rS} \rho'_{S\chi}}{\rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right] \\
&\quad + b_{6k}(t, X_t) D(t) \left[\frac{r(t)+f}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} + \frac{e}{2\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi^3(t)}} \right] \\
&\quad + b_{8k}(t, X_t) \left[\frac{\theta(t) \rho_{r\chi}}{\rho'_{\chi\chi}} + \frac{r(t) \chi(t) + \chi(t) f - e}{\sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}} - \frac{\rho'_{S\chi} (r(t) + \theta(t) \sigma_S \rho_{rS} - \mu_S)}{\sigma_S \rho'_{\chi\chi} \sqrt{1-\rho_{rS}^2}} \right]
\end{aligned}$$

A3.19 $b_{84}(t, X_t)$

$$b_{84}(t, X_t) = I(D(t), t, r(t)) = D(t) \left\{ \frac{\theta(t) \rho_{r\gamma} + r(t) \gamma(t) - \theta(t) \rho_{r\chi} \rho''_{\chi\gamma} - \rho''_{\chi\gamma} [r(t) \chi(t) + \chi(t) f - e]}{\rho''_{\gamma\gamma} + \eta \rho''_{\gamma\gamma} - \rho'_{\chi\chi} \rho''_{\gamma\gamma} - \sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{\chi(t)}} - \frac{(\rho'_{\chi\chi} \rho''_{S\gamma} - \rho'_{S\chi} \rho''_{\chi\gamma}) [r(t) + \theta(t) \sigma_S \rho_{rS} - \mu_S]}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right\},$$

$$\frac{\partial b_{84}(t, X_t)}{\partial x_1} = D(t) \left[\frac{\gamma(t)}{\eta \rho''_{\gamma\gamma}} - \frac{\rho''_{\chi\gamma} \sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho''_{S\gamma}}{\sigma_S \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} + \frac{\rho'_{S\chi} \rho''_{\chi\gamma}}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right],$$

$$\frac{\partial b_{84}(t, X_t)}{\partial x_2} = D(t) \left[\frac{\rho_{r\gamma}}{\rho''_{\gamma\gamma}} - \frac{\rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho_{rS} \rho''_{S\gamma}}{\rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} + \frac{\rho_{rS} \rho'_{S\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right],$$

$$\frac{\partial b_{84}(t, X_t)}{\partial x_6} = -\frac{\rho''_{\chi\gamma} D(t)}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} \left[\frac{r(t)+f}{2\sqrt{\chi(t)}} + \frac{e}{2\sqrt{\chi^3(t)}} \right], \quad \frac{\partial b_{84}(t, X_t)}{\partial x_7} = \frac{D(t) r(t)}{\eta \rho''_{\gamma\gamma}},$$

