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# STOCHASTIC DEFLATOR FOR AN ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

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## Abstract

In this paper, we implement a stochastic deflator with four economic and financial risk factors: interest rates, stock prices, default intensities, and convenience yields. We examine the deflator with different financial assets, such as stocks, zero-coupon bonds, vanilla options, and corporate coupon bonds. Our numerical results show the reliability of the deflator approach in pricing financial derivatives.

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## 1. INTRODUCTION

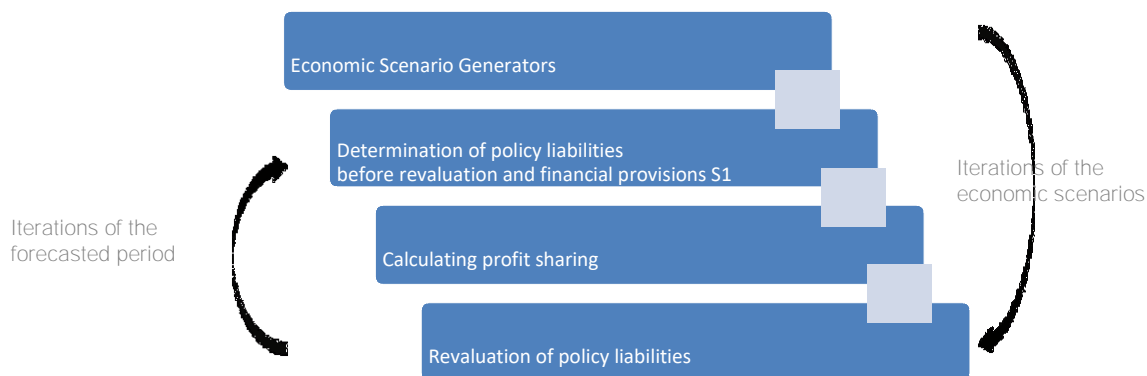
The Arrow-Debreu model of general equilibrium introduced the existence of an equilibrium in which the allocation of consumption and production is Pareto optimal with a system of prices for contingent commodities.<sup>1</sup> Their works have inspired tremendous research in fields of macroeconomics, financial economics, and asset pricing theory. Based on the concept of Arrow-Debreu securities, researchers had developed the fundamental theorems of asset pricing, which the second theorem tells us that an arbitrage-free market is complete if and only if the equivalent martingale measure is unique.<sup>2</sup>

In the case of Brownian diffusion, the Girsanov's Theorem enables us to change probability measure from a physical world to a risk-neutral world. Under risk-neutral measure, we have a closed-form solution for Black-Scholes options pricing model. However, we wouldn't always have analytical solutions for various classes of stochastic processes, which motivates us to study numerical methods for approximating solutions.

Due to the complicatedness of life insurance contracts and interactions among economic and financial risk factors, a reliable tool for asset/liability management (ALM) and calculations of reserves would be demanded. In practice, “economic scenario generators” assist insurers in pricing insurance contracts and managing long-term risk<sup>3</sup>.

The usual pricing scheme is as follows.

Fig. 1 - Calculating the best estimate reserve for a life insurance contract



Usually, economic scenarios are computed under a risk-neutral measure; the actualization process is then simply the risk-free rate and it makes the calculation quite easy, numerically speaking.

But, we like to point out that many “unusual” scenarios occur (e.g. 10-year rate  $\geq 50\%$ ) under risk-neutral measure, which increases the difficulty to justify the calibration of “reaction functions” embedded in the ALM-projection model used to compute cash flows. For example, the lapse rate is often a function of the difference between the revalorization rate of the contract and a reference rate; the parameters are calibrated observing “usual” values of economic parameters but may become difficult later to

<sup>1</sup> See, for example, Arrow and Debreu (1954), Geanakoplos (1989), and Mas-Colell et al. (1995) Chapter 19.

<sup>2</sup> See, for example, Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (1994), and Shreve (2004) Chapter 5.4.

<sup>3</sup> See, for example, Varnell (2011), Laurent et al. (2016) Chapters 3,4, and 5, and Pedersen et al. (2016).

justify for atypical values of economic risk factors. We could use a stochastic deflator in a way to address this problem, using only scenarios under physical measure<sup>4</sup>. But the numerical calculations become tedious due to the complexity of the deflator, mixing a risk-free rate process and a density of change of measure between physical and risk-neutral measure.

In this paper, we apply the deflator approach initiated by Dastarac and Sauveplane (2010) in the same context and include the processes of default and convenience yield from Longstaff et al. (2005) to calculate prices for financial derivatives. We compare the values calculated from the deflator approach with the values suggested by analytical formulas in simple cases. Our numerical results show the reliability of the deflator approach for quite simple financial derivatives. Our ultimate goal is then to use this deflator to compute best estimates for a life insurance contract, that will be done in a further work.

The remainder of the paper is organized as follows. Section 2 shows the deflator approach. Section 3 discusses the implementation of time discretization. Section 4 presents the numerical results. Section 5 concludes.

## 2. DEFLATOR APPROACH

Before discussing and deriving the general form of deflator, we need to generate correlated Brownian motions for the processes we include in our model. In our model, we consider the processes of interest rates, market prices of risk, stock prices, default intensities, and convenience yields. Sections 2 and 3 discuss the technical details of implementations of the deflator and time discretization. Readers who familiar with stochastic deflator and time discretization could directly skip to numerical results in Section 4.

### 2.1. GENERATE CORRELATED BROWNIAN MOTIONS

Let the Brownian motion part of each process  $\mathbf{W}_{ESG}$  and the correlations matrix  $\mathbf{C}_{ESG}$  among interest rates, stock prices, default intensities, and convenience yields be as follows.

$$\mathbf{W}_{ESG} = \begin{bmatrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{bmatrix}, \quad \mathbf{C}_{ESG} = \begin{matrix} & \begin{matrix} W_r & W_S & W_\chi & W_\gamma \end{matrix} \\ \begin{matrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{matrix} & \begin{bmatrix} 1 & \rho_{rS} & \rho_{r\chi} & \rho_{r\gamma} \\ \rho_{rS} & 1 & \rho_{S\chi} & \rho_{S\gamma} \\ \rho_{r\chi} & \rho_{S\chi} & 1 & \rho_{\chi\gamma} \\ \rho_{r\gamma} & \rho_{S\gamma} & \rho_{\chi\gamma} & 1 \end{bmatrix} \end{matrix} \quad (1)$$

Denote  $r$  interest rate;  $S$  stock price;  $\chi$  default intensity;  $\gamma$  convenience yield;  $W_i, i=r, S, \chi, \gamma$  Brownian motion part of each process; and  $\rho_{jk}, j \neq k$  correlation between each two processes. In order to generate correlated  $W_i, i=r, S, \chi, \gamma$ , we require four independent Brownian motions  $W_i, i=0,1,2,3$ . Following is the construction

---

<sup>4</sup> See, for example, Bonnin et al. (2014), Borel-Mathurin et al. (2015), and Vedani et al. (2017).

of  $\mathbf{W}_{ESG}$ , details of the calculations is provided in the Appendix 1 of Supplementary materials.

$$\mathbf{W}_{ESG} = \begin{bmatrix} W_r \\ W_S \\ W_\chi \\ W_\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{rS} & \sqrt{1-\rho_{rS}^2} & 0 & 0 \\ \rho_{r\chi} & \rho'_{S\chi} & \rho''_{\chi\chi} & 0 \\ \rho_{r\gamma} & \rho''_{S\gamma} & \rho''_{\chi\gamma} & \rho''_{\gamma\gamma} \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} \quad (2)$$

where

$$\rho'_{S\chi} = \frac{\rho_{S\chi} - \rho_{rS}\rho_{r\chi}}{\sqrt{1-\rho_{rS}^2}}, \quad \rho''_{\chi\chi} = \sqrt{\frac{1-\rho_{rS}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}{1-\rho_{rS}^2}},$$

$$\rho''_{\chi\gamma} = \frac{\rho_{\chi\gamma} - \rho_{r\chi}\rho_{r\gamma} - \rho_{S\chi}\rho_{S\gamma} - \rho_{rS}^2\rho_{\chi\gamma} + \rho_{rS}\rho_{r\chi}\rho_{S\gamma} + \rho_{rS}\rho_{r\gamma}\rho_{S\chi}}{\sqrt{1+\rho_{rS}^4 - 2\rho_{rS}^3\rho_{r\chi}\rho_{S\chi} - 2\rho_{rS}^2 + \rho_{rS}^2\rho_{r\chi}^2 + \rho_{rS}^2\rho_{S\chi}^2 - \rho_{r\chi}^2 - \rho_{S\chi}^2 + 2\rho_{rS}\rho_{r\chi}\rho_{S\chi}}},$$

$$\rho''_{\gamma\gamma} = \sqrt{1-\rho_{r\gamma}^2 - \rho_{S\gamma}^2 - \rho_{\chi\gamma}^2}.$$

## 2.2. GENERAL FORM OF DEFLATOR WITH FIVE FACTORS<sup>5</sup>

Let  $r(t)$ ,  $B(t)$ ,  $P(t, T, r(t))$ ,  $S(t)$ ,  $\chi(t)$ ,  $\gamma(t)$ ,  $D(t)$  be processes of interest rate, short-term saving, zero coupon bond of no risk with maturity  $T$ , stock price, default density, convenience yield, and deflator respectively. Denote  $E(\cdot)$  expectation under physical measure and  $E^Q(\cdot)$  expectation under risk-neutral measure. Let a discount process  $\delta(t)$  equal  $e^{-\int_0^t r(s)ds}$ . For a nonnegative random variable  $X$ , we would like to have  $E^Q(\delta(t)X) = E[D(t)X]$  (i.e.  $D(t) = \delta(t)\frac{dQ}{dP}$ ). We describe the dynamics of each process in the following paragraphs, in a quite general Markovian framework.

### 2.2.1. Dynamics of each process

$$dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW_r(t) \quad (3)$$

$$dB(t) = B(t)r(t)dt \quad (4)$$

Equations 3 and 4 are the dynamics of interest rate and short-term saving respectively, where  $\alpha(t, r(t))$  and  $\beta(t, r(t))$  are the drift term and diffusion term of interest rate process respectively and  $B(t)r(t)$  is the drift term of short-term saving process.

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<sup>5</sup> For discussions of stochastic deflator in insurance, see, for example, Dastarac and Sauveplane (2010) and Caja and Planchet (2011). For a reference of stochastic calculus related to Itô's lemma and Girsanov's Theorem, see Shreve (2004) Chapters 4 and 5.

Let the process of zero coupon bond of no risk with maturity  $T$  be  $\frac{dP(t,T,r(t))}{P(t,T,r(t))} = \tilde{\mu}(t,r(t))dt + \tilde{\sigma}(t,r(t))dW_r(t)$ . We would like to derive the drift term  $\tilde{\mu}(t,r(t))$  and the diffusion term  $\tilde{\sigma}(t,r(t))$  for  $P(t,T,r(t))$ . Since  $P(t,T,r(t))$  is a function of  $t$  and  $r(t)$ , by Itô formula, we have

$$dP(t,T,r(t)) = \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} dr(t) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} dr(t)dr(t)$$

where  $\frac{\partial P}{\partial t}$  is the first partial derivative of  $P(t,T,r(t))$  with respect to  $t$ ,  $\frac{\partial P}{\partial r}$  and  $\frac{\partial^2 P}{\partial r^2}$  are the first and second partial derivative of  $P(t,T,r(t))$  with respect to  $r(t)$ . Thus, the dynamics of  $P(t,T,r(t))$  could also be written as

$$dP(t,T,r(t)) = \left( P_t + \alpha(t,r(t))P_r + \frac{1}{2}P_{rr}\beta^2(t,r(t)) \right) dt + P_r\beta(t,r(t))dW_r(t),$$

$P_t = \frac{\partial P}{\partial t}$ ,  $P_r = \frac{\partial P}{\partial r}$ ,  $P_{rr} = \frac{\partial^2 P}{\partial r^2}$ . Comparing these two representations of  $dP(t,T,r(t))$ , we have  $P(t,T,r(t))\tilde{\sigma}(t,r(t)) = P_r\beta(t,r(t))$ . As a result, we could calculate  $\tilde{\sigma}(t,r(t))$  by  $\frac{P_r\beta(t,r(t))}{P(t,T,r(t))}$ .

