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► **To cite this version:**

Mark Rijnen, Benjamin J.J.B. Biemond, Nathan van de Wouw, Alessandro Saccon, Henk Nijmeijer.  
Hybrid Systems with State-Triggered Jumps: Sensitivity-Based Stability Analysis with Application  
to Trajectory Tracking. 2018. hal-01727842v1

**HAL Id: hal-01727842**

**<https://hal.science/hal-01727842v1>**

Preprint submitted on 9 Mar 2018 (v1), last revised 4 Sep 2018 (v2)

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# Hybrid Systems with State-Triggered Jumps: Sensitivity-Based Stability Analysis with Application to Trajectory Tracking

Mark Rijnen, Benjamin Biemond,  
Nathan van de Wouw, Alessandro Saccon, Henk Nijmeijer

**Abstract**—The definition of asymptotic stability for a trajectory of a hybrid system with state-triggered jumps is not straightforward. Nearby solutions jump at close but non-coincident times, making the standard notion of closeness, based on vector difference, unsuitable to compare trajectories point-wise in time. We propose a notion of stability and a constructive stability proof based on sensitivity analysis, showing how the results find also application to the tracking problem for hybrid systems. A key role in the analysis is played by a time-triggered linear system, associated to the discontinuous trajectory of interest, whose uniform asymptotic stability suffices to guarantee the asymptotic stability of the original discontinuous trajectory. As an illustrative example, the stability analysis is applied to guarantee closed-loop stable tracking for a trajectory with velocity jumps of a 2 DoF mechanical system with unilateral constraints, where jumps correspond to partially elastic impacts with the boundary of the constrained configuration space.

**Index Terms**—hybrid systems, stability, trajectory tracking, trajectories with jumps, sensitivity analysis

## I. INTRODUCTION

**T**HIS paper studies the problem of defining and assessing local asymptotic stability of a trajectory of a hybrid dynamical system. These systems show both continuous (flow) and discrete (jump) dynamics [1]. Our analysis concerns, in particular, hybrid systems with (time- and) state-triggered jumps, where the state trajectory becomes discontinuous under the effect of the discrete dynamics. We will refer to this class of systems as hybrid systems with state-triggered jumps.

Hybrid systems with state-triggered jumps can be used to describe and analyze the behavior of mechanical systems with unilateral contact constraints employed in robotics and multi-body dynamics [2]. Although other approaches worth considering for modeling and simulation of mechanical systems with unilateral constraints are available [3], [4], the hybrid system formalism [1] has been shown to be a viable modeling technique that allows for the description and control of, e.g., juggling and walking robots [5]–[7]. Whereas the stabilization

of jumping trajectories of a mechanical system with unilateral constraints is the main motivation for our investigation, the approach and obtained results are applicable to this whole class of hybrid systems and are therefore presented as such.

In earlier investigations [8]–[10], tracking problems for hybrid systems have been solved under the assumption that the jump times of the system and reference trajectory coincide. In that case, standard Lyapunov methods can be employed in terms of the classical Euclidean tracking error to perform stability analysis. The requirement that the jump times of the trajectories coincide with those of a reference trajectory is however stringent and this coincidence can generally not be assumed: this is not the case, in general, for hybrid systems with state-triggered jumps and, in particular, for hybrid systems that represent mechanical systems with unilateral constraints.

When reference and closed-loop jump times do not coincide, the Euclidean error between two trajectories shows a big increase whenever the system trajectory jumps and the reference trajectory does not or vice versa. This phenomenon is usually referred to as “peaking” [11]–[13]. A few approaches have been recently proposed in the literature to deal with the mismatch in the jump times by defining stability on the basis of a different notion of error/distance between jumping trajectories. In [14], the times belonging to the infinitesimal intervals around the jump times are neglected in defining the tracking problem while; in [15], the reference trajectory together with a *mirrored* version of it has been used to construct a new error notion for the tracking problem of a ball in a polyhedral billiard. In [1, Section 5.3], the concept of graphical closeness of solutions is considered. In [13], [16], [17], the authors propose to simplify the stability analysis and tracking control design by suggesting to use a distance function between two trajectories that is required to be invariant to the discrete dynamics describing the jumps of these trajectories. Instead of taking care of the state-triggered discrete dynamics using a notion of error, in [18], the authors use so-called gluing functions tailored to a system of interest to perform a state-transformation turning the hybrid dynamics into (piecewise) continuous dynamics, removing the state jumps.

