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COMPLETELY MONOTONIC DEGREES FOR A DIFFERENCE BETWEEN THE LOGARITHMIC AND PSI FUNCTIONS

FENG QI AND AI-QI LIU

ABSTRACT. In the paper, the authors firstly present a concise proof for complete monotonicity of a function involving a difference between the logarithmic and psi functions, secondly compute completely monotonic degree of the above-mentioned function, and finally pose several conjectures on completely monotonic degrees of remainders and their derivatives for the asymptotic formula of the logarithm of the classical Euler gamma function.

1. PRELIMINARIES

Recall from [16, Chapter XIII], [31, Chapter 1], and [32, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I, \quad n \in \{0\} \cup \mathbb{N}.$$

Theorem 12b in [32, p. 161] states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that

$$(1) \quad f(x) = \int_0^\infty e^{-xt} d\mu(t), \quad x \in (0, \infty),$$

where $\mu(t)$ is non-decreasing and the above integral converges for $x \in (0, \infty)$. In other words or simply speaking, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of a non-negative measure.

In [2, pp. 374–375, Theorem 1] and [8, Theorem 1], it was verified that the function

$$(2) \quad x^\alpha [\ln x - \psi(x)], \quad \alpha \in \mathbb{R}$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$, where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is called the psi function and $\Gamma(z)$ denotes the classical Euler gamma function which can be defined [1, 8, 18, 30] by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

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or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n!n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Let $f(x)$ be a completely monotonic function on $(0, \infty)$ and denote $f(\infty) = \lim_{x \rightarrow \infty} f(x) \geq 0$. When the function $x^\alpha[f(x) - f(\infty)]$ is completely monotonic on $(0, \infty)$ if and only if $0 \leq \alpha \leq r \in \mathbb{R}$, we say that the number r is the completely monotonic degree of $f(x)$ with respect to $x \in (0, \infty)$; if the function $x^\alpha[f(x) - f(\infty)]$ is completely monotonic on $(0, \infty)$ for all $\alpha \in \mathbb{R}$, then we say that the completely monotonic degree of $f(x)$ with respect to $x \in (0, \infty)$ is ∞ . For convenience, a notation $\deg_{\text{cm}}^x[f(x)]$ was designed to denote the completely monotonic degree r of $f(x)$ with respect to $x \in (0, \infty)$. For more information on completely monotonic degree and its properties, please refer to the papers [4, 10, 12, 13, 14, 15, 19, 20, 23, 24, 25, 26, 28, 29] and closely related references therein.

The necessary and sufficient condition $\alpha \leq 1$ for the function (2) to be completely monotonic on $(0, \infty)$ means that the completely monotonic degree of the completely monotonic function $\ln x - \psi(x)$ on $(0, \infty)$ is

$$\deg_{\text{cm}}^x[\ln x - \psi(x)] = 1.$$

In [30, Theorem 1.7], it was proved that the function

$$x^2[\psi(x) - \ln x] + \frac{x}{2}$$

is strictly decreasing and convex on $(0, \infty)$ and, as $x \rightarrow \infty$, tends to $-\frac{1}{12}$. In [3, Theorem 1], it was discovered that the function

$$(3) \quad \Phi(x) = x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}$$

is completely monotonic on $(0, \infty)$.

In the second section of this paper, motivated by the second proof of [8, Theorem 1] and with the aid of some conclusions in [33], we will present a concise proof, which is simpler and shorter than the corresponding proof in [3, Theorem 1], for complete monotonicity of the function $\Phi(x)$ on $(0, \infty)$.

In the third section of this paper, we will compute the completely monotonic degree of the completely monotonic function

$$(4) \quad \phi(x) = \frac{\Phi(x)}{x^2} = \psi(x) - \ln x + \frac{1}{2x} + \frac{1}{12x^2}$$

on $(0, \infty)$.

In the fourth section of this paper, we will pose conjectures on completely monotonic degrees of remainders for the asymptotic formula of $\ln \Gamma(x)$ and for polygamma functions $\psi^{(k)}(x)$ for $k \geq 0$.

