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► **To cite this version:**

Feng Qi, Ai-Qi Liu. NOTES ON COMPLETE MONOTONICITY RELATED TO THE DIFFERENCE OF THE PSI AND LOGARITHMIC FUNCTIONS: Difference of the psi and logarithmic functions. 2018. hal-01728682v1

HAL Id: hal-01728682

<https://hal.science/hal-01728682v1>

Preprint submitted on 11 Mar 2018 (v1), last revised 19 Jan 2020 (v2)

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NOTES ON COMPLETE MONOTONICITY RELATED TO THE DIFFERENCE OF THE PSI AND LOGARITHMIC FUNCTIONS

FENG QI AND AI-QI LIU

ABSTRACT. In the paper, the authors simply survey complete monotonicity of several functions involving the difference of the psi and logarithmic functions and present a concise proof for complete monotonicity of a function involving the difference of the psi and logarithmic functions.

1. MOTIVATIONS AND MAIN RESULTS

Recall from [10, Chapter XIII], [14, Chapter 1], and [15, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x \in I$ and $n \in \{0\} \cup \mathbb{N}$. Theorem 12b in [15, p. 161] states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a positive measure on $[0, \infty)$ such that the above integral converges on $(0, \infty)$.

In [2, pp. 374–375, Theorem 1] and [8, Theorem 1], it was verified that the function

$$x^\alpha [\ln x - \psi(x)]$$

for $\alpha \in \mathbb{R}$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$.

In [13, Theorem 1.7], it was proved that the function

$$x^2[\psi(x) - \ln x] + \frac{x}{2}$$

is strictly decreasing and convex on $(0, \infty)$ and, as $x \rightarrow \infty$, tends to $-\frac{1}{12}$.

In [3, Theorem 1], it was discovered that the function

$$x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}$$

is completely monotonic on $(0, \infty)$.

In [1, p. 259, 6.3.18] and [11, p. 140, 5.11.2], the asymptotic formula

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}$$

2010 *Mathematics Subject Classification.* Primary 26A48; Secondary 33B15, 44A10.

Key words and phrases. complete monotonicity; difference; psi function; logarithmic function; concise proof.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

$$= \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots$$

as $z \rightarrow \infty$ in $|\arg z| < \pi$ was listed, where B_{2n} denotes the Bernoulli numbers.

The above conclusions motivate us to guess that the functions

$$F_n(x) = (-1)^{n+1} \left(x^{2n} [\psi(x) - \ln x] + \frac{1}{2} x^{2n-1} + \sum_{k=1}^n \frac{B_{2k}}{2k} x^{2(n-k)} \right)$$

for $n \geq 2$ are also completely monotonic on $(0, \infty)$. Actually, this guess is wrong, because

$$\begin{aligned} [F_2(x)]' &= -\frac{1}{6}x[6x^3\psi'(x) + 24x^2\psi(x) - 24x^2\ln x - 6x^2 + 9x + 1] \\ &\rightarrow \begin{cases} 0, & x \rightarrow 0^+ \\ \frac{24\gamma - \pi^2 - 4}{6}, & x \rightarrow 1 \\ \frac{96\gamma - 8\pi^2 - 43 + 96\ln 2}{3}, & x \rightarrow 2 \end{cases} \\ &= \begin{cases} 0 \\ -0.0027380\dots \\ -0.0006673\dots \end{cases} \end{aligned}$$

shows that the derivative $[F_2(x)]'$ is not monotonic on $(0, \infty)$, where $\gamma = 0.577\dots$ denotes the Euler–Mascheroni constant.

In this paper, motivated by the second proof of [8, Theorem 1], we will present a concise proof, which is simpler than the corresponding proof in [3, Theorem 1], for complete monotonicity of the function $x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}$ on $(0, \infty)$.

Theorem 1. *The function*

$$x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}$$

is completely monotonic on $(0, \infty)$, with the limits

$$\lim_{x \rightarrow 0} \left(x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12} \right) = \frac{1}{12} \quad (1)$$

and

$$\lim_{x \rightarrow \infty} \left(x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12} \right) = 0. \quad (2)$$

2. A CONCISE PROOF OF THEOREM 1

In [11, p. 140, 5.9.13], it was listed that

$$\psi(z) = \ln z + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) e^{-tz} dt.$$

By integration by part, this formula can be reformulated as

$$\begin{aligned} \psi(x) - \ln x + \frac{1}{x} &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt \\ &= -\frac{1}{x} \left\{ [h(t)e^{-xt}] \Big|_{t=0}^{t=\infty} - \int_0^\infty h'(t)e^{-xt} dt \right\} \end{aligned}$$

$$= \frac{1}{2x} + \frac{1}{x} \int_0^{\infty} h'(t)e^{-xt} dt,$$

where the function

$$h(t) = \begin{cases} \frac{1}{t} - \frac{1}{e^t - 1}, & t \neq 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

satisfies

$$\lim_{t \rightarrow 0^+} h'(t) = -\frac{1}{12}, \quad \lim_{t \rightarrow \infty} h'(t) = 0, \quad (3)$$

and is concave on $(-\infty, 0)$ and convex on $(0, \infty)$. For information on the function $h(t)$, please refer to [16, Theorem 1] and [4, 5, 6, 7, 9, 12]. Consequently, integrating by part again yields

$$\begin{aligned} x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12} &= \frac{1}{12} + x \int_0^{\infty} h'(t)e^{-xt} dt \\ &= \frac{1}{12} - \int_0^{\infty} h'(t) d(e^{-xt}) = \int_0^{\infty} h''(t)e^{-xt} dt. \end{aligned}$$

Accordingly, by virtue of [15, p. 161, Theorem 12b] mentioned at the beginning of this paper, the convexity of the function $h(t)$ on $(0, \infty)$ implies that the function $x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}$ is completely monotonic on $(0, \infty)$, while the limits in (3) imply the limits (1) and (2) immediately. The proof of Theorem 1 is complete.

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