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Time Average in Micromagnetism

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Abstract - In this paper we study a model of ferromagnetic material governed by a nonlinear Landau-Lifschitz equation coupled with Maxwell equations. We prove the existence of weak solutions. Then we prove that all points of the ω -limit set of any trajectories are solutions of the stationary model. Furthermore we derive rigorously the quasistatic model by an appropriate time average method.

1 Introduction.

In this paper we study the following system

$$\frac{\partial u}{\partial t} + u \wedge \frac{\partial u}{\partial t} = 2u \wedge H_e \quad \text{in } \mathbb{R}^+ \times \Omega, \quad (1.1)$$

where $H_e = \Delta u + H - \varphi(u)$,

$$\mu_0 \frac{\partial}{\partial t} (H + \bar{u}) + \text{curl } E = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \quad (1.2)$$

$$\varepsilon_0 \frac{\partial E}{\partial t} - \text{curl } H + \sigma \mathbf{1}_\Omega (E + f) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \quad (1.3)$$

with the associated boundary conditions and initial data

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ E(0, x) = E_0(x) & \text{in } \mathbb{R}^3, \\ H(0, x) = H_0(x) & \text{in } \mathbb{R}^3. \end{array} \right. \quad (1.4)$$

We assume that

$$|u_0(x)| = 1 \quad \text{in } \Omega, \quad (1.5)$$

$$\text{div} (H_0 + \bar{u}_0) = 0 \quad \text{in } \mathbb{R}^3.$$

In the above equations Ω is a smooth bounded open domain of \mathbb{R}^3 , ν the unit normal on $\partial\Omega$, $\mathbf{1}_\Omega$ is the characteristic function of Ω , \bar{u} is the extension of u by zero outside Ω .

This system of equations which couples the Landau-Lifschitz equation with Maxwell's equations describes electromagnetic waves propagation in a ferromagnetic medium confined to the domain Ω .

In the ferromagnetic model the magnetic moment denoted by u links the magnetic field H with the magnetic induction B through the relationship $B = \mu_0(H + \bar{u})$. Moreover u is a vector field which takes its values on S^2 the unit sphere of \mathbb{R}^3 . The conductivity of the body Ω is

denoted by $\sigma \in \mathbb{R}^{+*}$, the anisotropic term is patterned by $\varphi(u)$ where $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the gradient of Φ a positively defined quadratic form of \mathbb{R}^3 , f is a source term supported by $\mathbb{R}^+ \times \Omega$. Finally ε_0 is the dielectric permittivity and μ_0 is the magnetic permeability.

This model is described in detail in [3], [11] and [15].

Remark 1.1 *When the solution of (1.1) is regular enough, this equation is equivalent to*

$$\frac{\partial u}{\partial t} = u \wedge H_e - u \wedge (u \wedge H_e) \text{ in } \mathbb{R}^+ \times \Omega. \quad (1.6)$$

In [14] A. Visintin establishes the existence of weak solutions of the system (1.6),(1.2)-(1.5). When H_e reduces to Δu , F. Alouges and A. Soyeur show in [2] the existence and the non uniqueness of the solutions of (1.1). F. Labbé establishes in [10] the non uniqueness of the solution for the quasistatic model. Numerical studies are carried on by P. Joly and O. Vacus in [9], and by F. Alouges in the steady state case in [1]. At least in the case when H_e reduces to H and $\Omega = \mathbb{R}^3$, J.L. Joly, G. Métivier and J. Rauch obtain existence and uniqueness results for the solutions of (1.6), (1.2), (1.3), (1.4).

Notations : in the sequel we denote $\mathbb{H}^1 = (H^1)^3$ and $\mathbb{L}^2 = (L^2)^3$.

2 Statement of the results.

Let us assume that

$$\left. \begin{aligned} u_0 \in \mathbb{H}^1(\Omega), H_0 \in \mathbb{L}^2(\mathbb{R}^3), E_0 \in \mathbb{L}^2(\mathbb{R}^3), f \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega), \\ |u_0| = 1 \text{ a.e.}, \operatorname{div}(H_0 + \bar{u}_0) = 0. \end{aligned} \right\} (\mathcal{H})$$

Definition 2.1 *We say that (u, E, H) is a weak solution of (1.1)-(1.5) if*

1. (u, E, H) verifies

$$\begin{aligned} u \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)), \frac{\partial u}{\partial t} \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega), |u(t, x)| = 1 \text{ a. e.}, \\ E \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)), H \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)). \end{aligned} \quad (2.1)$$

2. For all $\Psi \in C^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$,

$$\begin{aligned} \int_{\mathbb{R}^+ \times \Omega} \left(\frac{\partial u}{\partial t}(t, x) + u(t, x) \wedge \frac{\partial u}{\partial t}(t, x) \right) \cdot \Psi(t, x) dx dt = \\ -2 \int_{\mathbb{R}^+ \times \Omega} \sum_{i=1}^3 \left(u(t, x) \wedge \frac{\partial u}{\partial x_i}(t, x) \right) \cdot \frac{\partial \Psi}{\partial x_i}(t, x) dx dt \\ +2 \int_{\mathbb{R}^+ \times \Omega} u(t, x) \wedge \left(H(t, x) - \varphi(u(t, x)) \right) \cdot \Psi(t, x) dx dt. \end{aligned} \quad (2.2)$$

3. $u(0, x) = u_0(x)$ in the trace sense.

4. For all $\Psi \in \mathbb{H}^1(\mathbb{R}^+ \times \mathbb{R}^3)$,

$$\begin{aligned} & - \int_{\mathbb{R}^+ \times \mathbb{R}^3} \left(H(t, x) + \bar{u}(t, x) \right) \cdot \frac{\partial \Psi}{\partial t}(t, x) dt dx + \int_{\mathbb{R}^+ \times \mathbb{R}^3} E(t, x) \cdot \text{curl } \Psi(t, x) dx dt = \\ & \int_{\mathbb{R}^3} H_0(x) \cdot \Psi(0, x) dx + \int_{\Omega} u_0(x) \cdot \Psi(0, x) dx. \end{aligned} \quad (2.3)$$

5. For all $\Psi \in \mathbb{H}^1(\mathbb{R}^+ \times \mathbb{R}^3)$,

$$\begin{aligned} & - \int_{\mathbb{R}^+ \times \mathbb{R}^3} E(t, x) \cdot \frac{\partial \Psi}{\partial t}(t, x) dx dt - \int_{\mathbb{R}^+ \times \mathbb{R}^3} H(t, x) \cdot \text{curl } \Psi(t, x) dx dt \\ & + \sigma \int_{\mathbb{R}^+ \times \Omega} \left(E(t, x) + f(t, x) \right) \cdot \Psi(t, x) dx dt = \int_{\mathbb{R}^3} E_0(x) \cdot \Psi(0, x) dx. \end{aligned} \quad (2.4)$$

6. For all $t > 0$, we have the following energy estimate :

$$\begin{aligned} \mathcal{E}(t) + \int_0^t \int_{\Omega} \left| \frac{\partial u}{\partial t}(t, x) \right|^2 dx dt + \frac{\sigma}{\mu_0} \int_0^t \int_{\Omega} |E(t, x)|^2 dx dt & \leq \mathcal{E}(0) \\ & + \frac{\sigma}{\mu_0} \int_0^t \int_{\Omega} |f(t, x)|^2 dx dt \end{aligned} \quad (2.5)$$

where

$$\mathcal{E}(t) = \int_{\Omega} \left(|\nabla u(t, x)|^2 + 2\Phi(u(t, x)) \right) dx + \int_{\mathbb{R}^3} \left(|H(t, x)|^2 + \frac{\varepsilon_0}{\mu_0} |E(t, x)|^2 \right) dx.$$

Theorem 2.1 Under the assumption (\mathcal{H}) , there exists at least one weak solution of (1.1)-(1.5).

This theorem is established in section 3 using a Galerkin approximation for a relaxed problem.

Definition 2.2 Let u be a weak solution of (1.1)-(1.5). We call ω -limit set of the trajectory u the following set

$$\omega(u) = \left\{ v \in \mathbb{H}^1(\Omega), \exists t_n, \lim t_n = +\infty, u(t_n, \cdot) \rightharpoonup v \text{ in } \mathbb{H}^1(\Omega) \text{ weakly} \right\}$$

From the energy estimate (2.5), for any u , $\omega(u)$ is non empty.