2. COMPLETE MONOTONICITY OF THE FUNCTION $\Phi(x)$

The first main result in this paper can be stated as the following theorem.

Theorem 1. *The function $\Phi(x)$ defined in (3) is completely monotonic on $(0, \infty)$, with the limits*

$$(5) \quad \lim_{x \rightarrow 0^+} \Phi(x) = \frac{1}{12} \quad \text{and} \quad \lim_{x \rightarrow \infty} \Phi(x) = 0.$$

Proof. In [18, p. 140, 5.9.13], it was listed that

$$\psi(z) = \ln z + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) e^{-tz} dt.$$

By integration by parts, this formula can be reformulated as

$$\begin{aligned} \psi(x) - \ln x + \frac{1}{x} &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-xt} dt \\ &= -\frac{1}{x} \left\{ [h(t)e^{-xt}] \Big|_{t=0}^{t=\infty} - \int_0^\infty h'(t)e^{-xt} dt \right\} = \frac{1}{2x} + \frac{1}{x} \int_0^\infty h'(t)e^{-xt} dt, \end{aligned}$$

where the function

$$h(t) = \begin{cases} \frac{1}{t} - \frac{1}{e^t - 1}, & t \neq 0 \\ \frac{1}{2}, & t = 0 \end{cases}$$

is convex on $(0, \infty)$, with the limits

$$(6) \quad \lim_{t \rightarrow 0^+} h'(t) = -\frac{1}{12} \quad \text{and} \quad \lim_{t \rightarrow \infty} h'(t) = 0.$$

For detailed information on the function $h(t)$, please refer to [5, 6, 7, 21] and closely related references therein. Consequently, integrating by parts again yields

$$\Phi(x) = \frac{1}{12} + x \int_0^\infty h'(t)e^{-xt} dt = \frac{1}{12} - \int_0^\infty h'(t) d(e^{-xt}) = \int_0^\infty h''(t)e^{-xt} dt.$$

Accordingly, by virtue of [32, p. 161, Theorem 12b] mentioned at the beginning of this paper and with the help of the convexity for the function $h(t)$ on $(0, \infty)$, we immediately see that the function $\Phi(x)$ is completely monotonic on $(0, \infty)$, while, in light of the limits in (6), we readily derive the limits in (5). The proof of Theorem 1 is complete. \square

3. COMPLETELY MONOTONIC DEGREE OF THE FUNCTION $\phi(x)$

The second main result in this paper can be stated as the following theorem.

Theorem 2. *The completely monotonic degree of the completely monotonic function $\phi(x)$ defined in (4) on $(0, \infty)$ is*

$$(7) \quad \deg_{\text{cm}}^x[\phi(x)] = 2.$$

Proof. Since the relation $\Phi(x) = x^2\phi(x)$ and the function $\phi(x)$ is completely monotonic on $(0, \infty)$, see [22, Theorem 1], then Theorem 1 in [3] and Theorem 1 in this paper mean

$$(8) \quad \deg_{\text{cm}}^x[\phi(x)] \geq 2.$$

On the other hand side, if $x^\alpha\phi(x)$ is completely monotonic on $(0, \infty)$, then its first derivative is not positive, that is,

$$\alpha x^{\alpha-1}\phi(x) + x^\alpha\phi'(x) \leq 0$$

which can be rearranged as

$$\alpha \leq -\frac{x\phi'(x)}{\phi(x)} = -\frac{x^3[\psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}]}{x^2[\psi(x) - \ln x] + \frac{x}{2} + \frac{1}{12}} \rightarrow 2$$

for $x \rightarrow 0^+$, where we used the recursion relation $\psi'(x+1) = \psi'(x) - \frac{1}{x^2}$ and the first limit in (5). This means that

$$(9) \quad \deg_{\text{cm}}^x[\phi(x)] \leq 2.$$

Combining (8) with (9) concludes that the completely monotonic degree of the function $\phi(x)$ on $(0, \infty)$ is 2. The proof of Theorem 2 is complete. \square

4. CONJECTURES

The third main result in this paper is to pose conjectures on completely monotonic degrees of remainders for the asymptotic formula of $\ln \Gamma(x)$ and on completely monotonic degrees of derivatives of these remainders.