In this paper, we employ the notion of error introduced in [19] to define and analyze asymptotic stability of discontinuous trajectories and to propose a possible solution to the problem of tracking a reference trajectory for hybrid systems with state-triggered jumps. The employed error notion is based on extending the (reference) trajectory about the jump times

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and considering the distance between the state of a trajectory and that at the particular segment of the extended (reference) trajectory that has encountered the same number of jumps. This error will not, in particular, show any peaking and presents the basis for an effective trajectory tracking control approach named *reference spreading control* [19]–[21].

**The key contribution of this paper is to show that a linear time-triggered hybrid system that emerges from the sensitivity analysis as presented in [19] can be used to assess the local stability of a discontinuous trajectory of a state-triggered hybrid system similarly as the standard time-varying linearization can be used to assess the local asymptotic stability of a time-varying trajectory of a continuous-time nonlinear control system.** In [19], the discussion was limited to the case of a single jump event. In [20], the error definition was adapted to accommodate multiple jump events by employing the hybrid time domain [1]. Results presented in this paper are also using this solution concept.

This paper is organized as follows. In Section II, hybrid systems with state-triggered jumps are reviewed and the problem definition is precisely stated together with key regularity assumptions. Section III reviews the concept of extended reference trajectory, that leads to the error notion used to define stability of discontinuous trajectories. This section also presents the main result of this paper: the ability to infer stability of a discontinuous trajectory of a hybrid system with state-triggered jumps by analysis of the stability of an associated time-triggered linear system (the hybrid linearization). Section IV applies the obtained results to a mechanical system with a unilateral constraint. Conclusions are presented in Section V.

## II. PRELIMINARIES AND PROBLEM STATEMENT

### A. Hybrid systems

A hybrid dynamical system can be represented schematically as in Fig. 1a. The system has a state  $x \in \mathbb{R}^n$  that continuously evolves according to a control vector field  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ , parameterized by the external input  $u \in \mathbb{R}^m$ . Continuous evolution is only possible when the state  $x$  at a given time is in a closed set  $C \subseteq \mathbb{R}^n$  called the *flow set*. Explicitly, the state evolution satisfies the differential equation

$$\dot{x} = f(x, u), \quad x \in C. \quad (1a)$$

A *jump* in the state can occur whenever the state reaches a set  $D \subseteq \mathbb{R}^n$ , called the *jump set*. A jump implies an instantaneous state change according to the jump map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

In this paper, we consider unique solutions of hybrid systems. Aside from some basic regularity assumptions on the flow map  $f$ , uniqueness requires that whenever a jump occurs ( $x \in D$ ) evolving in  $C$  is also no longer possible. A simple way to enforce this is to assume that  $D \subseteq \partial C$ , with  $\partial C$  denoting the boundary of  $C$ , together with some transversality assumptions on the continuous flow to avoid that the flow is tangent to  $D$  when it reaches it (grazing). Transversality plays an important role in this paper and will be discussed in more detail in the problem formulation in Section II-B.

We largely adopt the hybrid system notation from [1]. In particular, we employ the notion of *hybrid time*, which merges

regular time  $t \in \mathbb{R}$  with discrete time  $j \in \mathbb{N}$ . Discrete time should be thought of as a jump counter, indicating how many times the state has jumped, so that the state jumps abide

$$x(t, j + 1) = g(x(t, j)) \quad x \in D. \quad (1b)$$

All these notions are standard and we refer to [1, Definition 2.6] for the definition of a solution to (1a), (1b).

To allow for time-varying vector fields, jump maps, and time-varying flow and jump sets, the hybrid dynamical systems that we consider in this paper are (with slight abuse of notation) written as follows:

$$\dot{x} = f(x, u, t, j), \quad x \in C(t, j), \quad (2a)$$

$$x^+ = g(x^-, t, j), \quad x \in D(t, j), \quad (2b)$$

where  $x^+ := x(t, j + 1)$  and  $x^- := x(t, j)$ . We refer to (2) as a *Nonlinear State-Triggered Hybrid System (NSTHS)* and represent it as shown in Figure 1b.