Theorem 2.2 Under the assumption (\mathcal{H}) , if u is a weak solution of (1.1)-(1.5), each point v in $\omega(u)$ is a weak solution of the steady state system

$$v \in H^1(\Omega), |v| = 1 \text{ a.e.}, \quad (2.6)$$

$$\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(v \wedge \frac{\partial v}{\partial x_i} \right) + v \wedge (H - \varphi(v)) = 0 \text{ in } \Omega, \quad (2.7)$$

$$\begin{cases} H \in \mathbb{L}^2(\mathbb{R}^3), \\ \text{curl } H = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \text{div } (H + \bar{v}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \end{cases} \quad (2.8)$$

Remark 2.1 As v lies in $\mathbb{H}^1(\Omega)$, Δv lies in $\mathbb{H}^{-1}(\Omega)$ so the product $v \wedge \Delta v$ makes sense in $W^{-1,t}(\Omega)$ with $\frac{1}{t} = \frac{1}{2} + \frac{1}{6}$ (see J. Simon [13]). Moreover from the equation (2.7) this product belongs to $\mathbb{L}^2(\Omega)$.

Theorem 2.2 is proved in section 4. The limit process for v is carried out thanks to the estimate

$$\int_{\mathbb{R}^+} \int_{\Omega} \left| \frac{\partial u}{\partial t}(t, x) \right|^2 dx dt < +\infty.$$

On the other hand an averaging technique is used to justify the limit for H .

The last part of this article is devoted to the validation when ε_0 and μ_0 go to zero of the quasi-stationary model. We suppose for this result that the source term f is zero.

Let us assume that

$$\left. \begin{aligned} u_0 \in \mathbb{H}^1(\Omega), H_0 \in \mathbb{L}^2(\mathbb{R}^3), E_0 \in \mathbb{L}^2(\mathbb{R}^3), \\ |u_0| = 1 \text{ a.e.}, \operatorname{div}(H_0 + \bar{u}_0) = 0. \end{aligned} \right\} (\mathcal{H}_q)$$

Definition 2.3 We say that u is a weak solution of the quasi-stationary model if

1. u satisfies

$$u \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)), \frac{\partial u}{\partial t} \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega), |u| = 1 \text{ a.e.} \quad (2.9)$$

2. For all $\Psi \in C^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$,

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \Omega} \left(\frac{\partial u}{\partial t}(t, x) + u(t, x) \right) \wedge \frac{\partial u}{\partial t}(t, x) \cdot \Psi(t, x) dx dt = \\ & -2 \int_{\mathbb{R}^+ \times \Omega} \sum_{i=1}^3 u(t, x) \wedge \frac{\partial u}{\partial x_i}(t, x) \cdot \frac{\partial \Psi}{\partial x_i}(t, x) dx dt \\ & + 2 \int_{\mathbb{R}^+ \times \Omega} u(t, x) \wedge (H(t, x) - \varphi(u(t, x))) \cdot \Psi(t, x) dx dt, \end{aligned} \quad (2.10)$$

3. $u(0, x) = u_0(x)$ in the trace sense.

4. For all $t \in \mathbb{R}^+$, $H(t, x)$ is the unique solution of

$$\left\{ \begin{aligned} \operatorname{curl} H(t, \cdot) &= 0, \\ \operatorname{div}(H(t, \cdot) + \bar{u}(t, \cdot)) &= 0, \\ H(t, \cdot) &\in \mathbb{L}^2(\mathbb{R}^3). \end{aligned} \right. \quad (2.11)$$

5. For all t we have the following energy estimate

$$\mathcal{E}_q(t) + \int_0^t \int_{\Omega} \left| \frac{\partial u}{\partial t}(t, x) \right|^2 dx dt \leq \mathcal{E}_q(0), \quad (2.12)$$

where

$$\mathcal{E}_q(t) = \int_{\Omega} \left(|\nabla u(t, x)|^2 + 2\Phi(u(t, x)) \right) dx + \int_{\mathbb{R}^3} |H(t, x)|^2 dx.$$

Theorem 2.3 *We consider two sequences $(\varepsilon^n)_n$ and $(\mu^n)_n$ which tend to zero as $n \rightarrow +\infty$ and such that μ^n/ε^n remains bounded.*

Under the assumption (\mathcal{H}_q) if u^n denote a weak solution of (1.1)-(1.5) with $\varepsilon_0 = \varepsilon^n$ and $\mu_0 = \mu^n$, there exists a subsequence still denoted $(u^n)_n$ such that u^n tends to a limit u in $L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ weak \star where u is a solution of the quasi-stationary model (2.9)-(2.12).

This result is obtained via a time average process on H which avoid the high frequency oscillations of H .

Proposition 2.1 *Every point of the ω -limit set of any trajectory of (2.9)-(2.11) is solution of the steady state model (2.7).*

This last result is straightforward from the estimate

$$\int_{\mathbb{R}^+} \int_{\Omega} \left| \frac{\partial u}{\partial t}(t, x) \right|^2 dx dt < +\infty$$

and from the continuity of the map $u \mapsto H$ given by (2.11).

3 Proof of the existence.

The main point is to establish that $|u| = 1$ almost everywhere. In order to construct a solution which satisfies this condition we first solve a relaxed problem \mathcal{P}_λ where u^λ takes its values in \mathbb{R}^3 . The penalization term takes the form $\frac{1}{\lambda}(|u| - 1)u$, λ tends to 0.

In fact instead of (1.1) we solve the following equation

$$\frac{\partial u^\lambda}{\partial t} - u^\lambda \wedge \frac{\partial u^\lambda}{\partial t} - 2\Delta u^\lambda - 2\varphi(u^\lambda) + \frac{1}{\lambda}(|u^\lambda|^2 - 1)u^\lambda = 2H. \quad (3.1)$$

By a Galerkin process we construct a solution of (3.1) satisfying an energy estimate, that allows us to pass to the limit as λ goes to zero. This limit u takes its values on S^2 and by a suitable test function we show that u satisfies (1.1).

First step. Resolution of (3.1).

Let us recall that the eigenfunctions of the operator $A = -\Delta + I$ with domain

$$D(A) = \{u \in \mathbb{H}^2(\Omega), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

build an orthonormal basis $\{\varphi_k\}_k$ in $\mathbb{L}^2(\Omega)$ and an orthogonal basis in $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^2(\Omega)$.

We denote V_N the N dimensional vector space spanned by $\{\varphi_k\}_{1 \leq k \leq N}$.

Now we introduce the Hilbert space

$$\mathbb{H}_{\text{curl}}(\mathbb{R}^3) = \{\psi \in \mathbb{L}^2(\mathbb{R}^3), \text{curl } \psi \in \mathbb{L}^2(\mathbb{R}^3)\}$$

We denote $\{\psi_k\}_k$ an hilbertian basis of $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$ orthonormal in $\mathbb{L}^2(\mathbb{R}^3)$ and W_N the N dimensional vector space spanned by $\{\psi_k\}_{1 \leq k \leq N}$.

In the approximate problem we seek (u_N, H_N, E_N) in $V_N \times W_N \times W_N$ such that

$$u_N(t, x) = \sum_{k=1}^N v_k(t) \varphi_k(x),$$

$$H_N(t, x) = \sum_{k=1}^N h_k(t) \psi_k(x),$$

$$E_N(t, x) = \sum_{k=1}^N e_k(t) \psi_k(x),$$

which satisfies

1. For any Φ_N in V_N ,

$$\begin{aligned} & \int_{\Omega} \left(\frac{\partial u_N}{\partial t}(t, x) - u_N(t, x) \wedge \frac{\partial u_N}{\partial t}(t, x) \right) \cdot \Phi_N(x) dx + 2 \int_{\Omega} \nabla u_N(t, x) \cdot \nabla \Phi_N(x) dx \\ & + \frac{4}{\lambda} \int_{\Omega} (|u_N(t, x)|^2 - 1) u_N(t, x) \cdot \Phi_N(x) dx \\ & - 2 \int_{\Omega} \left(H_N(t, x) - \varphi(u_N(t, x)) \right) \cdot \Phi_N(x) dx = 0. \end{aligned} \quad (3.2)$$

2. For any Ψ_N in W_N ,

$$\mu_0 \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (H_N(t, x) + \bar{u}_N(t, x)) \cdot \Psi_N(x) dx + \int_{\mathbb{R}^3} E_N(t, x) \cdot \text{curl } \Psi_N(x) dx = 0. \quad (3.3)$$

3. For any Θ_N in W_N

$$\begin{aligned} & \varepsilon_0 \int_{\mathbb{R}^3} \frac{\partial E_N}{\partial t}(t, x) \cdot \Theta_N(x) dx - \int_{\mathbb{R}^3} H_N(t, x) \cdot \text{curl } \Theta_N(x) dx \\ & + \sigma \int_{\Omega} (E_N(t, x) + f(t, x)) \cdot \Theta_N(x) dx = 0. \end{aligned} \quad (3.4)$$

4. With the initial data

$$\begin{cases} u_N(0) = \Pi_{V_N}(u_0), \\ E_N(0) = \Pi_{W_N}(E_0), \\ H_N(0) = \Pi_{W_N}(H_0), \end{cases} \quad (3.5)$$

where Π_{V_N} (resp. Π_{W_N}) denotes the orthogonal projection on V_N (resp. W_N).

