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FIXED POINTS IN SMOOTH CALOGERO-MOSER SPACES

by

CÉDRIC BONNAFÉ & RUSLAN MAKSIMAU

Abstract. — We prove that every irreducible component of the fixed point variety under the action of μ_d in a smooth Calogero-Moser space is isomorphic to a Calogero-Moser space associated with another reflection group.

Consider a finite subgroup $W \subset \mathbf{GL}_{\mathbb{C}}(V)$ of automorphisms of a finite dimensional complex vector space V , generated by reflections and let c be a complex valued function on the set of conjugacy classes of reflections of W . To the triple (V, W, c) , Etingof and Ginzburg [8] have associated an affine variety $\mathcal{Z}_c(V, W)$, the *Calogero-Moser space*, which is defined as the spectrum of the center of the rational Cherednik algebra \mathbf{H}_c “at $t = 0$ ” (see Section 1 for a precise definition). The algebra \mathbf{H}_c carries a \mathbb{Z} -grading (i.e. a \mathbb{C}^{\times} -action), which induces a \mathbb{C}^{\times} -action on the Calogero-Moser space $\mathcal{Z}_c(V, W)$.

We define a *reflection subquotient* of (V, W) to be a pair (V', W') where V' is a subspace of V , $W' \subset \mathbf{GL}_{\mathbb{C}}(V')$ is generated by reflections and there exists a subgroup H of W which stabilizes V' and whose image in $\mathbf{GL}_{\mathbb{C}}(V')$ is exactly W' .

Note that any element in $\mathbf{GL}_{\mathbb{C}}(V)$ normalizing W acts on $\mathcal{Z}_c(V, W)$. In [4, Conjecture FIX], Rouquier and the first author proposed the following conjecture:

Conjecture F. *Let $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ be an element of finite order and normalizing W and let \mathcal{X} be an irreducible component of $\mathcal{Z}_c(V, W)^{\sigma}$ (endowed with its reduced scheme structure). Then there exists a reflection subquotient (V', W') of W and a complex valued function c' on the set of conjugacy classes of reflections in W' such that $\mathcal{X} \simeq \mathcal{Z}_{c'}(V', W')$, as varieties endowed with a \mathbb{C}^{\times} -action.*

Note that the conjecture stated in [4, Conjecture FIX] does not give an explicit description of c' in terms of c : it is just mentioned that the map $c \mapsto c'$ should be linear. Note also that, as stated, it might be a little bit optimistic: maybe one should replace \mathcal{X} by its normalization (as it is not clear whether \mathcal{X} is normal or not). A special case of an element σ normalizing W is when σ is a root of unity, viewed as the corresponding homothety on V . The aim of this paper is to prove the following result:

Theorem. *Conjecture F holds if $\mathcal{Z}_c(V, W)$ is smooth and σ is a root of unity.*

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The proof is by case-by-case analysis, as the triples (V, W, c) such that $\mathcal{Z}_c(V, W)$ is smooth are classified (see the works of Etingof-Ginzburg [8], Gordon [10], Bellamy [1]). The classification and the conjecture can be easily reduced to the case where W acts irreducibly on V/V^W and in this case, the smoothness of $\mathcal{Z}_c(V, W)$ implies that W is of type $G(l, 1, n)$ or G_4 in Shephard-Todd classification. The case of G_4 can be handled by computer calculations (see Section 5), while the infinite family case will be handled by using an isomorphism between $\mathcal{Z}_c(V, W)$ and a quiver variety.

1. Notation and main result

All along this note, we will abbreviate $\otimes_{\mathbb{C}}$ as \otimes . By an algebraic variety, we mean a reduced scheme of finite type over \mathbb{C} .

Set-up. We fix in this paper a \mathbb{C} -vector space V of finite dimension n and a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\text{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}$$

and we assume that

$$W = \langle \text{Ref}(W) \rangle.$$

We also fix an element $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ of finite order normalizing W .

We set $\varepsilon : W \rightarrow \mathbb{C}^{\times}$, $w \mapsto \det(w)$. If $s \in \text{Ref}(W)$, we denote by α_s^{\vee} and α_s two elements of V and V^* respectively such that $V^s = \text{Ker}(\alpha_s)$ and $V^{*s} = \text{Ker}(\alpha_s^{\vee})$, where α_s^{\vee} is viewed as a linear form on V^* .

1.A. Rational Cherednik algebra at $t = 0$. — All along this note, we fix a function $c : \text{Ref}(W) \rightarrow \mathbb{C}$ which is invariant under conjugacy. We define the \mathbb{C} -algebra \mathbf{H}_c to be the quotient of the algebra $T(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $T(V \oplus V^*)$ with the group W) by the relations

$$(\mathcal{H}_c) \quad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \sum_{s \in \text{Ref}(W)} (\varepsilon(s) - 1) c_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^{\vee}, x \rangle}{\langle \alpha_s^{\vee}, \alpha_s \rangle} s, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. The algebra \mathbf{H}_c is called the *rational Cherednik algebra at $t = 0$* .

The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \rightarrow \mathbf{H}_c$ and $\mathbb{C}[V^*] \rightarrow \mathbf{H}_c$. Recall [8, Corollary 4.4] that we have an isomorphism of \mathbb{C} -vector spaces

$$(1.1) \quad \mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_c$$

induced by multiplication (this is the so-called *PBW-decomposition*).

We denote by \mathbf{Z}_c the center of \mathbf{H}_c : it is well-known [8, Lemma 3.5] that \mathbf{Z}_c is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes_{\mathbb{C}} \mathbb{C}[V^*]^W$), and which is a free \mathbf{P} -module of rank $|W|$. We denote by \mathcal{Z}_c the algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_c]$ is \mathbf{Z}_c : this is the *Calogero-Moser space* associated with the datum (V, W, c) . If necessary, we will write $\mathcal{Z}_c(V, W)$ for \mathcal{Z}_c .

Example 1.2. — Let $l \geq 1$, let ζ be a primitive l -th root of unity and let $\mu_l = \langle \zeta \rangle$. We assume in this example that $n = 1$ and $W = \langle t \rangle \simeq \mu_l$, where $t(v) = \zeta v$ for all $v \in V$. Then $\text{Ref}(W) = \{t, t^2, \dots, t^{l-1}\}$ and we set for simplification

$$k_j = \frac{1}{l} \sum_{i=1}^{l-1} \zeta^{-i(j-1)} c_{ti}$$

($0 \leq j \leq l-1$). Note that $k_0 + k_1 + \dots + k_{l-1} = 0$. Then

$$\mathcal{Z}_c \simeq \{(x, y, e) \in \mathbb{C}^3 \mid \prod_{i=0}^{l-1} (e - lk_i) = xy\}$$

(see for instance [4, Theorem 18.2.2]). In particular, \mathcal{Z}_c is smooth if and only if

$$\prod_{0 \leq i < j \leq l-1} (k_i - k_j) \neq 0.$$

Note also that the inclusion $\mathbf{P} \subset \mathbf{Z}_c$ corresponds to the morphism of algebraic varieties $\mathcal{Z}_c \rightarrow \mathbb{C}^2, (x, y, e) \mapsto (x, y)$.

The \mathbb{C}^\times -action is given by the formula

$$\xi \cdot (x, y, e) = (\xi^l x, \xi^{-l} y, e),$$

where $\xi \in \mathbb{C}^\times$ and $(x, y, e) \in \mathcal{Z}_c$. ■

1.B. Action of the normalizer. — The normalizer $N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ of W in $\mathbf{GL}_{\mathbb{C}}(V)$ acts on V naturally, on V^* by the contragredient action and on W by conjugation. This endows $T(V \oplus V^*) \rtimes W$ with an action of $N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ and it is easily checked that the bunch of relations (\mathcal{H}_c) is stable under this action. So \mathbf{H}_c inherits an action of $N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$. In particular, its centre \mathbf{Z}_c also inherits such an action, and this defines an action of $N_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ on the Calogero-Moser space \mathcal{Z}_c .

Theorem 1.3. — Assume that \mathcal{Z}_c is smooth and that σ is a root of unity, and let \mathcal{X} be an irreducible component of \mathcal{Z}_c^σ . Then there exists a reflection subquotient (V', W') of (V, W) and a complex-valued map c' on the set of conjugacy classes of reflections of W' such that

$$\mathcal{X} \simeq \mathcal{Z}_{c'}(V', W'),$$

as varieties endowed with a \mathbb{C}^\times -action.

Remark 1.4. — Note that if \mathcal{X} is reduced to a point, then the theorem is easy because in this case, $\mathcal{X} \simeq \mathcal{Z}_0(0, 1)$. ■

This Theorem will be proved in Sections 3 and 5.

1.C. Filtration of the group algebra. — If $w \in W$, we set $\text{cod}(w) = \text{codim}_{\mathbb{C}}(V^w)$. We define a filtration $\mathcal{F}_{\bullet}(\mathbb{C}W)$ of the group algebra of W as follows: let

$$\mathcal{F}_i(\mathbb{C}W) = \bigoplus_{\text{cod}(w) \leq i} \mathbb{C}w.$$

Then

$$\mathbb{C}\text{Id}_V = \mathcal{F}_0(\mathbb{C}W) \subset \mathcal{F}_1(\mathbb{C}W) \subset \dots \subset \mathcal{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathcal{F}_{n+1}(\mathbb{C}W) = \dots$$

is a filtration of $\mathbb{C}W$. For any subalgebra A of $\mathbb{C}W$, we set $\mathcal{F}_i(A) = A \cap \mathcal{F}_i(\mathbb{C}W)$, so that

$$\mathbb{C}\text{Id}_V = \mathbb{C}\mathcal{F}_0(A) \subset \mathcal{F}_1(A) \subset \dots \subset \mathcal{F}_n(A) = A = \mathcal{F}_{n+1}(A) = \dots$$

is also a filtration of A .