**Remark 1.** In (2), one could define a new state  $(x, t, j)$  showing that (2) is just a special case of (1). However, we found that keeping the hybrid time  $(t, j)$  explicit leads to a more intuitive understanding of the stability analysis and reference spreading control. In Section III, we show that the control law is of the form  $u = u(x, t, j)$ , depending therefore explicitly on the continuous flow state  $x$  and hybrid time  $(t, j)$ .

Another reason to keep  $(t, j)$  explicitly in (2) is that otherwise the proposed definition of discontinuous trajectory stability would require to treat  $(t, j)$  differently than the other part of the state (otherwise, when time would be included in the state, a state perturbation would also perturb time).  $\triangle$

In Section II-B and Appendix B, we will use the sets

$$C_j := \bigcup_{t \in \mathbb{R}} C(t, j) \times \{t\}, \quad (3)$$

$$D_j := \bigcup_{t \in \mathbb{R}} D(t, j) \times \{t\}, \quad (4)$$

to ease the derivations of the results. Note that  $C_j$  and  $D_j \subseteq \mathbb{R}^{n+1}$ . For a given  $j \in \mathbb{N}$ ,  $C(t, j)$  and  $D(t, j)$  can be interpreted as the “slices” of  $C_j$  and  $D_j$  at time  $t$ .

We adopt basic regularity assumptions for (2). To be precise, the flow map  $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^n$  is assumed to be locally Lipschitz with respect to the state  $x$  on the set  $\mathbb{R}^n$  and input  $u$  in  $\mathbb{R}^m$ , continuous and bounded in  $t$  for each  $x$  and

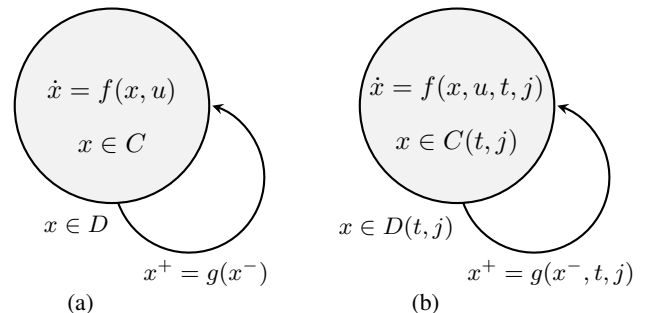


Fig. 1. Hybrid system with one mode of execution without (a) and with (b) explicit dependency on hybrid time  $(t, j)$ .

$u$  and every fixed  $j$ . The jump map  $g : \mathbb{R}^n \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^n$  is assumed to be continuous with respect to  $x$  and  $t$ .

For the NSTHS (2), one may aim at designing a control law to achieve tracking of a given reference trajectory. A generic time-varying state feedback to achieve this goal is

$$u = \kappa(x, t, j). \quad (5)$$

The *closed-loop* NSTHS (cl-NSTHS) resulting from substituting (5) into (2) is given by

$$\dot{x} = f_{cl}(x, t, j) \quad x \in C(t, j) \quad (6a)$$

$$x^+ = g(x^-, t, j) \quad x \in D(t, j), \quad (6b)$$

where  $f_{cl}(x, t, j) := f(x, \kappa(x, t, j), t, j)$ .

Suppose  $x(t, j)$  is the solution to (6) for a given initial condition  $x(t_0, 0) = x_0 \in \text{int } C(t_0, 0)$  (int means interior). The hybrid domain of  $x(t, j)$  is written as (cf. [1, Definition 2.3])

$$\text{dom } x = \bigcup_{j=0}^{J_x-1} I_x^j \times \{j\}, \quad J_x \in \mathbb{N} \cup \{\infty\}, \quad (7)$$