In [22, Theorem 1], among other things, the function $\phi(x)$ and three functions

$$\ln x - \frac{1}{2x} - \psi(x), \quad \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}, \quad \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \psi'(x)$$

were proved to be completely monotonic on $(0, \infty)$. What are the completely monotonic degrees of these three functions on $(0, \infty)$? We guess that

$$(10) \quad \deg_{\text{cm}}^x \left[\ln x - \frac{1}{2x} - \psi(x) \right] = 1,$$

$$(11) \quad \deg_{\text{cm}}^x \left[\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \psi'(x) \right] = 2,$$

$$(12) \quad \deg_{\text{cm}}^x \left[\psi'(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \right] = 4.$$

In [2, Theorem 8] and [11, Theorem 2], the functions

$$(13) \quad R_n(x) = (-1)^n \left[\ln \Gamma(x) - \left(x - \frac{1}{2} \right) \ln x + x - \frac{1}{2} \ln(2\pi) - \sum_{k=1}^n \frac{B_{2k}}{(2k-1)2k} \frac{1}{x^{2k-1}} \right]$$

for $n \geq 0$ were proved to be completely monotonic on $(0, \infty)$. Consequently, the functions $(-1)^m [R_n(x)]^{(m)}$ for $m, n \geq 0$ are completely monotonic on $(0, \infty)$. See also [9, Section 1.4] and [17, Theorem 3.1]. The functions $(-1)^n R_n(x)$ for $n \geq 0$ are remainders for the asymptotic formula of $\ln \Gamma(x)$. See [1, p. 257, 6.1.40] and [18, p. 140, 5.11.1].

In [15, Theorem 2.1], the completely monotonic degree of the function $R_n(x)$ for $n \geq 0$ on $(0, \infty)$ was proved to be at least n . The equations (7), (10), (11), and (12) and plotting by the well-known software Wolfram Mathematica imply that

$$\begin{aligned} \deg_{\text{cm}}^x [(-1)^0 [R_0(x)]^{(0)}] &= 0, & \deg_{\text{cm}}^x [(-1)^0 [R_1(x)]^{(0)}] &= 1, \\ \deg_{\text{cm}}^x [(-1)^0 [R_2(x)]^{(0)}] &= 2, & \deg_{\text{cm}}^x [(-1)^0 [R_3(x)]^{(0)}] &= 4, \\ \deg_{\text{cm}}^x [(-1)^0 [R_4(x)]^{(0)}] &= 6, & \deg_{\text{cm}}^x [(-1)^0 [R_5(x)]^{(0)}] &= 8, \\ \deg_{\text{cm}}^x [(-1)^1 [R_0(x)]^{(1)}] &= 1, & \deg_{\text{cm}}^x [(-1)^1 [R_1(x)]^{(1)}] &= 2, \\ \deg_{\text{cm}}^x [(-1)^1 [R_2(x)]^{(1)}] &= 3, & \deg_{\text{cm}}^x [(-1)^1 [R_3(x)]^{(1)}] &= 5, \\ \deg_{\text{cm}}^x [(-1)^1 [R_4(x)]^{(1)}] &= 7, & \deg_{\text{cm}}^x [(-1)^1 [R_5(x)]^{(1)}] &= 9, \\ \deg_{\text{cm}}^x [(-1)^2 [R_0(x)]^{(2)}] &= 1, & \deg_{\text{cm}}^x [(-1)^2 [R_1(x)]^{(2)}] &= 2, \\ \deg_{\text{cm}}^x [(-1)^2 [R_2(x)]^{(2)}] &= 4, & \deg_{\text{cm}}^x [(-1)^2 [R_3(x)]^{(2)}] &= 6, \end{aligned}$$