Now, assume that $\mathcal{Z}_c(V, W)$ is smooth, let σ be a root of unity and let \mathcal{X} be an irreducible component of $\mathcal{Z}_c(V, W)^{\sigma}$. According to Theorem 1.3, there exists a reflection subquotient (V', W') of (V, W) , a complex-valued map c' on the set of conjugacy classes of reflections of W' and a \mathbb{C}^{\times} -equivariant isomorphism $i : \mathcal{Z}_{c'}(V', W') \xrightarrow{\sim} \mathcal{X}$. We will view i as a closed immersion $i : \mathcal{Z}_{c'}(V', W') \hookrightarrow \mathcal{Z}_c(V, W)$. By [10, Corollary 5.8], the smoothness of $\mathcal{Z}_c(V, W)$ implies that there exists a bijection between $\text{Irr}(W)$ and $\mathcal{Z}_c(V, W)^{\mathbb{C}^{\times}}$. As the fixed point variety of a finite group in a smooth complex algebraic variety is still smooth, this means that \mathcal{X} is smooth, and so $\mathcal{Z}_{c'}(V', W')$ is smooth. Applying again [10, Corollary 5.8], we get another bijection between $\text{Irr}(W')$ and $\mathcal{Z}_{c'}(V', W')^{\mathbb{C}^{\times}}$. As $\mathcal{X}^{\mathbb{C}^{\times}} \subset \mathcal{Z}_c(V, W)^{\mathbb{C}^{\times}}$, this gives an injective map $i_{\mathcal{X}} : \text{Irr}(W') \hookrightarrow \text{Irr}(W)$, depending on \mathcal{X} and the choice of i . This allows to define a surjective morphism of algebras

$$i_{\mathcal{X}}^* : Z(\mathbb{C}W) \longrightarrow Z(\mathbb{C}W')$$

as follows: if $\chi \in \text{Irr}(W)$, let e_{χ}^W denote the corresponding primitive central idempotent of $\mathbb{C}W$ and set

$$i_{\mathcal{X}}^*(e_{\chi}^W) = \begin{cases} e_{\chi'}^{W'} & \text{if } \chi' \in \text{Irr}(W') \text{ is such that } i_{\mathcal{X}}(\chi') = \chi, \\ 0 & \text{if } \chi \text{ does not belong to the image of } i_{\mathcal{X}}. \end{cases}$$

Using results of Shan and the first author [5], Theorem 1.3 has the following consequence:

Corollary 1.5. — *If $i \geq 0$, then $i_{\mathcal{X}}^*(\mathcal{F}_i(Z(\mathbb{C}W))) \subset \mathcal{F}_i(Z(\mathbb{C}W'))$.*

Proof. — If \mathcal{Y} is a complex algebraic variety endowed with a \mathbb{C}^{\times} -action, we denote by $H_{\mathbb{C}^{\times}}^i(\mathcal{Y})$ the i -th group of equivariant cohomology, with coefficients in \mathbb{C} . Let \hbar be an indeterminate and identify $H_{\mathbb{C}^{\times}}(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ as usual, with \hbar homogeneous of degree 2. Since $\mathcal{Z}_c(V, W)$ is smooth, it follows from [5, Theorem A] that

$$\begin{cases} H_{\mathbb{C}^{\times}}^{2i+1}(\mathcal{Z}_c(V, W)) = 0 & \text{if } i \geq 0, \\ H_{\mathbb{C}^{\times}}^{2\bullet} \simeq \text{Rees}_{\mathcal{F}}(Z(\mathbb{C}W)) & \text{as } \mathbb{C}[\hbar]\text{-algebras,} \end{cases}$$

where $\text{Rees}_{\mathcal{F}}(Z(\mathbb{C}W)) = \bigoplus_{i \geq 0} \hbar^i \mathcal{F}_i(Z(\mathbb{C}W)) \subset \mathbb{C}[\hbar] \otimes_{\mathbb{C}} Z(\mathbb{C}W)$ is the Rees algebra associated with the filtration $\mathcal{F}_{\bullet}(Z(\mathbb{C}W))$. Using the analogous result for $\mathcal{Z}_{c'}(V', W')$ and the functoriality of equivariant cohomology, we get a morphism of algebras $i^* : \text{Rees}_{\mathcal{F}}(Z(\mathbb{C}W)) \longrightarrow \text{Rees}_{\mathcal{F}}(Z(\mathbb{C}W'))$.

Now, we will use not only [5, Theorem A] but also its proof: the proof goes by restriction to the fixed point subvariety $\mathcal{Z}_c(V, W)^{\mathbb{C}^{\times}}$ and identifying $H_{\mathbb{C}^{\times}}^{2\bullet}(\mathcal{Z}_c(V, W)^{\mathbb{C}^{\times}})$ with $\mathbb{C}[\hbar] \otimes Z(\mathbb{C}W)$ using [10, Corollary 5.8]. As the same strategy holds for $\mathcal{Z}_{c'}(V', W')$, the

functoriality of equivariant cohomology implies that the map i^* fits into a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Rees}_{\mathcal{F}}(Z(\mathbb{C}W)) & \xrightarrow{i^*} & \mathrm{Rees}_{\mathcal{F}}(Z(\mathbb{C}W')) \\
 \downarrow & & \downarrow \\
 \mathbb{C}[\hbar] \otimes_{\mathbb{C}} Z(\mathbb{C}W) & \xrightarrow{\mathrm{Id}_{\mathbb{C}[\hbar]} \otimes_{\mathbb{C}} i^*} & \mathbb{C}[\hbar] \otimes_{\mathbb{C}} Z(\mathbb{C}W').
 \end{array}$$

This shows the corollary. \square

2. Preliminaries on quiver varieties

2.A. Quiver varieties. — Let Q_l denote the cyclic quiver with l vertices, defined as follows:

- Vertices: $i \in \mathbb{Z}/l\mathbb{Z}$;
- Arrows: $y_i : i \rightarrow i+1, i \in \mathbb{Z}/l\mathbb{Z}$.

We denote by \overline{Q}_l the double quiver of Q_l that is, the quiver obtained from Q_l by adding an arrow $x_i : i+1 \rightarrow i$ for all $i \in \mathbb{Z}/l\mathbb{Z}$ (see Figure 1).

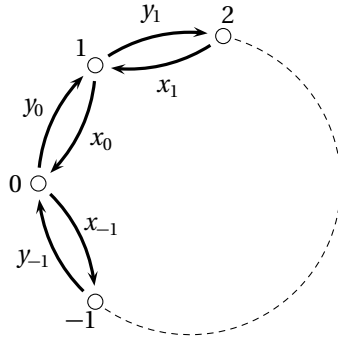


Fig. 1. The quiver \overline{Q}_l

Now, if $d = (d_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ is a family of elements of \mathbb{N} , we denote by $\mathbf{GL}(d)$ the direct product

$$\mathbf{GL}(d) = \prod_{i \in \mathbb{Z}/l\mathbb{Z}} \mathbf{GL}_{d_i}(\mathbb{C}),$$

by $\Delta\mathbb{C}^\times$ the image of \mathbb{C}^\times in $\mathbf{GL}(d)$ through the diagonal embedding $\Delta : \mathbb{C}^\times \hookrightarrow \mathbf{GL}(d)$ and we set

$$\mathbf{PGL}(d) = \mathbf{GL}(d) / \Delta\mathbb{C}^\times.$$

The group $\mathbf{PGL}(d)$ acts on the variety $\mathrm{Rep}(\overline{Q}_l, d)$ of representations of \overline{Q}_l in the family of vector spaces $(\mathbb{C}^{d_i})_{i \in \mathbb{Z}/l\mathbb{Z}}$. The orbits are the isomorphism classes of representations of \overline{Q}_l of dimension vector d . We denote by

$$\begin{aligned}
 \mu_d : \quad \mathrm{Rep}(\overline{Q}_l, d) & \longrightarrow \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} \mathrm{Mat}_{d_i}(\mathbb{C}) \\
 (X_i, Y_i)_{i \in \mathbb{Z}/l\mathbb{Z}} & \longmapsto (X_i Y_i - Y_{i-1} X_{i-1})_{i \in \mathbb{Z}/l\mathbb{Z}}
 \end{aligned}$$

the corresponding *moment map*. Finally, if $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ is a family of complex numbers, we denote by $I_\theta(d)$ the family $(\theta_i \text{Id}_{\mathbb{C}^{d_i}})_{i \in \mathbb{Z}/l\mathbb{Z}}$ and by $\mathcal{O}_\theta(d)$ the closed subvariety of $\text{Mat}_{d_0}(\mathbb{C})$ consisting of matrices of rank ≤ 1 and trace $-\sum_{i \in \mathbb{Z}/l\mathbb{Z}} \theta_i d_i$. Finally, we set

$$\mathcal{Y}_\theta(d) = \mu_d^{-1}(I_\theta(d) + \mathcal{O}_\theta(d)) \quad \text{and} \quad \mathcal{X}_\theta(d) = \mathcal{Y}_\theta(d) // \mathbf{PGL}(d).$$

We will usually denote an element $(X_i, Y_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ of $\mathcal{Y}_\theta(d)$ by (X, Y) , where $X = (X_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$ and $Y = (Y_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$. Note that $\mathcal{Y}_\theta(d)$ is endowed with a \mathbb{C}^\times -action: if $\xi \in \mathbb{C}^\times$, we set

$$\xi \cdot (X, Y) = (\xi^{-1}X, \xi Y).$$

This action commutes with the action of $\mathbf{GL}(d)$ and the moment map is constant on \mathbb{C}^\times -orbits, so it induces a \mathbb{C}^\times -action on $\mathcal{X}_\theta(d)$.

Remark 2.1. — We extend the definition of $\mathcal{X}_\theta(d)$ to the case where $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ by the convention that $\mathcal{X}_\theta(d) = \emptyset$ whenever at least one of the d_i 's is negative. ■

2.B. Action of the affine Weyl group. — If $l \geq 2$, let W_l^{aff} denote the affine Weyl group of type \tilde{A}_{l-1} . It is the Coxeter group with associated Coxeter system $(W_l^{\text{aff}}, S_l^{\text{aff}})$, where $S_l^{\text{aff}} = \{s_i \mid i \in \mathbb{Z}/l\mathbb{Z}\}$ and the Coxeter graph is given by

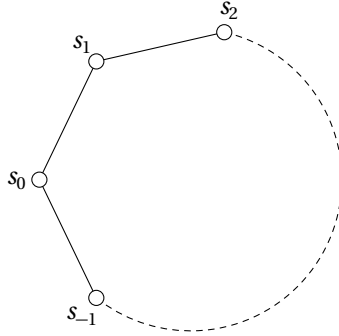


Fig. 2. Coxeter graph of $(W_l^{\text{aff}}, S_l^{\text{aff}})$

We extend this notation to the case where $l = 1$ by setting $W_1^{\text{aff}} = 1$.

Consider the Lie algebra $\mathfrak{g}_l = \mathfrak{sl}_l(\mathbb{C})$ and its affine version $\widehat{\mathfrak{g}}_l = \widehat{\mathfrak{sl}}_l(\mathbb{C}) = \mathfrak{sl}_l(\mathbb{C})[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra formed by the diagonal matrices and set $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$.

The \mathbb{C} -vector space $\widehat{\mathfrak{h}}^*$ has a basis $(\alpha_0, \alpha_1, \dots, \alpha_{l-1}, \Lambda_0)$, where $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ are the simple roots of $\widehat{\mathfrak{g}}_l$ and Λ_0 is such that Λ_0 annihilates \mathfrak{h} and ∂ and $\Lambda_0(\mathbf{1}) = 1$. Denote by R_l^{aff} and R_l the affine and the non-affine root lattices respectively (i.e., R_l^{aff} is the \mathbb{Z} -lattice generated by $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ and R_l is the sublattice generated by $\alpha_1, \dots, \alpha_{l-1}$.)