Let us remark that $v \mapsto v - u \wedge v$ is one to one in \mathbb{R}^3 so the equation (3.2) can be solve for the derivative in time. Then by Cauchy Picard theorem there exists a local solution of (3.2)-(3.5).

The following *a priori* estimates show that, in fact, the approximate solution is global in time.

Taking $\Phi_N = \frac{\partial u_N}{\partial t}$ in (3.2) one has

$$\begin{aligned} \int_{\Omega} \left| \frac{\partial u_N}{\partial t}(t, x) \right|^2 dx + \frac{d}{dt} \int_{\Omega} |\nabla u_N(t, x)|^2 dx + \frac{1}{\lambda} \frac{d}{dt} \int_{\Omega} (|u_N(t, x)|^2 - 1)^2 dx \\ + 2 \frac{d}{dt} \int_{\Omega} \Phi(u_N(t, x)) = \int_{\Omega} \frac{\partial u_N}{\partial t}(t, x) \cdot H_N(t, x) \end{aligned} \quad (3.6)$$

Now we put $\Psi_n = H_N$ in (3.3)

$$\frac{\mu_0}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |H_N(t, x)|^2 dx + \int_{\mathbb{R}^3} \operatorname{curl} E_N(t, x) \cdot H_N(t, x) dx = -\mu_0 \int_{\Omega} \frac{\partial u_N}{\partial t}(t, x) \cdot H_N(t, x) dx \quad (3.7)$$

In the same way taking $\Theta_N = E_N$ in (3.4),

$$\begin{aligned} \frac{1}{2} \varepsilon_0 \frac{d}{dt} \int_{\mathbb{R}^3} |E_N(t, x)|^2 dx - \int_{\mathbb{R}^3} H_N(t, x) \cdot \operatorname{curl} E_N(t, x) dx \\ + \sigma \int_{\Omega} \left(|E_N(t, x)|^2 + f(t, x) \cdot E_N(t, x) \right) dx = 0 \end{aligned} \quad (3.8)$$

Combining (3.6), (3.7) and (3.8) we derive the following estimate through Young inequality

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} |\nabla u_N(t, x)|^2 dx + \frac{1}{\lambda} \int_{\Omega} (|u_N(t, x)|^2 - 1)^2 dx + \int_{\Omega} \Phi(u_N(t, x)) dx \right\} \\ + \frac{1}{2} \frac{d}{dt} \left\{ \int_{\mathbb{R}^3} (|H_N(t, x)|^2 + \frac{\varepsilon_0}{\mu_0} |E_N(t, x)|^2) dx \right\} \\ + \int_{\Omega} \left| \frac{\partial u_N}{\partial t}(t, x) \right|^2 dx + \frac{\sigma}{\mu_0} \int_{\Omega} |E_N(t, x)|^2 dx \leq \frac{\sigma}{\mu_0} \int_{\Omega} |f(t, x)|^2 dx \end{aligned}$$

As $\Phi(u_N)$ is non negative we obtain the following bound for u_0 in $H^1(\Omega)$, E_0 and H_0 in $L^2(\mathbb{R}^3)$ and f in $L^2(\mathbb{R}^+ \times \Omega)$:

There exists constants k_i independant of N and λ such that

$$\begin{aligned} \|\nabla u_N\|_{L^\infty(\mathbb{R}^+; L^2(\Omega))} \leq k_1, \quad \left\| \frac{\partial u_N}{\partial t} \right\|_{L^2(\mathbb{R}^+ \times \Omega)} \leq k_2, \quad \|u_N\|_{L^\infty(\mathbb{R}^+; L^4(\Omega))} \leq k_3, \\ \|E_N\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} \leq k_4, \quad \|H_N\|_{L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))} \leq k_5. \end{aligned}$$

So we can suppose that there exists a subsequence still denoted (u_N, H_N, E_N) such that when N goes to $+\infty$,

$$\begin{aligned} u_N \rightharpoonup u^\lambda \quad & \text{in } L^\infty(\mathbb{R}^+; H^1(\Omega)) \text{ weak } \star, \\ \frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\lambda}{\partial t} \quad & \text{in } L^2(\mathbb{R}^+; L^2(\Omega)) \text{ weak } , \\ E_N \rightharpoonup E^\lambda \quad & \text{in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)) \text{ weak } \star, \\ H_N \rightharpoonup H^\lambda \quad & \text{in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)) \text{ weak } \star. \end{aligned}$$

And according to Aubin's Lemma

$$u_N \rightarrow u^\lambda \text{ in } L^4(0, T; \mathbb{L}^4(\Omega)) \text{ strong for all } T,$$

Taking the limit in the equation (3.2)-(3.5) we obtain

1. For any Φ in $\mathbb{H}^1(\Omega)$

$$\begin{aligned} & \int_{\Omega} \frac{\partial u^\lambda}{\partial t}(t, x) \cdot \Phi(x) dx - \int_{\Omega} u^\lambda(t, x) \wedge \frac{\partial u^\lambda}{\partial t}(t, x) \cdot \Phi(x) dx \\ & + 2 \int_{\Omega} \nabla u^\lambda(t, x) \cdot \nabla \Phi(x) dx + \frac{4}{\lambda} \int_{\Omega} (|u^\lambda(t, x)|^2 - 1) u^\lambda(t, x) \cdot \Phi(x) dx \\ & - 2 \int_{\Omega} (H^\lambda(t, x) - \varphi(u^\lambda(t, x))) \cdot \Phi(x) dx = 0 \text{ in } L^2(\mathbb{R}_t^+). \end{aligned} \quad (3.9)$$

2. For any Ψ in $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$,

$$\mu_0 \left\langle \frac{\partial H^\lambda}{\partial t} + \frac{\partial \bar{u}^\lambda}{\partial t}, \Psi \right\rangle + \int_{\mathbb{R}^3} E^\lambda(t, x) \cdot \text{curl } \Psi(x) dx = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+). \quad (3.10)$$

3. For any Θ in $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$

$$\begin{aligned} \varepsilon_0 \left\langle \frac{\partial E^\lambda}{\partial t}, \Theta \right\rangle - \int_{\mathbb{R}^3} H^\lambda(t, x) \cdot \text{curl } \Theta(x) dx \\ + \sigma \int_{\Omega} (E^\lambda(t, x) + f(t, x)) \cdot \Theta(x) dx = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+) \end{aligned} \quad (3.11)$$

4. With the initial data

$$\begin{aligned} u^\lambda(0) &= u_0 \text{ in } \mathbb{L}^2(\Omega), \\ E^\lambda(0) &= E_0, \quad H^\lambda(0) = H_0 \text{ in } (H_{\text{curl}}(\mathbb{R}^3))'. \end{aligned} \quad (3.12)$$

As the L^2 (resp. L^∞) norm is lower semi continuous for the weak (resp. weak \star) topology we obtain the energy estimate

$$\begin{aligned} \forall t > 0, \quad \mathcal{E}_\lambda(t) + \int_0^t \int_{\Omega} \left| \frac{\partial u^\lambda}{\partial t}(t, x) \right|^2 dx dt + \frac{\sigma}{2\mu_0} \int_0^t \int_{\Omega} |E^\lambda(t, x)|^2 dx dt \\ \leq \frac{\sigma}{2\mu_0} \int_0^t \int_{\Omega} |f(t, x)|^2 dx dt + \mathcal{E}_\lambda(0), \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \mathcal{E}_\lambda(t) &= \int_{\Omega} |\nabla u^\lambda(t, x)|^2 dx + \frac{1}{\lambda} \int_{\Omega} (|u^\lambda(t, x)|^2 - 1)^2 dx + \int_{\Omega} \Phi(u^\lambda(t, x)) dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} \left(|H^\lambda(t, x)|^2 + \frac{\varepsilon_0}{\mu_0} |E^\lambda(t, x)|^2 \right) dx. \end{aligned}$$

Second step. Limit as λ tends to 0.

We first note that as $|u_0| = 1$, $\mathcal{E}_\lambda(0)$ does not depend on λ .