Next step is to derive the drift term  $\tilde{\mu}(t,r(t))$ .  $\delta(t)$  equals  $e^{-\int_0^t r(s)ds}$ ,  $I(t)$  equals  $\int_0^t r(s)ds$ , and  $f(x) = e^{-x}$ . Then,  $dI(t)$  is equal to  $r(t)dt$ ,  $f'(x) = -f(x)$ , and  $f''(x) = f(x)$ . By Itô formula again,

$$d\delta(t) = df(I(t)) = f'(I(t))dI(t) + \frac{1}{2}f''(I(t))dI(t)dI(t) = -r(t)\delta(t)dt.$$

$d[\delta(t)P(t,T,r(t))] = \delta(t)dP(t,T,r(t)) + P(t,T,r(t))d\delta(t) + d\delta(t)dP(t,T,r(t))$  by Itô product rule. We introduce the process of market price of risk  $\theta(t)$  here, and let  $\theta(t)$  equal  $\frac{\tilde{\mu}(t,r(t)) - r(t)}{\tilde{\sigma}(t,r(t))}$  and  $d\tilde{W}_r(t) = \theta(t)dt + dW_r(t)$ . Then

$$\begin{aligned} d[\delta(t)P(t,T,r(t))] &= [\tilde{\mu}(t,r(t)) - r(t)]\delta(t)P(t,T,r(t))dt + \tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))dW_r(t) \\ &= \tilde{\sigma}(t,r(t))\delta(t)P(t,T,r(t))d\tilde{W}_r(t) \end{aligned}$$

Let  $\frac{d\mathbf{Q}'}{d\mathbf{P}} = Z(t) = \mathbf{exp}\left\{-\int_0^t \theta(s) dW_r - \frac{1}{2} \int_0^t \theta^2(s) ds\right\}$ ,  $Z = Z(T)$ , and assume  $E\left(\int_0^T \theta^2(s) Z^2(s) ds\right) < \infty$ . Then by Girsanov's Theorem,  $E(Z) = 1$ , and  $\tilde{W}_r(t)$ ,  $0 \leq t \leq T$ , under probability measure  $\mathbf{Q}'$  is a Brownian motion. Thus, we could rewrite

$$d\left[\delta(t)P(t, T, r(t))\right] = \tilde{\sigma}(t, r(t))\delta(t)P(t, T, r(t))d\tilde{W}_r(t)$$

and

$$\frac{dP(t, T, r(t))}{P(t, T, r(t))} = (\tilde{\mu}(t, r(t)) - \tilde{\sigma}(t, r(t))\theta(t))dt + \tilde{\sigma}(t, r(t))d\tilde{W}_r(t).$$

$P(t, T, r(t))$  is the zero coupon bond of no risk with maturity  $T$ , so the drift term of  $\frac{dP(t, T, r(t))}{P(t, T, r(t))}$  under probability measure  $\mathbf{Q}'$  is equal to  $r(t)$ .<sup>6</sup> Hence, we have  $\tilde{\mu}(t, r(t))$  equal  $r(t) + \tilde{\sigma}(t, r(t))\theta(t)$ .

$$\begin{aligned} \frac{dP(t, T, r(t))}{P(t, T, r(t))} &= [r(t) + \tilde{\sigma}(t, r(t))\theta(t)]dt + \tilde{\sigma}(t, r(t))dW_r(t), \quad \tilde{\sigma}(t, r(t)) = \frac{P_r \beta(t, r(t))}{P(t, r(t))} \\ dP(t, T, r(t)) &= P(t, T, r(t))[r(t) + \tilde{\sigma}(t, r(t))\theta(t)]dt + P(t, T, r(t))\tilde{\sigma}(t, r(t))dW_r(t) \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \mu_s(t)dt + \sigma_s(t)dW_s(t) \\ dS(t) &= S(t)\mu_s(t)dt + S(t)\sigma_s(t)dW_s(t) \\ &= S(t)\mu_s(t)dt + S(t)\sigma_s(t)\rho_{rs}dW_r(t) + S(t)\sigma_s(t)\sqrt{1 - \rho_{rs}^2}dW_1(t) \end{aligned} \quad (6)$$

Equations 5 and 6 show the dynamics of zero coupon bond  $P(t, T, r(t))$  and stock price  $S(t)$ . Following the model settings in Longstaff et al. (2005), we let the processes of default density  $\chi(t)$  and convenience yield  $\gamma(t)$  as follows.

$$\begin{aligned} d\chi(t) &= [e - f\chi(t)]dt + \sigma_\chi \sqrt{\chi(t)}dW_\chi(t); \quad e, f, \sigma_\chi > 0 \\ &= [e - f\chi(t)]dt + \sigma_\chi \rho_{r\chi} \sqrt{\chi(t)}dW_r(t) + \sigma_\chi \rho'_{s\chi} \sqrt{\chi(t)}dW_1(t) + \sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)}dW_2(t) \end{aligned} \quad (7)$$

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<sup>6</sup> We could verify this by the intuition of change of measure with  $\delta(t)$  and the solution of  $\delta(t)P(t, T, r(t))$  being exponential functions. Alternatively, we could plug  $\tilde{\mu}(t, r(t))$  back to  $d[\delta(t)P(t, T, r(t))]$  to verify the statement.

$$d\gamma(t) = \eta dW_\gamma(t) = \eta\rho_{r\gamma} dW_r(t) + \eta\rho_{s\gamma}'' dW_1(t) + \eta\rho_{\chi\gamma}'' dW_2(t) + \eta\rho_{\gamma\gamma}'' dW_3(t), \quad \eta > 0 \quad (8)$$

We are now able to derive the general form of deflator. First, let

$$dD(t) = \Omega(D(t), t, r(t))dt + \Phi(D(t), t, r(t))dW_r(t) + \Psi(D(t), t, r(t))dW_1(t) + \Gamma(D(t), t, r(t))dW_2(t) + I(D(t), t, r(t))dW_3(t). \quad (9)$$

We would like to have  $D(t)S(t)$ ,  $D(t)B(t)$ ,  $D(t)P(t, T, r(t))$ ,  $D(t)\chi(t)$ , and  $D(t)\gamma(t)$  be  $\mathbf{P}$ -martingale. By Itô product rule again, we have  $d[D(t)X(t)] = X(t)dD(t) + D(t)dX(t) + dX(t)dD(t)$  for a stochastic process  $X(t)$ . We derive  $\Omega(D(t), t, r(t))$ ,  $\Phi(D(t), t, r(t))$ ,  $\Psi(D(t), t, r(t))$ ,  $\Gamma(D(t), t, r(t))$ , and  $I(D(t), t, r(t))$  step by step in the Appendix 2 of Supplementary materials.

Let

$$\begin{aligned} K_\Psi(t, r(t)) &= \frac{\Psi(D(t), t, r(t))}{D(t)} = \frac{r(t) + \theta(t)\sigma_s(t)\rho_{rs} - \mu_s(t)}{\sigma_s(t)\sqrt{1 - \rho_{rs}^2}}, \\ K_\Gamma(t, r(t)) &= \frac{\Gamma(D(t), t, r(t))}{D(t)} = \frac{\theta(t)\rho_{r\chi} + r(t)\chi(t) - e + f\chi(t)}{\rho_{\chi\chi}' + \sigma_\chi\rho_{\chi\chi}'\sqrt{\chi(t)}} + \frac{\rho_{s\chi}'[\mu_s(t) - r(t) - \theta(t)\sigma_s(t)\rho_{rs}]}{\rho_{\chi\chi}''\sigma_s(t)\sqrt{1 - \rho_{rs}^2}}, \\ K_I(t, r(t)) &= \frac{I(D(t), t, r(t))}{D(t)} = \frac{\rho_{r\gamma}''\theta(t)}{\rho_{\gamma\gamma}''} + \frac{r(t)\gamma(t)}{\eta\rho_{\gamma\gamma}''} - \frac{\rho_{\chi\gamma}''\rho_{r\chi}\theta(t)}{\rho_{\gamma\gamma}''\rho_{\chi\chi}'} + \frac{\rho_{\chi\gamma}''[e - r(t)\chi(t) - f\chi(t)]}{\rho_{\gamma\gamma}''\rho_{\chi\chi}'\sigma_\chi\sqrt{\chi(t)}}, \\ &\quad + \frac{(\rho_{s\gamma}''\rho_{\chi\chi}' - \rho_{\chi\gamma}''\rho_{s\chi}')[\mu_s(t) - r(t) - \rho_{rs}\theta(t)\sigma_s(t)]}{\rho_{\gamma\gamma}''\rho_{\chi\chi}'\sigma_s(t)\sqrt{1 - \rho_{rs}^2}}, \end{aligned}$$

we have the general form of deflator  $D(t)$  as follows.

$$\begin{aligned} D(t) &= D(0) \exp \left\{ -\int_0^t r(s) ds - \int_0^t \frac{1}{2} \left[ \theta^2(s) + K_\Psi^2(t, r(t)) + K_\Gamma^2(t, r(t)) + K_I^2(t, r(t)) \right] ds \right\} \\ &\quad \times \exp \left[ -\int_0^t \theta(s) dW_r(t) + \int_0^t K_\Psi(t, r(t)) dW_1(t) + \int_0^t K_\Gamma(t, r(t)) dW_2(t) + \int_0^t K_I(t, r(t)) dW_3(t) \right] \end{aligned} \quad (10)$$

### 3. IMPLEMENTATION OF TIME DISCRETIZATION

In order to implement the deflator approach, we need to discretize time steps for each process. We discuss the time discretization here. We adopt the Euler method, the Milstein method, and the simplified Second Milstein method for time discretization in our model.<sup>7</sup>

<sup>7</sup>For references of time discretization, see, for example, Kloeden and Platen (1992), Iacus (2009), and Glasserman (2013).



Denote a stochastic process  $X(t)$  with its dynamics  $dX(t) = b_X(t, X(t))dt + \sigma_X(t, X(t))dW_X(t)$  where  $W_X(t)$  is the Brownian part of  $X(t)$ . We partition the time  $[0, T]$  into  $N$  segments with each length equaling  $(T-0)/N$ , then we have a time discretization  $\Pi_N = \Pi_N([0, T])$  with  $0 = t_0 < t_1 < \dots < t_N = T$ .

### 3.1. EULER METHOD

In Euler method, we approximate  $X(t)$  by  $Y_i$  discretely, in which  $Y_{i+1} = Y_i + b_X(t_i, Y_i)(t_{i+1} - t_i) + \sigma_X(t_i, Y_i)(W_{i+1} - W_i)$ ,  $i = 0, 1, \dots, N-1$ ,  $W_i$  is the value of a Brownian motion at time period  $i$ , and  $Y_0$  is equal to  $X(0)$ . Denote  $\Delta t_i = t_{i+1} - t_i = (T-0)/N$  and  $\Delta W_{k,i} = W_{k,t_{i+1}} - W_{k,t_i}$ ,  $k = r, 1, 2, 3$ . We approximate  $r(t)$ ,  $B(t)$ ,  $P(t, T, r(t))$ ,  $S(t)$ ,  $\chi(t)$ ,  $\gamma(t)$ ,  $D(t)$  by the Euler method as follows.