Following [17], we define two actions of W_l^{aff} : a non-linear one on $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, and a linear one on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$. If $l = 1$, there is nothing to define so we may assume that $l \geq 2$. If $d = (d_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ and if $j \in \mathbb{Z}/l\mathbb{Z}$, we set $s_j(d) = (d'_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$, where

$$d'_i = \begin{cases} d_i & \text{if } i \neq j, \\ \delta_{j0} + d_{i+1} + d_{i-1} - d_i & \text{if } i = j. \end{cases}$$

Remark 2.2. — We can identify $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ with the root lattice R_l^{aff} by

$$d \mapsto \sum_{i \in \mathbb{Z}/l\mathbb{Z}} d_i \alpha_i.$$

Beware, the action considered here is not the usual action of W_l^{aff} on the root lattice. When we have $w(d) = d'$ with respect to the action define above, this corresponds to $w(\Lambda_0 - d) = \Lambda_0 - d'$ for the usual action of W_l^{aff} on $\widehat{\mathfrak{h}}^*$. ■

If $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we set $s_j(\theta) = (\theta'_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$, where

$$\theta'_i = \begin{cases} \theta_i & \text{if } i \notin \{j-1, j, j+1\}, \\ \theta_j + \theta_i & \text{if } i \in \{j-1, j+1\}, \\ -\theta_i & \text{if } i = j. \end{cases}$$

It is readily seen that these definitions on generators extend to an action of the whole group W_l^{aff} . We also define a pairing $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}} \rightarrow \mathbb{C}$, $(d, \theta) \mapsto d \cdot \theta$, where

$$d \cdot \theta = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} d_i \theta_i.$$

Then

$$(2.3) \quad s_j(d) \cdot s_j(\theta) = (d \cdot \theta) - \delta_{j0} \theta_0.$$

It is proved in [17, Corollary 3.6] that

$$(2.4) \quad \mathcal{X}_{s_j(\theta)}(s_j(d)) \simeq \mathcal{X}_\theta(d) \quad \text{if } \theta_j \neq 0.$$

Note that this isomorphism takes into account the convention of Remark 2.1.

Recall also from [8, §11] the following result, which follows from the fact that $\mathcal{X}_\theta(n\delta_l)$ is isomorphic to some Calogero-Moser space.

Lemma 2.5. — *If $n \geq 0$, then $\mathcal{X}_\theta(n\delta_l)$ is normal and irreducible of dimension $2n$.*

Recall that the affine Weyl group has another presentation. We have $W_l^{\text{aff}} = W_l \ltimes R_l$, where $W_l = \mathfrak{S}_l$ is the non-affine Weyl group, that is, the subgroup generated by s_1, \dots, s_{l-1} . For each $\alpha \in R_l$, denote by t_α the image of α in W_l^{aff} . Each element of W_l^{aff} can be written in a unique way in the form $w \cdot t_\alpha$, where $w \in W_l$ and $\alpha \in R_l$. Let δ_l denote the constant family $\delta = (1)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. In the following lemma we identify $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ with R_l^{aff} .

Lemma 2.6. — *Assume $\alpha \in R_l$ and $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. Then we have $t_\alpha(d) \equiv d - \alpha \pmod{\mathbb{Z}\delta_l}$.*

Proof. — This statement is a partial case of [13, (6.5.2)] (see also Remark 2.2). □

Lemma 2.7. — *For each $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, there exists a unique $n \in \mathbb{Z}$ such that d and $n\delta_l$ are in the same W_l^{aff} -orbit.*

Proof. — Let us prove the existence of n . Set $\alpha = d - d_0\delta_l$. Then α is clearly an element of R_l . Then by Lemma 2.6, the element $t_\alpha(d)$ is of the form $n\delta_l$.

Let us prove the uniqueness of n . Assume that $n_1\delta_l$ and $n_2\delta_l$ are in the same W_l^{aff} -orbit. Since $n_1\delta_l$ is W_l -stable, the W_l^{aff} -orbit of $n_1\delta_l$ coincides with the R_l -orbit of $n_1\delta_l$. So there exists $\alpha \in R_l$ such that $t_\alpha(n_1\delta_l) = n_2\delta_l$. By Lemma 2.6, this forces $\alpha = 0$ and $n_1 = n_2$. \square

Consider the \mathbb{Z} -linear map

$$R_l^{\text{aff}} \rightarrow \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}, \quad d \mapsto \bar{d},$$

given by

$$(\bar{\alpha}_r)_i = 2\delta_{i,r} - \delta_{i,r+1} - \delta_{i,r-1}.$$

The kernel of this map is $\mathbb{Z}\delta_l$. Set

$$\Sigma(\theta) = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} \theta_i.$$

Lemma 2.8. — *For each $\alpha \in R_l$ and $\theta \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we have $t_\alpha(\theta) = \theta + \Sigma(\theta)\bar{\alpha}$.*

Proof. — The W_l^{aff} -action on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ defined above coincides with the (usual) action of W_l^{aff} on the dual of the span of $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ in $\hat{\mathfrak{h}}^*$. The statement follows from [13, (6.5.2)]. \square

We denote by $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ the set of $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ which belong to the orbit of some $n\delta_l$, with $n \geq 0$. This set has a more precise combinatorial description. For example, it will follow from Proposition 4.4 that an element $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if d is a residue of some Young diagram. In particular, the set $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ is contained in $\mathbb{N}^{\mathbb{Z}/l\mathbb{Z}}$.

Lemma 2.9. — *Assume that $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is such that there exists a simple representation in $\mathcal{Y}_\theta(d)$. Then*

- (a) *the variety $\mathcal{X}_\theta(d)$ is normal and irreducible,*
- (b) *we have $d \in (\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$.*

Proof. — Let $n \in \mathbb{Z}$ and $w \in W_l^{\text{aff}}$ be such that $n\delta_l = w(d)$. Let us show that (2.4) implies an isomorphism $\mathcal{X}_\theta(d) \simeq \mathcal{X}_{w(\theta)}(n\delta_l)$. (In particular, we must have $n \geq 0$.) The only difficulty that we have is caused by the condition $\theta_i \neq 0$ in (2.4). But [7, Lemma 7.2] implies that we can find a sequence of reflections such that $s_{i_1} s_{i_2} \dots s_{i_r}(d) = n\delta_l$ and such that we can apply a sequence of isomorphisms (2.4) with no danger to get $\theta_{i_j} = 0$ while applying s_{i_j} . \square

2.C. Smoothness. — The following result is proved in [11, Lemma 4.3 and its proof] (which is based on [15, Theorem 1.2] and [7, Theorem 1.2]):

Theorem 2.10. — *Let $n \geq 0$. Then the following are equivalent:*

- (1) *The variety $\mathcal{X}_\theta(n\delta_l)$ is smooth.*
- (2) *Every element of $\mathcal{Y}_\theta(n\delta_l)$ defines a simple representation of the quiver \overline{Q}_l .*
- (3) *The family θ satisfies*

$$\Sigma(\theta) \prod_{\substack{1 \leq i \leq j \leq l-1 \\ -(n-1) \leq k \leq n-1}} ((\theta_i + \theta_{i+1} + \cdots + \theta_j) + k\Sigma(\theta)) \neq 0.$$

If these equivalent conditions hold, then $\mathbf{PGL}(n\delta_l)$ acts freely on $\mathcal{Y}_\theta(n\delta_l)$, and so, as a set, $\mathcal{X}_\theta(n\delta_l)$ identifies with $\mathbf{GL}(n\delta_l)$ -orbits in $\mathcal{Y}_\theta(n\delta_l)$.

2.D. Fixed points. — For every $m \geq 1$, the group of m -th roots of unity μ_m acts on $\mathcal{X}_\theta(n\delta_l)$ (this is the restriction of the \mathbb{C}^\times -action defined in §2.A). We aim to compute the fixed point variety $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ whenever $\mathcal{X}_\theta(n\delta_l)$ is smooth. Note first that μ_l acts trivially on $\mathcal{X}_\theta(n\delta_l)$: indeed, if $\zeta \in \mu_l$ and if $(X, Y) \in \mathcal{Y}_\theta(n\delta_l)$, then $\zeta \cdot (X, Y) = g_0(X, Y)$, where $g_0 = (\zeta^i \text{Id}_{\mathbb{C}^n})_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbf{GL}(n\delta_l)$. This means that, in order to describe $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$, we may replace μ_m by $\langle \mu_m, \mu_l \rangle$ or, in other words, we may assume that l divides m .

Hypothesis. *We fix in this subsection a non-zero natural number k and we set $m = kl$.*

If $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we define $\theta[k]$ to be the element $(\theta[k]_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ such that

$$\theta[k]_j = \theta_i$$

if $j \equiv i \pmod{l}$. Roughly speaking, $\theta[k]$ is the concatenation of k copies of θ .

We denote by $\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ the set of $d = (d_j)_{j \in \mathbb{Z}/m\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}$ such that

$$n = \sum_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \pmod{l}}} d_j$$

for all $i \in \mathbb{Z}/l\mathbb{Z}$. Set also $\mathcal{E}(k, l, n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. If $d \in \mathcal{E}(k, l, n)$ and $(X, Y) \in \text{Rep}(\overline{Q}_m, d)$, we set

$$i_k^{(d)}(X, Y) = (X', Y'),$$

where $X'_i = \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \pmod{l}}} X_j$ and $Y'_i = \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \pmod{l}}} Y_j$. By the definition of $\mathcal{E}(k, l, n)$, we have

$X'_i, Y'_i \in \text{Mat}_n(\mathbb{C})$. In other words, $(X', Y') \in \text{Rep}(\overline{Q}_l, n\delta_l)$: it is clear that $i_k^{(d)} : \text{Rep}(\overline{Q}_m, d) \hookrightarrow \text{Rep}(\overline{Q}_l, n\delta_l)$ is a closed immersion.

Lemma 2.11. — *If $d \in \mathcal{E}(k, l, n)$ and $(X, Y) \in \mathcal{Y}_{\theta[k]}(d)$, then $i_k^{(d)}(X, Y) \in \mathcal{Y}_\theta(n\delta_l)$.*

Proof. — Write $i_k^{(d)}(X, Y) = (X', Y')$ and let $i \in \mathbb{Z}/l\mathbb{Z}$. Write p_0 for the rank one matrix $X_0 Y_0 - Y_{-1} X_{-1} - \theta[k]_0 \text{Id}_{\mathbb{C}^{d_0}}$. Then

$$\begin{aligned} X'_i Y'_i - Y'_{i-1} X'_{i-1} &= \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \pmod{l}}} (X_j Y_j - Y_{j-1} X_{j-1}) \\ &= \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \pmod{l}}} \theta[k]_j \text{Id}_{\mathbb{C}^{d_j}} + \delta_{0i} p_0 \\ &= \theta_i \text{Id}_{\mathbb{C}^n} + \delta_{0i} p_0, \end{aligned}$$

as desired. \square

It is also clear that, if (X, Y) and (\tilde{X}, \tilde{Y}) are two elements of $\mathcal{Y}_{\theta[k]}(d)$ which are conjugate under $\mathbf{GL}(d)$, then $i_k^{(d)}(X, Y)$ and $i_k^{(d)}(\tilde{X}, \tilde{Y})$ are conjugate under $\mathbf{GL}(n\delta_l)$. This means that $i_k^{(d)}$ induces a morphism of algebraic varieties still denoted by

$$i_k^{(d)} : \mathcal{X}_{\theta[k]}(d) \longrightarrow \mathcal{X}_{\theta}(n\delta_l).$$

For $d \in \mathcal{E}(k, l, n)$, set

$$\mathcal{X}_d = i_k^{(d)}(\mathcal{X}_{\theta[k]}(d)) \subset \mathcal{X}_{\theta}(n\delta_l).$$

For the moment, we allow the possibility that the varieties $\mathcal{X}_{\theta[k]}(d)$ and \mathcal{X}_d are empty. Set

$$\mathcal{E}(k, l, n)_{\neq \emptyset} = \{d \in \mathcal{E}(k, l, n); \mathcal{X}_d \neq \emptyset\}.$$

But we will show in Corollary 4.13 that we have $\mathcal{E}(k, l, n)_{\neq \emptyset} = \mathcal{E}(k, l, n)$.

