# Fixed points in smooth Calogero-Moser spaces

Cédric Bonnafé, Ruslan Maksimau

# FIXED POINTS IN SMOOTH CALOGERO-MOSER SPACES

*by*

CÉDRIC BONNAFÉ & RUSLAN MAKSIMAU

***Abstract.*** — We prove that every irreducible component of the fixed point variety under the action of $\boldsymbol{\mu}_d$ in a smooth Caloger-Moser space is isomorphic to a Calogero-Moser space associated with another reflection group.

Consider a finite subgroup $W \subset \mathbf{GL}_{\mathbb{C}}(V)$ of automorphisms of a finite dimensional complex vector space $V$, generated by reflections and let $c$ be a complex valued function on the set of conjugacy classes of reflections of $W$. To the triple $(V, W, c)$, Etingof and Ginzburg [**8**] have associated an affine variety $\mathscr{Z}_c(V, W)$, the *Calogero-Moser space*, which is defined as the spectrum of the center of the rational Cherednik algebra $\mathbf{H}_c$ *"at* $t = 0$*"* (see Section 1 for a precise definition). The algebra $\mathbf{H}_c$ carries a $\mathbb{Z}$-grading (i.e. a $\mathbb{C}^\times$-action), which induces a $\mathbb{C}^\times$-action on the Calogero-Moser space $\mathscr{Z}_c(V, W)$.

We define a *reflection subquotient* of $(V, W)$ to be a pair $(V', W')$ where $V'$ is a subspace of $V$, $W' \subset \mathbf{GL}_{\mathbb{C}}(V')$ is generated by reflections and there exists a subgroup $H$ of $W$ which stabilizes $V'$ and whose image in $\mathbf{GL}_{\mathbb{C}}(V')$ is exactly $W'$.

Note that any element in $\mathbf{GL}_{\mathbb{C}}(V)$ normalizing $W$ acts on $\mathscr{Z}_c(V, W)$. In [**4**, Conjecture FIX], Rouquier and the first author proposed the following conjecture:

> **Conjecture F.** *Let $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ be an element of finite order and normalizing $W$ and let $\mathscr{X}$ be an irreducible component of $\mathscr{Z}_c(V, W)^\sigma$ (endowed with its reduced scheme structure). Then there exists a reflection subquotient $(V', W')$ of $W$ and a complex valued function $c'$ on the set of conjugacy classes of reflections in $W'$ such that $\mathscr{X} \simeq \mathscr{Z}_{c'}(V', W')$, as varieties endowed with a $\mathbb{C}^\times$-action.*

Note that the conjecture stated in [**4**, Conjecture FIX] does not give an explicit description of $c'$ in terms of $c$: it is just mentioned that the map $c \mapsto c'$ should be linear. Note also that, as stated, it might be a little bit optimistic: maybe one should replace $\mathscr{X}$ by its normalization (as it is not clear whether $\mathscr{X}$ is normal or not). A special case of an element $\sigma$ normalizing $W$ is when $\sigma$ is a root of unity, viewed as the corresponding homothety on $V$. The aim of this paper is to prove the following result:

**Theorem.** *Conjecture F holds if $\mathscr{Z}_c(V, W)$ is smooth and $\sigma$ is a root of unity.*

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The proof is by case-by-case analysis, as the triples $(V, W, c)$ such that $\mathscr{Z}_c(V, W)$ is smooth are classified (see the works of Etingof-Ginzburg [**8**], Gordon [**10**], Bellamy [**1**]). The classification and the conjecture can be easily reduced to the case where $W$ acts irreducibly on $V/V^W$ and in this case, the smoothness of $\mathscr{Z}_c(V, W)$ implies that $W$ is of type $G(l, 1, n)$ or $G_4$ in Shephard-Todd classification. The case of $G_4$ can be handled by computer calculations (see Section 5), while the infinite family case will be handled by using an isomorphism between $\mathscr{Z}_c(V, W)$ and a quiver variety.

## 1. Notation and main result

All along this note, we will abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$. By an algebraic variety, we mean a reduced scheme of finite type over $\mathbb{C}$.

**Set-up.** *We fix in this paper a $\mathbb{C}$-vector space $V$ of finite dimension $n$ and a finite subgroup $W$ of $\mathbf{GL}_{\mathbb{C}}(V)$. We set*

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n-1\}$$

*and we assume that*

$$W = \langle \mathrm{Ref}(W) \rangle.$$

*We also fix an element $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ of finite order normalizing $W$.*

We set $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$. If $s \in \mathrm{Ref}(W)$, we denote by $\alpha_s^\vee$ and $\alpha_s$ two elements of $V$ and $V^*$ respectively such that $V^s = \mathrm{Ker}(\alpha_s)$ and $V^{*s} = \mathrm{Ker}(\alpha_s^\vee)$, where $\alpha_s^\vee$ is viewed as a linear form on $V^*$.

**1.A. Rational Cherednik algebra at $t = 0$. —** All along this note, we fix a function $c : \mathrm{Ref}(W) \to \mathbb{C}$ which is invariant under conjugacy. We define the $\mathbb{C}$-algebra $\mathbf{H}_c$ to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V \oplus V^*)$ with the group $W$) by the relations

$$(\mathscr{H}_c) \qquad \begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = \displaystyle\sum_{s \in \mathrm{Ref}(W)} (\varepsilon(s) - 1) c_s \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s, \end{cases}$$

for all $x, x' \in V^*$, $y, y' \in V$. The algebra $\mathbf{H}_c$ is called the *rational Cherednik algebra at $t = 0$*.

The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_c$ and $\mathbb{C}[V^*] \to \mathbf{H}_c$. Recall [**8**, Corollary 4.4] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_c \tag{1.1}$$

induced by multiplication (this is the so-called *PBW-decomposition*).

We denote by $\mathbf{Z}_c$ the center of $\mathbf{H}_c$: it is well-known [**8**, Lemma 3.5] that $\mathbf{Z}_c$ is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and which is a free $\mathbf{P}$-module of rank $|W|$. We denote by $\mathscr{Z}_c$ the algebraic variety whose ring of regular functions $\mathbb{C}[\mathscr{Z}_c]$ is $\mathbf{Z}_c$: this is the *Calogero-Moser space* associated with the datum $(V, W, c)$. If necessary, we will write $\mathscr{Z}_c(V, W)$ for $\mathscr{Z}_c$.

***Example 1.2***. — Let $l \geqslant 1$, let $\zeta$ be a primitive $l$-th root of unity and let $\boldsymbol{\mu}_l = \langle \zeta \rangle$. We assume in this example that $n = 1$ and $W = \langle t \rangle \simeq \boldsymbol{\mu}_l$, where $t(v) = \zeta v$ for all $v \in V$. Then $\mathrm{Ref}(W) = \{t, t^2, \ldots, t^{l-1}\}$ and we set for simplification

$$k_j = \frac{1}{l} \sum_{i=1}^{l-1} \zeta^{-i(j-1)} c_{t^i}$$

($0 \leqslant j \leqslant l-1$). Note that $k_0 + k_1 + \cdots k_{l-1} = 0$. Then

$$\mathscr{Z}_c \simeq \{(x, y, e) \in \mathbb{C}^3 \mid \prod_{i=0}^{l-1} (e - l\, k_i) = x\, y\}$$

(see for instance [**4**, Theorem 18.2.2]). In particular, $\mathscr{Z}_c$ is smooth if and only if

$$\prod_{0 \leqslant i < j \leqslant l-1} (k_i - k_j) \neq 0.$$

Note also that the inclusion $\mathbf{P} \subset \mathbf{Z}_c$ corresponds to the morphism of algebraic varieties $\mathscr{Z}_c \to \mathbb{C}^2, (x, y, e) \mapsto (x, y)$.

The $\mathbb{C}^\times$-action is given by the formula

$$\xi \cdot (x, y, e) = (\xi^l x, \xi^{-l} y, e),$$

where $\xi \in \mathbb{C}^\times$ and $(x, y, e) \in \mathscr{Z}_c$. ■

**1.B. Action of the normalizer. —** The normalizer $N_{\mathbf{GL}_\mathbb{C}(V)}(W)$ of $W$ in $\mathbf{GL}_\mathbb{C}(V)$ acts on $V$ naturally, on $V^*$ by the contragredient action and on $W$ by conjugation. This endows $\mathrm{T}(V \oplus V^*) \rtimes W$ with an action of $N_{\mathbf{GL}_\mathbb{C}(V)}(W)$ and it is easily checked that the bunch of relations $(\mathscr{H}_c)$ is stable under this action. So $\mathbf{H}_c$ inherits an action of $N_{\mathbf{GL}_\mathbb{C}(V)}(W)$. In particular, its centre $\mathbf{Z}_c$ also inherits such an action, and this defines an action of $N_{\mathbf{GL}_\mathbb{C}(V)}(W)$ on the Calogero-Moser space $\mathscr{Z}_c$.

***Theorem 1.3***. — *Assume that $\mathscr{Z}_c$ is smooth and that $\sigma$ is a root of unity, and let $\mathscr{X}$ be an irreducible component of $\mathscr{Z}_c^\sigma$. Then there exists a reflection subquotient $(V', W')$ of $(V, W)$ and a complex-valued map $c'$ on the set of conjugacy classes of reflections of $W'$ such that*

$$\mathscr{X} \simeq \mathscr{Z}_{c'}(V', W'),$$

*as varieties endowed with a $\mathbb{C}^\times$-action.*

***Remark 1.4***. — Note that if $\mathscr{X}$ is reduced to a point, then the theorem is easy because in this case, $\mathscr{X} \simeq \mathscr{Z}_0(0, 1)$. ■

This Theorem will be proved in Sections 3 and 5.

**1.C. Filtration of the group algebra. —** If $w \in W$, we set $\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$. We define a filtration $\mathscr{F}_\bullet(\mathbb{C}W)$ of the group algebra of $W$ as follows: let

$$\mathscr{F}_i(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \leqslant i} \mathbb{C}w.$$

Then

$$\mathbb{C}\mathrm{Id}_V = \mathscr{F}_0(\mathbb{C}W) \subset \mathscr{F}_1(\mathbb{C}W) \subset \cdots \subset \mathscr{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathscr{F}_{n+1}(\mathbb{C}W) = \cdots$$

is a filtration of $\mathbb{C}W$. For any subalgebra $A$ of $\mathbb{C}W$, we set $\mathscr{F}_i(A) = A \cap \mathscr{F}_i(\mathbb{C}W)$, so that

$$\mathbb{C}\mathrm{Id}_V = \mathbb{C}\mathscr{F}_0(A) \subset \mathscr{F}_1(A) \subset \cdots \subset \mathscr{F}_n(A) = A = \mathscr{F}_{n+1}(A) = \cdots$$

is also a filtration of $A$.