$$\begin{aligned}
\deg_{\text{cm}}^x [(-1)^2[R_4(x)]^{(2)}] &= 8, & \deg_{\text{cm}}^x [(-1)^2[R_5(x)]^{(2)}] &= 10, \\
\deg_{\text{cm}}^x [(-1)^3[R_0(x)]^{(3)}] &= 2, & \deg_{\text{cm}}^x [(-1)^3[R_1(x)]^{(3)}] &= 3, \\
\deg_{\text{cm}}^x [(-1)^3[R_2(x)]^{(3)}] &= 5, & \deg_{\text{cm}}^x [(-1)^3[R_3(x)]^{(3)}] &= 7, \\
\deg_{\text{cm}}^x [(-1)^3[R_4(x)]^{(3)}] &= 9, & \deg_{\text{cm}}^x [(-1)^3[R_5(x)]^{(3)}] &= 11, \\
\deg_{\text{cm}}^x [(-1)^4[R_0(x)]^{(4)}] &= 3, & \deg_{\text{cm}}^x [(-1)^4[R_1(x)]^{(4)}] &= 4, \\
\deg_{\text{cm}}^x [(-1)^4[R_2(x)]^{(4)}] &= 6, & \deg_{\text{cm}}^x [(-1)^4[R_3(x)]^{(4)}] &= 8, \\
\deg_{\text{cm}}^x [(-1)^4[R_4(x)]^{(4)}] &= 10, & \deg_{\text{cm}}^x [(-1)^4[R_5(x)]^{(4)}] &= 12, \\
\deg_{\text{cm}}^x [(-1)^5[R_0(x)]^{(5)}] &= 4, & \deg_{\text{cm}}^x [(-1)^5[R_1(x)]^{(5)}] &= 5, \\
\deg_{\text{cm}}^x [(-1)^5[R_2(x)]^{(5)}] &= 7, & \deg_{\text{cm}}^x [(-1)^5[R_3(x)]^{(5)}] &= 9, \\
\deg_{\text{cm}}^x [(-1)^5[R_4(x)]^{(5)}] &= 11, & \deg_{\text{cm}}^x [(-1)^5[R_5(x)]^{(5)}] &= 13.
\end{aligned}$$

What are the completely monotonic degrees of the completely monotonic functions $(-1)^m[R_n(x)]^{(m)}$ for $m, n \geq 0$ on $(0, \infty)$? We conjecture that

(i) when $m = 0$, the completely monotonic degrees of $R_n(x)$ on $(0, \infty)$ satisfy

$$\deg_{\text{cm}}^x[R_0(x)] = 0, \quad \deg_{\text{cm}}^x[R_1(x)] = 1,$$

and

$$\deg_{\text{cm}}^x[R_n(x)] = 2(n-1), \quad n \geq 2;$$

(ii) when $m = 1$, the completely monotonic degrees of $-R'_n(x)$ on $(0, \infty)$ satisfy

$$\deg_{\text{cm}}^x[-R'_0(x)] = 1, \quad \deg_{\text{cm}}^x[-R'_1(x)] = 2,$$

and

$$\deg_{\text{cm}}^x[-R'_n(x)] = 2n-1, \quad n \geq 2;$$

(iii) when $m \geq 2$, the completely monotonic degrees of $(-1)^m[R_n(x)]^{(m)}$ satisfy

$$\deg_{\text{cm}}^x[(-1)^m[R_0(x)]^{(m)}] = m-1, \quad \deg_{\text{cm}}^x[(-1)^m[R_1(x)]^{(m)}] = m,$$

and

$$\deg_{\text{cm}}^x[(-1)^m[R_n(x)]^{(m)}] = m+2(n-1), \quad n \geq 2.$$

5. REMARKS

Finally we give several remarks on the significance and importance of results obtained in this paper.

Remark 1. The significance of completely monotonic functions can be seen from the above equation (1) cited from [32, Theorem 12b].