The next result describes the fixed point variety $\mathcal{X}_{\theta}(n\delta_l)^{\mu_m}$: it is the main step towards the proof of Theorem 1.3.

Theorem 2.12. — *Assume that $\mathcal{X}_{\theta}(n\delta_l)$ is smooth. Then:*

- (a) $\mathcal{X}_{\theta}(n\delta_l)^{\mu_m}$ is smooth.
- (b) We have

$$\mathcal{X}_{\theta}(n\delta_l)^{\mu_m} = \coprod_{d \in \mathcal{E}(k, l, n)_{\neq \emptyset}} \mathcal{X}_d$$

and \mathcal{X}_d are exactly the irreducible components of $\mathcal{X}_{\theta}(n\delta_l)^{\mu_m}$.

- (c) If $d \in \mathcal{E}(k, l, n)_{\neq \emptyset}$, then $i_k^{(d)} : \mathcal{X}_{\theta[k]}(d) \rightarrow \mathcal{X}_d$ is an isomorphism.

Proof. — (a) follows from the fact already mentioned in the proof of Corollary 1.5 that the fixed point variety under the action of a finite group on a smooth variety is again smooth.

We will now prove (b). Let ξ be a primitive m -th root of unity and let $\zeta = \xi^k$ (it is a primitive l -th root of unity). Let g_0 denote the element $(\zeta^i \text{Id}_{\mathbb{C}^n})_{i \in \mathbb{Z}/l\mathbb{Z}}$ in $\mathbf{GL}(n\delta_l)$. This element satisfies

$$(*) \quad g_0(X, Y) = (\zeta^{-1} X, \zeta Y)$$

for all $(X, Y) \in \mathcal{Y}_{\theta}(n\delta_l)$. We will use the following variety:

$$\mathcal{Z} = \{(X, Y, g) \in \mathcal{Y}_{\theta}(n\delta_l) \times \mathbf{GL}(n\delta_l) \mid g_0(X, Y) = (\zeta^{-1} X, \zeta Y) \text{ and } g p_{X, Y} = p_{X, Y}\},$$

where $p_{X,Y} = ([X_i, Y_i] - \theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$: it is 0 except on the 0-th component, where it is equal to a rank one matrix of trace $-n\Sigma(\theta)$ (recall from Theorem 2.10 that $\Sigma(\theta) \neq 0$ because $\mathcal{X}_\theta(n\delta_l)$ is assumed to be smooth).

We denote by $p' : \mathcal{Z} \rightarrow \mathcal{Y}_\theta(n\delta_l)$ the projection on the first two factors, by $p : \mathcal{Z} \rightarrow \mathcal{X}_\theta(n\delta_l)$ the map induced by p' and by $q : \mathcal{Z} \rightarrow \mathbf{GL}(n\delta_l)$ the projection on the third factor. Note that $\mathbf{PGL}(n\delta_l)$ acts on \mathcal{Z} by conjugacy on the three factors, and so the maps p' and q are equivariant for this action.

First, it is clear that $p(\mathcal{Z}) \subset \mathcal{X}_\theta(n\delta_l)^{\mu_m}$. Conversely, if $(X, Y) \in \mathcal{Y}_\theta(n\delta_l)$ is a representative of an element of $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$, then there exists $g \in \mathbf{PGL}(n\delta_l)$ such that ${}^g(X, Y) = (\xi^{-1}X, \xi Y)$. In particular, ${}^g p_{X,Y} = p_{X,Y}$. So g stabilizes the image of $p_{X,Y}$, which is of dimension 1: this means that g acts by a non-zero scalar ω on $\text{Im}(p_{X,Y})$. Setting $g' = \omega^{-1}g$, we see that $(X, Y, g) \in \mathcal{Z}$. So we have proved the following fact:

Fact 1. $p(\mathcal{Z}) = \mathcal{X}_\theta(n\delta_l)^{\mu_m}$.

The next fact follows from the freeness of the action of $\mathbf{PGL}(n\delta_l)$ on $\mathcal{Y}_\theta(n\delta_l)$ (see Theorem 2.10).

Fact 2. *If (X, Y, g) and (X, Y, g') belong to \mathcal{Z} , then $g = g'$ and $g^k = g_0$.*

Indeed, the hypothesis implies that ${}^g(X, Y) = {}^{g'}(X, Y)$ and ${}^{g^k}(X, Y) = {}^{g_0}(X, Y)$. So there exists two non-zero complex numbers α and β such that $g' = \alpha g$ and $g^k = \beta g_0$. But the condition $g p_{X,Y} = g' p_{X,Y} = g_0 p_{X,Y}$ forces $\alpha = \beta = 1$ because $p_{X,Y} \neq 0$.

Now, let \mathcal{C} denote the set of elements $g \in \mathbf{GL}(n\delta_l)$ such that $g^k = g_0$. Since g_0 is central in $\mathbf{GL}(n\delta_l)$, \mathcal{C} is a disjoint union of finitely many semisimple conjugacy classes (which are the irreducible components of \mathcal{C}). We fix $g \in \mathcal{C}$ such that $q^{-1}(g) \neq \emptyset$ and we denote by \mathcal{C}_g its conjugacy class in $\mathbf{GL}(n\delta_l)$. We will describe $q^{-1}(g)$. For this, let $E = \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} \mathbb{C}^n$ and let $E_j = E^{\xi^{-j}g}$ (for $j \in \mathbb{Z}/m\mathbb{Z}$). In other words, E_j is the ξ^j -eigenspace of g in E : it is contained in the i -th component of E , where i is the element of $\mathbb{Z}/l\mathbb{Z}$ such that $j \equiv i \pmod{l}$. Let $d_j = \dim_{\mathbb{C}}(E_j)$ and let $d = (d_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$.

If $(X, Y, g) \in \mathcal{Z}$, then X sends E_j to E_{j-1} (we denote by $X_j : E_j \rightarrow E_{j-1}$ the induced map) while Y sends E_j to E_{j+1} (we denote by $Y_j : E_j \rightarrow E_{j+1}$ the induced map). By choosing a basis of every E_j , the family $(X_j, Y_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ defines a representation of the quiver \overline{Q}_m , of dimension vector d . This means that we have defined a map $q^{-1}(g) \rightarrow \text{Rep}(\overline{Q}_m, d)$, which is clearly a closed immersion, and whose image is $\mathcal{Y}_{\theta[k]}(d)$ (because $\text{Im}(p_{X,Y})$ is contained in E_0).

Note that by construction we have $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$. Moreover, by Lemma 2.9 (b) and Theorem 2.10 we also have $d \in (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. This means that d is an element of $\mathcal{E}(k, l, n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. Moreover, it is clear that the set of elements d that can appear as above (from some $g \in \mathcal{C}$ such that $q^{-1}(g) \neq \emptyset$) is $\mathcal{E}(k, l, n)_{\neq \emptyset}$.

This shows the following fact.

Fact 3. *With the above notation, $q^{-1}(g) \simeq \mathcal{Y}_{\theta[k]}(d)$. In particular, $q^{-1}(g)$ is irreducible.*

Fact 3 and its proof imply the following.

Fact 4. *We have $p(q^{-1}(\mathcal{C}_g)) = \mathcal{X}_d$.*

Denote by π the projection $\pi : \mathcal{Y}_\theta(n\delta_l) \rightarrow \mathcal{X}_\theta(n\delta_l)$. Fact 2 implies that the map $p' : \mathcal{Z} \rightarrow \pi^{-1}(\mathcal{X}_\theta(n\delta_l)^{\mu_m})$ is bijective. Moreover, since $\pi^{-1}(\mathcal{X}_\theta(n\delta_l)^{\mu_m})$ is smooth (the group $\mathbf{PGL}(n\delta_l)$ acts freely on $\mathcal{Y}_\theta(n\delta_l)$), this bijection is an isomorphism of algebraic varieties. Let

$$(p')^{-1} : \pi^{-1}(\mathcal{X}_\theta(n\delta_l)^{\mu_m}) \rightarrow \mathcal{Z}$$

be its inverse.

Consider the morphism $\tilde{\tau}: \pi^{-1}(\mathcal{X}_\theta(n\delta_l)^{\mu_m}) \rightarrow \mathbf{GL}(n\delta_l)$ given by $\tilde{\tau} = q \circ (p')^{-1}$. Since we have $\pi^{-1}(\mathcal{X}_\theta(n\delta_l)^{\mu_m}) // \mathbf{GL}(n\delta_l) = \mathcal{X}_\theta(n\delta_l)^{\mu_m}$, we get a morphism $\tau: \mathcal{X}_\theta(n\delta_l)^{\mu_m} \rightarrow \mathbf{GL}(n\delta_l) // \mathbf{GL}(n\delta_l)$, where the group $\mathbf{GL}(n\delta_l)$ acts on itself by conjugation.

Moreover, since we clearly have $\tau^{-1}(\mathcal{C}_g) = p(q^{-1}(\mathcal{C}_g))$, Fact 4 implies the following.

Fact 5. *We have $\tau^{-1}(\mathcal{C}_g) = \mathcal{X}_d$. In particular \mathcal{X}_d is closed.*

This completes the proof of (b). Now we prove (c). The group $\mathbf{GL}(n\delta_l)$ acts transitively on \mathcal{C}_g . So we get an isomorphism of varieties

$$q^{-1}(g) // C_{\mathbf{GL}(n\delta_l)}(g) \xrightarrow{\sim} q^{-1}(\mathcal{C}_g) // \mathbf{GL}(n\delta_l).$$

Now, if $d = (\dim_{\mathbb{C}}(E^{\xi^{-j}g}))_{j \in \mathbb{Z}/m\mathbb{Z}}$, $q^{-1}(g)$ identifies with $\mathcal{Y}_{\theta[k]}(d)$ and $C_{\mathbf{GL}(n\delta_l)}(g)$ identifies with $\mathbf{GL}(d)$. Glueing this with Facts 3 and 4, we get an isomorphism of varieties

$$\mathcal{X}_{\theta[k]}(d) \xrightarrow{\sim} \mathcal{X}_d,$$

which is the map $i_k^{(d)}$. This proves (c). \square

Corollary 2.13. — *If $k > n$, then we have $\mathcal{X}_\theta(n\delta_l)^{\mu_m} = \mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$.*

Proof. — Take $d \in \mathcal{E}(k, l, n)_{\neq \emptyset}$. Since $k > n$, some component of d must be zero. This implies that \mathbb{C}^\times acts trivially on $\mathcal{X}_{\theta[k]}(d)$ because the action of \mathbb{C}^\times is induced by elements of $\mathbf{GL}(d)$.