Now, assume that $\mathscr{Z}_c(V,W)$ is smooth, let $\sigma$ be a root of unity and let $\mathscr{X}$ be an irreducible component of $\mathscr{Z}_c(V,W)^\sigma$. According to Theorem 1.3, there exists a reflection subquotient $(V',W')$ of $(V,W)$, a complex-valued map $c'$ on the set of conjugacy classes of reflections of $W'$ and a $\mathbb{C}^\times$-equivariant isomorphism $i : \mathscr{Z}_{c'}(V',W') \xrightarrow{\sim} \mathscr{X}$. We will view $i$ as a closed immersion $i : \mathscr{Z}_{c'}(V',W') \hookrightarrow \mathscr{Z}_c(V,W)$. By [**10**, Corollary 5.8], the smoothness of $\mathscr{Z}_c(V,W)$ implies that there exists a bijection between $\mathrm{Irr}(W)$ and $\mathscr{Z}_c(V,W)^{\mathbb{C}^\times}$. As the fixed point variety of a finite group in a smooth complex algebraic variety is still smooth, this means that $\mathscr{X}$ is smooth, and so $\mathscr{Z}_{c'}(V',W')$ is smooth. Applying again [**10**, Corollary 5.8], we get another bijection between $\mathrm{Irr}(W')$ and $\mathscr{Z}_{c'}(V',W')^{\mathbb{C}^\times}$. As $\mathscr{X}^{\mathbb{C}^\times} \subset \mathscr{Z}_c(V,W)^{\mathbb{C}^\times}$, this gives an injective map $i_{\mathscr{X}} : \mathrm{Irr}(W') \hookrightarrow \mathrm{Irr}(W)$, depending on $\mathscr{X}$ and the choice of $i$. This allows to define a surjective morphism of algebras

$$i_{\mathscr{X}}^* : \mathrm{Z}(\mathbb{C}W) \twoheadrightarrow \mathrm{Z}(\mathbb{C}W')$$

as follows: if $\chi \in \mathrm{Irr}(W)$, let $e_\chi^W$ denote the corresponding primitive central idempotent of $\mathbb{C}W$ and set

$$i_{\mathscr{X}}^*(e_\chi^W) = \begin{cases} e_{\chi'}^{W'} & \text{if } \chi' \in \mathrm{Irr}(W') \text{ is such that } i_{\mathscr{X}}(\chi') = \chi, \\ 0 & \text{if } \chi \text{ is does not belong to the image of } i_{\mathscr{X}}. \end{cases}$$

Using results of Shan and the first author [**5**], Theorem 1.3 has the following consequence:

***Corollary 1.5.*** — *If $i \geqslant 0$, then $i_{\mathscr{X}}^*(\mathscr{F}_i(\mathrm{Z}(\mathbb{C}W))) \subset \mathscr{F}_i(\mathrm{Z}(\mathbb{C}W'))$.*

*Proof.* — If $\mathscr{Y}$ is a complex algebraic variety endowed with a $\mathbb{C}^\times$-action, we denote by $\mathrm{H}^i_{\mathbb{C}^\times}(\mathscr{Y})$ the $i$-th group of equivariant cohomology, with coefficients in $\mathbb{C}$. Let $\hbar$ be an indeterminate and identify $\mathrm{H}_{\mathbb{C}^\times}(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ as usual, with $\hbar$ homogeneous of degree 2. Since $\mathscr{Z}_c(V,W)$ is smooth, it follows from [**5**, Theorem A] that

$$\begin{cases} \mathrm{H}^{2i+1}_{\mathbb{C}^\times}(\mathscr{Z}_c(V,W)) = 0 & \text{if } i \geqslant 0, \\ \mathrm{H}^{2\bullet}_{\mathbb{C}^\times} \simeq \mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) & \text{as } \mathbb{C}[\hbar]\text{-algebras,} \end{cases}$$

where $\mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) = \bigoplus_{i \geqslant 0} \hbar^i \mathscr{F}_i(\mathrm{Z}(\mathbb{C}W)) \subset \mathbb{C}[\hbar] \otimes_{\mathbb{C}} \mathrm{Z}(\mathbb{C}W)$ is the Rees algebra associated with the filtration $\mathscr{F}_\bullet(\mathrm{Z}(\mathbb{C}W))$. Using the analogous result for $\mathscr{Z}_{c'}(V',W')$ and the functoriality of equivariant cohomology, we get a morphism of algebras $i^* : \mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) \longrightarrow \mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W'))$.

Now, we will use not only [**5**, Theorem A] but also its proof: the proof goes by restriction to the fixed point subvariety $\mathscr{Z}_c(V,W)^{\mathbb{C}^\times}$ and identifying $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_c(V,W)^{\mathbb{C}^\times})$ with $\mathbb{C}[\hbar] \otimes \mathrm{Z}(\mathbb{C}W)$ using [**10**, Corollary 5.8]. As the same strategy holds for $\mathscr{Z}_{c'}(V',W')$, the

functoriality of equivariant cohomology implies that the map $i^*$ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) & \xrightarrow{\quad i^*\quad} & \mathrm{Rees}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W')) \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}[\hbar]\otimes_{\mathbb{C}}\mathrm{Z}(\mathbb{C}W) & \xrightarrow{\mathrm{Id}_{\mathbb{C}[\hbar]}\otimes_{\mathbb{C}} i^*_{\mathscr{X}}} & \mathbb{C}[\hbar]\otimes_{\mathbb{C}}\mathrm{Z}(\mathbb{C}W'). \end{array}$$

This shows the corollary. □

## 2. Preliminaries on quiver varieties

**2.A. Quiver varieties. —** Let $Q_l$ denote the cyclic quiver with $l$ vertices, defined as follows:

- Vertices: $i \in \mathbb{Z}/l\mathbb{Z}$;
- Arrows: $y_i : i \longrightarrow i+1$, $i \in \mathbb{Z}/l\mathbb{Z}$.

We denote by $\overline{Q}_l$ the double quiver of $Q_l$ that is, the quiver obtained from $Q_l$ by adding an arrow $x_i : i+1 \to i$ for all $i \in \mathbb{Z}/l\mathbb{Z}$ (see Figure 1).

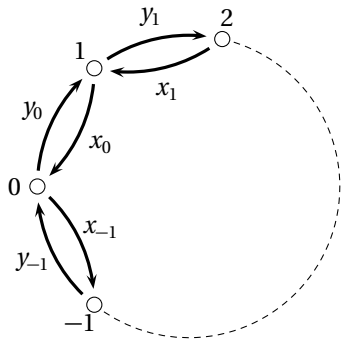



**Fig. 1.** The quiver $\overline{Q}_l$

Now, if $d = (d_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$ is a family of elements of $\mathbb{N}$, we denote by $\mathbf{GL}(d)$ the direct product

$$\mathbf{GL}(d) = \prod_{i\in\mathbb{Z}/l\mathbb{Z}} \mathbf{GL}_{d_i}(\mathbb{C}),$$

by $\Delta\mathbb{C}^\times$ the image of $\mathbb{C}^\times$ in $\mathbf{GL}(d)$ through the diagonal embedding $\Delta : \mathbb{C}^\times \hookrightarrow \mathbf{GL}(d)$ and we set

$$\mathbf{PGL}(d) = \mathbf{GL}(d)/\Delta\mathbb{C}^\times.$$

The group $\mathbf{PGL}(d)$ acts on the variety $\mathrm{Rep}(\overline{Q}_l, d)$ of representations of $\overline{Q}_l$ in the family of vector spaces $(\mathbb{C}^{d_i})_{i\in\mathbb{Z}/l\mathbb{Z}}$. The orbits are the isomorphism classes of representations of $\overline{Q}_l$ of dimension vector $d$. We denote by

$$\begin{array}{rccc} \mu_d : & \mathrm{Rep}(\overline{Q}_l, d) & \longrightarrow & \bigoplus_{i\in\mathbb{Z}/l\mathbb{Z}} \mathrm{Mat}_{d_i}(\mathbb{C}) \\ & (X_i, Y_i)_{i\in\mathbb{Z}/l\mathbb{Z}} & \longmapsto & (X_iY_i - Y_{i-1}X_{i-1})_{i\in\mathbb{Z}/l\mathbb{Z}} \end{array}$$

the corresponding *moment map*. Finally, if $\theta = (\theta_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$ is a family of complex numbers, we denote by $\mathrm{I}_\theta(d)$ the family $(\theta_i \mathrm{Id}_{\mathbb{C}^{d_i}})_{i\in\mathbb{Z}/l\mathbb{Z}}$ and by $\mathscr{O}_\theta(d)$ the closed subvariety of $\mathrm{Mat}_{d_0}(\mathbb{C})$ consisting of matrices of rank $\leqslant 1$ and trace $-\sum_{i\in\mathbb{Z}/l\mathbb{Z}} \theta_i d_i$. Finally, we set

$$\mathscr{Y}_\theta(d) = \mu_d^{-1}(\mathrm{I}_\theta(d) + \mathscr{O}_\theta(d)) \qquad \text{and} \qquad \mathscr{X}_\theta(d) = \mathscr{Y}_\theta(d)/\!/\mathbf{PGL}(d).$$

We will usually denote an element $(X_i, Y_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$ of $\mathscr{Y}_\theta(d)$ by $(X, Y)$, where $X = (X_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$ and $Y = (Y_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$. Note that $\mathscr{Y}_\theta(d)$ is endowed with a $\mathbb{C}^\times$-action: if $\xi \in \mathbb{C}^\times$, we set

$$\xi \cdot (X, Y) = (\xi^{-1}X, \xi Y).$$

This action commutes with the action of $\mathbf{GL}(d)$ and the moment map is constant on $\mathbb{C}^\times$-orbits, so it induces a $\mathbb{C}^\times$-action on $\mathscr{X}_\theta(d)$.

***Remark 2.1***. — We extend the definition of $\mathscr{X}_\theta(d)$ to the case where $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ by the convention that $\mathscr{X}_\theta(d) = \varnothing$ whenever at least one of the $d_i$'s is negative. ■

**2.B. Action of the affine Weyl group. —** If $l \geqslant 2$, let $W_l^{\mathrm{aff}}$ denote the affine Weyl group of type $\tilde{A}_{l-1}$. It is the Coxeter group with associated Coxeter system $(W_l^{\mathrm{aff}}, S_l^{\mathrm{aff}})$, where $S_l^{\mathrm{aff}} = \{s_i \mid i \in \mathbb{Z}/l\mathbb{Z}\}$ and the Coxeter graph is given by

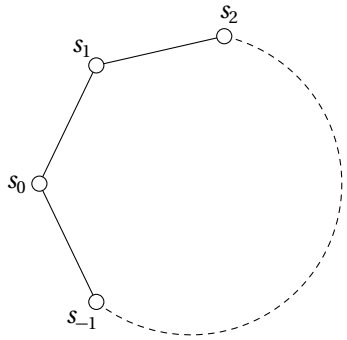



**Fig. 2.** Coxeter graph of $(W_l^{\mathrm{aff}}, S_l^{\mathrm{aff}})$

We extend this notation to the case where $l = 1$ by setting $W_1^{\mathrm{aff}} = 1$.

Consider the Lie algebra $\mathfrak{g}_l = \mathfrak{sl}_l(\mathbb{C})$ and its affine version $\widehat{\mathfrak{g}}_l = \widehat{\mathfrak{sl}}_l(\mathbb{C}) = \mathfrak{sl}_l(\mathbb{C})[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra formed by the diagonal matrices and set $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$.

The $\mathbb{C}$-vector space $\widehat{\mathfrak{h}}^*$ has a basis $(\alpha_0, \alpha_1, \ldots, \alpha_{l-1}, \Lambda_0)$, where $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ are the simple roots of $\widehat{\mathfrak{g}}_l$ and $\Lambda_0$ is such that $\Lambda_0$ annihilates $\mathfrak{h}$ and $\partial$ and $\Lambda_0(\mathbf{1}) = 1$. Denote by $R_l^{\mathrm{aff}}$ and $R_l$ the affine and the non-affine root lattices respectively (i.e., $R_l^{\mathrm{aff}}$ is the $\mathbb{Z}$-lattice generated by $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$ and $R_l$ is the sublattice generated by $\alpha_1, \ldots, \alpha_{l-1}$.)