- $dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW_r(t)$ , then  $r_{i+1} = r_i + \alpha(t_i, r_i)\Delta t_i + \beta(t_i, r_i)\Delta W_{r,i}$ .
- $dB(t) = B(t)r(t)dt$ , then  $B_{i+1} = B_i(1 + r_i\Delta t_i)$ .
- $dP(t, T, r(t)) = P(t, T, r(t))[r(t) + \tilde{\sigma}(t, r(t))\theta(t)]dt + P(t, T, r(t))\tilde{\sigma}(t, r(t))dW_r(t)$ ,  
then  $P_{i+1} = P_i \left\{ 1 + [r_i + \tilde{\sigma}(t_i, r_i)\theta(t_i)]\Delta t_i + \tilde{\sigma}(t_i, r_i)\Delta W_{r,i} \right\}$ .
- $dS(t) = S(t)\mu_S(t)dt + S(t)\sigma_S(t)\rho_{rS}dW_r(t) + S(t)\sigma_S(t)\sqrt{1 - \rho_{rS}^2}dW_1(t)$ ,  
then  $S_{i+1} = S_i \left[ 1 + \mu_S(t_i)\Delta t_i + \sigma_S(t_i)\rho_{rS}\Delta W_{r,i} + \sigma_S(t_i)\sqrt{1 - \rho_{rS}^2}\Delta W_{1,i} \right]$ .
- $d\chi(t) = [e - f\chi(t)]dt + \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)}dW_r(t) + \sigma_\chi\rho'_{S\chi}\sqrt{\chi(t)}dW_1(t) + \sigma_\chi\rho''_{\chi\chi}\sqrt{\chi(t)}dW_2(t)$ ,  
then  $\chi_{i+1} = \chi_i + (e - f\chi_i)\Delta t_i + \sigma_\chi\rho_{r\chi}\sqrt{\chi_i}\Delta W_{r,i} + \sigma_\chi\rho'_{S\chi}\sqrt{\chi_i}\Delta W_{1,i} + \sigma_\chi\rho''_{\chi\chi}\sqrt{\chi_i}\Delta W_{2,i}$ .
- $d\gamma(t) = \eta\rho_{r\gamma}dW_r(t) + \eta\rho''_{S\gamma}dW_1(t) + \eta\rho''_{\chi\gamma}dW_2(t) + \eta\rho''_{\gamma\gamma}dW_3(t)$ ,  
then  $\gamma_{i+1} = \gamma_i + \eta(\rho_{r\gamma}\Delta W_{r,i} + \rho''_{S\gamma}\Delta W_{1,i} + \rho''_{\chi\gamma}\Delta W_{2,i} + \rho''_{\gamma\gamma}\Delta W_{3,i})$ .
- $\frac{dD(t)}{D(t)} = -r(t)dt - \theta(t)dW_r(t) + K_\Psi(t, r(t))dW_1(t) + K_\Gamma(t, r(t))dW_2(t) + K_I(t, r(t))dW_3(t)$ ,  
then  $D_{i+1} = D_i \left[ 1 - r_i\Delta t_i - \theta(t_i)\Delta W_{r,i} + K_\Psi(t_i, r_i)\Delta W_{1,i} + K_\Gamma(t_i, r_i)\Delta W_{2,i} + K_I(t_i, r_i)\Delta W_{3,i} \right]$ .

### 3.2. MILSTEIN METHOD

Denote  $\sigma_x = \frac{\partial \sigma_X(t, x)}{\partial x}$ , we approximate  $X(t)$  by  $Y_i$  discretely as  $Y_{i+1} = Y_i + b_X(t_i, Y_i)(t_{i+1} - t_i) + \sigma_X(t_i, Y_i)(W_{i+1} - W_i) + \frac{1}{2}\sigma_X(t_i, Y_i)\sigma_x(t_i, Y_i)[(W_{i+1} - W_i)^2 - (t_{i+1} - t_i)]$ .

Thus, we approximate  $r(t)$ ,  $B(t)$ ,  $P(t, T, r(t))$ ,  $S(t)$ ,  $\chi(t)$ ,  $\gamma(t)$ ,  $D(t)$  by the Milstein method as follows.

$$\bullet dr(t) = \alpha(t, r(t))dt + \beta(t, r(t))dW_r(t),$$

$$\text{then } r_{i+1} = r_i + \alpha(t_i, r_i)\Delta t_i + \beta(t_i, r_i)\Delta W_{r,i} + \frac{1}{2}\beta(t_i, r_i)\beta_r(t_i, r_i)\left[(\Delta W_{r,i})^2 - \Delta t_i\right].$$

$$\bullet dB(t) = B(t)r(t)dt, \text{ then } B_{i+1} = B_i(1 + r_i\Delta t_i).$$

$$\bullet dP(t, T, r(t)) = P(t, T, r(t))\left[r(t) + \tilde{\sigma}(t, r(t))\theta(t)\right]dt + P(t, T, r(t))\tilde{\sigma}(t, r(t))dW_r(t),$$

$$\text{then } P_{i+1} = P_i \left\{ 1 + [r_i + \tilde{\sigma}(t_i, r_i)\theta(t_i)]\Delta t_i + \tilde{\sigma}(t_i, r_i)\Delta W_{r,i} + \frac{1}{2}\tilde{\sigma}^2(t_i, r_i)\left[(\Delta W_{r,i})^2 - \Delta t_i\right] \right\}.$$

$$\bullet dS(t) = S(t)\mu_s(t)dt + S(t)\sigma_s(t)\rho_{rs}dW_r(t) + S(t)\sigma_s(t)\sqrt{1 - \rho_{rs}^2}dW_1(t),$$

$$\text{then } S_{i+1} = S_i \left\{ 1 + \mu_s(t_i)\Delta t_i + \sigma_s(t_i)\rho_{rs}\Delta W_{r,i} + \sigma_s(t_i)\sqrt{1 - \rho_{rs}^2}\Delta W_{1,i} \right. \\ \left. + \frac{1}{2}[\sigma_s(t_i)\rho_{rs}]^2\left[(\Delta W_{r,i})^2 - \Delta t_i\right] + \frac{1}{2}[\sigma_s(t_i)\sqrt{1 - \rho_{rs}^2}]^2\left[(\Delta W_{1,i})^2 - \Delta t_i\right] \right\}.$$

$$\bullet d\chi(t) = (e - f\chi(t))dt + \sigma_\chi\rho_{r\chi}\sqrt{\chi(t)}dW_r(t) + \sigma_\chi\rho'_{s\chi}\sqrt{\chi(t)}dW_1(t) + \sigma_\chi\rho'_{xx}\sqrt{\chi(t)}dW_2(t),$$

$$\text{then } \chi_{i+1} = \chi_i + (e - f\chi_i)\Delta t_i + \sigma_\chi\rho_{r\chi}\sqrt{\chi_i}\Delta W_{r,i} + \sigma_\chi\rho'_{s\chi}\sqrt{\chi_i}\Delta W_{1,i} + \sigma_\chi\rho'_{xx}\sqrt{\chi_i}\Delta W_{2,i} \\ + \frac{1}{4}(\sigma_\chi\rho_{r\chi})^2\left[(\Delta W_{r,i})^2 - \Delta t_i\right] + \frac{1}{4}(\sigma_\chi\rho'_{s\chi})^2\left[(\Delta W_{1,i})^2 - \Delta t_i\right] + \frac{1}{4}(\sigma_\chi\rho'_{xx})^2\left[(\Delta W_{2,i})^2 - \Delta t_i\right].$$

$$\bullet d\gamma(t) = \eta\rho_{r\gamma}dW_r(t) + \eta\rho''_{s\gamma}dW_1(t) + \eta\rho''_{\chi\gamma}dW_2(t) + \eta\rho''_{\gamma\gamma}dW_3(t),$$

$$\text{then } \gamma_{i+1} = \gamma_i + \eta\left(\rho_{r\gamma}\Delta W_{r,i} + \rho''_{s\gamma}\Delta W_{1,i} + \rho''_{\chi\gamma}\Delta W_{2,i} + \rho''_{\gamma\gamma}\Delta W_{3,i}\right).$$

$$\bullet \frac{dD(t)}{D(t)} = -r(t)dt - \theta(t)dW_r(t) + K_\Psi(t, r(t))dW_1(t) + K_\Gamma(t, r(t))dW_2(t) + K_I(t, r(t))dW_3(t),$$

$$\text{then } D_{i+1} = D_i \left[ 1 - r_i\Delta t - \theta(t_i)\Delta W_{r,i} + K_\Psi(t_i, r_i)\Delta W_{1,i} + K_\Gamma(t_i, r_i)\Delta W_{2,i} + K_I(t_i, r_i)\Delta W_{3,i} \right] \\ - \frac{1}{2}D_i\theta^2(t_i)\left[(\Delta W_{r,i})^2 - \Delta t_i\right] + \frac{1}{2}D_iK_\Psi^2(t_i, r_i)\left[(\Delta W_{1,i})^2 - \Delta t_i\right] \\ + \frac{1}{2}D_iK_\Gamma^2(t_i, r_i)\left[(\Delta W_{2,i})^2 - \Delta t_i\right] + \frac{1}{2}D_iK_I^2(t_i, r_i)\left[(\Delta W_{3,i})^2 - \Delta t_i\right].$$

### 3.3. SIMPLIFIED SECOND MILSTEIN METHOD

We advance to multi-dimensional case in this sub-section. Let  $X_t$  be multi-dimensional stochastic processes with the dynamics  $dX_t = a(t, X_t)dt + b(t, X_t)dW_t$ , where  $X_t$  is a  $d \times 1$  vector,  $a(t, X_t)$  is a  $d \times 1$  vector,  $b(t, X_t)$  is a  $d \times m$  matrix, and  $W_t$  is a  $d \times 1$  vector.  $d$  is the number of different stochastic processes in  $X_t$ , and  $m$  is the number of independent Brownian motions involved in  $X_t$ .

For a continuously twice differentiable function  $f(t, x_{d \times 1})$ , we could write  $df(t, X_t)$  by Itô formula for multi-dimensional case as follows.

$$df(t, X_t) = \left[ \frac{\partial f(t, X_t)}{\partial t} + \sum_{i=1}^d \frac{\partial f(t, X_t)}{\partial x_i} a_i(t, X_t) + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f(t, X_t)}{\partial x_i \partial x_j} \Sigma_{t,ij} \right] dt + \sum_{i=1}^d \sum_{k=1}^m b_{ik}(t, X_t) \frac{\partial f(t, X_t)}{\partial x_i} dW_{t,k}, \quad \Sigma_t = b(t, X_t) b^T(t, X_t) \quad (11)$$

In equation 11,  $a_i(t, X_t)$  is the element of  $i^{\text{th}}$  row of  $a(t, X_t)$ ,  $b_{ik}(t, X_t)$  is the element of  $b(t, X_t)$  at its  $i^{\text{th}}$  row and  $k^{\text{th}}$  column,  $b^T(t, X_t)$  is the transpose of  $b(t, X_t)$ ,  $\Sigma_{t,ij}$  is the element of  $\Sigma_t$  at its  $i^{\text{th}}$  row and  $j^{\text{th}}$  column, and  $W_{t,k}$  is the element of  $k^{\text{th}}$  row of  $W_t$ . Next, we introduce operators  $L^0$  and  $L^k$  and rewrite  $df(t, X_t)$  for multi-dimensional case.