Completely monotonic degree is a new notion introduced in very recent years. See [4, 10, 12, 13, 14, 15, 19, 20, 23, 24, 25, 26, 28, 29] and closely related references. This new notion can be used to measure and differentiate complete monotonicity more accurately. For example, the functions $\frac{1}{x^\alpha}$ and $\frac{1}{x^\beta}$ for $\alpha, \beta > 0$ and $\alpha \neq \beta$ are both completely monotonic on $(0, \infty)$, but they are different completely monotonic functions. How to quantitatively measure their difference? How to quantitatively differentiate them? The notion of completely monotonic degrees is useful: the completely monotonic degrees of these functions $\frac{1}{x^\alpha}$ and $\frac{1}{x^\beta}$ for $\alpha, \beta > 0$ and $\alpha \neq \beta$ are $\alpha > 0$ and $\beta > 0$ respectively.

On the other hand, the classical gamma function $\Gamma(z)$ and polygamma functions $\psi^{(k)}$ for $k \geq 0$ and $x > 0$ are fundamental in mathematics and mathematical sciences. The asymptotic expansions of $\ln \Gamma(z)$ and $\psi^{(k)}$ for $k \geq 0$ have extensive applications in almost all mathematical sciences. The remainders $(-1)^m [R_n(x)]^{(m)}$ for $m, n \geq 0$ in (13) have been proved to be completely monotonic on $(0, \infty)$ by several mathematicians in Germany, Denmark, Romania, Cyprus, China, and so on. How to quantitatively differentiate complete monotonicity of these remainders? This is the main contribution of this paper.

The functions $\ln x - \psi(x)$ and $\psi(x) - \ln x + \frac{1}{2x} + \frac{1}{12x^2}$, which are simpler but applied extensively, and their complete monotonicity on $(0, \infty)$ have been studied for many years by many mathematicians in many top mathematical journals. In this paper, we tried and succeeded to strengthen and differentiate these known conclusions.

What are motivations of advances for pure mathematics? Questions! Problems! Guesses! Conjectures! Hypotheses! Therefore, it is very important to pose correct and significant problems for pure mathematics. To pose correct and significant problems is difficult, even very difficult, because it needs deep and intelligent insight into mathematics. We believe that we posed correct and significant problems in this paper.

Remark 2. This paper is a largely expanded version of the preprint [27].

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REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.
- [2] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), no. 217, 373–389; available online at <https://doi.org/10.1090/S0025-5718-97-00807-7>.
- [3] C.-P. Chen, F. Qi, and H. M. Srivastava, *Some properties of functions related to the gamma and psi functions*, Integral Transforms Spec. Funct. **21** (2010), no. 2, 153–164; available online at <https://doi.org/10.1080/10652460903064216>.
- [4] B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897; available online at <https://doi.org/10.1016/j.amc.2012.03.075>.
- [5] B.-N. Guo and F. Qi, *A simple proof of logarithmic convexity of extended mean values*, Numer. Algorithms **52** (2009), no. 1, 89–92; available online at <https://doi.org/10.1007/s11075-008-9259-7>.
- [6] B.-N. Guo and F. Qi, *Properties and applications of a function involving exponential functions*, Commun. Pure Appl. Anal. **8** (2009), no. 4, 1231–1249; available online at <https://doi.org/10.3934/cpaa.2009.8.1231>.
- [7] B.-N. Guo and F. Qi, *The function $(b^x - a^x)/x$: Logarithmic convexity and applications to extended mean values*, Filomat **25** (2011), no. 4, 63–73; available online at <https://doi.org/10.2298/FIL1104063G>.
- [8] B.-N. Guo and F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function*, Bull. Korean Math. Soc. **47** (2010), no. 1, 103–111; available online at <https://doi.org/10.4134/bkms.2010.47.1.103>.
- [9] B.-N. Guo, F. Qi, J.-L. Zhao, and Q.-M. Luo, *Sharp inequalities for polygamma functions*, Math. Slovaca **65** (2015), no. 1, 103–120; available online at <https://doi.org/10.1515/ms-2015-0010>.