Following [**17**], we define two actions of $W_l^{\mathrm{aff}}$: a non-linear one on $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, and a linear one on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$. If $l = 1$, there is nothing to define so we may assume that $l \geqslant 2$. If $d = (d_i)_{i\in\mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ and if $j \in \mathbb{Z}/l\mathbb{Z}$, we set $s_j(d) = (d_i')_{i\in\mathbb{Z}/l\mathbb{Z}}$, where

$$d_i' = \begin{cases} d_i & \text{if } i \neq j, \\ \delta_{j0} + d_{i+1} + d_{i-1} - d_i & \text{if } i = j. \end{cases}$$

***Remark 2.2***. — We can identify $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ with the root lattice $R_l^{\mathrm{aff}}$ by

$$d \mapsto \sum_{i\in\mathbb{Z}/l\mathbb{Z}} d_i\alpha_i.$$

Beware, the action considered here is not the usual action of $W_l^{\mathrm{aff}}$ on the root lattice. When we have $w(d) = d'$ with respect to the action define above, this corresponds to $w(\Lambda_0 - d) = \Lambda_0 - d'$ for the usual action of $W_l^{\mathrm{aff}}$ on $\widehat{\mathfrak{h}}^*$. ■

If $\theta = (\theta_i)_{i\in\mathbb{Z}/l\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we set $s_j(\theta) = (\theta'_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$, where

$$\theta'_i = \begin{cases} \theta_i & \text{if } i \notin \{j-1, j, j+1\}, \\ \theta_j + \theta_i & \text{if } i \in \{j-1, j+1\}, \\ -\theta_i & \text{if } i = j. \end{cases}$$

It is readily seen that these definitions on generators extend to an action of the whole group $W_l^{\mathrm{aff}}$. We also define a pairing $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}} \to \mathbb{C}$, $(d, \theta) \mapsto d \cdot \theta$, where

$$d \cdot \theta = \sum_{i\in\mathbb{Z}/l\mathbb{Z}} d_i\theta_i.$$

Then

$$s_j(d) \cdot s_j(\theta) = (d \cdot \theta) - \delta_{j0}\theta_0. \tag{2.3}$$

It is proved in [**17**, Corollary 3.6] that

$$\mathscr{X}_{s_j(\theta)}(s_j(d)) \simeq \mathscr{X}_\theta(d) \qquad \text{if } \theta_j \neq 0. \tag{2.4}$$

Note that this isomorphism takes into account the convention of Remark 2.1.

Recall also from [**8**, §11] the following result, which follows from the fact that $\mathscr{X}_\theta(n\delta_l)$ is isomorphic to some Calogero-Moser space.

***Lemma 2.5***. — *If $n \geqslant 0$, then $\mathscr{X}_\theta(n\delta_l)$ is normal and irreducible of dimension $2n$.*

Recall that the affine Weyl group has another presentation. We have $W_l^{\mathrm{aff}} = W_l \ltimes R_l$, where $W_l = \mathfrak{S}_l$ is the non-affine Weyl group, that is, the subgroup generated by $s_1, \ldots, s_{l-1}$. For each $\alpha \in R_l$, denote by $t_\alpha$ the image of $\alpha$ in $W_l^{\mathrm{aff}}$. Each element of $W_l^{\mathrm{aff}}$ can be written in a unique way in the form $w \cdot t_\alpha$, where $w \in W_l$ and $\alpha \in R_l$. Let $\delta_l$ denote the constant family $\delta = (1)_{i\in\mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. In the following lemma we identify $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ with $R_l^{\mathrm{aff}}$.

***Lemma 2.6***. — *Assume $\alpha \in R_l$ and $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. Then we have $t_\alpha(d) \equiv d - \alpha \mod \mathbb{Z}\delta_l$.*

*Proof*. — This statement is a partial case of [**13**, (6.5.2)] (see also Remark 2.2). □

***Lemma 2.7***. — *For each $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, there exists a unique $n \in \mathbb{Z}$ such that $d$ and $n\delta_l$ are in the same $W_l^{\mathrm{aff}}$-orbit.*

*Proof.* — Let us prove the existence of $n$. Set $\alpha = d - d_0\delta_l$. Then $\alpha$ is clearly an element of $R_l$. Then by Lemma 2.6, the element $t_\alpha(d)$ is of the form $n\delta_l$.

Let us prove the uniqueness of $n$. Assume that $n_1\delta_l$ and $n_2\delta_l$ are in the same $W_l^{\mathrm{aff}}$-orbit. Since $n_1\delta_l$ is $W_l$-stable, the $W_l^{\mathrm{aff}}$-orbit of $n_1\delta_l$ coincides with the $R_l$-orbit of $n_1\delta_l$. So there exists $\alpha \in R_l$ such that $t_\alpha(n_1\delta_l) = n_2\delta_l$. By Lemma 2.6, this forces $\alpha = 0$ and $n_1 = n_2$. □

Consider the $\mathbb{Z}$-linear map

$$R_l^{\mathrm{aff}} \to \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}, \quad d \mapsto \overline{d},$$

given by

$$(\overline{\alpha_r})_i = 2\delta_{i,r} - \delta_{i,r+1} - \delta_{i,r-1}.$$

The kernel of this map is $\mathbb{Z}\delta_l$. Set

$$\Sigma(\theta) = \sum_{i \in \mathbb{Z}/l\mathbb{Z}} \theta_i.$$

***Lemma 2.8.*** — *For each $\alpha \in R_l$ and $\theta \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we have $t_\alpha(\theta) = \theta + \Sigma(\theta)\overline{\alpha}$.*

*Proof.* — The $W_l^{\mathrm{aff}}$-action on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ defined above coincides with the (usual) action of $W_l^{\mathrm{aff}}$ on the dual of the span of $\alpha_0, \alpha_1, \dots, \alpha_{l-1}$ in $\widehat{\mathfrak{h}}^*$. The statement follows from [**13**, (6.5.2)]. □

We denote by $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ the set of $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ which belong to the orbit of some $n\delta_l$, with $n \geqslant 0$. This set has a more precise combinatorial description. For example, it will follow from Proposition 4.4 that an element $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if $d$ is a residue of some Young diagram. In particular, the set $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ is contained in $\mathbb{N}^{\mathbb{Z}/l\mathbb{Z}}$.

***Lemma 2.9.*** — *Assume that $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is such that there exists a simple representation in $\mathscr{Y}_\theta(d)$. Then*

*(a) the variety $\mathscr{X}_\theta(d)$ is normal and irreducible,*
*(b) we have $d \in (\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$.*

*Proof.* — Let $n \in \mathbb{Z}$ and $w \in W_l^{\mathrm{aff}}$ be such that $n\delta_l = w(d)$. Let us show that (2.4) implies an isomorphism $\mathscr{X}_\theta(d) \simeq \mathscr{X}_{w(\theta)}(n\delta_l)$. (In particular, we must have $n \geqslant 0$.) The only difficulty that we have is caused by the condition $\theta_i \neq 0$ in (2.4). But [**7**, Lemma 7.2] implies that we can find a sequence of reflections such that $s_{i_1} s_{i_2} \dots s_{i_r}(d) = n\delta_l$ and such that we can apply a sequence of isomorphisms (2.4) with no danger to get $\theta_{i_j} = 0$ while applying $s_{i_j}$. □

**2.C. Smoothness. —** The following result is proved in [**11**, Lemma 4.3 and its proof] (which is based on [**15**, Theorem 1.2] and [**7**, Theorem 1.2]):

***Theorem 2.10***. — *Let $n \geqslant 0$. Then the following are equivalent:*

(1) *The variety $\mathscr{X}_\theta(n\delta_l)$ is smooth.*
(2) *Every element of $\mathscr{Y}_\theta(n\delta_l)$ defines a simple representation of the quiver $\overline{Q}_l$.*
(3) *The family $\theta$ satisfies*

$$\Sigma(\theta) \prod_{\substack{1 \leqslant i \leqslant j \leqslant l-1 \\ -(n-1) \leqslant k \leqslant n-1}} \big((\theta_i + \theta_{i+1} + \cdots + \theta_j) + k\Sigma(\theta)\big) \neq 0.$$

*If these equivalent conditions hold, then $\mathbf{PGL}(n\delta_l)$ acts freely on $\mathscr{Y}_\theta(n\delta_l)$, and so, as a set, $\mathscr{X}_\theta(n\delta_l)$ identifies with $\mathbf{GL}(n\delta_l)$-orbits in $\mathscr{Y}_\theta(n\delta_l)$.*

**2.D. Fixed points. —** For every $m \geqslant 1$, the group of $m$-th roots of unity $\boldsymbol{\mu}_m$ acts on $\mathscr{X}_\theta(n\delta_l)$ (this is the restriction of the $\mathbb{C}^\times$-action defined in §2.A). We aim to compute the fixed point variety $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$ whenever $\mathscr{X}_\theta(n\delta_l)$ is smooth. Note first that $\boldsymbol{\mu}_l$ acts trivially on $\mathscr{X}_\theta(n\delta_l)$: indeed, if $\zeta \in \boldsymbol{\mu}_l$ and if $(X,Y) \in \mathscr{Y}_\theta(n\delta_l)$, then $\zeta \cdot (X,Y) = {}^{g_0}(X,Y)$, where $g_0 = (\zeta^i \operatorname{Id}_{\mathbb{C}^n})_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbf{GL}(n\delta_l)$. This means that, in order to describe $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$, we may replace $\boldsymbol{\mu}_m$ by $\langle \boldsymbol{\mu}_m, \boldsymbol{\mu}_l \rangle$ or, in other words, we may assume that $l$ divides $m$.

> **Hypothesis.** *We fix in this subsection a non-zero natural number $k$ and we set $m = kl$.*

If $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we define $\theta[k]$ to be the element $(\theta[k]_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ such that

$$\theta[k]_j = \theta_i$$

if $j \equiv i \mod l$. Roughly speaking, $\theta[k]$ is the concatenation of $k$ copies of $\theta$.

We denote by $\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ the set of $d = (d_j)_{j \in \mathbb{Z}/m\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}$ such that

$$n = \sum_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \mod l}} d_j$$

for all $i \in \mathbb{Z}/l\mathbb{Z}$. Set also $\mathscr{E}(k,l,n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. If $d \in \mathscr{E}(k,l,n)$ and $(X,Y) \in \operatorname{Rep}(\overline{Q}_m, d)$, we set

$$i_k^{(d)}(X,Y) = (X', Y'),$$

where $X'_i = \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \mod l}} X_j$ and $Y'_i = \bigoplus_{\substack{j \in \mathbb{Z}/m\mathbb{Z} \\ j \equiv i \mod l}} Y_j$. By the definition of $\mathscr{E}(k,l,n)$, we have $X'_i,\ Y'_i \in \operatorname{Mat}_n(\mathbb{C})$. In other words, $(X', Y') \in \operatorname{Rep}(\overline{Q}_l, n\delta_l)$: it is clear that $i_k^{(d)} : \operatorname{Rep}(\overline{Q}_m, d) \hookrightarrow \operatorname{Rep}(\overline{Q}_l, n\delta_l)$ is a closed immersion.

***Lemma 2.11***. — *If $d \in \mathscr{E}(k,l,n)$ and $(X,Y) \in \mathscr{Y}_{\theta[k]}(d)$, then $i_k^{(d)}(X,Y) \in \mathscr{Y}_\theta(n\delta_l)$.*

*Proof.* — Write $i_k^{(d)}(X,Y)=(X',Y')$ and let $i\in\mathbb{Z}/l\mathbb{Z}$. Write $p_0$ for the rank one matrix $X_0Y_0-Y_{-1}X_{-1}-\theta[k]_0\,\mathrm{Id}_{\mathbb{C}^{d_0}}$. Then

$$\begin{aligned}X_i'Y_i'-Y_{i-1}'X_{i-1}' &= \bigoplus_{\substack{j\in\mathbb{Z}/m\mathbb{Z}\\ j\equiv i \mod l}}(X_jY_j-Y_{j-1}X_{j-1})\\ &= \bigoplus_{\substack{j\in\mathbb{Z}/m\mathbb{Z}\\ j\equiv i \mod l}}\theta[k]_j\,\mathrm{Id}_{\mathbb{C}^{d_j}}+\delta_{0i}p_0\\ &= \theta_i\,\mathrm{Id}_{\mathbb{C}^n}+\delta_{0i}p_0,\end{aligned}$$

as desired. □

It is also clear that, if $(X,Y)$ and $(\widetilde{X},\widetilde{Y})$ are two elements of $\mathscr{Y}_{\theta[k]}(d)$ which are conjugate under $\mathbf{GL}(d)$, then $i_k^{(d)}(X,Y)$ and $i_k^{(d)}(\widetilde{X},\widetilde{Y})$ are conjugate under $\mathbf{GL}(n\delta_l)$. This means that $i_k^{(d)}$ induces a morphism of algebraic varieties still denoted by

$$i_k^{(d)}:\mathscr{X}_{\theta[k]}(d)\longrightarrow\mathscr{X}_\theta(n\delta_l).$$

For $d\in\mathscr{E}(k,l,n)$, set

$$\mathscr{X}_d=i_k^{(d)}(\mathscr{X}_{\theta[k]}(d))\subset\mathscr{X}_\theta(n\delta_l).$$

For the moment, we allow the possibility that the varieties $\mathscr{X}_{\theta[k]}(d)$ and $\mathscr{X}_d$ are empty. Set

$$\mathscr{E}(k,l,n)_{\neq\varnothing}=\{d\in\mathscr{E}(k,l,n);\ \mathscr{X}_d\neq\varnothing\}.$$

But we will show in Corollary 4.13 that we have $\mathscr{E}(k,l,n)_{\neq\varnothing}=\mathscr{E}(k,l,n)$.