Since $i_k^{(d)}$ is \mathbb{C}^\times -equivariant and

$$\mathcal{X}_\theta(n\delta_l)^{\mu_m} = \coprod_{d \in \mathcal{E}(k, l, n)_{\neq \emptyset}} i_k^{(d)}(\mathcal{X}_{\theta[k]}(d))$$

(by Theorem 2.12(b)), the \mathbb{C}^\times -action on the variety $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ is also trivial. \square

3. Proof of Theorem 1.3

First, the explicit description of \mathcal{Z}_c whenever $n = 1$ (see Example 1.2) allows to prove easily Theorem 1.3 in this case. Also, let us decompose V as

$$V = V^W \oplus V_1 \oplus \cdots \oplus V_r,$$

where the V_i are the non-trivial irreducible components of V as a CW-module. Then W decomposes as $W = W_1 \times \cdots \times W_r$, where (V_i, W_i) is then a reflection subquotient (subgroup) of W . Let c_i denote the restriction of c to W_i . Then

$$\mathcal{Z}_c(V, W) \simeq V^W \times V^{*W} \times \mathcal{Z}_{c_1}(V_1, W_1) \times \cdots \times \mathcal{Z}_{c_r}(V_r, W_r)$$

so that the proof of Theorem 1.3 is easily reduced to the case where W acts irreducibly on V/V^W .

In this case, the smoothness of $\mathcal{Z}_c(V, W)$ implies that W is of type G_4 or $G(l, 1, n)$ for some $l \geq 1$. The case of type G_4 will be handled by computer calculations: see Section 5 for details. This means that we can work under the following hypothesis:

Hypothesis. From now on, and until the end of this section, we assume that $n \geq 2$, that $V = \mathbb{C}^n$ and that $W = G(l, 1, n)$. We also assume that σ is an m -th root of unity, where $m = kl$ for some $k \geq 1$.

Recall that $G(l, 1, n)$ is the group of monomial matrices with coefficients in μ_l (the group of l -th root of unity in \mathbb{C}^\times) and recall that $n \geq 2$.

3.A. Quiver varieties vs Calogero-Moser spaces. — We fix a primitive l -th root of unity ζ . We denote by s the permutation matrix corresponding to the transposition $(1, 2)$ and we set

$$t = \text{diag}(\zeta, 1, \dots, 1) \in W.$$

Then $s, t, t^2, \dots, t^{l-1}$ is a set of representatives of conjugacy classes of reflections of W . We set for simplification

$$a = c_s \quad \text{and} \quad k_j = \frac{1}{l} \sum_{i=1}^{l-1} \zeta^{-i(j-1)} c_{t^i}$$

for $j \in \mathbb{Z}/l\mathbb{Z}$. Then

$$(3.1) \quad k_0 + \dots + k_{l-1} = 0 \quad \text{and} \quad c_{t^i} = \sum_{j \in \mathbb{Z}/l\mathbb{Z}} \zeta^{i(j-1)} k_j$$

for $1 \leq i \leq l-1$. Finally, if $i \in \mathbb{Z}/l\mathbb{Z}$, we set

$$(3.2) \quad \theta_i = \begin{cases} k_{-i} - k_{1-i} & \text{if } i \neq 0, \\ -a + k_0 - k_1 & \text{if } i = 0. \end{cases}$$

and $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$.

The following result is proved in [11, Theorem 3.10]. (Note that our k_i is related with Gordon's H_i via $H_i = k_{-i} - k_{1-i}$.)

Theorem 3.3. — With the above notation, there is a \mathbb{C}^\times -equivariant isomorphism of varieties

$$\mathcal{X}_c \xrightarrow{\sim} \mathcal{X}_\theta(n\delta_l).$$

The theorem above is also true for $n = 1$. But in this case, the variety \mathcal{X}_c has no parameter a . On the other hand, the variety $\mathcal{X}_\theta(\delta_l)$ is independent of θ_0 .

Putting Theorems 2.10 and 3.3 together, one gets:

Corollary 3.4. — The variety \mathcal{X}_c is smooth if and only if

$$a \prod_{\substack{0 \leq i \neq j \leq l-1 \\ 0 \leq r \leq n-1}} (k_i - k_j - ra) \neq 0.$$

Proof of Theorem 1.3. — Assume now that \mathcal{Z}_c is smooth and let \mathcal{X} be an irreducible component of $\mathcal{Z}_c^{\mu_m}$. Using the isomorphism of Theorem 3.3, we see that

$$\mathcal{Z}_c^{\mu_m} \xrightarrow{\sim} \mathcal{X}_{\theta}(n\delta_l)^{\mu_m}.$$

So, by Theorem 2.12, there exists $d \in \mathcal{E}(k, l, n)_{\neq \emptyset}$ such that $\mathcal{X} \simeq \mathcal{X}_{\theta[k]}(d)$. By Lemma 2.7, there exists $r \geq 0$ and $w \in W_m^{\text{aff}}$ such that $w(d) = r\delta_m$. So it follows from the isomorphism (2.4) that

$$\mathcal{X} \simeq \mathcal{X}_{w(\theta[k])}(r\delta_m)$$

(see also Lemma 2.9 and its proof). Using Theorem 3.3 in the other way, we see that there exists a complex valued function c' on the set of conjugacy classes of reflections of $W' = G(m, 1, r)$ such that $\mathcal{X}_{w(\theta[k])}(r\delta_m) \simeq \mathcal{Z}_{c'}(W')$. Moreover, as the action of W_m^{aff} on $\mathbb{C}^{\mathbb{Z}/m\mathbb{Z}}$ is linear, this implies that the map $c \mapsto c'$ is linear.

This proves almost every statement of Theorem 1.3, except that W' can be realized as a reflection subquotient of (V, W) . For this, we need the following fact.

Fact. *We have $kr \leq n$.*

Indeed, the element $d - r\delta_m$ is in the W_m^{aff} -orbit of 0. Then Proposition 4.4 implies that $d - r\delta_m$ is a residue of some m -core. In particular, each coordinates of $d - r\delta_m$ must be positive. Since the sum of the coordinates of d is nl , this gives $nl - rkl \geq 0$. This implies $kr \leq n$.

Now, using the fact proved above, let x denotes the matrix

$$x = \text{diag}(\underbrace{\zeta, 1, \dots, 1}_{k \text{ terms}}) \cdot M_{(1,2,\dots,k)} \in \mathbf{GL}_k(\mathbb{C}),$$

where $M_{(1,2,\dots,k)}$ is the permutation matrix associated with the cycle $(1, 2, \dots, k)$. Now, let g be the matrix

$$g = \text{diag}(\underbrace{x, \dots, x}_{r \text{ terms}}, \text{Id}_{\mathbb{C}^{n-rk}}) \in G(l, 1, n).$$

Let V' denote the ξ -eigenspace of g , where ξ is a primitive m -th root of unity such that $\xi^k = \zeta$. Then $C_W(g)$ acts on V' , V' is of dimension r and, if we denote by K the kernel of this action, then $C_W(g)/K \simeq G(m, 1, r)$. So $(V', G(m, 1, r))$ is a reflection subquotient of (V, W) . \square

Remark 3.5. — The formula (3.2) above yields a bijection between $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ and the set of $(l+1)$ -tuples of complex numbers (a, k_0, \dots, k_{l-1}) such that $k_0 + k_1 + \dots + k_{l-1} = 0$. Moreover, the W_l^{aff} -action on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ (see Section 2.B) translates to the action on the $(l+1)$ -tuples in the following way

$$s_r(a, k_0, \dots, k_{l-r}, k_{l-r+1}, \dots, k_{l-1}) = (a, k_0, \dots, k_{l-r+1}, k_{l-r}, \dots, k_{l-1}), \quad \text{for } r = 1, 2, \dots, l-1$$

and

$$s_0(a, k_0, k_1, k_2, \dots, k_{l-1}) = (a, k_1 + a, k_0 - a, k_2, \dots, k_{l-1}).$$

In particular, we see that the subgroup W_l of W_l^{aff} acts by permutation of the parameters k_0, k_1, \dots, k_{l-1} . \blacksquare

The next result follows from [2], but we provide here a different proof (which works only in type $G(l, 1, n)$).

Corollary 3.6. — *A permutation of parameters k_0, k_1, \dots, k_{l-1} does not change the Calogero-Moser space \mathcal{X}_c (up to an isomorphism of algebraic varieties).*

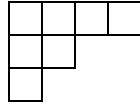
Proof. — The subgroup W_l of W_l^{aff} stabilizes δ_l . Then by (2.4), for each $w \in W_l$, $n \in \mathbb{Z}_{>0}$ and $\theta \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we have $\mathcal{X}_\theta(n\delta_l) \simeq \mathcal{X}_{w(\theta)}(n\delta_l)$. This proves the statement by Theorem 3.3 and Remark 3.5 \square

4. Combinatorics

In this subsection, we aim to make the statement of Theorem 1.3 more precise in the case where $W = G(l, 1, n)$: we wish to describe precisely the map $c \mapsto c'$ in terms of the combinatorics of partitions, cores, l -quotients, etc.

4.A. Partitions and cores. — Let l and n be positive integers. A *partition* is a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers (with no fixed length) such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. Set $|\lambda| = \sum_{i=1}^r \lambda_i$. If $|\lambda| = n$, we say that λ is a partition of n .

Denote by \mathcal{P} (resp. $\mathcal{P}[n]$) be the set of all partitions (resp. the set of all partitions of n). By convention, $\mathcal{P}[0]$ contains one (empty) partition. We will identify partitions with Young diagrams. The partition λ corresponds to a Young diagram with r lines such that the i th line contains λ_i boxes. For example the partition $(4, 2, 1)$ corresponds to the Young diagram

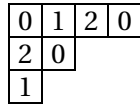


Let b be a box of a Young diagram in the line r and column s . The l -*residue* of the box b is the number $s - r$ modulo l . (We also say that the integer $s - r$ is the ∞ -residue of the box b .) Then we obtain a map

$$\text{Res}_l: \mathcal{P} \rightarrow \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}, \quad \lambda \mapsto \text{Res}_l(\lambda),$$

such that for each $i \in \mathbb{Z}/l\mathbb{Z}$ the number of boxes with residue i in λ is $(\text{Res}_l(\lambda))_i$. (Similarly, we obtain a map $\text{Res}_\infty: \mathcal{P} \rightarrow \mathbb{Z}^{\mathbb{Z}}$.)