The estimate (3.13) allows us to suppose via the extraction of a subsequence that when λ goes to 0

$$\begin{aligned} u^\lambda &\rightharpoonup u && \text{in } L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \text{ weak } \star, \\ \frac{\partial u^\lambda}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } \mathbb{L}^2(\mathbb{R}^+ \times \Omega) \text{ weakly,} \\ u^\lambda &\rightarrow u && \text{in } L^2((0, T); \mathbb{L}^2(\Omega)) \text{ strongly for all } T > 0 \text{ and a.e.,} \\ E^\lambda &\rightharpoonup E && \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)) \text{ weak } \star, \\ H^\lambda &\rightharpoonup H && \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)) \text{ weak } \star. \end{aligned}$$

• We remark, and it is the main point of the proof, that $|u| = 1$ a.e. in $\mathbb{R}^+ \times \Omega$, as $u^\lambda \rightarrow u$ a.e.

• Now we derive the equation satisfied by u by taking in (3.9) $\Phi = u^\lambda(t, x) \wedge \xi(t, x)$ where ξ is any test function given in $\mathbb{L}_{loc}^2(\mathbb{R}^+; \mathbb{H}^2(\Omega))$.

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial u^\lambda}{\partial t}(t, x) \cdot (u^\lambda(t, x) \wedge \xi(t, x)) dx dt \\ &- \int_0^T \int_\Omega u^\lambda(t, x) \wedge \frac{\partial u^\lambda}{\partial t}(t, x) \cdot (u^\lambda(t, x) \wedge \xi(t, x)) dx dt \\ &+ 2 \int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial u^\lambda}{\partial x_i}(t, x) \cdot \frac{\partial}{\partial x_i} (u^\lambda(t, x) \wedge \xi(t, x)) dx dt \tag{3.14} \\ &- 2 \int_0^T \int_\Omega \left(H^\lambda(t, x) - \varphi(u^\lambda(t, x)) \right) \cdot (u^\lambda(t, x) \wedge \xi(t, x)) dx dt \\ &+ \frac{4}{\lambda} \int_0^T \int_\Omega (|u^\lambda(t, x)|^2 - 1) u^\lambda(t, x) \cdot (u^\lambda(t, x) \wedge \xi(t, x)) dx dt = 0 \end{aligned}$$

The last term of the left-hand side of (3.14) vanishes identically. Furthermore we remark that

$$\frac{\partial u^\lambda}{\partial x_i} \cdot \frac{\partial}{\partial x_i} (u^\lambda \wedge \xi) = -(u^\lambda \wedge \frac{\partial u^\lambda}{\partial x_i}) \cdot \frac{\partial \xi}{\partial x_i}.$$

Now we can take the limit when λ goes to 0 to obtain

$$\begin{aligned} &\int_0^T \int_\Omega \left(\frac{\partial u}{\partial t}(t, x) - u(t, x) \wedge \frac{\partial u}{\partial t}(t, x) \right) \cdot (u(t, x) \wedge \xi(t, x)) dx dt \\ &- 2 \int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial \xi}{\partial x_i}(t, x) \cdot \left(u(t, x) \wedge \frac{\partial u}{\partial x_i}(t, x) \right) dx dt \\ &- 2 \int_0^T \int_\Omega \left(H(t, x) - \varphi(u(t, x)) \right) \cdot (u(t, x) \wedge \xi(t, x)) dx dt = 0, \end{aligned}$$

that is

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(\frac{\partial u}{\partial t}(t, x) + u(t, x) \wedge \frac{\partial u}{\partial t}(t, x) \right) \cdot \xi(t, x) dx dt \\ & + 2 \int_0^T \int_{\Omega} \sum_{i=1}^3 \left(u(t, x) \wedge \frac{\partial u}{\partial x_i}(t, x) \right) \cdot \frac{\partial \xi}{\partial x_i}(t, x) dx dt \end{aligned} \quad (3.15)$$

$$- 2 \int_0^T \int_{\Omega} u(t, x) \wedge (H(t, x) - \varphi(u(t, x))) \cdot \xi(t, x) dx dt = 0$$

as

$$\frac{\partial u}{\partial t} \cdot (u \wedge \xi) = - (u \wedge \frac{\partial u}{\partial t}) \cdot \xi, \text{ and } - (u \wedge \frac{\partial u}{\partial t}) \cdot (u \wedge \xi) = - \frac{\partial u}{\partial t} \cdot \xi$$

since $|u| = 1$ a.e. in $\mathbb{R}^+ \times \Omega$.

• Moreover as the L^2 (resp. L^∞) norm is lower semi continuous for the weak (resp. weak \star) the energy estimate (3.13) remains valid for $|u_0| = 1$.

• Next from (3.15) we derive that

$$\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(u \wedge \frac{\partial u}{\partial x_i} \right) \text{ belongs to } L_{loc}^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))$$

so $u \wedge \frac{\partial u}{\partial \nu}$ makes sense in $L_{loc}^2(\mathbb{R}^+; \mathbb{H}^{-1/2}(\partial\Omega))$.

Moreover as $|u|^2 = 1$, one has $u \cdot \frac{\partial u}{\partial \nu} = 0$. So from the equality

$$\frac{\partial u}{\partial \nu} = \left(u \cdot \frac{\partial u}{\partial \nu} \right) u + u \wedge \left(u \wedge \frac{\partial u}{\partial \nu} \right) = \frac{\partial u}{\partial \nu}$$

which is valid in $H^{-1-\eta}(\partial\Omega)$ for any $\eta > 0$ according to the product of function in sobolev spaces (see L. Hörmander [6]) so in fact

$$\frac{\partial u}{\partial \nu} \text{ makes sense in } L_{loc}^2(\mathbb{R}^+; H^{-1-\eta}(\partial\Omega)) \text{ for any } \eta > 0.$$

• As the Maxwell equations are linear, it is straightforward to take the limit in (3.10) and (3.11) to obtain (2.3) and (2.4).

4 Description of the ω -limit set.

Consider a weak solution u of (1.1)-(1.5). From the energy estimate (2.5), the ω -limit set $\omega(u)$ is not empty. We denote u_∞ a point of this set.

Hence there exists a sequence $(t_n)_{n \geq 1}$, with $\lim_{n \rightarrow +\infty} t_n = +\infty$ such that $u(t_n, \cdot)$ tends to u_∞ in $\mathbb{H}^1(\Omega)$ weak, in $\mathbb{L}^2(\Omega)$ strong, and almost everywhere in Ω . In particular one has $|u| = 1$ a.e. in Ω .

First step. Let be a a non negative real number. For s in $(-a, a)$ and x in Ω we define for n large enough

$$U_n(s, x) = u(t_n + s, x).$$

The sequence $(U_n)_{n \geq 1}$ tends to u_∞ in $\mathbb{L}^2((-a, a) \times \Omega)$ strongly and in $L^2((-a, a); \mathbb{H}^1(\Omega))$ weakly. In fact following [12], we have the estimate

$$\begin{aligned} \frac{1}{2a} \int_{-a}^a \int_{\Omega} |U_n(s, x) - u(t_n, x)|^2 dx ds &= \frac{1}{2a} \int_{-a}^a \int_{\Omega} \left| \int_0^s \frac{\partial u}{\partial t}(t_n + \tau, x) d\tau \right|^2 dx ds \\ &\leq \frac{1}{2a} \int_{-a}^a |s| \int_{\Omega} \int_{t_n-a}^{+\infty} \left| \frac{\partial u}{\partial t}(\tau, x) \right|^2 d\tau dx ds \\ &\leq a \int_{t_n-a}^{+\infty} \int_{\Omega} \left| \frac{\partial u}{\partial t}(\tau, x) \right|^2 dx d\tau. \end{aligned}$$

Now, as $\frac{\partial u}{\partial t}$ lies in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega)$, one gets

$$\lim_{n \rightarrow +\infty} \frac{1}{2a} \int_{-a}^a \int_{\Omega} |U_n(s, x) - u(t_n, x)|^2 dx ds = 0.$$

Since $u(t_n, \cdot)$ tends to u_∞ in $\mathbb{L}^2(\Omega)$ strongly, U_n tends to u_∞ in $L^2((-a, a); \mathbb{L}^2(\Omega))$ strongly.

Moreover we obviously see that the sequence $(\nabla U_n)_{n \geq 1}$ is bounded in $\mathbb{L}^2((-a, a) \times \Omega)$ so there exists a subsequence still noted $(U_n)_{n \geq 1}$ such that U_n tends to u_∞ in $L^2((-a, a); \mathbb{H}^1(\Omega))$ weakly, in $L^2((-a, a); \mathbb{L}^2(\Omega))$ strongly and almost everywhere in Ω .