The next result describes the fixed point variety $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$: it is the main step towards the proof of Theorem 1.3.

***Theorem 2.12***. — *Assume that $\mathscr{X}_\theta(n\delta_l)$ is smooth. Then:*

(a) *$\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$ is smooth.*

(b) *We have*

$$\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}=\coprod_{d\in\mathscr{E}(k,l,n)_{\neq\varnothing}}\mathscr{X}_d$$

*and $\mathscr{X}_d$ are exactly the irreducible components of $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$.*

(c) *If $d\in\mathscr{E}(k,l,n)_{\neq\varnothing}$, then $i_k^{(d)}:\mathscr{X}_{\theta[k]}(d)\to\mathscr{X}_d$ is an isomorphism.*

*Proof.* — (a) follows from the fact already mentioned in the proof of Corollary 1.5 that the fixed point variety under the action of a finite group on a smooth variety is again smooth.

We will now prove (b). Let $\xi$ be a primitive $m$-th root of unity and let $\zeta=\xi^k$ (it is a primitive $l$-th root of unity). Let $g_0$ denote the element $(\zeta^i\,\mathrm{Id}_{\mathbb{C}^n})_{i\in\mathbb{Z}/l\mathbb{Z}}$ in $\mathbf{GL}(n\delta_l)$. This element satisfies

$$(*)\qquad {}^{g_0}(X,Y)=(\zeta^{-1}X,\zeta Y)$$

for all $(X,Y)\in\mathscr{Y}_\theta(n\delta_l)$. We will use the following variety:

$$\mathscr{Z}=\{(X,Y,g)\in\mathscr{Y}_\theta(n\delta_l)\times\mathbf{GL}(n\delta_l)\mid {}^g(X,Y)=(\xi^{-1}X,\xi Y)\text{ and }gp_{X,Y}=p_{X,Y}\},$$

where $p_{X,Y} = ([X_i, Y_i] - \theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$: it is 0 except on the 0-th component, where it is equal to a rank one matrix of trace $-n\Sigma(\theta)$ (recall from Theorem 2.10 that $\Sigma(\theta) \neq 0$ because $\mathscr{X}_\theta(n\delta_l)$ is assumed to be smooth).

We denote by $p' : \mathscr{Z} \to \mathscr{Y}_\theta(n\delta_l)$ the projection on the first two factors, by $p : \mathscr{Z} \to \mathscr{X}_\theta(n\delta_l)$ the map induced by $p'$ and by $q : \mathscr{Z} \to \mathbf{GL}(n\delta_l)$ the projection on the third factor. Note that $\mathbf{PGL}(n\delta_l)$ acts on $\mathscr{Z}$ by conjugacy on the three factors, and so the maps $p'$ and $q$ are equivariant for this action.

First, it is clear that $p(\mathscr{Z}) \subset \mathscr{X}_\theta(n\delta_l)^{\mu_m}$. Conversely, if $(X, Y) \in \mathscr{Y}_\theta(n\delta_l)$ is a representative of an element of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$, then there exists $g \in \mathbf{PGL}(n\delta_l)$ such that ${}^g(X, Y) = (\xi^{-1}X, \xi Y)$. In particular, ${}^g p_{X,Y} = p_{X,Y}$. So $g$ stabilizes the image of $p_{X,Y}$, which is of dimension 1: this means that $g$ acts by a non-zero scalar $\omega$ on $\mathrm{Im}(p_{X,Y})$. Setting $g' = \omega^{-1}g$, we see that $(X, Y, g) \in \mathscr{Z}$. So we have proved the following fact:

**Fact 1.** *$p(\mathscr{Z}) = \mathscr{X}_\theta(n\delta_l)^{\mu_m}$.*

The next fact follows from the freeness of the action of $\mathbf{PGL}(n\delta_l)$ on $\mathscr{Y}_\theta(n\delta_l)$ (see Theorem 2.10).

**Fact 2.** *If $(X, Y, g)$ and $(X, Y, g')$ belong to $\mathscr{Z}$, then $g = g'$ and $g^k = g_0$.*

Indeed, the hypothesis implies that ${}^g(X, Y) = {}^{g'}(X, Y)$ and ${}^{g^k}(X, Y) = {}^{g_0}(X, Y)$. So there exists two non-zero complex numbers $\alpha$ and $\beta$ such that $g' = \alpha g$ and $g^k = \beta g_0$. But the condition $g p_{X,Y} = g' p_{X,Y} = g_0 p_{X,Y}$ forces $\alpha = \beta = 1$ because $p_{X,Y} \neq 0$.

Now, let $\mathscr{C}$ denote the set of elements $g \in \mathbf{GL}(n\delta_l)$ such that $g^k = g_0$. Since $g_0$ is central in $\mathbf{GL}(n\delta_l)$, $\mathscr{C}$ is a disjoint union of finitely many semisimple conjugacy classes (which are the irreducible components of $\mathscr{C}$). We fix $g \in \mathscr{C}$ such that $q^{-1}(g) \neq \varnothing$ and we denote by $\mathscr{C}_g$ its conjugacy class in $\mathbf{GL}(n\delta_l)$. We will describe $q^{-1}(g)$. For this, let $E = \bigoplus_{i \in \mathbb{Z}/l\mathbb{Z}} \mathbb{C}^n$ and let $E_j = E^{\xi^{-j} g}$ (for $j \in \mathbb{Z}/m\mathbb{Z}$). In other words, $E_j$ is the $\xi^j$-eigenspace of $g$ in $E$: it is contained in the $i$-th component of $E$, where $i$ is the element of $\mathbb{Z}/l\mathbb{Z}$ such that $j \equiv i \bmod l$. Let $d_j = \dim_{\mathbb{C}}(E_j)$ and let $d = (d_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$.

If $(X, Y, g) \in \mathscr{Z}$, then $X$ sends $E_j$ to $E_{j-1}$ (we denote by $X_j : E_j \to E_{j-1}$ the induced map) while $Y$ sends $E_j$ to $E_{j+1}$ (we denote by $Y_j : E_j \to E_{j+1}$ the induced map). By choosing a basis of every $E_j$, the family $(X_j, Y_j)_{j \in \mathbb{Z}/m\mathbb{Z}}$ defines a representation of the quiver $\overline{Q}_m$, of dimension vector $d$. This means that we have defined a map $q^{-1}(g) \longrightarrow \mathrm{Rep}(\overline{Q}_m, d)$, which is clearly a closed immersion, and whose image is $\mathscr{Y}_{\theta[k]}(d)$ (because $\mathrm{Im}(p_{X,Y})$ is contained in $E_0$).

Note that by construction we have $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$. Moreover, by Lemma 2.9 $(b)$ and Theorem 2.10 we also have $d \in (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. This means that $d$ is an element of $\mathscr{E}(k, l, n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. Moreover, it is clear that the set of elements $d$ that can apper as above (from some $g \in \mathscr{C}$ such that $q^{-1}(g) \neq \varnothing$) is $\mathscr{E}(k, l, n)_{\neq \varnothing}$.

This shows the following fact.

**Fact 3.** *With the above notation, $q^{-1}(g) \simeq \mathscr{Y}_{\theta[k]}(d)$ . In particular, $q^{-1}(g)$ is irreducible.*

Fact 3 and its proof imply the following.

**Fact 4.** *We have $p(q^{-1}(\mathscr{C}_g)) = \mathscr{X}_d$.*

Denote by $\pi$ the projection $\pi\colon \mathscr{Y}_\theta(n\delta_l) \to \mathscr{X}_\theta(n\delta_l)$. Fact 2 implies that the map $p'\colon \mathscr{Z} \to \pi^{-1}(\mathscr{X}_\theta(n\delta_l)^{\mu_m})$ is bijective. Moreover, since $\pi^{-1}(\mathscr{X}_\theta(n\delta_l)^{\mu_m})$ is smooth (the group $\mathbf{PGL}(n\delta_l)$ acts freely on $\mathscr{Y}_\theta(n\delta_l)$), this bijection is an isomorphism of algebraic varieties. Let

$$(p')^{-1}\colon \pi^{-1}(\mathscr{X}_\theta(n\delta_l)^{\mu_m}) \to \mathscr{Z}$$

be its inverse.

Consider the morphism $\widetilde{\tau}\colon \pi^{-1}(\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}) \to \mathbf{GL}(n\delta_l)$ given by $\widetilde{\tau} = q \circ (p')^{-1}$. Since we have $\pi^{-1}(\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m})/\!/\mathbf{GL}(n\delta_l) = \mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$, we get a morphism $\tau\colon \mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m} \to \mathbf{GL}(n\delta_l)/\!/\mathbf{GL}(n\delta_l)$, where the group $\mathbf{GL}(n\delta_l)$ acts on itself by conjugation.

Moreover, since we clearly have $\tau^{-1}(\mathscr{C}_g) = p(q^{-1}(\mathscr{C}_g))$, Fact 4 implies the following.

**Fact 5.** *We have $\tau^{-1}(\mathscr{C}_g) = \mathscr{X}_d$. In particular $\mathscr{X}_d$ is closed.*

This completes the proof of $(b)$. Now we prove $(c)$. The group $\mathbf{GL}(n\delta_l)$ acts transitively on $\mathscr{C}_g$. So we get an isomorphism of varieties

$$q^{-1}(g)/\!/C_{\mathbf{GL}(n\delta_l)}(g) \xrightarrow{\sim} q^{-1}(\mathscr{C}_g)/\!/\mathbf{GL}(n\delta_l).$$

Now, if $d = (\dim_{\mathbb{C}}(E^{\xi^{-j}g}))_{j\in\mathbb{Z}/m\mathbb{Z}}$, $q^{-1}(g)$ identifies with $\mathscr{Y}_{\theta[k]}(d)$ and $C_{\mathbf{GL}(n\delta_l)}(g)$ identifies with $\mathbf{GL}(d)$. Glueing this with Facts 3 and 4, we get an isomorphism of varieties

$$\mathscr{X}_{\theta[k]}(d) \xrightarrow{\sim} \mathscr{X}_d,$$

which is the map $i_k^{(d)}$. This proves (c). □

***Corollary 2.13.*** — *If $k > n$, then we have $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m} = \mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$.*

*Proof.* — Take $d \in \mathscr{E}(k,l,n)_{\neq\varnothing}$. Since $k > n$, some component of $d$ must be zero. This implies that $\mathbb{C}^\times$ acts trivially on $\mathscr{X}_{\theta[k]}(d)$ because the action of $\mathbb{C}^\times$ is induced by elements of $\mathbf{GL}(d)$.