with  $I_x^j$  the closed (continuous-time) interval between the  $j$ -th and  $(j+1)$ -th jump events and  $J_x$  the number of time intervals ( $J_x = \infty$ , when an infinite number of jumps occur). The first interval  $I_x^0$  starts at  $t_0$ . If  $J_x < \infty$ , the last interval  $I_x^{J_x-1}$  ends at  $t_f \in \mathbb{R} \cup \{\infty\}$ . The  $j$ -th jump time is denoted  $t_j$  so that

$$I_x^j = [t_j, t_{j+1}] \quad (8)$$

and also

$$t_j = \min I_x^j. \quad (9)$$

The set of jump times associated to  $x(t, j)$  is denoted

$$E_x := \bigcup_{j=1}^{J_x-1} \{t_j\} \times \{j-1\}. \quad (10)$$

We conclude this section by introducing several notational conventions used to indicate the flow map. We will use  $\mathcal{H}$  to indicate the closed-loop hybrid system defined by the quadruple  $(f_{cl}, g, C, D)$ , representing (6). Sometimes, we regard the solution to  $\mathcal{H}$  starting from  $x_0$  at hybrid time  $(t_0, 0)$ , just as a function of continuous time  $t$  instead of hybrid time  $(t, j) \geq (t_0, 0)$ . To this end, with a slight abuse of notation, we will write  $x(t)$  or, more explicitly,  $\phi_{\mathcal{H}}(t, t_0, x_0)$  to indicate

$$x(t) = \phi_{\mathcal{H}}(t, t_0, x_0) := x(t, j_{\mathcal{H}}(t, t_0, x_0)), \quad (11)$$

with  $x(t, j)$  as in (6) and  $j_{\mathcal{H}}(t, t_0, x_0) := \max_j(t, j)$ , with  $(t, j) \in \text{dom } x$ , indicating the discrete time corresponding to assuming that, at time  $t$ , all discrete-time transitions have already occurred. We will also employ

$$\varphi_j(t, s, x) \quad (12)$$

to denote the flow (with no jumps) of the time-varying vector field  $f_{cl}(\cdot, \cdot, j)$  in (6a) in the time interval  $[s, t]$  with initial condition  $x(s) = x$ . Employing (11) and (12), and assuming to have just one jump at a time (this assumption will be stated more formally in the next section), one can write with no ambiguity  $x(t, j) = \varphi_j(t, t_j, x(t_j, j)) = \varphi_j(t, t_j, \phi_{\mathcal{H}}(t_j, t_0, x_0))$ , as  $j_{\mathcal{H}}(t_j, t_0, x_0) = j$  by definition of  $t_j$ . Furthermore, it holds that  $x(t_j, j) = \phi_{\mathcal{H}}(t_j, t_0, x_0) = g(x(t_j, j-1), t_j, j-1)$ .

## B. Problem formulation

This paper focuses on establishing local asymptotic stability of a known reference trajectory with jumps, denoted  $\alpha(t, j)$  in the following. We assume both that  $\alpha$  is  $t$ -complete (i.e.,  $\sup_t \text{dom } \alpha = \infty$ ) and that it is the unique solution to (2) for input  $u = \mu(t)$ , assumed to be continuous, and initial condition  $\alpha(t_0, 0) = \alpha_0 \in \text{int } C(t_0, 0)$ . The  $j$ -th event time of  $\alpha$  is denoted  $\tau_j$ , with  $\tau_0 = t_0$ , and  $I_{\alpha}^j = [\tau_j, \tau_{j+1}]$  is the  $j$ -th time interval between two consecutive events. The number of intervals is  $J_{\alpha}$  and the event time set is  $E_{\alpha}$ . Roughly speaking, our stability analysis applies to a reference trajectory that is  $t$ -complete, non-Zeno, has a bounded dwell time in between jumps, and intersects the jump set transversally. Some minimal regularity conditions on the hybrid system are also required. More precisely, the following assumptions are considered.