Second step. We consider a \mathcal{C}^∞ non negative function ρ_a supported by $(-a, a)$ satisfying

$$\begin{aligned} \rho_a(\tau) &= 1 \text{ for } \tau \in (-a + 1, a - 1), \\ 0 &\leq \rho_a(\tau) \leq 1, \quad |\rho'_a(\tau)| \leq 2. \end{aligned}$$

We set

$$H_a^n(x) = \frac{1}{2a} \int_{-a}^a H(t_n + s, x) \rho_a(s) ds$$

and

$$E_a^n(x) = \frac{1}{2a} \int_{-a}^a E(t_n + s, x) \rho_a(s) ds.$$

From the estimate (2.5), E and H are bounded in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))$. Then H_a^n and E_a^n are bounded in $\mathbb{L}^2(\mathbb{R}^3)$ independently of n and a . So by extracting a subsequence we may suppose that $(E_a^n, H_a^n)_{n \geq 1}$ converges in $\mathbb{L}^2(\mathbb{R}^3)$ weakly to (E_a, H_a) when n goes to $+\infty$.

Third step. In the weak formulation (2.2) we take as test function $\rho_a(t - t_n)\Psi(x)$ where Ψ is a function lying in $\mathcal{D}(\bar{\Omega})$. We obtain after the change of chart $s = t - t_n$

$$\begin{aligned} &\frac{1}{2a} \int_{-a}^a \int_{\Omega} \left(\frac{\partial U_n}{\partial t}(s, x) + U_n(s, x) \wedge \frac{\partial U_n}{\partial t}(s, x) \right) \cdot \Psi(x) \rho_a(s) dx ds \\ &\quad + 2 \frac{1}{2a} \int_{-a}^a \int_{\Omega} \sum_{i=1}^3 \left(U_n(s, x) \wedge \frac{\partial U_n}{\partial x_i}(s, x) \right) \cdot \frac{\partial \Psi}{\partial x_i} \rho_a(s) dx ds \tag{4.1} \\ &- 2 \frac{1}{2a} \int_{-a}^a \int_{\Omega} U_n \wedge \left(H(t_n + s, x) - \varphi(U_n(s, x)) \right) \cdot \Psi(x) \rho_a(s) dx ds = 0. \end{aligned}$$

To pass through the limit in (4.1) we bound separately each term of (4.1).

- First term.

$$\begin{aligned}
& \left| \frac{1}{2a} \int_{-a}^a \int_{\Omega} \frac{\partial U_n}{\partial t}(s, x) \cdot \Psi(x) \rho_a(s) dx ds \right| \\
& \leq \frac{1}{2a} \int_{-a}^a \rho_a(s) \left(\int_{\Omega} \left| \frac{\partial U_n}{\partial t}(s, x) \right|^2 dx \right)^{1/2} \left(\int_{\Omega} |\Psi(x)|^2 dx \right)^{1/2} ds \\
& \leq \frac{1}{\sqrt{2a}} \left(\int_{\Omega} |\Psi(x)| dx \right)^{1/2} \left(\int_{t_n-a}^{t_n+a} \int_{\Omega} \left| \frac{\partial u}{\partial t}(s, x) \right|^2 dx ds \right)^{1/2}
\end{aligned}$$

Since $\frac{\partial u}{\partial t}$ belongs to $\mathbb{L}^2(\mathbb{R}^+ \times \Omega)$, this last term tends to zero as n goes to $+\infty$. In the same way, as U_n takes its values on S^2 , one also has

$$\lim_{n \rightarrow +\infty} \frac{1}{2a} \int_{-a}^a \int_{\Omega} U_n(s, x) \wedge \frac{\partial U_n}{\partial t}(s, x) \rho_a(s) \cdot \Psi(x) dx ds = 0$$

- Second term.

As $(U_n)_{n \geq 1}$ tends to u_{∞} strongly in $\mathbb{L}^2((-a, a) \times \Omega)$, as $(\frac{\partial U_n}{\partial x_i})_{n \geq 1}$ tends to $\frac{\partial u_{\infty}}{\partial x_i}$ weakly in $\mathbb{L}^2((-a, a) \times \Omega)$ and since $\frac{\partial \Psi}{\partial x_i} \rho_a$ belongs to $\mathbb{L}^{\infty}((-a, a) \times \Omega)$, the second term of (4.1) tends to

$$2 \frac{1}{2a} \int_{-a}^a \rho_a(s) ds \int_{\Omega} \sum_{i=1}^3 \left(u_{\infty}(x) \wedge \frac{\partial u_{\infty}}{\partial x_i}(x) \right) \cdot \frac{\partial \Psi}{\partial x_i}(x) dx.$$

- Third term.

$$\begin{aligned}
& \frac{1}{2a} \int_{-a}^a \int_{\Omega} U_n(s, x) \wedge H(t_n + s, x) \cdot \Psi(x) \rho_a(s) dx ds \\
& = \frac{1}{2a} \int_{-a}^a \int_{\Omega} \left(U_n(s, x) - u_{\infty}(x) \right) \wedge H(t_n + s, x) \cdot \Psi(x) \rho_a(s) dx ds \quad (4.2) \\
& \quad + \frac{1}{2a} \int_{-a}^a \int_{\Omega} u_{\infty}(x) \wedge H(t_n + s, x) \cdot \Psi(x) \rho_a(s) dx ds.
\end{aligned}$$

The first term of (4.2) goes to zero as $(U_n - u_{\infty})_n$ tends strongly to zero in $\mathbb{L}^2((-a, a) \times \Omega)$ and as H is bounded in $L^{\infty}(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))$. The second term is equal to

$$\int_{\Omega} \left(u_{\infty}(x) \wedge H_a^n(x) \right) \cdot \Psi(x) dx,$$

and tends obviously to

$$\int_{\Omega} \left(u_{\infty}(x) \wedge H_a(x) \right) \cdot \Psi(x) dx.$$

As φ is linear, it is straightforward to take the limit in the last term.

So from equation (4.1) we derive that u_{∞} solve the equation

$$\begin{aligned}
& \int_{\Omega} \sum_{i=1}^3 \left(u_{\infty}(x) \wedge \frac{\partial u_{\infty}}{\partial x_i}(x) \right) \cdot \frac{\partial \Psi}{\partial x_i}(x) + \int_{\Omega} \left(u_{\infty}(x) \wedge \varphi(u_{\infty}(x)) \right) \cdot \Psi(x) dx \\
& \quad \frac{2a}{\int_{-a}^a \rho(s) ds} \int_{\Omega} \left(u_{\infty}(x) \wedge H_a(x) \right) \cdot \Psi(x) dx = 0. \quad (4.3)
\end{aligned}$$

Forth step. In order to obtain the desired result it remains to take the limit in (4.3) when a tends to $+\infty$.

We first remark that

$$\lim_{a \rightarrow +\infty} \frac{2a}{\int_{-a}^a \rho(s) ds} = 1.$$

Through estimate (2.5) and by definition of H_a , $(H_a)_{a \geq 1}$ is uniformly bounded in $\mathbb{L}^2(\mathbb{R}^3)$. Hence, by extraction we can suppose that H_a tends to H_∞ weakly in $\mathbb{L}^2(\mathbb{R}^3)$. So at the limit one has

$$-\int_{\Omega} \sum_{i=1}^3 \left(u_\infty(x) \wedge \frac{\partial u_\infty}{\partial x_i}(x) \right) \cdot \frac{\partial \Psi}{\partial x_i}(x) dx + \int_{\Omega} u_\infty(x) \wedge \left(H_\infty(x) - \varphi(u_\infty(x)) \right) \cdot \Psi(x) dx = 0$$

Fifth step. In order to derive the equation satisfied by H_∞ we first recall the equation verified by H_a^n and E_a^n .