Since $i_k^{(d)}$ is $\mathbb{C}^\times$-equivariant and

$$\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m} = \coprod_{d\in\mathscr{E}(k,l,n)_{\neq\varnothing}} i_k^{(d)}(\mathscr{X}_{\theta[k]}(d))$$

(by Theorem 2.12(b)), the $\mathbb{C}^\times$-action on the variety $\mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}$ is also trivial. □

## 3. Proof of Theorem 1.3

First, the explicit description of $\mathscr{Z}_c$ whenever $n = 1$ (see Example 1.2) allows to prove easily Theorem 1.3 in this case. Also, let us decompose $V$ as

$$V = V^W \oplus V_1 \oplus \cdots \oplus V_r,$$

where the $V_i$ are the non-trivial irreducible components of $V$ as a $\mathbb{C}W$-module. Then $W$ decomposes as $W = W_1 \times \cdots \times W_r$, where $(V_i, W_i)$ is then a reflection subquotient (subgroup) of $W$. Let $c_i$ denote the restriction of $c$ to $W_i$. Then

$$\mathscr{Z}_c(V,W) \simeq V^W \times V^{*W} \times \mathscr{Z}_{c_1}(V_1,W_1) \times \cdots \times \mathscr{Z}_{c_r}(V_r,W_r)$$

so that the proof of Theorem 1.3 is easily reduced to the case where $W$ acts irreducibly on $V/V^W$.

In this case, the smoothness of $\mathscr{Z}_c(V,W)$ implies that $W$ is of type $G_4$ or $G(l,1,n)$ for some $l \geqslant 1$. The case of type $G_4$ will be handled by computer calculations: see Section 5 for details. This means that we can work under the following hypothesis:

**Hypothesis.** *From now on, and until the end of this section, we assume that $n \geqslant 2$, that $V = \mathbb{C}^n$ and that $W = G(l,1,n)$. We also assume that $\sigma$ is an $m$-th root of unity, where $m = kl$ for some $k \geqslant 1$.*

Recall that $G(l,1,n)$ is the group of monomial matrices with coefficients in $\boldsymbol{\mu}_l$ (the group of $l$-th root of unity in $\mathbb{C}^\times$) and recall that $n \geqslant 2$.

**3.A. Quiver varieties vs Calogero-Moser spaces. —** We fix a primitive $l$-th root of unity $\zeta$. We denote by $s$ the permutation matrix corresponding to the transposition $(1,2)$ and we set

$$t = \mathrm{diag}(\zeta, 1, \dots, 1) \in W.$$

Then $s$, $t$, $t^2, \dots, t^{l-1}$ is a set of representatives of conjugacy classes of reflections of $W$. We set for simplification

$$a = c_s \qquad \text{and} \qquad k_j = \frac{1}{l}\sum_{i=1}^{l-1} \zeta^{-i(j-1)} c_{t^i}$$

for $j \in \mathbb{Z}/l\mathbb{Z}$. Then

$$k_0 + \cdots + k_{l-1} = 0 \qquad \text{and} \qquad c_{t^i} = \sum_{j \in \mathbb{Z}/l\mathbb{Z}} \zeta^{i(j-1)} k_j \tag{3.1}$$

for $1 \leqslant i \leqslant l-1$. Finally, if $i \in \mathbb{Z}/l\mathbb{Z}$, we set

$$\theta_i = \begin{cases} k_{-i} - k_{1-i} & \text{if } i \neq 0, \\ -a + k_0 - k_1 & \text{if } i = 0. \end{cases} \tag{3.2}$$

and $\theta = (\theta_i)_{i \in \mathbb{Z}/l\mathbb{Z}}$.

The following result is proved in [**11**, Theorem 3.10]. (Note that our $k_i$ is related with Gordon's $H_i$ via $H_i = k_{-i} - k_{1-i}$.)

***Theorem 3.3***. — *With the above notation, there is a $\mathbb{C}^\times$-equivariant isomorphism of varieties*

$$\mathscr{Z}_c \xrightarrow{\sim} \mathscr{X}_\theta(n\delta_l).$$

The theorem above is also true for $n = 1$. But in this case, the variety $\mathscr{Z}_c$ has no parameter $a$. On the other hand, the variety $\mathscr{X}_\theta(\delta_l)$ is independent of $\theta_0$.

Putting Theorems 2.10 and 3.3 together, one gets:

***Corollary 3.4***. — *The variety $\mathscr{Z}_c$ is smooth if and only if*

$$a \prod_{\substack{0 \leqslant i \neq j \leqslant l-1 \\ 0 \leqslant r \leqslant n-1}} (k_i - k_j - ra) \neq 0.$$

*Proof of Theorem 1.3.* — Assume now that $\mathscr{Z}_c$ is smooth and let $\mathscr{X}$ be an irreducible component of $\mathscr{Z}_c^{\boldsymbol{\mu}_m}$. Using the isomorphism of Theorem 3.3, we see that

$$\mathscr{Z}_c^{\boldsymbol{\mu}_m} \xrightarrow{\sim} \mathscr{X}_\theta(n\delta_l)^{\boldsymbol{\mu}_m}.$$

So, by Theorem 2.12, there exists $d \in \mathscr{E}(k,l,n)_{\neq\emptyset}$ such that $\mathscr{X} \simeq \mathscr{X}_{\theta[k]}(d)$. By Lemma 2.7, there exists $r \geqslant 0$ and $w \in W_m^{\mathrm{aff}}$ such that $w(d) = r\delta_m$. So it follows from the isomorphism (2.4) that

$$\mathscr{X} \simeq \mathscr{X}_{w(\theta[k])}(r\delta_m)$$

(see also Lemma 2.9 and its proof). Using Theorem 3.3 in the other way, we see that there exists a complex valued function $c'$ on the set of conjugacy classes of reflections of $W' = G(m,1,r)$ such that $\mathscr{X}_{w(\theta[k])}(r\delta_m) \simeq \mathscr{Z}_{c'}(W')$. Moreover, as the action of $W_m^{\mathrm{aff}}$ on $\mathbb{C}^{\mathbb{Z}/m\mathbb{Z}}$ is linear, this implies that the map $c \mapsto c'$ is linear.

This proves almost every statement of Theorem 1.3, except that $W'$ can be realized as a reflection subquotient of $(V,W)$. For this, we need the following fact.

**Fact.** *We have* $kr \leqslant n$.

Indeed, the element $d - r\delta_m$ is in the $W_m^{\mathrm{aff}}$-orbit of 0. Then Proposition 4.4 implies that $d - r\delta_m$ is a residue of some $m$-core. In particular, each coordinates of $d - r\delta_m$ must be positive. Since the sum of the coordinated of $d$ is $nl$, this gives $nl - rkl \geqslant 0$. This implies $kr \leqslant n$.

Now, using the fact proved above, let $x$ denotes the matrix

$$x = \mathrm{diag}(\underbrace{\zeta, 1, \dots, 1}_{k \text{ terms}}) \cdot M_{(1,2,\dots,k)} \in \mathbf{GL}_k(\mathbb{C}),$$

where $M_{(1,2,\dots,k)}$ is the permutation matrix associated with the cycle $(1,2,\dots,k)$. Now, let $g$ be the matrix

$$g = \mathrm{diag}(\underbrace{x, \dots, x}_{r \text{ terms}}, \mathrm{Id}_{\mathbb{C}^{n-rk}}) \in G(l,1,n).$$

Let $V'$ denote the $\xi$-eigenspace of $g$, where $\xi$ is a primitive $m$-th root of unity such that $\xi^k = \zeta$. Then $C_W(g)$ acts on $V'$, $V'$ is of dimension $r$ and, if we denote by $K$ the kernel of this action, then $C_W(g)/K \simeq G(m,1,r)$. So $(V', G(m,1,r))$ is a reflection subquotient of $(V,W)$. □

***Remark 3.5***. — The formula (3.2) above yields a bijection between $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ and the set of $(l+1)$-tuples of complex numbers $(a, k_0, \dots, k_{l-1})$ such that $k_0 + k_1 + \dots + k_{l-1} = 0$. Moreover, the $W_l^{\mathrm{aff}}$-action on $\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ (see Section 2.B) translates to the action on the $(l+1)$-tuples in the following way

$$s_r(a, k_0, \dots, k_{l-r}, k_{l-r+1}, \dots, k_{l-1}) = (a, k_0, \dots, k_{l-r+1}, k_{l-r}, \dots, k_{l-1}), \quad \text{for } r = 1,2,\dots,l-1$$

and

$$s_0(a, k_0, k_1, k_2, \dots, k_{l-1}) = (a, k_1 + a, k_0 - a, k_2, \dots, k_{l-1}).$$

In particular, we see that the subgroup $W_l$ of $W_l^{\mathrm{aff}}$ acts by permutation of the parameters $k_0, k_1, \dots, k_{l-1}$. ■

The next result follows from **[2]**, but we provide here a different proof (which works only in type $G(l,1,n)$).

***Corollary 3.6***. — *A permutation of parameters $k_0, k_1, \dots, k_{l-1}$ does not change the Calogero-Moser space $\mathscr{Z}_c$ (up to an isomorphism of algebraic varieties).*

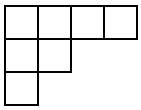
*Proof*. — The subgroup $W_l$ of $W_l^{\mathrm{aff}}$ stabilizes $\delta_l$. Then by (2.4), for each $w \in W_l$, $n \in \mathbb{Z}_{>0}$ and $\theta \in \mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$, we have $\mathscr{X}_\theta(n\delta_l) \simeq \mathscr{X}_{w(\theta)}(n\delta_l)$. This proves the statement by Theorem 3.3 and Remark 3.5 □

## 4. Combinatorics

In this subsection, we aim to make the statement of Theorem 1.3 more precise in the case where $W = G(l,1,n)$: we wish to describe precisely the map $c \mapsto c'$ in terms of the combinatorics of partitions, cores, $l$-quotients, etc.

**4.A. Partitions and cores. —** Let $l$ and $n$ be positive integers. A *partition* is a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers (with no fixed length) such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. Set $|\lambda| = \sum_{i=1}^{r} \lambda_i$. If $|\lambda| = n$, we say that $\lambda$ is a partition of $n$.

Denote by $\mathscr{P}$ (resp. $\mathscr{P}[n]$) be the set of all partitions (resp. the set of all partitions of $n$). By convention, $\mathscr{P}[0]$ contains one (empty) partition. We will identify partitions with Young diagrams. The partition $\lambda$ corresponds to a Young diagram with $r$ lines such that the $i$th line contains $\lambda_i$ boxes. For example the partition $(4,2,1)$ corresponds to the Young diagram



Let $b$ be a box of a Young diagram in the line $r$ and column $s$. The *$l$-residue* of the box $b$ is the number $s-r$ modulo $l$. (We also say that the integer $s-r$ is the $\infty$-residue of the box $b$). Then we obtain a map

$$\mathrm{Res}_l : \mathscr{P} \to \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}, \qquad \lambda \mapsto \mathrm{Res}_l(\lambda),$$

such that for each $i \in \mathbb{Z}/l\mathbb{Z}$ the number of boxes with residue $i$ in $\lambda$ is $(\mathrm{Res}_l(\lambda))_i$. (Similarly, we obtain a map $\mathrm{Res}_\infty : \mathscr{P} \to \mathbb{Z}^{\mathbb{Z}}$.)

***Example 4.1***. — For the partition $\lambda = (4,2,1)$ and $l = 3$ the residues of the boxes are

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 0 |
| 2 | 0 | | |
| 1 | | | |

In this case we have $\mathrm{Res}_l(\lambda) = (3,2,2)$ because there are three boxes with residue 0, two boxes with residue 1 and two boxes with residue 2. ■

We say that a box of a Young diagram is *removable* if it has no boxes on the right and on the bottom. For $\lambda, \mu \in \mathscr{P}$, we write $\mu \leqslant \lambda$ if the Young diagram of $\mu$ can be obtained from the Young diagram of $\lambda$ by removing a sequence of removable boxes.

***Definition 4.2***. — *We say that the partition $\lambda$ is an $l$-core if there is no partition $\mu \leqslant \lambda$ such that the Young diagram of $\mu$ differs from the Young diagram of $\lambda$ by $l$ boxes with $l$ different $l$-residues.*

See [**3**] for more details about the combinatorics of $l$-cores. Let $\mathscr{C}_l \subset \mathscr{P}$ be the set of $l$-cores. Set $\mathscr{C}_l[n] = \mathscr{P}[n] \cap \mathscr{C}_l$.