Example 4.1. — For the partition $\lambda = (4, 2, 1)$ and $l = 3$ the residues of the boxes are



In this case we have $\text{Res}_l(\lambda) = (3, 2, 2)$ because there are three boxes with residue 0, two boxes with residue 1 and two boxes with residue 2. ■

We say that a box of a Young diagram is *removable* if it has no boxes on the right and on the bottom. For $\lambda, \mu \in \mathcal{P}$, we write $\mu \leq \lambda$ if the Young diagram of μ can be obtained from the Young diagram of λ by removing a sequence of removable boxes.

Definition 4.2. — We say that the partition λ is an l -core if there is no partition $\mu \leq \lambda$ such that the Young diagram of μ differs from the Young diagram of λ by l boxes with l different l -residues.

See [3] for more details about the combinatorics of l -cores. Let $\mathcal{C}_l \subset \mathcal{P}$ be the set of l -cores. Set $\mathcal{C}_l[n] = \mathcal{P}[n] \cap \mathcal{C}_l$.

If a partition λ is not an l -core, then we can get a smaller Young diagram from its Young diagram by removing l boxes with different l -residues. We can repeat this operation again and again until we get an l -core. It is well-known, that the l -core that we get is independent of the choice of the boxes that we remove. Then we get an application

$$\text{Core}_l: \mathcal{P} \rightarrow \mathcal{C}_l.$$

If $\mu = \text{Core}_l(\lambda)$, we will say that the partition μ is the l -core of the partition λ .

Example 4.3. — The partition $(4, 2, 1)$ from the previous example is not a 3-core because it is possible to remove three bottom boxes. We get

$$\boxed{0} \boxed{1} \boxed{2} \boxed{0}$$

But this is still not a 3-core because we can remove three more boxes and we get

$$\boxed{0}$$

This shows that the partition (1) is the 3-core of the partition $(4, 2, 1)$. ■

As mentioned in Lemma 2.7, for each $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, the W_l^{aff} -orbit of d contains an element of the form $n\delta_l$ for a unique $n \in \mathbb{Z}$.

Proposition 4.4. — Let d be an element of $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. The following statements are equivalent.

- (a) d is of the form $d = \text{Res}_l(\lambda)$ for some l -core λ ,
- (b) d is in the W_l^{aff} -orbit of 0 .

Proof. — Consider the W_l^{aff} -action on \mathcal{C}_l as in [3, Sec. 3]. By construction, the map $\text{Res}_l: \mathcal{C}_l \rightarrow \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is W_l^{aff} -invariant. Moreover, the residue of the empty partition is zero. The stabilizer of the empty partition in W_l^{aff} is W_l and the stabilizer of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ in W_l^{aff} is also W_l . This shows that the map Res_l yields a bijection between the set of l -cores and the W_l^{aff} -orbit of 0 in $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. □

Denote by π the obvious surjection $\pi: \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \rightarrow \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} / \mathbb{Z}\delta_l$.

Proposition 4.5. — The chain of maps

$$\mathcal{C}_l \xrightarrow{\text{Res}_l} \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \xrightarrow{\pi} \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} / \mathbb{Z}\delta_l$$

yields a bijection between \mathcal{C}_l and $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} / \mathbb{Z}\delta_l$.

Proof. — We have seen in the proof of Proposition 4.4 that Res_l yields a bijection between \mathcal{C}_l and the W_l^{aff} -orbit of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. This proves the statement because the restriction of π to the W_l^{aff} -orbit of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is bijective by Lemma 2.7. □

For $d = (d_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ we set $|d| = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} d_i$. Set $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n] = \{d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}; |d| = n\}$. Denote by $\mathcal{C}_l[\equiv n]$ the subset of \mathcal{C}_l that contains only l -cores with the number of boxes congruent to n modulo l . Proposition 4.5 yields the following bijection $\varepsilon: \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n] \xrightarrow{\sim} \mathcal{C}_l[\equiv n]$.

Lemma 4.6. — Assume $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n]$. Then d is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if $|\varepsilon(d)| \leq n$.

Proof. — By construction, $\varepsilon(d)$ is the unique l -core $\lambda \in \mathcal{C}_l[\equiv n]$ such that $\text{Res}_l(\lambda)$ is congruent to d modulo δ_l . Write $d = \text{Res}_l(\lambda) + r\delta_l$. Since by Proposition 4.4, the element $\text{Res}_l(\lambda)$ is in the W_l^{aff} -orbit of 0, then d is in the W_l^{aff} -orbit of $r\delta_l$. Then, by definition, the element d is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if $r \geq 0$. Now, let us count the sum of the coordinates in the sides of the equality $d = \text{Res}_l(\lambda) + r\delta_l$. We obtain $n = |\lambda| + rl$. This shows that we have $r \geq 0$ if and only if $|\lambda| \leq n$. \square

Let ν be an l -core. Denote by $\mathcal{P}_{l,\nu}$ the set of partitions with the l -core ν . In particular we denote by $\mathcal{P}_{l,\emptyset}$ the set of partitions with trivial l -core. Let \mathcal{P}^l be the set of l -partitions (i.e., the set of l -tuples of partitions). If λ is a partition, we denote by $\beta_l(\lambda)$ its l -quotient (see for example [16, Sec. 2.2]).

Lemma 4.7. — If ν is an l -core, then β_l restricts to a bijection $\beta_{l,\nu}: \mathcal{P}_{l,\nu} \xrightarrow{\sim} \mathcal{P}^l$.

4.B. Description of $\mathcal{E}(k, l, n)$ in terms of cores. — Let k be a positive integer. Set $m = kl$. Set also $(\mathcal{C}_m)_{l,\nu} = \mathcal{P}_{l,\nu} \cap \mathcal{C}_m$.

Recall that we have a bijection $\varepsilon: \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl] \xrightarrow{\sim} \mathcal{C}_m[\equiv nl]$. Assume $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl]$.

Lemma 4.8. — We have $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ if and only if the l -core of $\varepsilon(d)$ is trivial, where the set $\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ is as in Section 2.D.

Proof. — By definition, $\varepsilon(d)$ is the unique m -core $\lambda \in \mathcal{C}_m[\equiv nl]$ such that $\text{Res}_m(\lambda)$ is equal to d modulo δ_m . On the other hand, by Proposition 4.4, the partition λ is an l -core if and only if $\text{Res}_l(\lambda)$ is a multiple of δ_l . By commutativity of the following diagram,

$$\begin{array}{ccc} & \mathcal{P} & \\ \text{Res}_m \swarrow & & \searrow \text{Res}_l \\ \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}} & \xrightarrow{\quad} & \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \end{array}$$

this condition is clearly equivalent to $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$. \square

Recall that the set $\mathcal{E}(k, l, n)$ was defined as the intersection $\mathcal{E}(k, l, n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. Combining Lemmas 4.6 and 4.8, we obtain the following.

Lemma 4.9. — The bijection $\varepsilon: \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl] \simeq \mathcal{C}_m[\equiv nl]$ restricts to a bijection

$$\varepsilon': \mathcal{E}(k, l, n) \rightarrow \mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}.$$

For $\lambda = (\lambda^0, \dots, \lambda^{l-1}) \in \mathcal{P}^l$ we set $|\lambda| = |\lambda^0| + \dots + |\lambda^{l-1}|$. Set also $\mathcal{P}^l[n] = \{\lambda \in \mathcal{P}^l; |\lambda| = n\}$. We say that an element of $\mathcal{P}^l[n]$ is an l -partition of n .

Now recall the bijection $\beta_{l,v}: \mathcal{P}_{l,v} \xrightarrow{\sim} \mathcal{P}^l$ obtained by restriction from β_l as in Lemma 4.7. By construction, it has the following properties.

Lemma 4.10. — (a) For each $\lambda \in \mathcal{P}_{l,\emptyset}$, we have $\lambda \in \mathcal{C}_m$ if and only if $\beta_{l,\emptyset}(\lambda) \in (\mathcal{C}_k)^l$.
 (b) For each $\lambda \in \mathcal{P}_{l,v}$, we have $|\lambda| = l \cdot |\beta_{l,v}(\lambda)| + |v|$.

Part (a) of the lemma above shows that $\beta_{l,\emptyset}$ can be restricted to a bijection $(\mathcal{C}_m)_{l,\emptyset} \simeq (\mathcal{C}_k)^l$. Moreover, part (b) shows that new bijection restricts to a bijection

$$\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset} \simeq (\mathcal{C}_k)^l[\equiv n, \leq n],$$

where

$$(\mathcal{C}_k)^l[\equiv n, \leq n] = \{\lambda \in (\mathcal{C}_k)^l; |\lambda| \leq n \text{ and } |\lambda| \equiv n \pmod{k}\}.$$

Combining this with Lemma 4.9, we get a bijection

$$\delta: \mathcal{E}(k, l, n) \rightarrow (\mathcal{C}_k)^l[\equiv n, \leq n].$$

This bijection (and Corollary 4.13 below) yields a parametrization of the irreducible components of $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ by the set $(\mathcal{C}_k)^l[\equiv n, \leq n]$.

4.C. Parametrization of the μ_m -fixed points. — We have proved in Theorem 2.12 that the subset $\mathcal{E}(k, l, n)_{\neq \emptyset}$ of $\mathcal{E}(k, l, n)$ parametrizes the irreducible components of $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$. (But we will show in Corollary 4.13 that we have $\mathcal{E}(k, l, n)_{\neq \emptyset} = \mathcal{E}(k, l, n)$.) Then the bijection ε' above gives another parametrization of the irreducible components of $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ in terms of $\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$. For each $\lambda \in \mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$, we set $\mathcal{X}_\lambda = \mathcal{X}_{\varepsilon'^{-1}(\lambda)}$.

This new parametrization has the following nice property.

Lemma 4.11. — Fix positive integers k_1 and k_2 such that k_1 divides k_2 . Set $m_1 = k_1 l$ and $m_2 = k_2 l$. Fix $\lambda_1 \in \mathcal{C}_{m_1}[\equiv nl] \cap \mathcal{C}_{m_1}[\leq nl] \cap (\mathcal{C}_{m_1})_{l,\emptyset}$ and $\lambda_2 \in \mathcal{C}_{m_2}[\equiv nl] \cap \mathcal{C}_{m_2}[\leq nl] \cap (\mathcal{C}_{m_2})_{l,\emptyset}$. Then we have $\mathcal{X}_{\lambda_2} \subset \mathcal{X}_{\lambda_1}$ if and only if $\text{Core}_{m_1}(\lambda_2) = \lambda_1$.