**Assumption 1** ( $t$ -complete, non-Zeno, bounded dwell time). *The reference trajectory  $\alpha$  is defined for all  $t > t_0$  ( $t$ -complete) and all the time intervals  $I_{\alpha}^j$  are of non-vanishing extent (non-Zeno), implying also that none of its jumps occur at the same time. When  $J_{\alpha} = \infty$ , the separation between two consecutive events is uniformly bounded (bounded dwell time).  $\blacktriangle$*

From the fact that the reference trajectory is non-Zeno it follows that the number of events  $J_{\alpha} - 1$  can only become infinite for time  $t$  approaching infinity. The nonzero time between jumps in the assumption can be guaranteed when  $g(\alpha(t, j), t, j) \notin D(t, j+1)$  for all  $(t, j) \in E_{\alpha}$ . Completeness implies that the trajectory  $\alpha$  does not show a finite escape time and is contained in  $C(t, j_{\mathcal{H}}(t, t_0, \alpha_0))$  for all time  $t \geq t_0$ .

Next, we require the jumps of  $\alpha(t, j)$  to be *transversal* to the boundary of  $C$ . To formalize this property, we make use of a so-called *guard function* that will be detailed next.

**Assumption 2** (Existence of a guard function). *Given Assumption 1, we assume there exist constants  $\varepsilon_{\gamma}, c_1 > 0$ , and a real valued (guard) function  $\gamma_{\alpha}(x, t, j)$ , continuously differentiable with respect to both  $x$  and  $t$  for each  $j \in \{0, 1, \dots, J_{\alpha} - 1\}$ , such that, firstly*

$$\begin{aligned} \gamma_{\alpha}(x, t, j) &> 0 & (x, t) &\in P_j \cap \text{int}(C_j) \\ \gamma_{\alpha}(x, t, j) &= 0 & (x, t) &\in P_j \cap \partial C_j =: Z_j \\ \gamma_{\alpha}(x, t, j) &< 0 & (x, t) &\in P_j \cap ((\mathbb{R}^n \times \mathbb{R}) \setminus C_j) \end{aligned} \quad (13)$$

where  $P_j := B_{\varepsilon_{\gamma}}(\alpha(\tau_{j+1}, j), \tau_{j+1})$  with  $B_{\varepsilon_{\gamma}}(x, t) := \{(u, v) \in \mathbb{R}^n \times \mathbb{R} \mid \|(u, v) - (x, t)\| < \varepsilon_{\gamma}\}$  and  $\text{int}(C_j)$ , and  $\partial C_j$  denote, respectively, the interior and boundary of  $C_j$ . Secondly, we require that  $\gamma_{\alpha}(\cdot, \cdot, j)$  can be thought of as the first coordinate in a  $C^1$ -diffeomorphism between  $B_{\varepsilon_{\gamma}}$  and an open neighborhood of the origin on  $\mathbb{R}^{n+1}$ . As a third condition, we impose that

$$Z_j \subset D_j \quad (14)$$

and, lastly, we assume that the guard function  $\gamma_{\alpha}(x, t, j)$  is such that its gradient can be upper bounded, i.e.

$$\|\mathbf{D}_1 \gamma_{\alpha}(\alpha(t, j), t, j)\| \leq c_1, \quad (15)$$

uniformly for every event time  $(t, j) = (\tau_{j+1}, j) \in E_{\alpha}$ . Herein,  $\mathbf{D}_i$  denotes differentiation with respect to the  $i$ -th argument.  $\blacktriangle$

**Remark 2.** Due to (14), the condition  $\gamma_\alpha(\cdot, \cdot, j) = 0$  in (13) implicitly represents the set of state and time pairs  $(x, t)$  within  $B_{\varepsilon_\gamma}(\alpha(\tau_{j+1}, j), \tau_{j+1})$  where a state reset has to occur.  $\gamma_\alpha$  is required to be a  $C^1$  diffeomorphism which ensures furthermore that, within each ball  $B_{\varepsilon_\gamma}(\alpha(\tau_{j+1}, j), \tau_{j+1})$ ,  $\gamma_\alpha(\cdot, \cdot, j) = 0$  also defines a dimension  $n$  embedded  $C^1$  sub-manifold.  $\triangle$

Let us now formalize the assumption on the transversality property of the desired trajectory  $\alpha$ .