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$$\frac{\partial b_{84}(t, X_t)}{\partial x_8} = \frac{\theta(t)\rho_{r\gamma}}{\rho_{\gamma\gamma}''} + \frac{r(t)\gamma(t)}{\eta\rho_{\gamma\gamma}''} - \frac{\theta(t)\rho_{r\chi}\rho_{\chi\gamma}''}{\rho_{\chi\chi}'\rho_{\gamma\gamma}''} - \frac{\rho_{\chi\gamma}'' [r(t)\chi(t) + \chi(t)f - e]}{\sigma_{\chi}\rho_{\chi\chi}'\rho_{\gamma\gamma}''\sqrt{\chi(t)}} - \frac{(\rho_{\chi\chi}'\rho_{S\gamma}'' - \rho_{S\chi}'\rho_{\chi\gamma}'')[r(t) + \theta(t)\sigma_S\rho_{rS} - \mu_S]}{\sigma_S\rho_{\chi\chi}'\rho_{\gamma\gamma}''\sqrt{1 - \rho_{rS}^2}},$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_1 \partial x_6} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_6 \partial x_1} = -\frac{\rho_{\chi\gamma}'' D(t)}{2\sigma_{\chi}\rho_{\chi\chi}'\rho_{\gamma\gamma}''\sqrt{\chi(t)}},$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_1 \partial x_7} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_7 \partial x_1} = \frac{D(t)}{\eta\rho_{\gamma\gamma}''},$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_1 \partial x_8} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_8 \partial x_1} = \frac{\gamma(t)}{\eta\rho_{\gamma\gamma}''} - \frac{\rho_{\chi\gamma}''\sqrt{\chi(t)}}{\sigma_{\chi}\rho_{\chi\chi}'\rho_{\gamma\gamma}''} - \frac{\rho_{S\gamma}''}{\sigma_S\rho_{\gamma\gamma}''\sqrt{1 - \rho_{rS}^2}} + \frac{\rho_{S\chi}'\rho_{\chi\gamma}''}{\sigma_S\rho_{\chi\chi}'\rho_{\gamma\gamma}''\sqrt{1 - \rho_{rS}^2}},$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_2 \partial x_8} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_8 \partial x_2} = \frac{\rho_{r\gamma}}{\rho_{\gamma\gamma}''} - \frac{\rho_{r\chi}\rho_{\chi\gamma}''}{\rho_{\chi\chi}'\rho_{\gamma\gamma}''} - \frac{\rho_{rS}\rho_{S\gamma}''}{\rho_{\gamma\gamma}''\sqrt{1 - \rho_{rS}^2}} + \frac{\rho_{rS}\rho_{S\chi}'\rho_{\chi\gamma}''}{\rho_{\chi\chi}'\rho_{\gamma\gamma}''\sqrt{1 - \rho_{rS}^2}},$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_6 \partial x_6} = \frac{\rho_{\chi\gamma}'' D(t)}{\sigma_{\chi}\rho_{\chi\chi}'\rho_{\gamma\gamma}''} \left[\frac{(r(t) + f)}{4\sqrt{\chi^3(t)}} + \frac{3e}{4\sqrt{\chi^5(t)}} \right]$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_6 \partial x_8} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_8 \partial x_6} = -\frac{\rho_{\chi\gamma}''}{\sigma_{\chi}\rho_{\chi\chi}'\rho_{\gamma\gamma}''} \left[\frac{(r(t) + f)}{2\sqrt{\chi(t)}} + \frac{e}{2\sqrt{\chi^3(t)}} \right],$$

$$\frac{\partial^2 b_{84}(t, X_t)}{\partial x_7 \partial x_8} = \frac{\partial^2 b_{84}(t, X_t)}{\partial x_8 \partial x_7} = \frac{r(t)}{\eta\rho_{\gamma\gamma}''},$$