Proof. — Consider the map

$$p: \mathbb{Z}^{\mathbb{Z}/m_2\mathbb{Z}} \rightarrow \mathbb{Z}^{\mathbb{Z}/m_1\mathbb{Z}}, \quad p(d)_i = \sum_{j \equiv i \pmod{m_1}} d_j.$$

Let $d^1 \in \mathcal{E}(k_1, l, n)$ and $d^2 \in \mathcal{E}(k_2, l, n)$ be such that $\varepsilon'(d^1) = \lambda_1$ and $\varepsilon'(d^2) = \lambda_2$. We have $\mathcal{X}_{d^1} = \mathcal{X}_{\lambda_1}$ and $\mathcal{X}_{d^2} = \mathcal{X}_{\lambda_2}$. It is clear from the definition that we have $\mathcal{X}_{d^2} \subset \mathcal{X}_{d^1}$ if and only if $p(d^2) = d^1$. Then the statement follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{E}(k_2, l, n) & \xrightarrow{p} & \mathcal{E}(k_1, l, n) \\ \varepsilon' \downarrow & & \varepsilon' \downarrow \\ \mathcal{C}_{m_2}[\equiv nl] \cap \mathcal{C}_{m_2}[\leq nl] \cap (\mathcal{C}_{m_2})_{l,\emptyset} & \xrightarrow{\text{Core}_{m_1}} & \mathcal{C}_{m_1}[\equiv nl] \cap \mathcal{C}_{m_1}[\leq nl] \cap (\mathcal{C}_{m_1})_{l,\emptyset}. \end{array}$$

□

Note that the set $\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$ considered above is nothing else but the image of $\mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$ by Core_m . In other words, the set $\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$ is the same thing as the set of m -cores of partitions of nl with trivial l -cores.

Proposition 4.12. — *The \mathbb{C}^\times -fixed points in $\mathcal{X}_\theta(n\delta_l)$ are parametrized by the set $\mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$. This parametrization $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times} = \{p_\mu; \mu \in \mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}\}$ can be chosen in such a way that for each $k \in \mathbb{Z}$, each $\lambda \in \mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$ and each $\mu \in \mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$ we have $p_\mu \in \mathcal{X}_\lambda$ if and only if λ is the m -core of μ .*

Proof. — Assume $k > n$. Then by Corollary 2.13, we have $\mathcal{X}_\theta(n\delta_l)^{\mu_m} = \mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$. Moreover, the set $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ is finite. We already know that $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ is in bijection with a subset of $\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$. Moreover, $k > n$ also implies that we have $\mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset} = \mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$. This shows that $\mathcal{X}_\theta(n\delta_l)^{\mu_m} = \mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ is in bijection with a subset of $\mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$. Moreover, by Lemma 4.11, this bijection is independent of the choice of k . But it is well-known (see [10]) that $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ and $\mathcal{P}^l[n] \simeq \mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$ have the same cardinalities. This shows that the bijection above is a bijection between $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ and $\mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$ (not just a subset of $\mathcal{P}[nl] \cap \mathcal{P}_{l,\emptyset}$).

The second statement follows from Lemma 4.11. \square

Corollary 4.13. — *For each $\lambda \in \mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$, we have $\mathcal{X}_\lambda \neq \emptyset$. In particular, we have $\mathcal{E}(k, l, n) = \mathcal{E}(k, l, n)_{\neq \emptyset}$.*

Proof. — Assume $\lambda \in \mathcal{C}_m[\equiv nl] \cap \mathcal{C}_m[\leq nl] \cap (\mathcal{C}_m)_{l,\emptyset}$. Then there exists $\mu \in \mathcal{P}[nl]$ such that $\text{Core}_m(\mu) = \lambda$. Then μ is also automatically in $\mathcal{P}_{l,\emptyset}$. Then \mathcal{X}_λ is not empty because we have $p_\mu \in \mathcal{X}_\lambda$ by Proposition 4.12. \square

Proposition 4.12 and the bijection $\beta_{l,\emptyset}$ yields a parametrization of $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ by $\mathcal{P}^l[n]$.

4.D. \mathbb{C}^\times -fixed points. — The parametrization of the points of $\mathcal{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ given in Proposition 4.12 is constructed as a parametrization of the irreducible components of $\mathcal{X}_\theta(n\delta_l)^{\mu_m}$ for a very big m . This parametrization can be seen in an " m -independent way" if we replace the quiver \overline{Q}_m with a very big m by an infinite quiver. This can be done in the following way.

Let \overline{Q}_∞ be the quiver defined in the same way as \overline{Q}_l with respect to the vertex set \mathbb{Z} instead of $\mathbb{Z}/l\mathbb{Z}$. The dimension vectors of the quiver \overline{Q}_∞ are in $\mathbb{Z}^{\mathbb{Z}}$. Set

$$\mathbb{Z}_{\text{fin}}^{\mathbb{Z}} = \{\hat{d} \in \mathbb{Z}^{\mathbb{Z}}; \hat{d} \text{ has finitely many non-zero components}\}.$$

Consider the map

$$p: \mathbb{Z}_{\text{fin}}^{\mathbb{Z}} \rightarrow \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}, \quad (p(\hat{d}))_i = \sum_{j \equiv i \pmod{l}} \hat{d}_j.$$

For each $\hat{d} \in \mathbb{Z}_{\text{fin}}^{\mathbb{Z}}$, we have a linear map $i_\infty^{(\hat{d})}: \text{Rep}(\overline{Q}_\infty, \hat{d}) \rightarrow \text{Rep}(\overline{Q}_l, p(\hat{d}))$, defined in the same way as in Section 2.D. Now, for each $\theta \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ we consider the element $\theta[\infty] \in \mathbb{C}^{\mathbb{Z}}$ given by $\theta[\infty]_i = \theta_{i \pmod{l}}$. Then we obtain a \mathbb{C}^\times -invariant morphism of algebraic varieties $i_\infty^{(\hat{d})}: \mathcal{X}_{\theta[\infty]}(\hat{d}) \rightarrow \mathcal{X}_\theta(p(\hat{d}))$. But the variety $\mathcal{X}_{\theta[\infty]}(\hat{d})$ is obviously \mathbb{C}^\times -stable because the \mathbb{C}^\times

action is induced by elements of $\mathbf{GL}(\widehat{d})$. Since $\mathcal{X}_{\theta[\infty]}(\widehat{d})$ is connected, it is a singleton (if it is not empty). Set

$$\mathcal{E}(\infty, l, d) = \{\widehat{d} \in \mathbb{Z}_{\text{fin}}^{\mathbb{Z}}; p(\widehat{d}) = d, \mathcal{X}_{\theta[\infty]}(\widehat{d}) \neq \emptyset\}.$$

If $\widehat{d} \in \mathcal{E}(\infty, l, d)$, the image of $i_{\infty}^{(\widehat{d})}$ is a \mathbb{C}^{\times} -fixed point in $\mathcal{X}_{\theta}(d)$. Let us call this point $p_{\widehat{d}}$.

Now, we assume $d = n\delta_l$, $n \geq 0$ and that $\mathcal{X}_{\theta}(d)$ is smooth. The construction of the parametrization of the \mathbb{C}^{\times} -fixed points in $\mathcal{X}_{\theta}(n\delta_l)$ given in Proposition 4.12 implies that each \mathbb{C}^{\times} -fixed point p_{μ} , $\mu \in \mathcal{P}[nl] \cap \mathcal{P}_{l, \emptyset}$, in $\mathcal{X}_{\theta}(n\delta_l)$ is of the form $p_{\widehat{d}}$ for a unique $\widehat{d} \in \mathcal{E}(\infty, l, n\delta_l)$. Moreover, this \widehat{d} is given by $\widehat{d} = \text{Res}_{\infty}(\mu)$.

4.E. Explicit construction of c' . — We know from Theorem 2.12 that each irreducible component \mathcal{X} of $\mathcal{X}_{\theta}(n\delta_l)^{\mu_m}$ is of the form $\mathcal{X} = \mathcal{X}_d$. Moreover, by Theorem 1.3 there exists a reflection subquotient (V', W') of (V, W) and a complex-valued map $c': W' \rightarrow \mathbb{C}$ such that $\mathcal{X} \simeq \mathcal{X}_{c'}(V', W')$. The reflection subquotient was constructed in the proof of Theorem 1.3. Now we give an explicit construction of c' .

Recall from §1.A that the parameter c is determined by the sequence of parameters $(a, k_0, k_1, \dots, k_{l-1})$. Similarly, the parameter c' is determined by $(a', k'_0, k'_1, \dots, k'_{m-1})$. We are going to give explicit expressions for $(a', k'_0, k'_1, \dots, k'_{m-1})$ in terms of $d \in \mathcal{E}(k, l, m)$ and $(a, k_0, k_1, \dots, k_{l-1})$.

First, we recall the construction of c' in the proof of Theorem 1.3. There is an element $w \in W_m^{\text{aff}}$ such that $w(d)$ is a multiple of δ_m . Then we put $\theta' = w(\theta[k])$. Finally, we can obtain $(a', k'_0, \dots, k'_{m-1})$ from θ' using (3.1) and (3.2).

This construction of c' is not canonical because there is no preferable choice of an element w . Note that δ_m is stable by the subgroup W_m of W_m^{aff} . So the element w can be replaced by any elements of $W_m w$. (By Remark 3.5, a different choice of w yields just a permutation of the parameters $k'_0, k'_1, \dots, k'_{m-1}$.) The affine Weyl group W_m^{aff} is of the form $W_m \ltimes R_m$, where R_m is the root lattice for \mathfrak{sl}_m . In particular, there is an element of $w' \in W_m w$ that is in R_m . More precisely, we have $w' = t_{\alpha}$, where $\alpha = d - d_0\delta_m$ (see the proof of Lemma 2.7).

Then, by Lemma 2.8, we have $w'(\theta[k]) = t_{\alpha}(\theta[k]) = \theta[k] + \Sigma(\theta[k]) \cdot \bar{\alpha} = \theta[k] + k\Sigma(\theta) \cdot \bar{d}$. (Where the notation $\bar{\alpha}$ and \bar{d} is as in Section 2.B.) This means that we have

$$\theta'_i = (\theta[k])_i + k\Sigma(\theta)(2d_i - d_{i-1} + d_{i+1}) = \theta_{(i \bmod l)} - ka(2d_i - d_{i-1} + d_{i+1})$$

Passing back to the k_i -notations, we get $a' = ka$ and

$$k'_r = k_{(r \bmod l)} + a \left(\left\lfloor \frac{r-1}{l} \right\rfloor - \frac{k-1}{2} + k(d_{1-r} - d_{-r}) \right)$$

for $r = 1, \dots, m-1, m$. (Here $k_0 = k_m$.)

Example 4.14. — Let us make the above formula explicit in the case $l = 1$. In this case we have $m = k$. Then we get $\theta = \theta_0 = -a$ and $\theta[k] = (-a, -a, \dots, -a)$ (k times). This gives

$$\theta'_i = -a - ak(2d_i - d_{i-1} - d_{i+1}),$$

for $i \in \mathbb{Z}/k\mathbb{Z}$.

$$k'_r = a \left(r - \frac{k+1}{2} + k(d_{1-r} - d_{-r}) \right)$$

for $r = 1, \dots, k-1, k$. (Here $k_0 = k_k$.) ■

Example 4.15. — Assume $l = 2$, $k = 2$ and $d = (0, 0)$. In this case c is determined by (a, k_0, k_1) such that $k_0 + k_1 = 0$. Put $k_1 = b$ (and then $k_0 = -b$). Let us make the formulas for k'_r explicit in this case. We have

$$\begin{cases} k'_0 &= -b + a/2, \\ k'_1 &= b - a/2, \\ k'_2 &= -b - a/2, \\ k'_3 &= b + a/2. \end{cases}$$

5. Type G_4

Hypothesis. In this section, and only in this section, we assume that $V = \mathbb{C}^2$, we identify $\mathbf{GL}_{\mathbb{C}}(V)$ with $\mathbf{GL}_2(\mathbb{C})$, we fix a primitive third root of unity ω , we set

$$s = \begin{pmatrix} \omega & 0 \\ \omega^2 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\omega^2 \\ 0 & \omega \end{pmatrix}$$

and we assume that $W = \langle s, t \rangle$, so that W is of type G_4 in Shephard-Todd classification. We denote by (y_1, y_2) the canonical basis of V and by (x_1, x_2) its dual basis.