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^d a_i(t, X_t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \Sigma_{t,ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad (12)$$

$$L^k = \sum_{i=1}^d b_{ik}(t, X_t) \frac{\partial}{\partial x_i}, \quad \forall k = 1, \dots, m \quad (13)$$

$$df(t, X_t) = L^0 f(t, X_t) dt + \sum_{k=1}^m L^k f(t, X_t) dW_{t,k} \quad (14)$$

We approximate  $X_t$  by  $Y_t$  discretely by simplified Second Milstein method, where  $Y_t$  is a  $d \times 1$  vector. For each  $i = 1, \dots, d$ ,

$$Y_{n+1,i} = Y_{n,i} + a_i(n, Y_n) \Delta t + \sum_{k=1}^m b_{ik}(n, Y_n) \Delta W_{n,k} + \frac{1}{2} L^0 a_i(n, Y_n) (\Delta t)^2 + \frac{1}{2} \sum_{k=1}^m [L^k a_i(n, Y_n) + L^0 b_{ik}(n, Y_n)] \Delta W_{n,k} \Delta t + \frac{1}{2} \sum_{k=1}^m \sum_{j=1}^m L^j b_{ik}(n, Y_n) (\Delta W_{n,j} \Delta W_{n,k} - V_{jk}) \quad (15)$$

$Y_{n+1,i}$  is the element of  $i^{\text{th}}$  row of  $Y_t$  in the time step  $n+1$ .  $V_{jk}$  is an independent random variable with probabilities  $\Pr(V_{jk} = \Delta t) = \Pr(V_{jk} = -\Delta t) = \frac{1}{2}$  for  $j < k$ ,  $V_{kj} = -V_{jk}$  for  $j > k$ , and  $V_{jk} = \Delta t$  for  $j = k$ . The following are the  $X_t$ ,  $a(t, X_t)$ ,  $W_t$ , and  $b(t, X_t)$  in our model.

$$X_t = \begin{bmatrix} r(t) \\ \theta(t) \\ B(t) \\ P(t, T, r(t)) \\ S(t) \\ \chi(t) \\ \gamma(t) \\ D(t) \end{bmatrix}, \quad a(t, X_t) = \begin{bmatrix} \alpha(t, r(t)) \\ a_\theta - b_\theta \theta(t) \\ B(t)r(t) \\ P(t, T, r(t)) [r(t) + \tilde{\sigma}(t, r(t)) \theta(t)] \\ S(t) \mu_s(t) \\ e - f \chi(t) \\ 0 \\ -r(t) D(t) \end{bmatrix}, \quad W_t = \begin{bmatrix} W_r(t) \\ W_1(t) \\ W_2(t) \\ W_3(t) \\ W_\theta(t) \end{bmatrix},$$

$$b(t, X_t) = \begin{bmatrix} \beta(t, r(t)) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\theta \sqrt{\theta(t)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P(t, T, r(t)) \tilde{\sigma}(t, r(t)) & 0 & 0 & 0 & 0 & 0 & 0 \\ S(t) \sigma_s(t) \rho_{rs} & S(t) \sigma_s(t) \sqrt{1 - \rho_{rs}^2} & 0 & 0 & 0 & 0 & 0 \\ \sigma_\chi \rho_{r\chi} \sqrt{\chi(t)} & \sigma_\chi \rho_{s\chi} \sqrt{\chi(t)} & \sigma_\chi \rho'_{\chi\chi} \sqrt{\chi(t)} & 0 & 0 & 0 & 0 \\ \eta \rho_{r\gamma} & \eta \rho'_{s\gamma} & \eta \rho'_{\chi\gamma} & \eta \rho'_{\gamma\gamma} & 0 & 0 & 0 \\ -\theta(t) D(t) & \Psi(D(t), t, r(t)) & \Gamma(D(t), t, r(t)) & \mathbf{I}(D(t), t, r(t)) & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

#### 4. NUMERICAL RESULTS

We implement the deflator approach with three methods for time discretization and adopt CIR interest rate model for short-term saving.<sup>8</sup> In addition, we also incorporate parallel computing with a variance technique, antithetic sampling in our algorithm.<sup>9</sup> In CIR interest rate model,  $dr(t) = [a_r - b_r r(t)] dt + \sigma_r \sqrt{r(t)} d\tilde{W}_r(t)$ ;  $a_r, b_r, \sigma_r > 0$ . The process of interest rate is defined under probability measure  $\mathbf{Q}'$ . In order to convert the process into physical measure  $\mathbf{P}$ , we have to consider the process of market price of risk  $\theta(t)$ . From Section 2.2.1, we let  $d\tilde{W}_r(t) = \theta(t) dt + dW_r(t)$ . Thus, we could rewrite  $dr(t)$  in  $\mathbf{P}$ -measure as  $dr(t) = [a_r - b_r r(t) + \theta(t) \sigma_r \sqrt{r(t)}] dt + \sigma_r \sqrt{r(t)} dW_r(t)$ .

Let  $\theta(t)$  also be CIR process here and  $W_\theta$  is an independent Brownian motion of  $W_i$ ,  $i = r, 1, 2, 3$ .<sup>10</sup> The dynamics of  $\theta(t)$  is

$$d\theta(t) = [a_\theta - b_\theta \theta(t)] dt + \sigma_\theta \sqrt{\theta(t)} dW_\theta(t); a_\theta, b_\theta, \sigma_\theta > 0.$$

In CIR interest rate model, the price of zero coupon bond of no risk with maturity  $T$ ,  $P(t, T, r(t))$ , is equal to  $e^{-r(t)C_P(t, T) - A_P(t, T)}$  where

$$C_P(t, T) = \frac{\sinh(\gamma_{CIR}(T-t))}{\gamma_{CIR} \cosh(\gamma_{CIR}(T-t)) + \frac{1}{2} b_r \sinh(\gamma_{CIR}(T-t))}, \quad \gamma_{CIR} = \frac{1}{2} \sqrt{b_r^2 + 2\sigma_r^2},$$

<sup>8</sup> Here we choose CIR interest rate model because the model has a closed-form formula for prices of zero-coupon bonds of no risk.

<sup>9</sup> The R codes are available from the authors by inquiry.

<sup>10</sup> Here we choose  $\theta(t)$  to be CIR process, so that  $\theta(t)$  would be positive in any time period  $t$ .

$$\text{and } A_p(t, T) = -\frac{2a_r}{\sigma_r^2} \ln \left[ \frac{\gamma_{CIR} e^{\frac{1}{2}b_r(T-t)}}{\gamma_{CIR} \cosh(\gamma_{CIR}(T-t)) + \frac{1}{2}b_r \sinh(\gamma_{CIR}(T-t))} \right].$$

Note that  $\sinh u = \frac{e^u - e^{-u}}{2}$ ,  $\cosh u = \frac{e^u + e^{-u}}{2}$ , and  $P(0, T, r(0)) = e^{-r(0)C_p(0, T) - A_p(0, T)}$ .<sup>11</sup>

To calculate the option price for stock under CIR interest rate process, we use the formula proposed by Kim (2002).<sup>12</sup> In Kim (2002), the process of CIR interest rate  $dr(t)$  under probability measure  $\mathbf{Q}'$  is as follows

$$dr(t) = [\kappa_{Kim} \theta_{Kim} - (\kappa_{Kim} + \delta_{Kim} \lambda_{Kim}) r(t)] dt + \delta_{Kim} \sqrt{r(t)} d\tilde{W}_r(t) \quad (13)$$

Let  $\lambda_{Kim} = 1$ , then  $\delta_{Kim} = \sigma_r$ ,  $\kappa_{Kim} = b_r - \sigma_r$ ,  $\theta_{Kim} = a_r / (b_r - \sigma_r)$ . Thus, we could calculate the price of an European call option of stock  $S$  with strike  $K$  and maturity  $T$  at time zero,  $Call_{Kim}(0, S(0), T, K)$ .

$$\begin{aligned} Call_{Kim}(0, S(0), T, K) = & \left[ S(0) \Phi(d_1) - K \exp\left(-\int_0^T r_t^* dt\right) \Phi(d_2) \right] \\ & + \delta_{Kim} C_0 \left[ S(0) \phi(d_1) - K \exp\left(-\int_0^T r_t^* dt\right) (\phi(d_2) - \sigma_S \sqrt{T} \Phi(d_2)) \right] \\ & + \delta_{Kim} C_1 \left[ d_2 S(0) \phi(d_1) - d_1 K \exp\left(-\int_0^T r_t^* dt\right) \phi(d_2) \right] + o(\delta_{Kim}), \end{aligned}$$

where  $\Phi(\cdot)$  and  $\phi(\cdot)$  are the cumulative density function and probability density function of the standard normal distribution respectively;

$$r_t^* = r_0 e^{-\kappa_{Kim} t} + \theta_{Kim} (1 - e^{-\kappa_{Kim} t}), \quad \exp\left(-\int_0^T r_t^* dt\right) = \exp\left[-\frac{(r_0 - \theta_{Kim})}{\kappa_{Kim}} (1 - e^{-\kappa_{Kim} T}) - \theta_{Kim} T\right];$$

$$d_1 = \frac{1}{\sigma_S \sqrt{T}} \left[ \ln \frac{S(0)}{K} + \frac{(r_0 - \theta_{Kim})}{\kappa_{Kim}} (1 - e^{-\kappa_{Kim} T}) + \theta_{Kim} T + \frac{\sigma_S^2}{2} T \right], \quad d_2 = d_1 - \sigma_S \sqrt{T};$$

$$C_0 = \frac{1}{\kappa_{Kim} \sigma_S \sqrt{T}} \left[ (r_0 - \theta_{Kim}) \left( \frac{1 - e^{-\kappa_{Kim} T}}{\kappa_{Kim}} - T e^{-\kappa_{Kim} T} \right) + \theta_{Kim} T \left( 1 - \frac{1 - e^{-\kappa_{Kim} T}}{\kappa_{Kim}} \right) \right];$$

<sup>11</sup> See, for example, Shreve (2004) Chapter 6.

<sup>12</sup> For references of option pricing under stochastic interest rates, see, for example, Shreve (2004) Chapter 9, and Brigo and Mercurio (2006) Chapter 3 and Appendix B.

$$C_1 = -\frac{\rho_{rS}}{\sigma_S T} C_{11}, \quad C_{11} = \frac{2\sqrt{\theta_{Kim}} \left[ (1+2e^{\kappa_{Kim} T}) \sqrt{r_0} - 3e^{\frac{\kappa_{Kim} T}{2}} \sqrt{r_0 - \theta_{Kim} (1 - e^{-\kappa_{Kim} T})} \right] + \psi_{Kim} \left[ \theta_{Kim} (1+2e^{\kappa_{Kim} T}) - r_0 \right]}{2e^{\kappa_{Kim} T} \kappa_{Kim}^2 \sqrt{\theta_{Kim}}},$$

$$\psi_{Kim} = \ln \left[ \frac{\theta_{Kim} (2e^{\kappa_{Kim} T} - 1) + r_0 + 2e^{\frac{\kappa_{Kim} T}{2}} \sqrt{\theta_{Kim}^2 (e^{\kappa_{Kim} T} - 1) + \theta_{Kim} r_0}}{(\sqrt{r_0} + \sqrt{\theta_{Kim}})^2} \right].$$

By Put-Call parity,  $Call(0, S(0), T, K) + Ke^{-\int_0^T r_u du} = Put(0, S(0), T, K) + S(0)$ . We could then calculate the price of the European put option at time zero  $Put(0, S(0), T, K)$  as  $Call(0, S(0), T, K) + KP(0, T, r(0)) - S(0)$ .