$$\begin{aligned}
L^0 b_{84}(t, X_t) &= \frac{\partial b_{84}(t, X_t)}{\partial t} + \sum_{i=1}^8 a_i(t, X_t) \frac{\partial b_{84}(t, X_t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^8 \Sigma_{t,ij} \frac{\partial^2 b_{84}(t, X_t)}{\partial x_i \partial x_j} \\
&= a_1(t, X_t) D(t) \left[\frac{\gamma(t)}{\eta \rho''_{\gamma\gamma}} - \frac{\rho''_{\chi\gamma} \sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho''_{S\gamma}}{\sigma_S \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} + \frac{\rho'_{S\chi} \rho''_{\chi\gamma}}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right] \\
&\quad + a_2(t, X_t) D(t) \left[\frac{\rho_{r\gamma}}{\rho''_{\gamma\gamma}} - \frac{\rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho_{rs} \rho''_{S\gamma}}{\rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} + \frac{\rho_{rs} \rho'_{S\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right] \\
&\quad - a_6(t, X_t) \frac{\rho''_{\chi\gamma} D(t)}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} \left[\frac{(r(t)+f)}{2\sqrt{\chi(t)}} + \frac{e}{2\sqrt{\chi^3(t)}} \right] \\
&\quad + a_7(t, X_t) \frac{D(t)r(t)}{\eta \rho''_{\gamma\gamma}} + a_8(t, X_t) \left\{ \frac{\theta(t) \rho_{r\gamma} + r(t) \gamma(t)}{\rho''_{\gamma\gamma}} - \frac{\theta(t) \rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} \right. \\
&\quad \left. - \frac{\rho''_{\chi\gamma} [r(t) \chi(t) + \chi(t) f - e]}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{\chi(t)}} \right. \\
&\quad \left. - \frac{(\rho'_{\chi\chi} \rho''_{S\gamma} - \rho'_{S\chi} \rho''_{\chi\gamma}) [r(t) + \theta(t) \sigma_S \rho_{rs} - \mu_S]}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right\} \\
&\quad - \Sigma_{t,16} \frac{\rho''_{\chi\gamma} D(t)}{2\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{\chi(t)}} + \Sigma_{t,17} \frac{D(t)}{\eta \rho''_{\gamma\gamma}} + \Sigma_{t,18} \left[\frac{\gamma(t)}{\eta \rho''_{\gamma\gamma}} - \frac{\rho''_{\chi\gamma} \sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho''_{S\gamma}}{\sigma_S \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right. \\
&\quad \left. + \frac{\rho'_{S\chi} \rho''_{\chi\gamma}}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right] \\
&\quad + \Sigma_{t,28} \left[\frac{\rho_{r\gamma}}{\rho''_{\gamma\gamma}} - \frac{\rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho_{rs} \rho''_{S\gamma}}{\rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} + \frac{\rho_{rs} \rho'_{S\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rs}^2}} \right] \\
&\quad + \frac{1}{2} \Sigma_{t,66} \frac{\rho''_{\chi\gamma} D(t)}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} \left[\frac{(r(t)+f)}{4\sqrt{\chi^3(t)}} + \frac{3e}{4\sqrt{\chi^5(t)}} \right] \\
&\quad - \Sigma_{t,68} \frac{\rho''_{\chi\gamma}}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} \left[\frac{(r(t)+f)}{2\sqrt{\chi(t)}} + \frac{e}{2\sqrt{\chi^3(t)}} \right] + \Sigma_{t,78} \frac{r(t)}{\eta \rho''_{\gamma\gamma}}
\end{aligned}$$

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$$\begin{aligned}
 L^k b_{84}(t, X_t) &= \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial b_{84}(t, X_t)}{\partial x_i} \\
 &= b_{1k}(t, X_t) D(t) \left[\frac{\gamma(t)}{\eta \rho''_{\gamma\gamma}} - \frac{\rho''_{\chi\gamma} \sqrt{\chi(t)}}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho''_{S\gamma}}{\sigma_S \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} + \frac{\rho'_{S\chi} \rho''_{\chi\gamma}}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right] \\
 &\quad + b_{2k}(t, X_t) D(t) \left[\frac{\rho_{r\gamma}}{\rho''_{\gamma\gamma}} - \frac{\rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho_{rS} \rho''_{S\gamma}}{\rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} + \frac{\rho_{rS} \rho'_{S\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right] \\
 &\quad - b_{6k}(t, X_t) \frac{\rho''_{\chi\gamma} D(t)}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma}} \left[\frac{(r(t)+f)}{2\sqrt{\chi(t)}} + \frac{e}{2\sqrt{\chi^3(t)}} \right] + b_{7k}(t, X_t) \frac{D(t)r(t)}{\eta \rho''_{\gamma\gamma}} \\
 &\quad + b_{8k}(t, X_t) \left\{ \frac{\theta(t) \rho_{r\gamma}}{\rho''_{\gamma\gamma}} + \frac{r(t) \gamma(t)}{\eta \rho''_{\gamma\gamma}} - \frac{\theta(t) \rho_{r\chi} \rho''_{\chi\gamma}}{\rho'_{\chi\chi} \rho''_{\gamma\gamma}} - \frac{\rho''_{\chi\gamma} [r(t) \chi(t) + \chi(t) f - e]}{\sigma_\chi \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{\chi(t)}} \right\} \\
 &\quad \left\{ - \frac{(\rho'_{\chi\chi} \rho''_{S\gamma} - \rho'_{S\chi} \rho''_{\chi\gamma}) [r(t) + \theta(t) \sigma_S \rho_{rS} - \mu_S]}{\sigma_S \rho'_{\chi\chi} \rho''_{\gamma\gamma} \sqrt{1-\rho_{rS}^2}} \right\}
 \end{aligned}$$