Note that there is only one W -orbit of reflection hyperplanes (let us call it Ω) and we set for simplifying $k_i = k_{\Omega, i}$, so that $k_0 + k_1 + k_2 = 0$. We also set $\rho = 1 + 2\omega$, so that $\rho^2 = -3$.

In this case, we will check Conjecture F even if \mathcal{Z}_c is not smooth (and for all elements $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ normalizing W , as any such elements belong to $\mathbb{C}^\times \cdot W$, as we will see in the proof below).

Theorem 5.1. — If W is of type G_4 , then Conjecture F holds.

Proof. — Let $\sigma \in \mathbf{GL}_2(\mathbb{C})$ be of finite order and normalizing W . The conjugacy class of s is

$$\{s, t, sts^{-1}, s^2ts^{-2} = tst^{-1}\},$$

and is the unique conjugacy class of reflections of determinant ω . Since $\sigma s \sigma^{-1}$ is a reflection of W of determinant ω , there exists $w \in W$ such that $w \sigma$ commutes with s . Since W acts trivially on \mathcal{Z}_c , we may, and we will, assume that $\sigma s \sigma^{-1} = s$. Now, $\sigma t \sigma^{-1}$ belongs to this conjugacy class and is different from s , so there exists $i \in \mathbb{Z}$ such that $s^i \sigma$ commutes with t . So, replacing σ by $s^i \sigma$, we may, and we will, assume that σ commutes with s and t . In other words, σ is a root of unity.

Since $Z(W) \simeq \mu_2$, we may also assume that the order of σ is even, equal to $2d$ for some $d \geq 2$ (because the case $d = 1$ is obvious). Let I_d denote the ideal of \mathbf{Z}_c generated by the $\sigma(z) - z$, where z runs over a set of generators of \mathbf{Z}_c . Then

$$\clubsuit \quad \mathbb{C}[\mathcal{Z}_c^\sigma] = \mathbf{Z}_c / \sqrt{I_d}.$$

Let ζ denote the root of unity by which σ acts: then $\sigma(z) - z = (\zeta^l - 1)z$ if z is homogeneous of \mathbb{Z} -degree l . This shows that I_d is generated by the generators of \mathbf{Z}_c of degree not divisible by $2d$. As we will see below, there is a set of generators of \mathbf{Z}_c whose \mathbb{Z} -degrees belong to the interval $[-6, 6]$. So, if $d \geq 4$, then $\mathcal{X}_c^\sigma = \mathcal{X}_c^{\mathbb{C}^\times}$, so it is a finite set and the Conjecture F is checked in this case. So we may assume that $d \in \{2, 3\}$.

In [6], Thiel and the first author have provided algorithms for computing presentations of the algebra \mathbf{Z}_c , and such algorithms have been included in the MAGMA package CHAMP developed by Thiel [20] (about the MAGMA software, see [18]). These algorithms provide an explicit presentation of the center \mathbf{Z}_c given as follows:

Generators:

$$Y_1, Y_2, X_1, X_2, \text{eu}, A, B, D;$$

Relations:

$$\left\{ \begin{array}{l} \text{(Z1)} \quad \text{eu}^4 + 4\rho Y_1 X_1 - AB + 3\rho \text{eu} D + 18(k_1^2 + k_1 k_2 + k_2^2) \text{eu}^2 \\ \quad + 756(k_1^2 k_2 + k_1 k_2^2) \text{eu} = 0 \\ \text{(Z2)} \quad \rho \text{eu}^3 A - 4X_1 \text{eu} B - DA - \rho Y_1 X_2 - 30\rho(k_1^2 + k_1 k_2 + k_2^2) \text{eu} A \\ \quad - 108\rho(k_1^2 k_2 + k_1 k_2^2) A = 0 \\ \text{(Z3)} \quad Y_2 A - \text{eu} B^2 + \rho Y_1 D + 54(k_1^2 + k_1 k_2 + k_2^2) Y_1 \text{eu} - 324(k_1^2 k_2 + k_1 k_2^2) Y_1 = 0 \\ \text{(Z4)} \quad \rho \text{eu}^3 B - 4X_1 Y_2 - \rho Y_1 \text{eu} A - BD - 30\rho(k_1^2 + k_1 k_2 + k_2^2) \text{eu} B \\ \quad - 108\rho(k_1^2 k_2 + k_1 k_2^2) B = 0 \\ \text{(Z5)} \quad \text{eu} A^2 - 4X_1 D - X_2 B + 72\rho(k_1^2 + k_1 k_2 + k_2^2) X_1 \text{eu} - 432\rho(k_1^2 k_2 + k_1 k_2^2) X_1 = 0 \\ \text{(Z6)} \quad Y_2 \text{eu}^3 + 3Y_1^2 A - 3Y_1 \text{eu}^2 B - B^3 + 3\rho Y_2 D + 144(k_1^2 + k_1 k_2 + k_2^2) Y_1 B \\ \quad + 18(k_1^2 + k_1 k_2 + k_2^2) Y_2 \text{eu} + 756(k_1^2 k_2 + k_1 k_2^2) Y_2 = 0 \\ \text{(Z7)} \quad 4\rho X_1 \text{eu}^2 A - A^3 - 16X_1^2 B + X_2 \text{eu}^3 + 3\rho X_2 D + 18(k_1^2 + k_1 k_2 + k_2^2) X_2 \text{eu} \\ \quad - 192\rho(k_1^2 + k_1 k_2 + k_2^2) X_1 A + 756(k_1^2 k_2 + k_1 k_2^2) X_2 = 0 \\ \text{(Z8)} \quad \text{eu}^6 + 4\rho Y_1 X_1 \text{eu}^2 + 2\rho D \text{eu}^3 - Y_2 X_2 + D^2 - 36(k_1^2 + k_1 k_2 + k_2^2) \text{eu}^4 \\ \quad + 12\rho(k_1^2 + k_1 k_2 + k_2^2) \text{eu} D + 216\rho(k_1^2 k_2 + k_1 k_2^2) D + 1080(k_1^2 k_2 + k_1 k_2^2) \text{eu}^3 \\ \quad + 1620(k_1^2 + k_1 k_2 + k_2^2)^2 \text{eu}^2 - 3888(k_1^2 k_2 + k_1 k_2^2)(k_1^2 + k_1 k_2 + k_2^2) \text{eu} \\ \quad - 34992(k_1^2 k_2 + k_1 k_2^2)^2 = 0 \\ \text{(Z9)} \quad \rho Y_1 A^2 + 4X_1 B^2 + 4\text{eu}^3 D - \rho Y_2 X_2 + 4\rho D^2 - 24\rho(k_1^2 + k_1 k_2 + k_2^2) \text{eu}^4 \\ \quad - 288(k_1^2 + k_1 k_2 + k_2^2) \text{eu} D - 576(k_1^2 + k_1 k_2 + k_2^2) Y_1 X_1 \\ \quad + 2160\rho(k_1^2 + k_1 k_2 + k_2^2)^2 \text{eu}^2 + 432\rho(k_1^2 k_2 + k_1 k_2^2) \text{eu}^3 - 864(k_1^2 k_2 + k_1 k_2^2) D \\ \quad + 20736\rho(k_1^2 k_2 + k_1 k_2^2)(k_1^2 + k_1 k_2 + k_2^2) \text{eu} + 46656\rho(k_1^2 k_2 + k_1 k_2^2)^2 = 0 \end{array} \right.$$

The generators have \mathbb{Z} -degrees given by the following map

$$(Y_1, Y_2, X_1, X_2, \text{eu}, A, B, D) \longmapsto (-4, -6, 4, 6, 0, 2, -2, 0).$$

Moreover, in this presentation, $\mathbb{C}[V]^W = \mathbb{C}[X_1, X_2]$ and $\mathbb{C}[V^*]^W = \mathbb{C}[Y_1, Y_2]$.

We can deduce from this that \mathcal{X}_c is smooth if and only if

$$(\diamond) \quad k_0 k_1 k_2 (k_0 - k_1)(k_0 - k_2)(k_1 - k_2) \neq 0.$$

First case: assume that $d = 2$. In this case, the above presentation shows that, whenever (\diamond) holds, then \mathcal{X}_c^σ has four irreducible components \mathcal{X} , p , q and r where \mathcal{X} has dimension 2 and is isomorphic to

$$(\heartsuit) \quad \mathcal{X} \simeq \{(x, y, e) \in \mathbb{C}^3 \mid e(e - 12k_0)(e - 12k_1)(e - 12k_2) = xy\},$$

and p , q and r are three points (which belong to $\mathcal{X}_c^{\mathbb{C}^\times}$). If (\diamond) does not hold, then \mathcal{X}_c^σ still contains an irreducible component of dimension 2 with the same description as above,

and a few other isolated points (the number depends on the exact values of the k_i). By Remark 1.4, we only need to consider the irreducible component \mathcal{X} . But it follows from (♥) and Example 1.2 that then \mathcal{X} is isomorphic to the Calogero-Moser associated with a cyclic group of order 4 acting on a one-dimensional space: as there is an element w of order 4 in W , one can choose for V' an eigenspace of w and for W' the cyclic group $\langle w \rangle$ and Conjecture F is checked in this case (note also that the map $c \mapsto c'$ is linear).

Second case: assume that $d = 3$. In this case, the above presentation shows that, whenever (♢) holds, then \mathcal{X}_c^σ has two irreducible components \mathcal{X} and p where \mathcal{X} has dimension 2 and is isomorphic to

$$(\spadesuit) \quad \mathcal{X} \simeq \{(x, y, e) \in \mathbb{C}^3 \mid (e - 6k_0)(e - 6k_1)(e - 6k_2)(e - 12k_0)(e - 12k_1)(e - 12k_2) = xy\},$$

and p is a point (which belongs to $\mathcal{X}_c^{\mathbb{C}^\times}$). If (♢) does not hold, then \mathcal{X}_c^σ still contains an irreducible component of dimension 2 with the same description as above, and maybe one extra point according to the values of the k_i . By Remark 1.4, we only need to consider the irreducible component \mathcal{X} . But it follows from (♠) and Example 1.2 that then \mathcal{X} is isomorphic to the Calogero-Moser associated with a cyclic group of order 6 acting on a one-dimensional space: as $-s \in W$ is of order 6, one can choose for V' the ω -eigenspace of s and for W' the cyclic group $\langle -s \rangle$ and Conjecture F is checked in this case (note also that the map $c \mapsto c'$ is linear). \square

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