In equation (2.3) we take $\Psi(t, x) = \theta_a(t - t_n) \nabla \xi(x)$ with ξ in $\mathcal{D}(\mathbb{R}^3)$ and θ_a is defined by

$$\theta_a(t) = \int_a^t \rho_a(s) ds.$$

We obtain that for every ξ in $\mathcal{D}(\mathbb{R}^3)$

$$-\int_{-a}^a \int_{\mathbb{R}^3} \left(H(t_n + s, x) + \bar{u}(t_n + s, x) \right) \cdot \nabla \xi(x) \rho_a(s) ds = \int_{\mathbb{R}^3} \left(H_0(x) + \bar{u}_0(x) \right) \cdot \nabla \xi(x) dx \theta_a(0).$$

As $\operatorname{div}(H_0 + \bar{u}_0) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$, we obtain after dividing by $2a$

$$\int_{\mathbb{R}^3} \left(H_a^n(x) + \frac{1}{2a} \int_{-a}^a \bar{u}(t_n + s, x) \rho_a(s) ds \right) \cdot \nabla \xi(x) dx = 0.$$

When n goes to $+\infty$ we obtain that

$$\int_{\mathbb{R}^3} \left(H_a(x) + \bar{u}_\infty(x) \right) \cdot \nabla \xi(x) dx = 0,$$

and so when a goes to infinity we get

$$\operatorname{div}(H_\infty + \bar{u}_\infty) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

Now we take $\Psi(t, x) = \rho_a(t - t_n) \xi(x)$ in (2.4). We obtain that

$$\begin{aligned} & \frac{1}{2a} \int_{-a}^a \int_{\mathbb{R}^3} E(t_n + s, x) \cdot \rho'_a(s) \xi(x) dx ds - \int_{\mathbb{R}^3} H_a^n(x) \cdot \operatorname{curl} \xi(x) dx \\ & + \sigma \int_{\Omega} E_a^n(x) \cdot \xi(x) dx + \sigma \frac{1}{2a} \int_{-a}^a \int_{\Omega} f(t_n + s, x) \cdot \rho_a(s) \xi(x) dx ds \\ & = \int_{\mathbb{R}^3} E_0(x) \cdot \xi(x) dx \rho_a(-t_n). \end{aligned} \quad (4.4)$$

For n large enough, the righthand side of (4.4) vanishes identically.

Let us bound the first term of (4.4). As ρ'_a is identically zero on $(-a + 1, a - 1)$ and is bounded by 2, one has

$$\left| \frac{1}{2a} \int_{-a}^a \int_{\mathbb{R}^3} E(t_n + s, x) \cdot \rho'_a(s) \xi(x) dx \right| \leq \frac{1}{a} \|\xi\|_{\mathbb{L}^2(\mathbb{R}^3)} \|E\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))}. \quad (4.5)$$

Moreover

$$\begin{aligned} & \left| \frac{1}{2a} \int_{-a}^a \int_{\mathbb{R}^3} f(t_n + s, x) \cdot \rho_a(s) \xi(x) dx ds \right| \\ & \leq \frac{1}{2a} \left(\int_{-a+t_n}^{a+t_n} \|f(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \right)^{1/2} \left(\int_{-a}^a \rho_a(s)^2 ds \right)^{1/2} \|\xi\|_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

that is

$$\left| \frac{1}{2a} \int_{-a}^a \int_{\mathbb{R}^3} f(t_n + s, x) \cdot \rho_a(s) \xi(x) dx ds \right| \leq \frac{1}{\sqrt{2a}} \left(\int_{-a+t_n}^{a+t_n} \|f(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \right)^{1/2} \|\xi\|_{\mathbb{L}^2(\Omega)} \quad (4.6)$$

since $0 \leq \rho_a(s) \leq 1$.

When n goes to infinity, by extraction of a subsequence the first term of the left-hand side of (4.4) tends to a real α_a satisfying

$$|\alpha_a| \leq \frac{1}{2a} \|E\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \|\xi\|_{\mathbb{L}^2(\Omega)} \quad (4.7)$$

Due to (4.6), the fourth term of the left-hand side of (4.4) goes to zero as

$$\int_{\mathbb{R}^+} \int_{\Omega} |f(t, x)|^2 dx dt < +\infty.$$

Hence we obtain

$$\alpha_a - \int_{\mathbb{R}^3} H_a(x) \cdot \text{curl } \xi(x) dx + \sigma \int_{\Omega} E_a(x) \cdot \xi(x) dx = 0.$$

Then taking the limit as a goes to infinity, one has from (4.7)

$$\int_{\mathbb{R}^3} H_\infty(x) \cdot \text{curl } \xi(x) dx = \sigma \int_{\Omega} E_\infty(x) \cdot \xi(x) dx. \quad (4.8)$$

In the same way, taking $\Psi(t, x) = \rho_a(t_n - t) \xi(x)$ in (2.3) we derive that

$$\int_{\mathbb{R}^3} E_\infty(x) \cdot \text{curl } \xi(x) dx = 0,$$

that is $\text{curl } E_\infty = 0$. So it is valid to take $\xi(x) = E_\infty(x)$ in (4.8) which leads to

$$\sigma \int_{\Omega} |E_\infty(x)|^2 dx = 0.$$

This (4.8) gives $\text{curl } H_\infty = 0$. Finally H_∞ is uniquely determined by

$$\begin{cases} \text{div } (H_\infty + \bar{u}_\infty) = 0 \text{ in } \mathbb{R}^3, \\ \text{curl } H_\infty = 0 \text{ in } \mathbb{R}^3, \\ H_\infty \in \mathbb{L}^2(\mathbb{R}^3). \end{cases}$$

Therefore u_∞ is a solution of the stationary model (2.6)-(2.8).

Remark 4.1 Following an idea of G. Métivier, it is possible to prove Theorem 2.2 without average Maxwell Equations. This is due to the fact that $H(t, \cdot) - H(u(t))$ tends to zero in L_{loc}^2 when t tends to $+\infty$ (see [8]).

5 Quasi-stationary model

The last part of this paper is devoted to the justification of the quasi-stationary model.

We recall that we suppose $f \equiv 0$.

We consider ε^n and μ^n such that ε^n , μ^n and ε^n/μ^n tend to zero. In the sequel we denote (u^n, H^n, E^n) a family of weak solutions of (1.1)-(1.5) with $\varepsilon_0 = \varepsilon^n$ and $\mu_0 = \mu^n$.

We recall the energy estimate satisfied by (u^n, H^n, E^n) .

$$\mathcal{E}^n(t) + \int_0^t \int_{\Omega} \left| \frac{\partial u^n}{\partial t}(t, x) \right|^2 dx dt + \frac{\sigma}{\mu^n} \int_0^t \int_{\Omega} |E^n(t, x)|^2 dx dt \leq \mathcal{E}^n(0) \quad (5.1)$$

where

$$\mathcal{E}^n(t) = \int_{\Omega} \left(|\nabla u^n(t, x)|^2 + 2\Phi(u^n(t, x)) \right) dx + \int_{\mathbb{R}^3} \left(|H^n(t, x)|^2 + \frac{\varepsilon^n}{\mu^n} |E^n(t, x)|^2 \right) dx.$$

Since ε^n/μ^n remains bounded, the right hand-side term of (5.1) remains bounded uniformly in n . Therefore, by the energy estimate (5.1), u^n is bounded in $L^\infty(\mathbb{R}^+; H^1(\Omega))$ and $\frac{\partial u^n}{\partial t}$ is bounded in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega)$ uniformly in n . Furthermore H^n and $\sqrt{\varepsilon^n/\mu^n} E^n$ are uniformly bounded in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))$. Extracting a subsequence we can suppose that

$$\begin{aligned} u^n &\rightharpoonup u && \text{in } L^\infty(\mathbb{R}^+; H^1(\Omega)) \text{ weak } \star, \\ u^n &\rightarrow u && \text{in } L^2((0, T); \mathbb{L}^2(\Omega)) \text{ strong for all } T > 0, \\ \frac{\partial u^n}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } L^2((0, T); \mathbb{L}^2(\Omega)) \text{ weak for all } T > 0. \end{aligned}$$

First step.

For any $a > 0$ we set

$$\begin{aligned} u_a^n(t, x) &:= \frac{1}{a} \int_0^a u^n(t + s, x) ds, \\ H_a^n(t, x) &:= \frac{1}{a} \int_0^a H^n(t + s, x) ds, \\ E_a^n(t, x) &:= \frac{1}{a} \int_0^a E^n(t + s, x) ds. \end{aligned} \quad (5.2)$$

Lemma 5.1 *For each $n \in \mathbb{N}$ and $a > 0$, (u_a^n, H_a^n, E_a^n) satisfies the following estimates.*

$$\|u_a^n\|_{L^\infty(\mathbb{R}^+; H^1(\Omega))} \leq \|u^n\|_{L^\infty(\mathbb{R}^+; H^1(\Omega))}, \quad (5.3)$$

$$\left\| \frac{\partial u_a^n}{\partial t} \right\|_{\mathbb{L}^2(\mathbb{R}^+ \times \Omega)} \leq \left\| \frac{\partial u^n}{\partial t} \right\|_{\mathbb{L}^2(\mathbb{R}^+ \times \Omega)}, \quad (5.4)$$

$$\|H_a^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \leq \|H^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))}, \quad (5.5)$$

$$\|E_a^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \leq \|E^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))}. \quad (5.6)$$

Proof. The estimates (5.3), (5.5) and (5.6) follow directly from the definition (5.2).