**Assumption 3** (Transversality). *Let Assumption 2 hold, implying the existence of the guard function  $\gamma_\alpha$ . We require the existence of a constant  $c_2 > 0$  such that*

$$\begin{aligned} D_1 \gamma_\alpha(\alpha(t, j), t, j) \cdot f(\alpha(t, j), \mu(t), t, j) + \\ D_2 \gamma_\alpha(\alpha(t, j), t, j) \cdot 1 \leq -c_2 \end{aligned} \quad (16)$$

for every event time  $(t, j) = (\tau_{j+1}, j) \in E_\alpha$ .  $\blacktriangle$

Aside from the above mentioned assumptions on the reference trajectory  $\alpha$ , we also pose conditions on the jump map  $g$  and vector field  $f$ . These conditions are related to the continuity properties with respect to their arguments and are given below. The condition on  $f$  makes use of the so-called extended reference trajectory  $\bar{\alpha}$  which abides (19) and will be discussed in more detail in the next section.

**Assumption 4** (Locally differentiable jump map). *Given Assumptions 1 and 2, we require, for all  $j \in \{0, 1, \dots, J_\alpha - 1\}$ , the jump map  $g(\cdot, \cdot, j)$  to be at least  $C^1$  in the open ball  $B_{\varepsilon_\gamma}(\alpha(\tau_{j+1}, j), \tau_{j+1})$ .  $\blacktriangle$*

**Assumption 5** (Uniform Lipschitz condition on the vector field). *Let  $f_{cl}(x, t, j) := f(x, \kappa(x, t, j), t, j)$  denote the closed-loop vector field where  $f$  and  $\kappa$  are Lipschitz with respect to state  $x$  and continuous and bounded in  $t$  for each  $j$  and  $\kappa(\alpha(t, j), t, j) = \mu(t)$ . We assume that, at least in a neighborhood of the reference trajectory  $\alpha$ ,  $f_{cl}(x, t, j)$  is Lipschitz with respect to  $x$ , uniformly in  $t$  and  $j$ . Namely, we assume we can choose  $\varepsilon_L > 0$ , independent of  $(t, j)$ , such that there exist  $L$  for which, for all  $j \in \{0, 1, \dots, J_\alpha - 1\}$ ,  $\|f_{cl}(x, t, j) - f_{cl}(y, t, j)\| < L\|x - y\|$ , for all  $t \in (\tau_j - \varepsilon_L, \tau_{j+1} + \varepsilon_L)$  and  $x, y \in B_{\varepsilon_L}(\bar{\alpha}(t, j))$ .  $\blacktriangle$*

**Remark 3.** Note how the Lipschitz constant is defined in the time intervals  $t \in (\tau_j - \varepsilon_L, \tau_{j+1} + \varepsilon_L)$  that are not contained in  $\text{dom } \alpha$ . Therefore, an “extended” reference trajectory  $\bar{\alpha}$  for which  $\text{dom } \alpha \subset \text{dom } \bar{\alpha}$  is used in its definition. This extended reference will be explained in more detail in the next section where it is used to define a measure of error.  $\triangle$

In order not to have to resort to notions as “pre”-asymptotic stability as introduced in [1, Chapter 7] for handling solutions close to the reference trajectory that are only defined on a finite time domain, we assume certain properties of the reference trajectory and the closed-loop hybrid system such that trajectories sufficiently close to  $\alpha(t, j)$  are also t-complete. These properties guarantee that trajectories in a neighborhood of  $\alpha(t, j)$  remain in the flow set away from the jump event times and that after each jump, flow is again possible. The properties essentially prevent the state to be blocked from further evolution by reaching the border of  $C(t, j)$  as it is

assumed that the vector field is directed inward for the points on the border that are close to  $\alpha$ . The idea is essentially quite simple, but the formulation below might appear involved at first sight. Furthermore, to account for any differences in jump time of the nearby solutions, the given properties need to hold on a larger time domain than  $\text{dom } \alpha$  (similarly as discussed in the remark above) and are consequently required of the extended reference trajectory  $\bar{\alpha}$  (see Section III-A for more details). The assumption is formalized below.