We approximate  $r(t)$ ,  $\theta(t)$ ,  $P(t, T, r(t))$  by the Euler method and the Milstein method as follows.

$$\bullet dr(t) = \left[ a_r - b_r r(t) + \theta(t) \sigma_r \sqrt{r(t)} \right] dt + \sigma_r \sqrt{r(t)} dW_r(t)$$

By Euler method,  $r_{i+1} = r_i + (a_r - b_r r_i + \theta_i \sigma_r \sqrt{r_i}) \Delta t_i + \sigma_r \sqrt{r_i} \Delta W_{r,i}$ ; and by Milstein method,  $r_{i+1} = r_i + (a_r - b_r r_i + \theta_i \sigma_r \sqrt{r_i}) \Delta t_i + \sigma_r \sqrt{r_i} \Delta W_{r,i} + \frac{1}{4} \sigma_r^2 \left[ (\Delta W_{r,i})^2 - \Delta t_i \right]$ .

$$\bullet d\theta(t) = \left[ a_\theta - b_\theta \theta(t) \right] dt + \sigma_\theta \sqrt{\theta(t)} dW_\theta(t)$$

By Euler method,  $\theta_{i+1} = \theta_i + (a_\theta - b_\theta \theta_i) \Delta t_i + \sigma_\theta \sqrt{\theta_i} \Delta W_{\theta,i}$ ; and by Milstein method,  $\theta_{i+1} = \theta_i + (a_\theta - b_\theta \theta_i) \Delta t_i + \sigma_\theta \sqrt{\theta_i} \Delta W_{\theta,i} + \frac{1}{4} \sigma_\theta^2 \left[ (\Delta W_{\theta,i})^2 - \Delta t_i \right]$ .

$$\bullet dP(t, T, r(t)) = P(t, T, r(t)) \left[ r(t) + \tilde{\sigma}(t, r(t)) \theta(t) \right] dt + P(t, T, r(t)) \tilde{\sigma}(t, r(t)) dW_r(t)$$

$$\tilde{\sigma}(t, r(t)) = \frac{P_r \beta(t, r(t))}{P(t, r(t))}, \quad \text{we could rewrite the process as}$$

$$dP(t, T, r(t)) = \left[ P(t, T, r(t)) r(t) + P_r \sigma_r \sqrt{r(t)} \theta(t) \right] dt + P_r \sigma_r \sqrt{r(t)} dW_r(t).$$

Note that  $P_r = -C_p(t, T) e^{-r(t)C_p(t, T) - A_p(t, T)}$  in CIR interest rate model. By Euler method,  $P_{i+1} = P_i + (P_r r_i + P_{r,t_i} \sigma_r \sqrt{r_i} \theta_i) \Delta t_i + P_{r,t_i} \sigma_r \sqrt{r_i} \Delta W_{r,i}$ ; and by Milstein method,  $P_{i+1} = P_i + (P_r r_i + P_{r,t_i} \sigma_r \sqrt{r_i} \theta_i) \Delta t_i + P_{r,t_i} \sigma_r \sqrt{r_i} \Delta W_{r,i}$ .

The details of implementation of simplified Second Milstein method is provided in the Appendix 3 of Supplementary materials.

#### 4.1. EXAMPLE WITH CIR MODEL AND GEOMETRIC BROWNIAN MOTION

The following is the settings describing the dynamics of each process in our example.<sup>13</sup>

$$T = 1, \Delta t = 0.01, \rho_{rS} = 0.6, \rho_{r\chi} = 0.7, \rho_{r\gamma} = 0.5, \rho_{S\chi} = 0.1, \rho_{S\gamma} = 0.3, \rho_{\chi\gamma} = 0.1$$

$$d\theta(t) = [0.05 - 0.01\theta(t)]dt + 0.01\sqrt{\theta(t)}dW_{\theta}(t), \theta(0) = 0.02$$

$$dr(t) = [0.02 - 0.04r(t)]dt + 0.01\sqrt{r(t)}d\tilde{W}_r(t), r(0) = 0.02$$

$$dS(t) = 0.1S(t)dt + 0.2S(t)dW_S(t), S(0) = 1$$

$$d\chi(t) = [0.05 - 0.1\chi(t)]dt + 0.1\sqrt{\chi(t)}dW_{\chi}(t), \chi(0) = 0.05$$

$$d\gamma(t) = 0.1dW_{\gamma}(t), \gamma(0) = 0.2$$

We could see in equation (12) that  $D(0)$  is equal to 1. The deflator approach tells us that for a nonnegative random variable  $X(t)$ , we would have  $E^Q[\delta(t)X(t)] = E[D(t)X(t)]$ . The price of a zero coupon bond of no risk with maturity  $T$  at time period  $T$  is equal to 1. We also calculate the price of Put option of  $S(T)$  with strike  $K$  equaling to 2. Thus, we would expect the following equations to hold.

$$D(0)S(0) = S(0) = E^Q[\delta(T)S(T)] = E[D(T)S(T)] = 1 \quad (14)$$

$$D(0)P(0,T,r(0)) = P(0,T,r(0)) = E^Q[\delta(T)P(T,T,r(T))] = E[D(T)P(T,T,r(T))] = E[D(T)] \quad (15)$$

$$D(0)Put(r(0),S(0)) = Put(r(0),S(0)) = E^Q[\delta(T)(K - S(T))^+] = E[D(T)(K - S(T))^+] \quad (16)$$

We compare the numerical results with Black-Scholes put option price,  $Put(r(0),S(0))$ . Tables 1, 2, and 3 show the numerical results. Figures 2, 3, 4, 5, 6, 7, 8, and 9 show the convergence of the approximations to expected value, i.e.  $S(0)$ ,  $P(0,T,r(0))$ ,  $Put(0,S(0),T,K)$ . We could see that the simplified Second Milstein method provides better approximations and converges faster than the Euler method and the Milstein method do. This could be explained by convergence order in which the simplified Second Milstein method has larger weak order of convergence.<sup>14</sup>

#### 4.2. EXAMPLE WITH CIR MODEL AND CORPORATE COUPON BOND

Longstaff et al. (2005) assumed the independence among interest rate, default intensity, and convenience yield. Thus, we let  $\rho_{r\chi} = 0$ ,  $\rho_{r\gamma} = 0$ ,  $\rho_{S\chi} = 0$ ,  $\rho_{S\gamma} = 0$ , and  $\rho_{\chi\gamma} = 0$ .<sup>15</sup> To accommodate the three risk factors (interest rate, default intensity, and convenience

<sup>13</sup> Here we provide a numerical example for the model, in which the chosen values for model settings could be different. In our example, there are strong positive correlations between interest rates and other factors (i.e. stock prices, default densities, and convenience yields), but weak positive correlations between each two of stock prices, default densities, and convenience yields.

<sup>14</sup> The Euler method and the Milstein method have weak order of convergence 1, and the simplified Second Milstein method has weak order of convergence 2, see, for example, Glasserman (2013) Chapter 6.

<sup>15</sup> We let  $dW_i dW_j = 0$  here, i.e. pairwise independence.

yield) with deflator, we let  $dB(t) = B(t)[r(t) + \chi(t) + \gamma(t)]dt$ .<sup>16</sup> The following is the formula provided in Longstaff et al. (2005).

$$\begin{aligned} CB(c, \omega, T) = & c \int_0^T A_{CB}(t) \exp(B_{CB}(t) \chi_0) C_{CB}(t) P(0, t, r(0)) e^{-\gamma_0 t} dt \\ & + A_{CB}(T) \exp(B_{CB}(T) \chi_0) C_{CB}(T) P(0, T, r(0)) e^{-\gamma_0 T} \\ & + (1 - \omega) \int_0^T \exp(B_{CB}(t) \chi_0) C_{CB}(t) P(0, t, r(0)) [G_{CB}(t) + H_{CB}(t) \chi_0] e^{-\gamma_0 t} dt \end{aligned}$$

$$A_{CB}(t) = \exp \left[ \frac{e_\chi (f_\chi + \phi)}{\sigma_\chi^2} t \right] \left( \frac{1 - \kappa}{1 - \kappa e^{\phi t}} \right)^{\frac{2e_\chi}{\sigma_\chi^2}}, \quad B_{CB}(t) = \frac{f_\chi - \phi}{\sigma_\chi^2} + \frac{2\phi}{\sigma_\chi^2 (1 - \kappa e^{\phi t})},$$

$$C_{CB}(t) = \exp \left( \frac{\eta^2 t^3}{6} \right), \quad G_{CB}(t) = \frac{e_\chi}{\phi} (e^{\phi t} - 1) \exp \left[ \frac{e_\chi (f_\chi + \phi)}{\sigma_\chi^2} t \right] \left( \frac{1 - \kappa}{1 - \kappa e^{\phi t}} \right)^{\frac{2e_\chi}{\sigma_\chi^2} + 1},$$

$$H_{CB}(t) = \exp \left[ \frac{e_\chi (f_\chi + \phi) + \phi \sigma_\chi^2}{\sigma_\chi^2} t \right] \left( \frac{1 - \kappa}{1 - \kappa e^{\phi t}} \right)^{\frac{2e_\chi}{\sigma_\chi^2} + 2}, \quad \kappa = (f_\chi + \phi) / (f_\chi - \phi),$$

$$\phi = \sqrt{2\sigma_\chi^2 + f_\chi^2}$$

In order to implement the deflator, we look at the original definition of  $CB(c, \omega, T)$ .

$$\begin{aligned} CB(c, \omega, T) = & E \left\{ c \int_0^T \exp \left[ - \int_0^t (r(s) + \chi(s) + \gamma(s)) ds \right] dt \right\} + E \left\{ \exp \left[ - \int_0^T (r(s) + \chi(s) + \gamma(s)) ds \right] \right\} \\ & + E \left\{ (1 - \omega) \int_0^T \chi_t \exp \left[ - \int_0^t (r(s) + \chi(s) + \gamma(s)) ds \right] dt \right\} \end{aligned} \quad (17)$$

For the time period  $t$  when a bond holder receives a coupon or a fraction of the par value of the bond (because of default), the payoff at that time period  $t$  is equal to  $c$  and  $(1 - \omega)$  respectively. Thus, we could implement the deflator as follows.

$$D(0)CB(c, \omega, T) = E[D(T)] + cE \left[ \int_0^T D(t) dt \right] + (1 - \omega)E \left[ \int_0^T \chi_t D(t) dt \right] \quad (18)$$

Table 4 and Figures 10 and 11 show the numerical results and convergence of the approximations.

#### 4.3. COMPARISONS OF NUMERICAL RESULTS WITH ANTITHETIC SAMPLING

Table 5 shows the differences of variances with/without antithetic sampling. Given the variance of a random variable  $X$ ,  $Var(X)$ , the variance of  $\frac{1}{n}X$  is equal to  $\frac{1}{n}Var(X)$ . In our model,  $D(t)$  is lognormal distributed. With the sample size being equal to  $n$ , the

<sup>16</sup> Recall that  $D(t) = Discount\ factor \cdot \frac{dQ}{dP}$ ,  $D(t)$  could not be the same given different discount factors with the same Radon-Nikodym derivative.



mean of  $D(t)$ ,  $E[D(t)]$ , equals  $\frac{1}{n} \sum_{n \text{ trials}} D(t)$ ; and its variance  $Var(E[D(t)])$  is equal to  $\frac{1}{n} Var(D(t))$ . We could calculate its 95% confidence interval as follows.<sup>17</sup>

$$CI_{95\%} = E[D(t)] + \frac{Var(E[D(t)])}{2} \pm t_{d.f.=n} \sqrt{\frac{Var(E[D(t)])}{n} + \frac{[Var(E[D(t)])]^2}{2(n-1)}} \quad (19)$$

Here  $t_{d.f.=n}$  is the  $t$  statistics with degree of freedom equalling  $n$ . For example, in our numerical results of Second Milstein method with antithetic sampling and sample size equalling 2500, the 95% confidence interval of  $E[D(T)]$  is equal to  $[0.95192, 0.95559]$ .