For (5.4) we write

$$\frac{\partial u_a^n}{\partial t}(t, x) = \frac{1}{a}(u^n(t+a, x) - u^n(t, x)) = \int_0^1 \frac{\partial u^n}{\partial t}(t + \theta a, x) d\theta,$$

so

$$\int_{\mathbb{R}^+} \left| \frac{\partial u_a^n}{\partial t}(s, x) \right|^2 ds \leq \int_{\mathbb{R}^+} \left(\int_0^1 \frac{\partial u^n}{\partial t}(t + \theta a, x) d\theta \right)^2 dt \leq \int_{\mathbb{R}^+} \left| \frac{\partial u^n}{\partial t}(s, x) \right|^2 ds.$$

That is

$$\int_{\mathbb{R}^+ \times \Omega} \left| \frac{\partial u_a^n}{\partial t}(s, x) \right|^2 ds dx \leq \int_{\mathbb{R}^+ \times \Omega} \left| \frac{\partial u^n}{\partial t}(s, x) \right|^2 ds dx.$$

Lemma 5.2 *For every $a > 0$ we have the following estimate*

$$\|u_a^n - u^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))} \leq \sqrt{a} \left\| \frac{\partial u^n}{\partial t} \right\|_{\mathbb{L}^2(\mathbb{R}^+ \times \Omega)}.$$

Proof. From the definition (5.2) one gets

$$\begin{aligned} u_a^n(t, x) - u^n(t, x) &= \frac{1}{a} \int_0^a (u^n(s+t, x) - u^n(t, x)) ds \\ &= \frac{1}{a} \int_0^a \int_0^s \frac{\partial u^n}{\partial t}(t + \tau, x) d\tau ds, \end{aligned}$$

so

$$\begin{aligned} |u_a^n(t, x) - u^n(t, x)|^2 &\leq \left| \frac{1}{a} \int_0^a \int_0^s \frac{\partial u^n}{\partial t}(t + \tau, x) d\tau ds \right|^2 \\ &\leq \left| \int_0^a \left| \frac{\partial u^n}{\partial t}(t + \tau, x) \right| d\tau \right|^2 \\ &\leq a \int_t^{t+a} \left| \frac{\partial u^n}{\partial t}(s, x) \right|^2 ds, \end{aligned}$$

hence

$$\int_{\Omega} |u_a^n(t, x) - u^n(t, x)|^2 dx \leq a \int_{\mathbb{R}^+} \int_{\Omega} \left| \frac{\partial u^n}{\partial t}(s, x) \right|^2 ds dx.$$

Second step. We choose $a_n = (\varepsilon^n \mu^n)^{\frac{1}{4}}$, and we denote in the sequel

$$u_n := u_{a_n}^n, \quad H_n := H_{a_n}^n, \quad \text{and} \quad E_n := E_{a_n}^n.$$

Thanks to the energy estimate (5.1) and Lemma 5.1, we can suppose after extraction of a subsequence that

$$\begin{aligned} u_n &\rightharpoonup u^\infty && \text{in } L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \text{ weak } \star, \\ u_n &\rightarrow u^\infty && \text{in } L^2((0, T); \mathbb{L}^2(\Omega)) \text{ strong for all } T > 0, \\ H_n &\rightharpoonup H^\infty && \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3)) \text{ weak } \star, \\ \frac{\partial u_n}{\partial t} &\rightharpoonup \frac{\partial u^\infty}{\partial t} && \text{in } \mathbb{L}^2(\mathbb{R}^+ \times \Omega) \text{ weak.} \end{aligned}$$

Furthermore Lemma 5.2 ensures that $u^\infty = u$ and $u_n(0, \cdot) \rightarrow u_0(\cdot)$ in $\mathbb{L}^2(\Omega)$ strong.

Third step. For t given in \mathbb{R}^+ we take $\Psi(s, x) = \mathbf{1}_{[t, t+a]}(s)\xi(x)$ in (2.1). After dividing by a_n we obtain that

$$\begin{aligned} & \int_{\Omega} \frac{\partial u_n}{\partial t}(t, x) \cdot \xi(x) dx + \int_{\Omega} \frac{1}{a_n} \int_0^{a_n} (u^n(t+s, x) \wedge \delta dt u^n(t+s, x)) \cdot \xi(x) ds dx \\ & + 2 \int_{\Omega} \frac{1}{a_n} \int_0^{a_n} \sum_{i=1}^3 \left(u^n(t+s, x) \wedge \frac{\partial u^n}{\partial x_i}(t+s, x) \right) \cdot \frac{\partial \xi}{\partial x_i}(x) ds dx \\ & - 2 \int_{\Omega} \frac{1}{a_n} \int_0^{a_n} u^n(t+s, x) \wedge \left(H^n(t+s, x) - \varphi(u^n(t+s, x)) \right) \cdot \xi(x) ds dx = 0. \end{aligned}$$

Multiplying this last formula by a test function $\rho(t)$, we obtain after integration

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \Omega} \frac{\partial u_n}{\partial t}(t, x) \cdot \xi(x) \rho(t) dx dt \\ & + \int_{\mathbb{R}^+ \times \Omega} \frac{1}{a_n} \int_0^{a_n} \left(u^n(t+s, x) \wedge \frac{\partial u^n}{\partial t}(t+s, x) \right) \cdot \xi(x) \rho(t) ds dx dt \\ & + \int_{\mathbb{R}^+ \times \Omega} \frac{1}{a_n} \int_0^{a_n} \sum_{i=1}^3 \left(u^n(t+s, x) \wedge \frac{\partial u^n}{\partial x_i}(t+s, x) \right) \cdot \frac{\partial \xi}{\partial x_i}(x) \rho(t) ds dx dt \\ & - \frac{2}{a_n} \int_{\mathbb{R}^+ \times \Omega} \int_t^{t+a_n} u^n(s, x) \wedge \left(H^n(s, x) - \varphi(u^n(s, x)) \right) \cdot \xi(x) \rho(t) ds dx dt = 0. \end{aligned} \tag{5.7}$$

As $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u^\infty}{\partial t}$ in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega)$ weakly, the first term of (5.7) tends to

$$\int_{\mathbb{R}^+} \int_{\Omega} \frac{\partial u^\infty}{\partial t}(t, x) \cdot \xi(x) \rho(t) dx dt.$$

Let us now study the second term.

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \Omega} \frac{1}{a_n} \int_0^{a_n} u^n(t+s, x) \wedge \frac{\partial u^n}{\partial t}(t+s, x) \cdot \xi(x) \rho(t) ds dx dt = \\ & \int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot u^n(t, x) \wedge \left(\frac{1}{a_n} \int_0^{a_n} \frac{\partial u^n}{\partial t}(t+s, x) ds \right) dx dt \\ & + \int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot \frac{1}{a_n} \int_0^{a_n} \left(u^n(t+s, x) - u^n(t, x) \right) \wedge \frac{\partial u^n}{\partial t}(s, x) ds dt dx. \end{aligned}$$

The definition of u_n shows that this is equal to

$$\begin{aligned} & \int_{\mathbb{R}^+ \times \Omega} \frac{1}{a_n} \int_0^{a_n} \left(u^n(t+s, x) \wedge \frac{\partial u^n}{\partial t}(t+s, x) \right) \cdot \xi(x) \rho(t) ds dx dt = \\ & \int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot \left(u^n(t, x) \wedge \frac{\partial u_n}{\partial t}(t, x) \right) dt dx \\ & + \int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot \frac{1}{a_n} \int_0^{a_n} \left(u^n(t+s, x) - u^n(t, x) \right) \wedge \frac{\partial u^n}{\partial t}(s, x) ds dt dx. \end{aligned} \tag{5.8}$$

The first term of (5.8) tends to

$$\int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot \left(u^\infty(t, x) \wedge \frac{\partial u^\infty}{\partial t}(t, x) \right) dt dx$$

as

$$u^n \rightarrow u^\infty \text{ in } L^2_{loc}(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ strongly}$$

and

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u^\infty}{\partial t} \text{ in } \mathbb{L}^2(\mathbb{R}^+ \times \Omega) \text{ weakly.}$$