If a partition $\lambda$ is not an $l$-core, then we can get a smaller Young diagram from its Young diagram by removing $l$ boxes with different $l$-residues. We can repeat this operation again and again until we get an $l$-core. It is well-known, that the $l$-core that we get is independent of the choice of the boxes that we remove. Then we get an application

$$\mathrm{Core}_l : \mathscr{P} \to \mathscr{C}_l.$$

If $\mu = \mathrm{Core}_l(\lambda)$, we will say that the partition $\mu$ is *the $l$-core* of the partition $\lambda$.

***Example 4.3***. — The partition $(4,2,1)$ from the previous example is not a 3-core because it is possible to remove three bottom boxed. We get

| 0 | 1 | 2 | 0 |
|---|---|---|---|

But this is still not a 3-core because we can remove three more boxes and we get

| 0 |
|---|

This shows that the partition $(1)$ is the 3-core of the partition $(4,2,1)$. ■

As mentioned in Lemma 2.7, for each $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$, the $W_l^{\mathrm{aff}}$-orbit of $d$ contains an element of the form $n\delta_l$ for a unique $n \in \mathbb{Z}$.

***Proposition 4.4***. — *Let $d$ be an element of $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. The following statements are equivalent.*

*(a) $d$ is of the form $d = \mathrm{Res}_l(\lambda)$ for some $l$-core $\lambda$,*

*(b) $d$ is in the $W_l^{\mathrm{aff}}$-orbit of $0$.*

*Proof*. — Consider the $W_l^{\mathrm{aff}}$-action on $\mathscr{C}_l$ as in [**3**, Sec. 3]. By construction, the map $\mathrm{Res}_l : \mathscr{C}_l \to \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is $W_l^{\mathrm{aff}}$-invariant. Moreover, the residue of the empty partition is zero. The stabilizer of the empty partition in $W_l^{\mathrm{aff}}$ is $W_l$ and the stabilizer of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ in $W_l^{\mathrm{aff}}$ is also $W_l$. This shows that the map $\mathrm{Res}_l$ yields a bijection between the set of $l$-cores and the $W_l^{\mathrm{aff}}$-orbit of 0 in $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. □

Denote by $\pi$ the obvious surjection $\pi : \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \to \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}/\mathbb{Z}\delta_l$.

***Proposition 4.5***. — *The chain of maps*

$$\mathscr{C}_l \xrightarrow{\mathrm{Res}_l} \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}} \xrightarrow{\pi} \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}/\mathbb{Z}\delta_l$$

*yields a bijection between $\mathscr{C}_l$ and $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}/\mathbb{Z}\delta_l$.*

*Proof*. — We have seen in the proof of Proposition 4.4 that $\mathrm{Res}_l$ yields a bijection between $\mathscr{C}_l$ and the $W_l^{\mathrm{aff}}$-orbit of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$. This proves the statement because the restriction of $\pi$ to the $W_l^{\mathrm{aff}}$-orbit of $0 \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ is bijective by Lemma 2.7. □

For $d = (d_i)_{i\in\mathbb{Z}/l\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}$ we set $|d| = \sum_{i\in\mathbb{Z}/l\mathbb{Z}} d_i$. Set $\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n] = \{d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}; |d| = n\}$. Denote by $\mathscr{C}_l[\equiv n]$ the subset of $\mathscr{C}_l$ that contains only $l$-cores with the number of boxes congruent to $n$ modulo $l$. Proposition 4.5 yields the following bijection $\varepsilon\colon \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n] \xleftrightarrow{\sim} \mathscr{C}_l[\equiv n]$.

***Lemma 4.6.*** — *Assume $d \in \mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}}[n]$. Then $d$ is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if $|\varepsilon(d)| \leqslant n$.*

*Proof.* — By construction, $\varepsilon(d)$ is the unique $l$-core $\lambda \in \mathscr{C}_l[\equiv n]$ such that $\mathrm{Res}_l(\lambda)$ is congruent to $d$ modulo $\delta_l$. Write $d = \mathrm{Res}_l(\lambda) + r\delta_l$. Since by Proposition 4.4, the element $\mathrm{Res}_l(\lambda)$ is in the $W_l^{\mathrm{aff}}$-orbit of 0, then $d$ is in the $W_l^{\mathrm{aff}}$-orbit of $r\delta_l$. Then, by definition, the element $d$ is in $(\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}})_+$ if and only if $r \geqslant 0$. Now, let us count the sum of the coordinates in the sides of the equality $d = \mathrm{Res}_l(\lambda) + r\delta_l$. We obtain $n = |\lambda| + rl$. This shows that we have $r \geqslant 0$ if and only if $|\lambda| \leqslant n$. □

Let $\nu$ be an $l$-core. Denote by $\mathscr{P}_{l,\nu}$ the set of partitions with the $l$-core $\nu$. In particular we denote by $\mathscr{P}_{l,\varnothing}$ the set of partitions with trivial $l$-core. Let $\mathscr{P}^l$ be the set of $l$-partitions (i.e., the set of $l$-tuples of partitions). If $\lambda$ is a partition, we denote by $\beta_l(\lambda)$ its *$l$-quotient* (see for example [**16**, Sec. 2.2]).

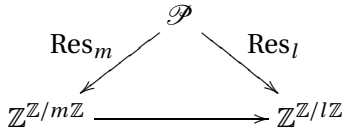
***Lemma 4.7.*** — *If $\nu$ is an $l$-core, then $\beta_l$ restricts to a bijection $\beta_{l,\nu}\colon \mathscr{P}_{l,\nu} \xleftrightarrow{\sim} \mathscr{P}^l$.*

**4.B. Description of $\mathscr{E}(k,l,n)$ in terms of cores.** — Let $k$ be a positive integer. Set $m = kl$. Set also $(\mathscr{C}_m)_{l,\nu} = \mathscr{P}_{l,\nu} \cap \mathscr{C}_m$.

Recall that we have a bijection $\varepsilon\colon \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl] \xleftrightarrow{\sim} \mathscr{C}_m[\equiv nl]$. Assume $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl]$.

***Lemma 4.8.*** — *We have $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ if and only if the $l$-core of $\varepsilon(d)$ is trivial, where the set $\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$ is as in Section 2.D.*

*Proof.* — By definition, $\varepsilon(d)$ is the unique $m$-core $\lambda \in \mathscr{C}_m[\equiv nl]$ such that $\mathrm{Res}_m(\lambda)$ is equal to $d$ modulo $\delta_m$. On the other hand, by Proposition 4.4, the partition $\lambda$ is an $l$-core if and only if $\mathrm{Res}_l(\lambda)$ is a multiple of $\delta_l$. By commutativity of the following diagram,




this condition is clearly equivalent to $d \in \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l]$. □

Recall that the set $\mathscr{E}(k,l,n)$ was defined as the intersection $\mathscr{E}(k,l,n) = \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[n\delta_l] \cap (\mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}})_+$. Combining Lemmas 4.6 and 4.8, we obtain the following.

***Lemma 4.9.*** — *The bijection $\varepsilon\colon \mathbb{Z}^{\mathbb{Z}/m\mathbb{Z}}[nl] \simeq \mathscr{C}_m[\equiv nl]$ restricts to a bijection*

$$\varepsilon'\colon \mathscr{E}(k,l,n) \to \mathscr{C}_m[\equiv nl] \cap \mathscr{C}_m[\leqslant nl] \cap (\mathscr{C}_m)_{l,\varnothing}.$$

For $\lambda=(\lambda^0,\dots,\lambda^{l-1})\in\mathscr{P}^l$ we set $|\lambda|=|\lambda^0|+\cdots+|\lambda^{l-1}|$. Set also $\mathscr{P}^l[n]=\{\lambda\in\mathscr{P}^l;\ |\lambda|=n\}$. We say that an element of $\mathscr{P}^l[n]$ is an $l$-partition of $n$.

Now recall the bijection $\beta_{l,\nu}\colon \mathscr{P}_{l,\nu}\xrightarrow{\sim}\mathscr{P}^l$ obtained by restriction from $\beta_l$ as in Lemma 4.7. By construction, it has the following properties.

***Lemma 4.10***. — (a) *For each $\lambda\in\mathscr{P}_{l,\varnothing}$, we have $\lambda\in\mathscr{C}_m$ if and only if $\beta_{l,\varnothing}(\lambda)\in(\mathscr{C}_k)^l$.*
(b) *For each $\lambda\in\mathscr{P}_{l,\nu}$, we have $|\lambda|=l\cdot|\beta_{l,\nu}(\lambda)|+|\nu|$.*

Part (a) of the lemma above shows that $\beta_{l,\emptyset}$ can be restricted to a bijection $(\mathscr{C}_m)_{l,\varnothing}\simeq(\mathscr{C}_k)^l$. Moreover, part (b) shows that new bijection restricts to a bijection

$$\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}\simeq(\mathscr{C}_k)^l[\equiv n,\leqslant n],$$

where

$$(\mathscr{C}_k)^l[\equiv n,\leqslant n]=\{\lambda\in(\mathscr{C}_k)^l;\ |\lambda|\leqslant n \text{ and } |\lambda|\equiv n \mod k\}.$$

Combining this with Lemma 4.9, we get a bijection

$$\delta\colon\mathscr{E}(k,l,n)\to(\mathscr{C}_k)^l[\equiv n,\leqslant n].$$

This bijection (and Corollary 4.13 below) yileds a parametrization of the irreducible components of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$ by the set $(\mathscr{C}_k)^l[\equiv n,\leqslant n]$.

**4.C. Parametrization of the $\mu_m$-fixed points. —** We have proved in Theorem 2.12 that the subset $\mathscr{E}(k,l,n)_{\neq\varnothing}$ of $\mathscr{E}(k,l,n)$ parametrizes the irreducible components of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$. (But we will show in Corollary 4.13 that we have $\mathscr{E}(k,l,n)_{\neq\varnothing}=\mathscr{E}(k,l,n)$.) Then the bijection $\varepsilon'$ above gives another parametrization of the irreducible components of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$ in terms of $\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$. For each $\lambda\in\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$, we set $\mathscr{X}_\lambda=\mathscr{X}_{\varepsilon'^{-1}(\lambda)}$.

This new parametrization has the following nice property.

***Lemma 4.11***. — *Fix positive integers $k_1$ and $k_2$ such that $k_1$ divides $k_2$. Set $m_1=k_1l$ and $m_2=k_2l$. Fix $\lambda_1\in\mathscr{C}_{m_1}[\equiv nl]\cap\mathscr{C}_{m_1}[\leqslant nl]\cap(\mathscr{C}_{m_1})_{l,\varnothing}$ and $\lambda_2\in\mathscr{C}_{m_2}[\equiv nl]\cap\mathscr{C}_{m_2}[\leqslant nl]\cap(\mathscr{C}_{m_2})_{l,\varnothing}$. Then we have $\mathscr{X}_{\lambda_2}\subset\mathscr{X}_{\lambda_1}$ if and only if* $\mathrm{Core}_{m_1}(\lambda_2)=\lambda_1$.