Suppose the weights of investment in a portfolio on stock, zero coupon bond of no risk with maturity  $T$ , and corporate coupon bond equal  $w_S$ ,  $w_P$ , and  $w_{CB}$  respectively. Theoretically, the variance of the portfolio is equal to  $\sum_{i=S,P,CB} w_i^2 Var(i) + 2 \sum_{j,k=S,P,CB; j \neq k} w_j w_k Cov(j,k)$ , where  $Cov(j,k)$  is the covariance between  $j$  and  $k$ . Given stochastic differential equations of two stochastic processes  $dX$  and  $dY$ , we could calculate  $Cov(X,Y)$  by  $dXdY$ . The multiplication of lognormal random variables is again lognormal distributed, and the sum of lognormal random variables most likely behaves as either normal or lognormal distributions (so that we could still calculate the confidence interval).<sup>18</sup> As a numerical example, let  $w_S$ ,  $w_P$ , and  $w_{CB}$  be 0.15, 0.65, and 0.2 respectively. Figure 12 shows the comparison of histograms with/without antithetic sampling of the portfolio.

## 5. CONCLUSION

In this paper, we derive the general form of deflator for four risk factors: interest rates, stock prices, default intensities, and convenience yields. We examine the deflator with different financial derivatives, comparing the numerical results with values calculated from closed-form formulas.

Our results indicate the reliability of the deflator for financial asset pricing, if the time discretization of the underlying stochastic processes is done carefully.

Except the benefit that we could compute best estimate value by simply averaging the multiplication of deflator and projected cash flows, the fact that we observe data only in physical world would provide the motivation for us to use deflator for the convenience to estimate parameters of “reaction functions” in an ALM projection model as in Chapter 4 of Laurent et al. (2016).

Further work would be to implement the deflator and compare the best estimate values of a life insurance contract under physical measure and risk-neutral measure.

<sup>17</sup> See, for example, Olsson (2005).

<sup>18</sup> See, for example, Dufresne (2004), Lo (2012), and Gulisashvili and Tankov (2016).

## 6. REFERENCES

- Arrow, Kenneth J., and Gerard Debreu. "Existence of an equilibrium for a competitive economy." *Econometrica*, Vol. 22, No. 3 (July 1954), 265-290.
- Bonnin, François, Frédéric Planchet, and Marc Juillard. "Best estimate calculations of savings contracts by closed formulas: application to the ORSA." *European Actuarial Journal*, Vol. 4, No. 1 (2014), 181-196.
- Brigo, Damiano, and Fabio Mercurio. *Interest Rate Models-Theory and Practice: With Smile, Inflation and Credit*. Springer-Verlag Berlin Heidelberg (2006).
- Caja, Anisa, and Frédéric Planchet. "La mesure du prix de marché du risque: quels outils pour une utilisation dans les modèles en assurance?" *Assurances et gestion des risques*, Vol. 78, No. 3-4 (January 2011), 251-281.
- Dastarac, Hugues, and Paul Sauveplane. "Les Déflateurs stochastiques: quelle utilisation en assurance?" *Mémoire d'actuaire*, ENSAE (2010).
- Delbaen, Freddy, and Walter Schachermayer. "A general version of the fundamental theorem of asset pricing." *Mathematische Annalen*, Vol. 300, No. 1 (September 1994), 463-520.
- Dufresne, Daniel. "The log-normal approximation in financial and other computations." *Advances in Applied Probability*, Vol. 36, No. 3 (September 2004), 747-773.
- Fabrice Borel-Mathurin, Pierre-Emmanuel Darpeix, Quentin Guibert, Stéphane Loisel. "Main Determinants of Profit Sharing Policy in the French Life Insurance Industry." *PSE Working Papers n°2015-16* (2015)
- Geanakoplos, John. "Arrow-Debreu Model of General Equilibrium." In: Eatwell J., Milgate M., Newman P. (eds.) *General Equilibrium*, The New Palgrave, London: Palgrave Macmillan (1989).
- Glasserman, Paul. *Monte Carlo methods in financial engineering*, New York City: Springer Science+Business Media (2013).
- Harrison, J. Michael, and David M. Kreps. "Martingales and arbitrage in multiperiod securities markets." *Journal of Economic theory*, Vol. 20, No. 3 (June 1979), 381-408.
- Gulisashvili, Archil, and Peter Tankov. "Tail behavior of sums and differences of log-normal random variables." *Bernoulli*, Vol. 22, No. 1 (2016), 444-493.
- Harrison, J. Michael, and Stanley R. Pliska. "Martingales and stochastic integrals in the theory of continuous trading." *Stochastic processes and their applications*, Vol. 11, No. 3 (August 1981), 215-260.
- Iacus, Stefano M. *Simulation and inference for stochastic differential equations: with R examples*, New York City: Springer Science+Business Media (2009).
- Kim, Yong-Jin. "Option pricing under stochastic interest rates: an empirical investigation." *Asia-Pacific Financial Markets*, Vol. 9, No. 1 (March 2002), 23-44.
- Kloeden, Peter E., and Eckhard Platen. *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag Berlin Heidelberg (1992).

Lo, Chi-Fai. "The Sum and Difference of Two Lognormal Random Variables." *Journal of Applied Mathematics*, Vol. 2012 (2012).

Longstaff, Francis A., Sanjay Mithal, and Eric Neis. "Corporate yield spreads: Default risk or liquidity? New evidence from the credit default swap market." *The Journal of Finance*, Vol. 60, No. 5 (September 2005), 2213-2253.

Laurent, Jean-Paul, Ragnar Norberg, and Frédéric Planchet (eds.). *Modelling in Life Insurance: A Management Perspective*, Basel: Springer International Publishing AG (2016).

Mas-Colell, Andreu, Michael Dennis Whinston, and Jerry R. Green. *Microeconomic theory*, New York: Oxford University Press (1995).

Olsson, Ulf. "Confidence Intervals for the Mean of a Log-Normal Distribution." *Journal of Statistics Education*, Vol. 13, No. 1 (2005).

Pedersen, Hal, Mary Pat Campbell, Stephan L. Christiansen, Samuel H. Cox, Daniel Finn, Ken Griffin, Nigel Hooker, Matthew Lightwood, Stephen M. Sonlin, and Chris Suchar. "Economic Scenario Generators: A Practical Guide." The Society of Actuaries (July 2016).

Shreve, Steven E. *Stochastic calculus for finance II: Continuous-time models*, New York City: Springer Science+Business Media (2004).

Varnell, E. M. "Economic scenario generators and Solvency II." *British Actuarial Journal*, Vol. 16, No. 1 (May 2011), 121-159.

Vedani, Julien, Nicole El Karoui, Stéphane Loisel, and Jean-Luc Prigent. "Market inconsistencies of market-consistent European life insurance economic valuations: pitfalls and practical solutions." *European Actuarial Journal*, Vol. 7, No. 1 (2017), 1-28.

7. TABLES AND FIGURES

Tab. 1. Stock

# of Simulations	$E[D(T)S(T)]$	$S(0)$
<b>Euler method</b>		
2500	0.600069755272291	
5000	0.393586742835856	
10000	0.37903361971978	
100000	0.657324756344535	1
250000	0.843102951645603	
500000	0.770462054324114	
1000000	1.29863524218499	
<b>Milstein method</b>		
2500	0.370434390825238	
5000	0.398022925866662	
10000	0.395876767138054	
100000	0.679818876796659	1
250000	0.771340365068307	
500000	0.824386955022253	
1000000	1.12292596930633	
<b>Second Milstein method</b>		
2500	0.954846533976465	
5000	1.00743887812346	
10000	1.02341198496773	
100000	1.00680833122829	1
250000	1.00231591266646	
500000	1.00268873753513	
1000000	0.999997325958791	

Tab. 2. Zero coupon bond of no risk with maturity  $T$

# of Simulations	$E[D(T)]$	$E[D(T)P(T,T,r(T))]$	$P(0,T,r(0))$
<b>Euler method</b>			
2500	0.48812434395029	0.488066560145415	
5000	0.308038513966897	0.308003081851407	
10000	0.325024763106268	0.324990036772998	
100000	0.625985956594468	0.625922227828396	0.970957220487724
250000	0.771419872927604	0.771346645830578	
500000	0.726182177029966	0.72611342647583	
1000000	1.18961111433854	1.18949806538064	
<b>Milstein method</b>			
2500	0.332402071677238	0.332366375930865	
5000	0.366969978004539	0.366931013968048	
10000	0.360964352720743	0.360926322954742	
100000	0.654420842856107	0.654354469594408	0.970957220487724
250000	0.746857198318386	0.746783090935181	
500000	0.824192927498608	0.82411348941373	
1000000	1.15968204198937	1.15956727600934	
<b>Second Milstein method</b>			
2500	0.952656631120277	0.952657151589584	
5000	0.986807314190818	0.986807555270611	
10000	1.00104803675261	1.00104821737114	
100000	0.97943249260489	0.979432483928667	0.970957220487724
250000	0.974542088257683	0.974542020575479	
500000	0.975959099152459	0.975959075808478	
1000000	0.973919234689554	0.97391925959157	

Tab. 3. Put option

# of Simulations	$E[D(T)(K - S(T))^+]$	Kim (2002)
<b>Euler method</b>		
2500	0.376179173414971	
5000	0.222535471469381	
10000	0.271038627789198	
100000	0.594654067016785	0.950345063621439
250000	0.699758896691695	
500000	0.681914718350244	
1000000	1.07921012174278	
<b>Milstein method</b>		
2500	0.294370393477994	
5000	0.335949588638566	
10000	0.32606847481562	
100000	0.629027012422739	0.950345063621439
250000	0.72239181263158	
500000	0.824011757155215	
1000000	1.19641443858681	
<b>Second Milstein method</b>		
2500	0.950440933233568	
5000	0.966323781894296	
10000	0.978829277799832	
100000	0.952030741956879	0.950345063621439
250000	0.94674239771515	
500000	0.949181442176693	
1000000	0.947797286021223	

ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

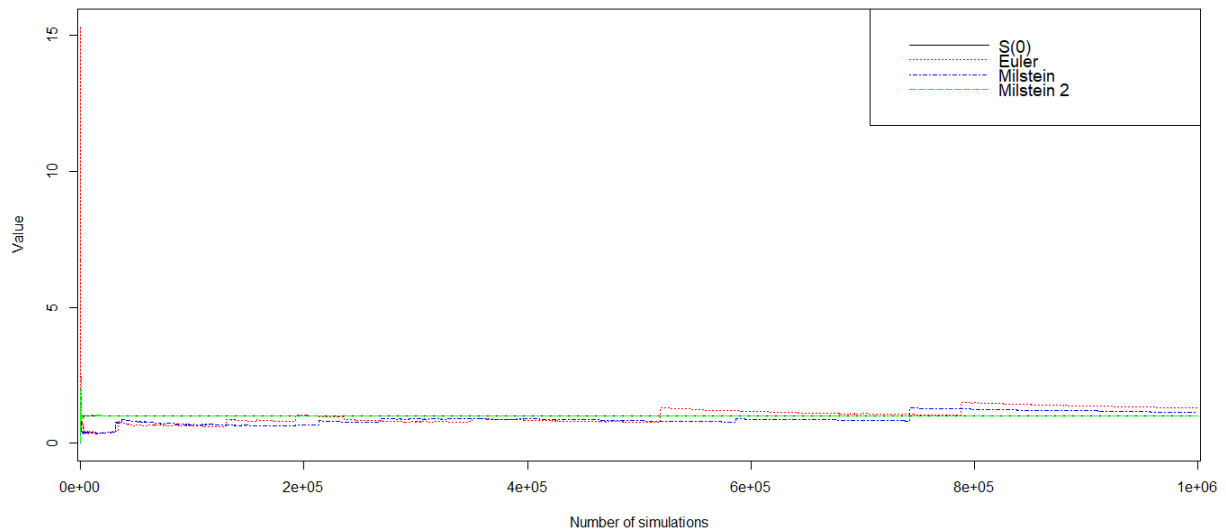
Tab. 4. Corporate coupon bond

# of Simulations	Deflator	Longstaff et al. (2005)
<b>Euler method</b>		
2500	0.861924054813666	
5000	0.972373894195048	
10000	0.904175407930738	
100000	0.877454293920751	0.858680337551511
250000	0.878602279971190	
500000	0.857615754345209	
1000000	0.907459377255768	
<b>Milstein method</b>		
2500	0.811300811125437	
5000	0.963754448602619	
10000	0.938308214385827	
100000	0.896750823102638	0.858680337551511
250000	0.879656712709339	
500000	0.874192077843085	
1000000	0.891953664767561	
<b>Second Milstein method</b>		
2500	0.838321934993519	
5000	0.867139374025251	
10000	0.881789095441836	
100000	0.865607112017938	0.858680337551511
250000	0.864357451331807	
500000	0.865196031950400	
1000000	0.865178960263938	

Tab. 5. Comparisons of numerical results with antithetic sampling

Sample size $n = 1000000$					
	$D(T)$	$D(T)S(T)$	$D(T)P(T, T, r(T))$	$D(T)(K - S(T))^+$	CB
<b>Euler method</b>					
$Var_{original}$	273480.434	340681.898	273426.081	219495.036	132.512
$Var_{antithetic}$	175562.779	221021.187	175531.642	135907.953	4040.719
<b>Milstein method</b>					
$Var_{original}$	22417.520	27288.816	22412.939	18276.458	3285.788
$Var_{antithetic}$	172987.167	132384.322	172952.076	219810.389	983.915
<b>Second Milstein method</b>					
$Var_{original}$	10.914	13.437	10.914	9.691	12.719
$Var_{antithetic}$	5.473	6.735	5.473	4.865	1.181
<b>Portfolio with Second Milstein method</b>					
$Var_{original}$	7.315				
$Var_{antithetic}$	3.669				

Fig. 2 - Stock



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Fig. 3 - Stock, number of simulations less than 5000

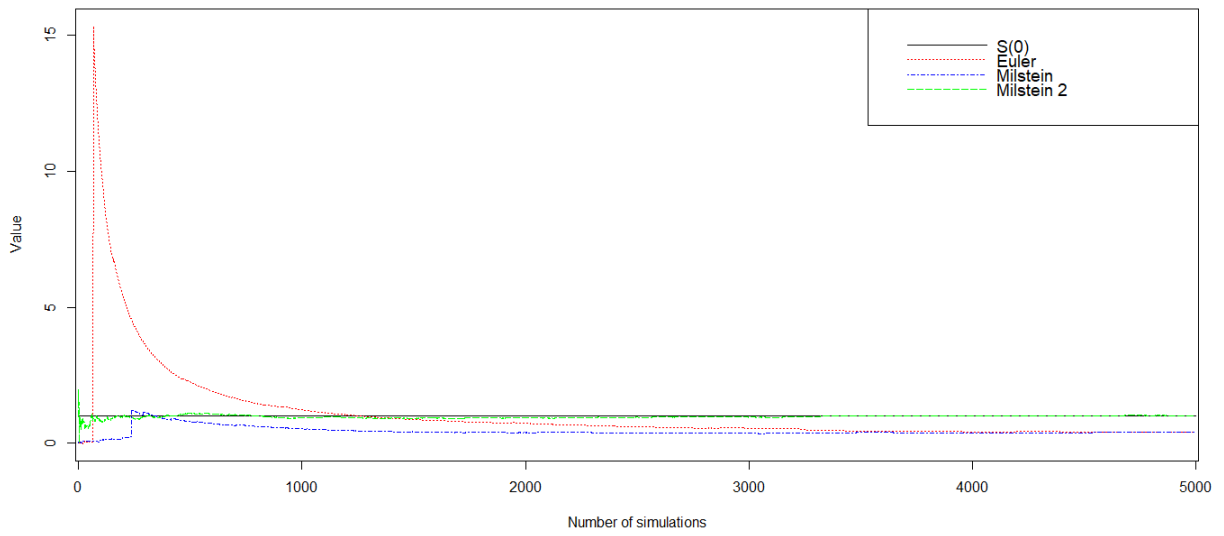


Fig. 4 - Zero coupon bond,  $E[D(T)]$

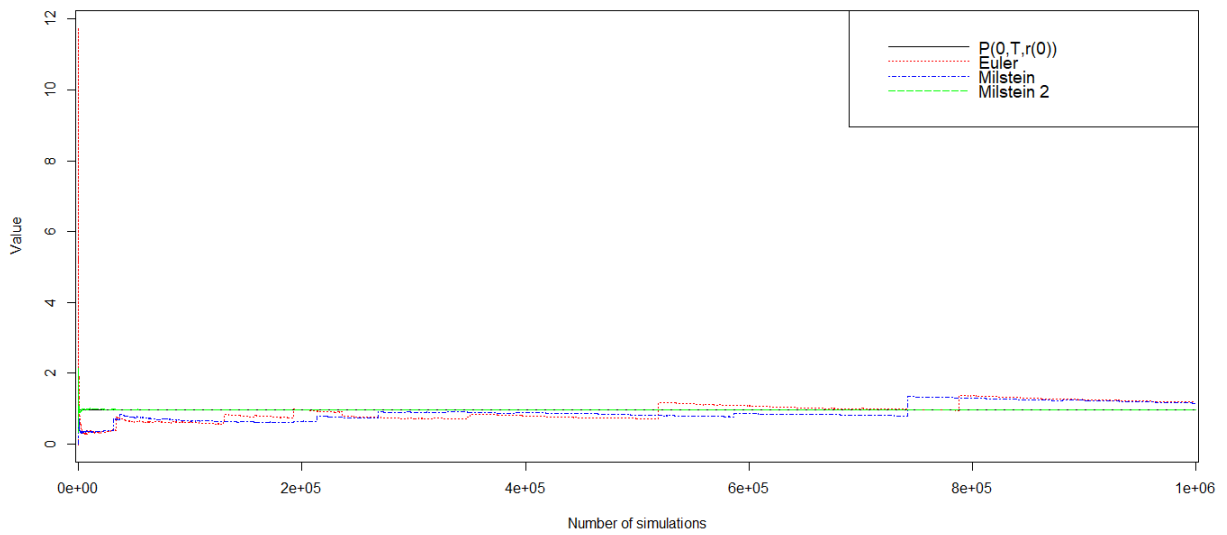




Fig. 5 - Zero coupon bond,  $E[D(T)]$ , number of simulations less than 5000

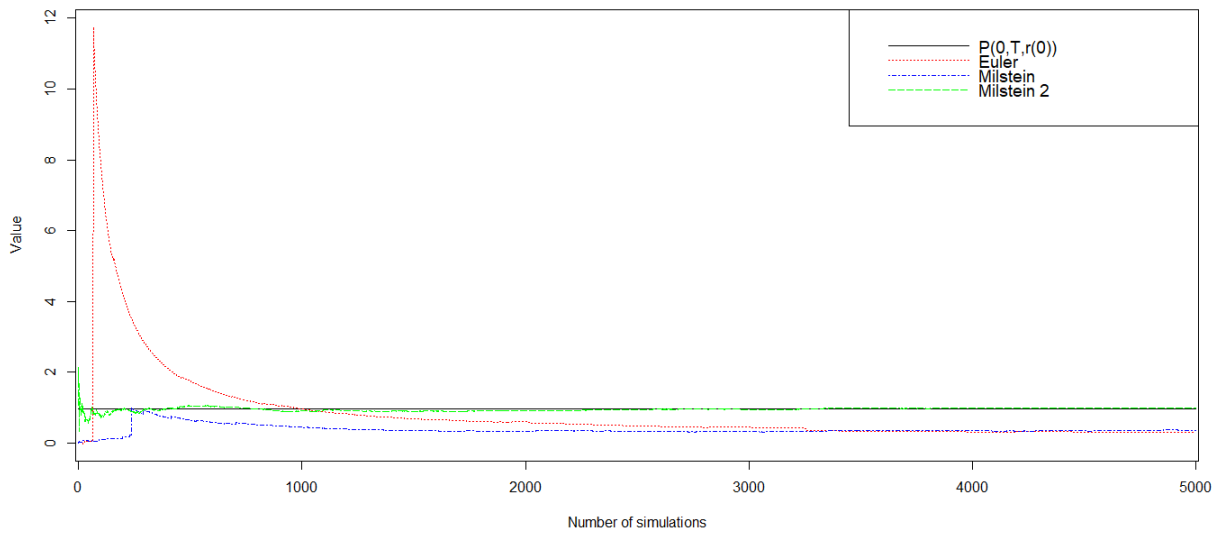
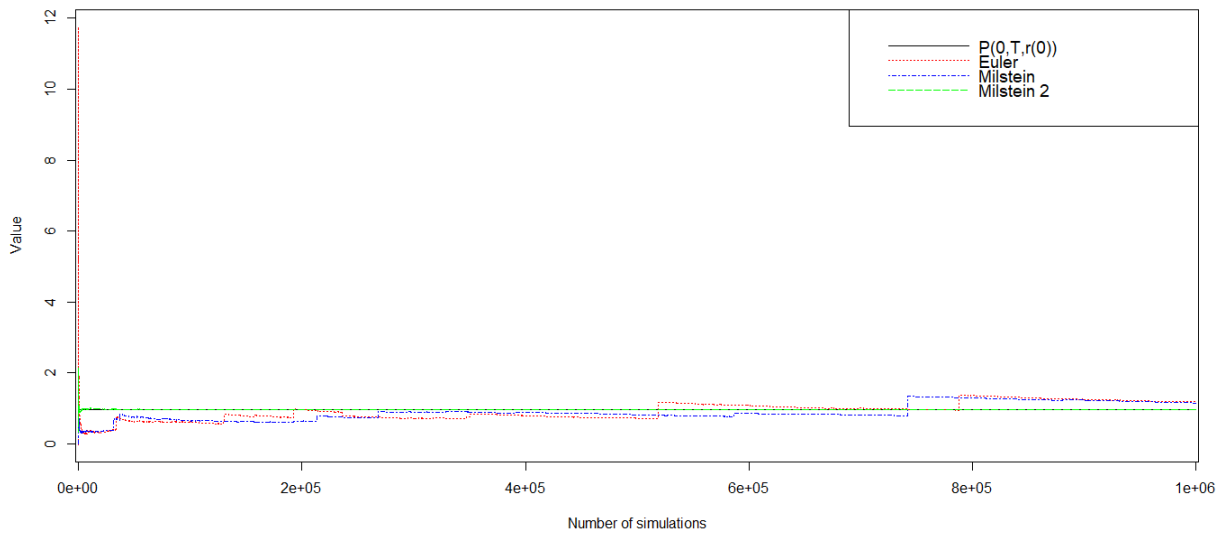