Now we prove that the second term goes to zero. We use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} A &:= \left| \int_{\mathbb{R}^+ \times \Omega} \rho(t) \xi(x) \cdot \frac{1}{a_n} \int_0^{a_n} (u^n(t+s, x) - u^n(t, x)) \wedge \frac{\partial u^n}{\partial t}(t+s, x) ds dx dt \right| \\ A &\leq \|\xi\|_{\mathbb{L}^\infty(\Omega)} \|\rho\|_{\mathbb{L}^\infty(\mathbb{R}^+)} \frac{1}{a_n} \left\{ \int_{\mathbb{R}^+ \times \Omega} \int_0^{a_n} \left(\int_0^s \frac{\partial u^n}{\partial t}(t+\tau, x) d\tau \right)^2 dx dt ds \right\}^{\frac{1}{2}} \times \\ &\quad \left\{ \int_{\mathbb{R}^+ \times \Omega} \int_0^{a_n} \left| \frac{\partial u^n}{\partial t}(t+s, x) \right|^2 ds dx dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Now by the Cauchy-Schwarz inequality and Fubini theorem we get

$$A \leq \|\xi\|_{\mathbb{L}^\infty(\Omega)} \|\rho\|_{\mathbb{L}^\infty(\mathbb{R}^+)} \frac{1}{\sqrt{a_n}} \left\{ \int_{\mathbb{R}^+ \times \Omega} \int_0^{a_n} s \int_0^{a_n} \left| \frac{\partial u^n}{\partial t}(t+\tau, x) \right|^2 d\tau ds dt dx \right\}^{\frac{1}{2}} \left\| \frac{\partial u^n}{\partial t} \right\|_{\mathbb{L}^2(\mathbb{R}^+ \times \Omega)}.$$

So after integration

$$A \leq \frac{a_n}{\sqrt{2}} \|\xi\|_{\mathbb{L}^\infty(\Omega)} \|\rho\|_{\mathbb{L}^\infty(\mathbb{R}^+)} \left\| \frac{\partial u^n}{\partial t} \right\|_{\mathbb{L}^2(\mathbb{R}^+ \times \Omega)}^2.$$

Hence by the energy estimate (5.1), A tends to zero as a_n .

In the same way as in the previous section we obtain finally

$$\begin{aligned} &\int_{\mathbb{R}^+ \times \Omega} \left(\frac{\partial u^\infty}{\partial t}(t, x) + u^\infty(t, x) \wedge \frac{\partial u^\infty}{\partial t}(t, x) \right) \cdot \xi(x) \rho(t) dx dt \\ &\quad + 2 \int_{\mathbb{R}^+ \times \Omega} \sum_{i=1}^3 \left(u^\infty(t, x) \wedge \frac{\partial u^\infty}{\partial x_i}(t, x) \right) \cdot \frac{\partial \xi}{\partial x_i}(x) \rho(t) dx dt \tag{5.9} \\ &\quad - 2 \int_{\mathbb{R}^+ \times \Omega} u^\infty(t, x) \wedge \left(H^\infty(t, x) - \varphi(u^\infty(t, x)) \right) \cdot \xi(x) \rho(t) dx dt = 0. \end{aligned}$$

Fourth step. As for the study of the ω -limit set we can prove that

$$\operatorname{div} (H^\infty + \bar{u}^\infty) = 0.$$

Now it remains to obtain

$$\operatorname{curl} H^\infty = 0. \tag{5.10}$$

We recall that for all ξ in $\mathcal{D}(\mathbb{R}^3)$ and ρ in $\mathcal{D}([0, +\infty))$ we have according to (2.4) that

$$\begin{aligned} & - \int_{\mathbb{R}^+ \times \mathbb{R}^3} \varepsilon^n E^n(s, x) \cdot \frac{\partial \rho}{\partial t}(s) \xi(x) ds dx - \int_{\mathbb{R}^+ \times \mathbb{R}^3} H^n(s, x) \cdot \text{curl } \xi(x) \rho(s) dx ds \\ & + \sigma \int_{\mathbb{R}^+ \times \Omega} E^n(s, x) \cdot \rho(s) \xi(x) ds dx = \int_{\mathbb{R}^3} E_0(x) \cdot \xi(x) \rho(0) dx. \end{aligned} \quad (5.11)$$

Formally, the identity (5.10) is obtained taking $\rho = \mathbf{1}_{(t, t+a_n)}$ in (5.11). Unfortunately this function is not regular enough, so we introduce a regularised function ρ_δ .

For each $\delta > 0$ given, $0 < \delta < a_n$, we denote

$$\rho_\delta(s) = \begin{cases} 1 & \delta \leq s \leq a_n - \delta \\ 0 & s \leq 0 \text{ or } s \geq a_n \\ \text{linear} & 0 \leq s \leq \delta \text{ and } a_n - \delta \leq s \leq a_n \end{cases}$$

Now, for $\rho = \rho_\delta(s - t)$ equation (5.11) gives

$$\begin{aligned} & - \frac{\varepsilon^n}{a_n} \int_t^{t+\delta} \int_{\mathbb{R}^3} E^n(s, x) \frac{\partial \rho_\delta}{\partial t}(s - t) \cdot \xi(x) ds dx \\ & - \frac{\varepsilon^n}{a_n} \int_{t+a_n-\delta}^{t+a_n} \int_{\mathbb{R}^3} E^n(s, x) \frac{\partial \rho_\delta}{\partial t}(s - t) \cdot \xi(x) ds dx - \int_{\mathbb{R}^3} H_a^n(x) \cdot \text{curl } \xi(x) dx \\ & + \frac{\sigma}{a_n} \int_t^{t+a_n} \int_{\Omega} \rho_\delta(t - s) E^n(t, x) \cdot \xi(x) dx ds \\ & = - \frac{1}{a_n} \int_t^{t+a_n} \int_{\mathbb{R}^3} H^n(s, x) \cdot (1 - \rho_\delta(s - t)) \text{curl } \xi(x) dx ds. \end{aligned} \quad (5.12)$$

The two first terms of the left-hand side of (5.12) are bounded by

$$2 \frac{\varepsilon^n}{a_n} \|E^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \|\xi\|_{\mathbb{L}^2(\mathbb{R}^3)}. \quad (5.13)$$

The last term of the left-hand side of (5.12) is bounded by

$$\sigma \|E^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))} \|\xi\|_{\mathbb{L}^2(\Omega)}. \quad (5.14)$$

The right-hand side of (5.12) is bounded by

$$2 \frac{\delta}{a_n} \|H^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \|\text{curl } \xi\|_{\mathbb{L}^2(\mathbb{R}^3)}. \quad (5.15)$$

According to the energy estimate (5.1) we have

$$\|E^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3))} \leq k \sqrt{\frac{\mu^n}{\varepsilon^n}}$$

$$\|E^n\|_{L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))} \leq k \sqrt{\mu^n}$$

for some constant k

So by choosing $a_n = (\varepsilon^n \mu^n)^{\frac{1}{4}}$ and $\delta = a_n^2$ we get, for any test function φ

$$\int_{\mathbb{R}^3} H^\infty(t, x) \cdot \text{curl } \xi(x) \varphi(t) dx dt = 0.$$

Fifth step. Energy estimate.

By convexity and thanks to the definition (5.2), one has

$$\begin{aligned} & \int_{\Omega} |\nabla u_a^n(t, x)|^2 dx + 2 \int_{\Omega} \Phi(u_a^n(t, x)) + \int_{\mathbb{R}^3} |H_a^n(t, x)|^2 dx \\ & \leq \frac{1}{a} \int_0^a \left(\int_{\Omega} |\nabla u^n(t+s, x)|^2 dx + 2 \int_{\Omega} \Phi(u^n(t+s, x)) + \int_{\mathbb{R}^3} |H^n(t+s, x)|^2 dx \right) \\ & \leq \frac{1}{a} \int_0^a \mathcal{E}^n(t+s) ds. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_0^t \int_{\Omega} \left| \frac{\partial u_a^n}{\partial t}(s, x) \right|^2 dx ds &= \int_0^t \int_{\Omega} \left| \frac{1}{a} \int_0^a \frac{\partial u^n}{\partial t}(\tau+s, x) d\tau \right|^2 dx ds \\ &\leq \frac{1}{a} \int_0^a \int_0^{t+s} \int_{\Omega} \left| \frac{\partial u^n}{\partial t}(\tau, x) \right|^2 d\tau dx ds. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\Omega} |\nabla u_a^n(t, x)|^2 dx + 2 \int_{\Omega} \Phi(u_a^n(t, x)) + \int_{\mathbb{R}^3} |H_a^n(t, x)|^2 dx + \int_0^t \int_{\Omega} \left| \frac{\partial u_a^n}{\partial t}(s, x) \right|^2 dx ds \\ & \leq \frac{1}{a} \int_0^a \left(\mathcal{E}^n(t+s) + \int_0^{t+s} \int_{\Omega} \left| \frac{\partial u^n}{\partial t}(\tau, x) \right|^2 d\tau dx \right) ds \leq \mathcal{E}^n(0). \end{aligned}$$

Since ε^n/μ^n tends to zero, $\mathcal{E}^n(0)$ tends to $\mathcal{E}_q(0)$. Therefore using the semi continuity of the norms for the weak topology, we derive the desired energy estimate (2.12).

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