*Proof*. — Consider the map

$$p\colon\mathbb{Z}^{\mathbb{Z}/m_2\mathbb{Z}}\to\mathbb{Z}^{\mathbb{Z}/m_1\mathbb{Z}},\qquad p(d)_i=\sum_{j\equiv i \mod m_1} d_j.$$

Let $d^1\in\mathscr{E}(k_1,l,n)$ and $d^2\in\mathscr{E}(k_2,l,n)$ be such that $\varepsilon'(d^1)=\lambda_1$ and $\varepsilon'(d^2)=\lambda_2$. We have $\mathscr{X}_{d^1}=\mathscr{X}_{\lambda_1}$ and $\mathscr{X}_{d^2}=\mathscr{X}_{\lambda_2}$. It is clear from the definition that we have $\mathscr{X}_{d^2}\subset\mathscr{X}_{d^1}$ if and only if $p(d^2)=d^1$. Then the statement follows from the commutativity of the following diagram

$$\begin{array}{ccc}\mathscr{E}(k_2,l,n) & \xrightarrow{\ p\ } & \mathscr{E}(k_1,l,n)\\ \Big\downarrow{\scriptstyle\varepsilon'} & & \Big\downarrow{\scriptstyle\varepsilon'}\\ \mathscr{C}_{m_2}[\equiv nl]\cap\mathscr{C}_{m_2}[\leqslant nl]\cap(\mathscr{C}_{m_2})_{l,\varnothing} & \xrightarrow{\mathrm{Core}_{m_1}} & \mathscr{C}_{m_1}[\equiv nl]\cap\mathscr{C}_{m_1}[\leqslant nl]\cap(\mathscr{C}_{m_1})_{l,\varnothing}.\end{array}$$

□

Note that the set $\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$ considered above is nothing else but the image of $\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$ by $\mathrm{Core}_m$. In other words, the set $\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$ is the same thing as the set of $m$-cores of partitions of $nl$ with trivial $l$-cores.

***Proposition 4.12***. — *The $\mathbb{C}^\times$-fixed points in $\mathscr{X}_\theta(n\delta_l)$ are parametrized by the set $\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$. This parametrization $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}=\{p_\mu;\ \mu\in\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}\}$ can be chosen in such a way that for each $k\in\mathbb{Z}$, each $\lambda\in\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$ and each $\mu\in\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$ we have $p_\mu\in\mathscr{X}_\lambda$ if and only if $\lambda$ is the $m$-core of $\mu$.*

*Proof*. — Assume $k>n$. Then by Corollary 2.13, we have $\mathscr{X}_\theta(n\delta_l)^{\mu_m}=\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$. Moreover, the set $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ is finite. We already know that $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$ is in bijection with a subset of $\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$. Moreover, $k>n$ also implies that we have $\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}=\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$. This shows that $\mathscr{X}_\theta(n\delta_l)^{\mu_m}=\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ is in bijection with a subset of $\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$. Moreover, by Lemma 4.11, this bijection is independent of the choice of $k$. But it is well-known (see **[10]**) that $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ and $\mathscr{P}^l[n]\simeq\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$ have the same cardinalities. This shows that the bijection above is a bijection between $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ and $\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$ (not just a subset of $\mathscr{P}[nl]\cap\mathscr{P}_{l,\varnothing}$).

The second statement follows from Lemma 4.11. □

***Corollary 4.13***. — *For each $\lambda\in\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$, we have $\mathscr{X}_\lambda\neq\varnothing$. In particular, we have $\mathscr{E}(k,l,n)=\mathscr{E}(k,l,n)_{\neq\varnothing}$.*

*Proof*. — Assume $\lambda\in\mathscr{C}_m[\equiv nl]\cap\mathscr{C}_m[\leqslant nl]\cap(\mathscr{C}_m)_{l,\varnothing}$. Then there exists $\mu\in\mathscr{P}[nl]$ such that $\mathrm{Core}_m(\mu)=\lambda$. Then $\mu$ is also automatically in $\mathscr{P}_{l,\varnothing}$. Then $\mathscr{X}_\lambda$ is not empty because we have $p_\mu\in\mathscr{X}_\lambda$ by Proposition 4.12. □

Proposition 4.12 and the bijection $\beta_{l,\varnothing}$ yields a parametrization of $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ by $\mathscr{P}^l[n]$.

**4.D. $\mathbb{C}^\times$-fixed points. —** The parametrization of the points of $\mathscr{X}_\theta(n\delta_l)^{\mathbb{C}^\times}$ given in Proposition 4.12 is constructed as a parametrization of the irreducible components of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$ for a very big $m$. This parametrization can be seen in an "$m$-independent way" if we replace the quiver $\overline{Q}_m$ with a very big $m$ by an infinite quiver. This can be done in the following way.

Let $\overline{Q}_\infty$ be the quiver difined in the same way as $\overline{Q}_l$ with respect to the vertex set $\mathbb{Z}$ instead of $\mathbb{Z}/l\mathbb{Z}$. The dimention vectors of the quiver $\overline{Q}_\infty$ are in $\mathbb{Z}^{\mathbb{Z}}$. Set

$$\mathbb{Z}^{\mathbb{Z}}_{\mathrm{fin}}=\{\widehat{d}\in\mathbb{Z}^{\mathbb{Z}};\ \widehat{d}\text{ has finitely many non-zero components}\}.$$

Consider the map

$$p\colon\mathbb{Z}^{\mathbb{Z}}_{\mathrm{fin}}\to\mathbb{Z}^{\mathbb{Z}/l\mathbb{Z}},\quad (p(\widehat{d}))_i=\sum_{j\equiv i\mod l}\widehat{d}_j.$$

For each $\widehat{d}\in\mathbb{Z}^{\mathbb{Z}}_{\mathrm{fin}}$, we have a linear map $i_\infty^{(\widehat{d})}\colon\mathrm{Rep}(\overline{Q}_\infty,\widehat{d})\to\mathrm{Rep}(\overline{Q}_l,p(\widehat{d}))$, defined in the same way is in Section 2.D. Now, for each $\theta\in\mathbb{C}^{\mathbb{Z}/l\mathbb{Z}}$ we consider the element $\theta[\infty]\in\mathbb{C}^{\mathbb{Z}}$ given by $\theta[\infty]_i=\theta_{i\mod l}$. Then we obtain a $\mathbb{C}^\times$-invariant morphism of algebraic varieties $i_\infty^{(\widehat{d})}\colon\mathscr{X}_{\theta[\infty]}(\widehat{d})\to\mathscr{X}_\theta(p(\widehat{d}))$. But the variety $\mathscr{X}_{\theta[\infty]}(\widehat{d})$ is obviously $\mathbb{C}^\times$-stable because the $\mathbb{C}^\times$

action is induced by elements of $\mathbf{GL}(\widehat{d})$. Since $\mathscr{X}_{\theta[\infty]}(\widehat{d})$ is connected, it is a singleton (if it is not empty). Set

$$\mathscr{E}(\infty, l, d) = \{\widehat{d} \in \mathbb{Z}^{\mathbb{Z}}_{\mathrm{fin}};\ p(\widehat{d}) = d, \mathscr{X}_{\theta[\infty]}(\widehat{d}) \neq \varnothing\}.$$

If $\widehat{d} \in \mathscr{E}(\infty, l, d)$, the image of $i_\infty^{(\widehat{d})}$ is a $\mathbb{C}^\times$-fixed point in $\mathscr{X}_\theta(d)$. Let us call this point $p_{\widehat{d}}$.

Now, we assume $d = n\delta_l$, $n \geqslant 0$ and that $\mathscr{X}_\theta(d)$ is smooth. The construction of the parametrization of the $\mathbb{C}^\times$-fixed points in $\mathscr{X}_\theta(n\delta_l)$ given in Proposition 4.12 implies that each $\mathbb{C}^\times$-fixed point $p_\mu$, $\mu \in \mathscr{P}[nl] \cap \mathscr{P}_{l,\varnothing}$, in $\mathscr{X}_\theta(n\delta_l)$ is of the form $p_{\widehat{d}}$ for a unique $\widehat{d} \in \mathscr{E}(\infty, l, n\delta_l)$. Moreover, this $\widehat{d}$ is given by $\widehat{d} = \mathrm{Res}_\infty(\mu)$.

**4.E. Explicit construction of $c'$. —** We know from Theorem 2.12 that each irreducible component $\mathscr{X}$ of $\mathscr{X}_\theta(n\delta_l)^{\mu_m}$ is of the form $\mathscr{X} = \mathscr{X}_d$. Moreover, by Theorem 1.3 there exists a reflection subquotient $(V', W')$ of $(V, W)$ and a complex-valued map $c' \colon W' \to \mathbb{C}$ such that $\mathscr{X} \simeq \mathscr{Z}_{c'}(V', W')$. The reflection subquotient was constructed in the proof of Theorem 1.3. Now we give an explicit construction of $c'$.

Recall from §1.A that the parameter $c$ is determined by the sequence of parameters $(a, k_0, k_1, \dots, k_{l-1})$. Similarly, the parameter $c'$ is determined by $(a', k'_0, k'_1, \dots, k'_{m-1})$. We are going to give explicit expressions for $(a', k'_0, k'_1, \dots, k'_{m-1})$ in terms of $d \in \mathscr{E}(k, l, m)$ and $(a, k_0, k_1, \dots, k_{l-1})$.

First, we recall the construction of $c'$ in the proof of Theorem 1.3. There is an element $w \in W_m^{\mathrm{aff}}$ such that $w(d)$ is a multiple of $\delta_m$. Then we put $\theta' = w(\theta[k])$. Finally, we can obtain $(a', k'_0, \dots, k'_{m-1})$ from $\theta'$ using (3.1) and (3.2).

This construction of $c'$ is not canonical because there is no preferable choice of an element $w$. Note that $\delta_m$ is stable by the subgroup $W_m$ of $W_m^{\mathrm{aff}}$. So the element $w$ can be replaced by any elements of $W_m w$. (By Remark 3.5, a different choice of $w$ yields just a permutation of the parameters $k'_0, k'_1, \dots, k'_{m-1}$.) The affine Weyl group $W_m^{\mathrm{aff}}$ is of the form $W_m \ltimes R_m$, where $R_m$ is the root lattice for $\mathfrak{sl}_m$. In particular, there is an element of $w' \in W_m w$ that is in $R_m$. More precisely, we have $w' = t_\alpha$, where $\alpha = d - d_0\delta_m$ (see the proof of Lemma 2.7).

Then, by Lemma 2.8, we have $w'(\theta[k]) = t_\alpha(\theta[k]) = \theta[k] + \Sigma(\theta[k]) \cdot \overline{\alpha} = \theta[k] + k\Sigma(\theta) \cdot \overline{d}$. (Where the notation $\overline{\alpha}$ and $\overline{d}$ is as in Section 2.B.) This means that we have

$$\theta'_i = (\theta[k])_i + k\Sigma(\theta)(2d_i - d_{i-1} + d_{i+1}) = \theta_{(i \mod l)} - ka(2d_i - d_{i-1} + d_{i+1})$$

Passing back to the $k_i$-notations, we get $a' = ka$ and

$$k'_r = k_{(r \mod l)} + a\left(\left\lfloor \frac{r-1}{l} \right\rfloor - \frac{k-1}{2} + k(d_{1-r} - d_{-r})\right)$$

for $r = 1, \dots, m-1, m$. (Here $k_0 = k_m$.)