Fig. 6 - Zero coupon bond,  $E[D(T)P(T, T, r(T))]$



ECONOMIC SCENARIO GENERATOR WITH FIVE FACTORS

Fig. 7 - Zero coupon bond,  $E[D(T)P(T,T,r(T))]$ , number of simulations less than 5000

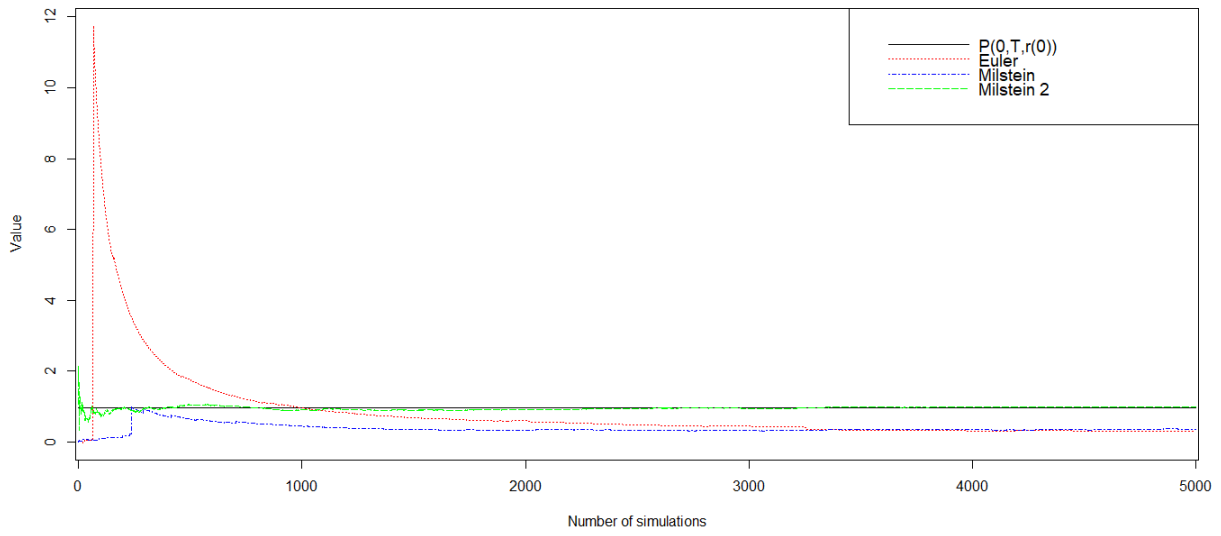


Fig. 8 - Put option,  $E[D(T)(K - S(T))^+]$

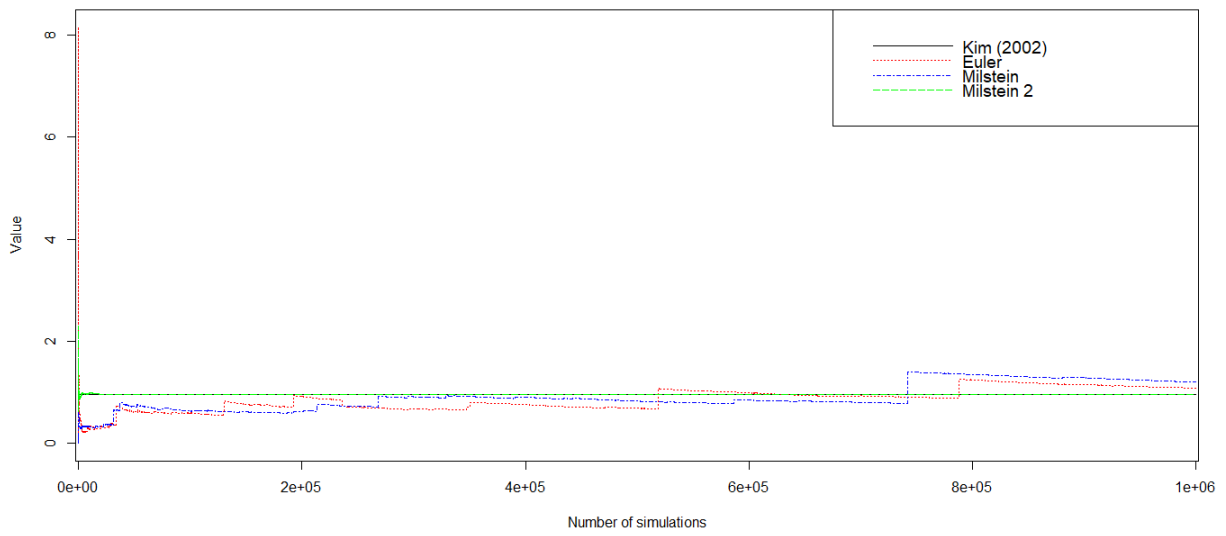


Fig. 9 - Put option,  $E\left[D(T)(K - S(T))^+\right]$ , number of simulations less than 5000

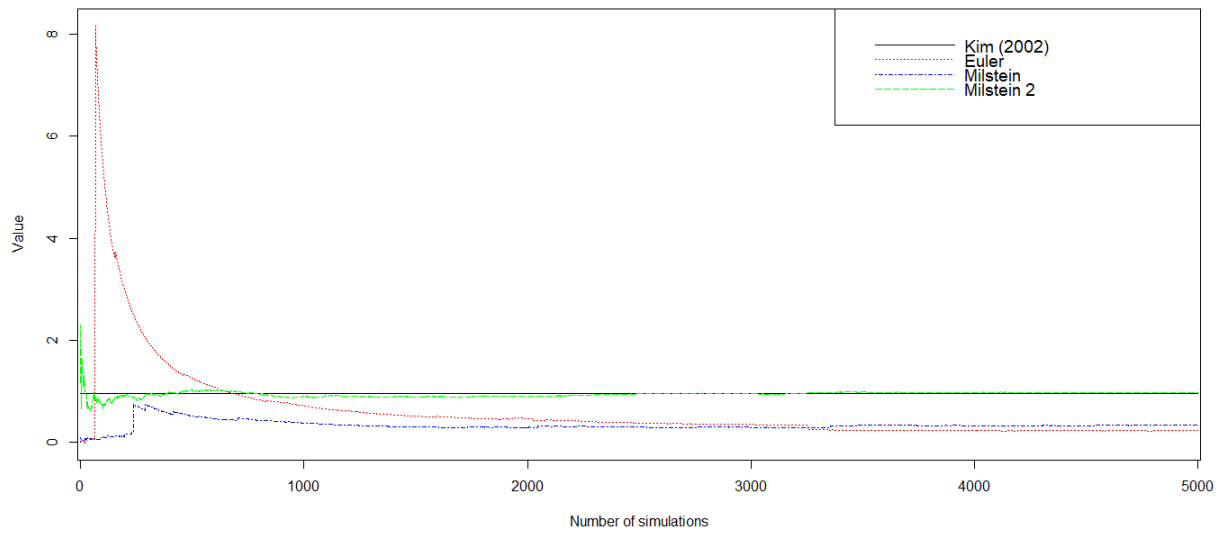


Fig. 10 - Corporate coupon bond

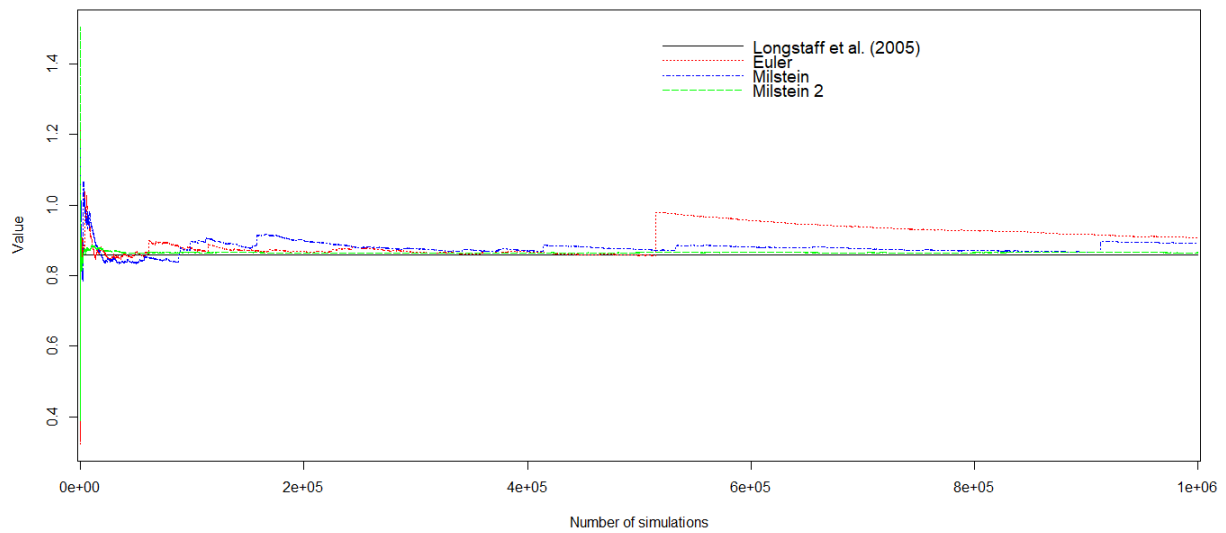
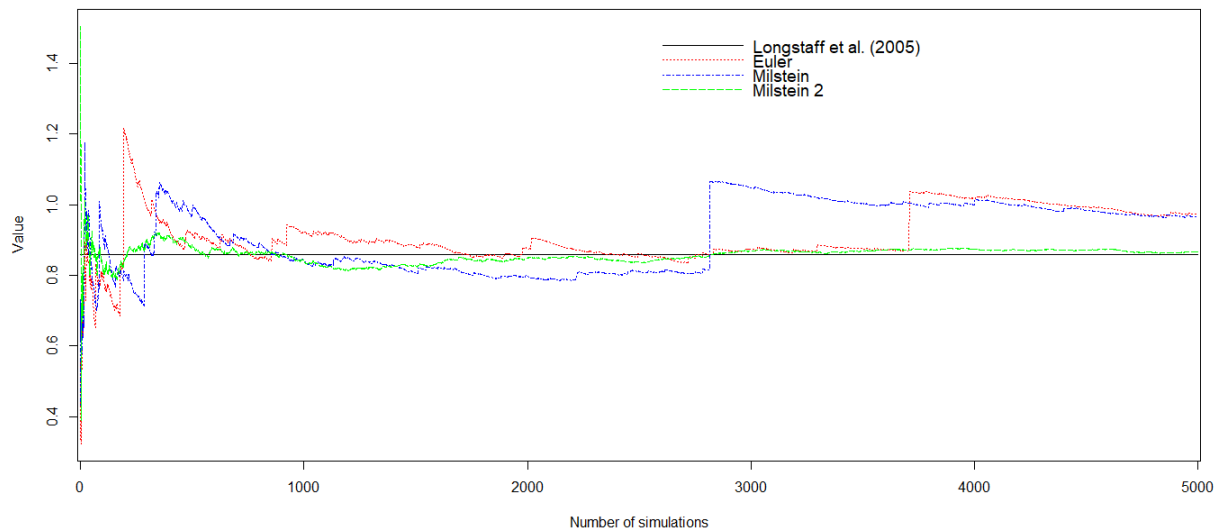


Fig. 11 - Corporate coupon bond, number of simulations less than 5000



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Fig. 12 - Histogram comparison with antithetic sampling