- [10] S. Koumandos, *Monotonicity of some functions involving the gamma and psi functions*, Math. Comp. **77** (2008), no. 264, 2261–2275; available online at <https://doi.org/10.1090/S0025-5718-08-02140-6>.
- [11] S. Koumandos, *Remarks on some completely monotonic functions*, J. Math. Anal. Appl. **324** (2006), no. 2, 1458–1461; available online at <http://dx.doi.org/10.1016/j.jmaa.2005.12.017>.
- [12] S. Koumandos and M. Lamprecht, *Some completely monotonic functions of positive order*, Math. Comp. **79** (2010), no. 271, 1697–1707; available online at <https://doi.org/10.1090/S0025-5718-09-02313-8>.
- [13] S. Koumandos and M. Lamprecht, *Complete monotonicity and related properties of some special functions*, Math. Comp. **82** (2013), no. 282, 1097–1120; available online at <https://doi.org/10.1090/S0025-5718-2012-02629-9>.
- [14] S. Koumandos and H. L. Pedersen, *Absolutely monotonic functions related to Euler’s gamma function and Barnes’ double and triple gamma function*, Monatsh. Math. **163** (2011), no. 1, 51–69; available online at <https://doi.org/10.1007/s00605-010-0197-9>.
- [15] S. Koumandos and H. L. Pedersen, *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function*, J. Math. Anal. Appl. **355** (2009), no. 1, 33–40; available online at <https://doi.org/10.1016/j.jmaa.2009.01.042>.
- [16] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
- [17] C. Mortici, *Very accurate estimates of the polygamma functions*, Asympt. Anal. **68** (2010), no. 3, 125–134; available online at <http://dx.doi.org/10.3233/ASY-2010-0983>.
- [18] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; available online at <http://dlmf.nist.gov/>.
- [19] F. Qi, *Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions*, Math. Inequal. Appl. **18** (2015), no. 2, 493–518; available online at <https://doi.org/10.7153/mia-18-37>.
- [20] F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at <https://doi.org/10.1186/s13660-019-1976-z>.
- [21] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory **159** (2016), 89–100; available online at <https://doi.org/10.1016/j.jnt.2015.07.021>.
- [22] F. Qi, R.-Q. Cui, C.-P. Chen, and B.-N. Guo, *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl. **310** (2005), no. 1, 303–308; available online at <http://dx.doi.org/10.1016/j.jmaa.2005.02.016>.
- [23] F. Qi and B.-N. Guo, *Lévy–Khintchine representation of Toader–Qi mean*, Math. Inequal. Appl. **21** (2018), no. 2, 421–431; available online at <https://doi.org/10.7153/mia-2018-21-29>.
- [24] F. Qi and B.-N. Guo, *The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function*, Quaest. Math. **41** (2018), no. 5, 653–664; available online at <https://doi.org/10.2989/16073606.2017.1396508>.
- [25] F. Qi and W.-H. Li, *Integral representations and properties of some functions involving the logarithmic function*, Filomat **30** (2016), no. 7, 1659–1674; available online at <https://doi.org/10.2298/FIL1607659Q>.
- [26] F. Qi and D. Lim, *Integral representations of bivariate complex geometric mean and their applications*, J. Comput. Appl. Math. **330** (2018), 41–58; available online at <https://doi.org/10.1016/j.cam.2017.08.005>.
- [27] F. Qi and A.-Q. Liu, *Notes on complete monotonicity related to the difference of the psi and logarithmic functions*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01728682>.
- [28] F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy–Khintchine representations of the weighted geometric mean and the logarithmic mean*, Mediterr. J. Math. **11** (2014), no. 2, 315–327; available online at <https://doi.org/10.1007/s00009-013-0311-z>.

- [29] F. Qi, X.-J. Zhang, and W.-H. Li, *The harmonic and geometric means are Bernstein functions*, Bol. Soc. Mat. Mex. (3) **23** (2017), no. 2, 713–736; available online at <https://doi.org/10.1007/s40590-016-0085-y>.
- [30] S.-L. Qiu and M. Vuorinen, *Some properties of the gamma and psi functions, with applications*, Math. Comp. **74** (2005), no. 250, 723–742; available online at <https://doi.org/10.1090/S0025-5718-04-01675-8>.
- [31] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions—Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
- [32] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [33] S.-Q. Zhang, B.-N. Guo, and F. Qi, *A concise proof for properties of three functions involving the exponential function*, Appl. Math. E-Notes **9** (2009), 177–183.

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