***Example 4.14***. — Let us make the above formula explicit in the case $l = 1$. In this case we have $m = k$. Then we get $\theta = \theta_0 = -a$ and $\theta[k] = (-a, -a, \dots -a)$ ($k$ times). This gives

$$\theta'_i = -a - ak(2d_i - d_{i-1} - d_{i+1}),$$

for $i \in \mathbb{Z}/k\mathbb{Z}$.

$$k'_r = a\left(r - \frac{k+1}{2} + k(d_{1-r} - d_{-r})\right)$$

for $r = 1, \dots, k-1, k$. (Here $k_0 = k_k$.) ■

***Example 4.15***. — Assume $l = 2$, $k = 2$ and $d = (0,0)$. In this case $c$ is determined by $(a, k_0, k_1)$ such that $k_0 + k_1 = 0$. Put $k_1 = b$ (and then $k_0 = -b$). Let us make the formulas for $k'_r$ explicit in this case. We have

$$\begin{cases} k'_0 &= -b + a/2, \\ k'_1 &= b - a/2, \\ k'_2 &= -b - a/2, \\ k'_3 &= b + a/2. \end{cases}$$

## 5. Type $G_4$

**Hypothesis.** *In this section, and only in this section, we assume that $V = \mathbb{C}^2$, we identify $\mathbf{GL}_{\mathbb{C}}(V)$ with $\mathbf{GL}_2(\mathbb{C})$, we fix a primitive third root of unity $\omega$, we set*

$$s = \begin{pmatrix} \omega & 0 \\ \omega^2 & 1 \end{pmatrix} \qquad \text{and} \qquad t = \begin{pmatrix} 1 & -\omega^2 \\ 0 & \omega \end{pmatrix}$$

*and we assume that $W = \langle s, t \rangle$, so that $W$ is of type $G_4$ in Shephard-Todd classification. We denote by $(y_1, y_2)$ the canonical basis of $V$ and by $(x_1, x_2)$ its dual basis.*

*Note that there is only one $W$-orbit of reflection hyperplanes (let us call it $\Omega$) and we set for simplifying $k_i = k_{\Omega,i}$, so that $k_0 + k_1 + k_2 = 0$. We also set $\rho = 1 + 2\omega$, so that $\rho^2 = -3$.*

In this case, we will check Conjecture F even if $\mathscr{Z}_c$ is not smooth (and for all elements $\sigma \in \mathbf{GL}_{\mathbb{C}}(V)$ normalizing $W$, as any such elements belong to $\mathbb{C}^\times \cdot W$, as we will see in the proof below).

***Theorem 5.1***. — *If $W$ is of type $G_4$, then Conjecture F holds.*

*Proof*. — Let $\sigma \in \mathbf{GL}_2(\mathbb{C})$ be of finite order and normalizing $W$. The conjugacy class of $s$ is

$$\{s, t, sts^{-1}, s^2ts^{-2} = tst^{-1}\},$$

and is the unique conjugacy class of reflections of determinant $\omega$. Since $\sigma s \sigma^{-1}$ is a reflection of $W$ of determinant $\omega$, there exists $w \in W$ such that $w\sigma$ commutes with $s$. Since $W$ acts trivially on $\mathscr{Z}_c$, we may, and we will, assume that $\sigma s \sigma^{-1} = s$. Now, $\sigma t \sigma^{-1}$ belongs to this conjugacy class and is different from $s$, so there exists $i \in \mathbb{Z}$ such that $s^i\sigma$ commutes with $t$. So, replacing $\sigma$ by $s^i\sigma$, we may, and we will, assume that $\sigma$ commutes with $s$ and $t$. In other words, $\sigma$ is a root of unity.

Since $\mathrm{Z}(W) \simeq \boldsymbol{\mu}_2$, we may also assume that the order of $\sigma$ is even, equal to $2d$ for some $d \geqslant 2$ (because the case $d = 1$ is obvious). Let $I_d$ denote the ideal of $\mathbf{Z}_c$ generated by the $\sigma(z) - z$, where $z$ runs over a set of generators of $\mathbf{Z}_c$. Then

$$(\clubsuit) \qquad \mathbb{C}[\mathscr{Z}_c^\sigma] = \mathbf{Z}_c / \sqrt{I_d}.$$

Let $\zeta$ denote the root of unity by which $\sigma$ acts: then $\sigma(z)-z=(\zeta^l-1)z$ if $z$ is homogeneous of $\mathbb{Z}$-degree $l$. This shows that $I_d$ is generated by the generators of $\mathbf{Z}_c$ of degree not divisible by $2d$. As we will see below, there is a set of generators of $\mathbf{Z}_c$ whose $\mathbb{Z}$-degrees belong to the interval $[-6,6]$. So, if $d\geqslant 4$, then $\mathscr{Z}_c^{\sigma}=\mathscr{Z}_c^{\mathbb{C}^\times}$, so it is a finite set and the Conjecture F is checked in this case. So we may assume that $d\in\{2,3\}$.

In [**6**], Thiel and the first author have provided algorithms for computing presentations of the algebra $\mathbf{Z}_c$, and such algorithms have been included in the MAGMA package CHAMP developed by Thiel [**20**] (about the MAGMA software, see [**18**]). These algorithms provide an explicit presentation of the center $\mathbf{Z}_c$ given as follows:

Generators:

$$Y_1,\ Y_2,\ X_1,\ X_2,\ \mathrm{eu},\ A,\ B,\ D;$$

Relations:

$$\begin{cases}
\text{(Z1)} & \mathrm{eu}^4+4\rho\,Y_1X_1-AB+3\rho\,\mathrm{eu}D+18(k_1^2+k_1k_2+k_2^2)\mathrm{eu}^2\\
& +756(k_1^2k_2+k_1k_2^2)\mathrm{eu}=0\\
\text{(Z2)} & \rho\,\mathrm{eu}^3A-4X_1\mathrm{eu}B-DA-\rho\,Y_1X_2-30\rho(k_1^2+k_1k_2+k_2^2)\mathrm{eu}A\\
& -108\rho(k_1^2k_2+k_1k_2^2)A=0\\
\text{(Z3)} & Y_2A-\mathrm{eu}B^2+\rho\,Y_1D+54(k_1^2+k_1k_2+k_2^2)Y_1\mathrm{eu}-324(k_1^2k_2+k_1k_2^2)Y_1=0\\
\text{(Z4)} & \rho\,\mathrm{eu}^3B-4X_1Y_2-\rho\,Y_1\mathrm{eu}A-BD-30\rho(k_1^2+k_1k_2+k_2^2)\mathrm{eu}B\\
& -108\rho(k_1^2k_2+k_1k_2^2)B=0\\
\text{(Z5)} & \mathrm{eu}A^2-4X_1D-X_2B+72\rho(k_1^2+k_1k_2+k_2^2)X_1\mathrm{eu}-432\rho(k_1^2k_2+k_1k_2^2)X_1=0\\
\text{(Z6)} & Y_2\mathrm{eu}^3+3Y_1^2A-3Y_1\mathrm{eu}^2B-B^3+3\rho\,Y_2D+144(k_1^2+k_1k_2+k_2^2)Y_1B\\
& +18(k_1^2+k_1k_2+k_2^2)Y_2\mathrm{eu}+756(k_1^2k_2+k_1k_2^2)Y_2=0\\
\text{(Z7)} & 4\rho\,X_1\mathrm{eu}^2A-A^3-16X_1^2B+X_2\mathrm{eu}^3+3\rho\,X_2D+18(k_1^2+k_1k_2+k_2^2)X_2\mathrm{eu}\\
& -192\rho(k_1^2+k_1k_2+k_2^2)X_1A+756(k_1^2k_2+k_1k_2^2)X_2=0\\
\text{(Z8)} & \mathrm{eu}^6+4\rho\,Y_1X_1\mathrm{eu}^2+2\rho\,D\mathrm{eu}^3-Y_2X_2+D^2-36(k_1^2+k_1k_2+k_2^2)\mathrm{eu}^4\\
& +12\rho(k_1^2+k_1k_2+k_2^2)\mathrm{eu}D+216\rho(k_1^2k_2+k_1k_2^2)D+1080(k_1^2k_2+k_1k_2^2)\mathrm{eu}^3\\
& +1620(k_1^2+k_1k_2+k_2^2)^2\mathrm{eu}^2-3888(k_1^2k_2+k_1k_2^2)(k_1^2+k_1k_2+k_2^2)\mathrm{eu}\\
& -34992(k_1^2k_2+k_1k_2^2)^2=0\\
\text{(Z9)} & \rho\,Y_1A^2+4X_1B^2+4\mathrm{eu}^3D-\rho\,Y_2X_2+4\rho\,D^2-24\rho(k_1^2+k_1k_2+k_2^2)\mathrm{eu}^4\\
& -288(k_1^2+k_1k_2+k_2^2)\mathrm{eu}D-576(k_1^2+k_1k_2+k_2^2)Y_1X_1\\
& +2160\rho(k_1^2+k_1k_2+k_2^2)^2\mathrm{eu}^2+432\rho(k_1^2k_2+k_1k_2^2)\mathrm{eu}^3-864(k_1^2k_2+k_1k_2^2)D\\
& +20736\rho(k_1^2k_2+k_1k_2^2)(k_1^2+k_1k_2+k_2^2)\mathrm{eu}+46656\rho(k_1^2k_2+k_1k_2^2)^2=0
\end{cases}$$

The generators have $\mathbb{Z}$-degrees given by the following map

$$(Y_1,Y_2,X_1,X_2,\mathrm{eu},A,B,D)\longmapsto(-4,-6,4,6,0,2,-2,0).$$

Moreover, in this presentation, $\mathbb{C}[V]^W=\mathbb{C}[X_1,X_2]$ and $\mathbb{C}[V^*]^W=\mathbb{C}[Y_1,Y_2]$.

We can deduce from this that $\mathscr{Z}_c$ is smooth if and only if

$$(\diamondsuit)\qquad k_0k_1k_2(k_0-k_1)(k_0-k_2)(k_1-k_2)\neq 0.$$

***First case: assume that*** $d=2$. In this case, the above presentation shows that, whenever $(\diamondsuit)$ holds, then $\mathscr{Z}_c^{\sigma}$ has four irreducible components $\mathscr{X}$, $p$, $q$ and $r$ where $\mathscr{X}$ has dimension 2 and is isomorphic to

$$(\heartsuit)\qquad \mathscr{X}\simeq\{(x,y,e)\in\mathbb{C}^3\mid e(e-12k_0)(e-12k_1)(e-12k_2)=xy\},$$

and $p$, $q$ and $r$ are three points (which belong to $\mathscr{Z}_c^{\mathbb{C}^\times}$). If $(\diamondsuit)$ does not hold, then $\mathscr{Z}_c^{\sigma}$ still contains an irreducible component of dimension 2 with the same description as above,

and a few other isolated points (the number depends on the exact values of the $k_i$). By Remark 1.4, we only need to consider the irreducible component $\mathscr{X}$. But it follows from ($\heartsuit$) and Example 1.2 that then $\mathscr{X}$ is isomorphic to the Calogero-Moser associated with a cyclic group of order 4 acting on a one-dimensional space: as there is an element $w$ of order 4 in $W$, one can choose for $V'$ an eigenspace of $w$ and for $W'$ the cyclic group $\langle w\rangle$ and Conjecture F is checked in this case (note also that the map $c \mapsto c'$ is linear).

***Second case: assume that*** $d = 3$**.** In this case, the above presentation shows that, whenever ($\diamondsuit$) holds, then $\mathscr{Z}_c^\sigma$ has two irreducible components $\mathscr{X}$ and $p$ where $\mathscr{X}$ has dimension 2 and is isomorphic to

$$(\spadesuit) \qquad \mathscr{X} \simeq \{(x,y,e)\in\mathbb{C}^3 \mid (e-6k_0)(e-6k_1)(e-6k_2)(e-12k_0)(e-12k_1)(e-12k_2) = xy\},$$

and $p$ is a point (which belongs to $\mathscr{Z}_c^{\mathbb{C}^\times}$). If ($\diamondsuit$) does not hold, then $\mathscr{Z}_c^\sigma$ still contains an irreducible component of dimension 2 with the same description as above, and maybe one extra point according to the values of the $k_i$. By Remark 1.4, we only need to consider the irreducible component $\mathscr{X}$. But it follows from ($\spadesuit$) and Example 1.2 that then $\mathscr{X}$ is isomorphic to the Calogero-Moser associated with a cyclic group of order 6 acting on a one-dimensional space: as $-s \in W$ is of order 6, one can choose for $V'$ the $\omega$-eigenspace of $s$ and for $W'$ the cyclic group $\langle -s\rangle$ and Conjecture F is checked in this case (note also that the map $c \mapsto c'$ is linear). □

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CÉDRIC BONNAFÉ, Institut Montpelliérain Alexander Grothendieck (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE
*E-mail :* cedric.bonnafe@umontpellier.fr

RUSLAN MAKSIMAU, Institut Montpelliérain Alexander Grothendieck (CNRS: UMR 5149), Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 MONTPELLIER Cedex, FRANCE
*E-mail :* ruslan.maksimau@umontpellier.fr