**Assumption 6** (Local existence of t-complete solutions). *Let Assumptions 1 to 3 hold. Every state-time pair  $(x, t) \in Z_j$  of the reference event  $(\alpha(\tau_{j+1}, j), \tau_{j+1})$ , with  $Z_j$  defined as in (13) and  $j \in \{0, 1, \dots, J_\alpha - 1\}$ , is mapped by the jump map  $g$  to the subsequent flow set, while avoiding the jump set, i.e.,*

$$(x, t) \in Z_j \implies (g(x, t, j), t) \in C_{j+1} \setminus D_{j+1}.$$

Furthermore, we assume that the extended vector field  $(f_{cl}(x, t, j), 1)$  satisfies

$$(f_{cl}(x, t, j), 1) \in \mathcal{T}_{(x,t)} C_j, \quad (17)$$

for each  $j \in \{0, 1, \dots, J_\alpha - 1\}$  and  $(x, t) \in \partial C_j \cap (U_j \setminus P_j)$ , with  $U_j$  defined below and  $P_j$  as in (13), and where  $\mathcal{T}_{(x,t)} C_j$  denotes the tangent cone to  $C_j \subset \mathbb{R}^{n+1}$  at  $(x, t)$  as defined, e.g., in [1, Definition 5.12]. In (17),  $U_j$  denotes the set of all state-time pairs  $(x, t)$  contained in a tube of size  $\varepsilon_C > 0$  about the (extended) reference trajectory  $\bar{\alpha}$ , that is

$$U_j := \bigcup_{t \in [\tau_j - \varepsilon_C, \tau_{j+1}]} (B_{\varepsilon_C}(\bar{\alpha}(t, j)) \times \{t\}). \quad (18)$$

In case  $J_\alpha$  is finite, by convention we assume  $P_j$  to be the empty set for  $j = J_\alpha - 1$  as no more jumps occur in  $\alpha$ .  $\blacktriangle$

In Assumption 6, posing the requirement of the vector field being directed inwards only on the set  $(U_j \setminus P_j) \cap \partial C_j$  allows to exclude the satisfaction of condition (17) in proximity of the subsequent event  $(\alpha(\tau_{j+1}, j), \tau_{j+1}) \in P_j$ . By Assumption 3 we actually require that the vector field pushes the reference state out of the flow set  $C_j$  in the case of a jump event.

**The problem considered in this paper is that of assessing the stability properties of a jumping reference trajectory  $\alpha(t, j)$  of the closed-loop state-triggered hybrid dynamical system (6) under Assumptions 1-6.**

The technical assumptions we adopted above reduce the applicability of the theory in this work to a subclass of hybrid systems and reference trajectories. Still, a large and relevant system class remains. One may for example think of mechanical systems with unilateral constraints, with as trajectory of interest, a motion with non-accumulating (partially) elastic impacts at non-zero speed that are separated in time.

In the next section, we will first introduce a notion of error (named *reference spreading error*) between two trajectories  $\alpha(t, j)$  and  $x(t, j)$ , inspired by [19]. Next we introduce a time-triggered linear system related to the NSTHS (2), reference trajectory  $\alpha$ , and the reference spreading error. Finally, we show that stability of the time-triggered linear hybrid system implies stability of the NSTHS about  $\alpha$ .

### III. SENSITIVITY-BASED STABILITY ANALYSIS OF JUMPING TRAJECTORIES

In this section, the stability properties of the reference trajectory  $\alpha(t, j)$  of the cl-NSTHS in (6) are analyzed. First, the notion of extended (reference) trajectory will be introduced to define a useful error measure on the basis of which a definition of stability of the jumping reference trajectory will be given. Secondly, a time-triggered linear hybrid system will be defined, the trajectories of which can be used to approximate those of the NSTHS starting near the reference trajectory. This section is concluded by showing that a switching controller can be designed using this linear hybrid system, as asymptotic stability of this system in closed-loop implies asymptotic stability of the reference trajectory of the cl-NSTHS in (6).

#### A. Error definition

Any difference between the initial conditions  $x_0$  and  $\alpha_0$  will most likely result in differences between the jump times of the reference and those of the closed-loop system. Therefore, as mentioned in the introduction, designing a tracking controller based on the Euclidean error defined as the difference between the current state  $x$  and  $\alpha$  (for the same  $t$  only) may easily result in poor tracking performance (see [15], [20] for simulation examples supporting this statement). Therefore, as suggested in [19], for every value of the jump counter  $j$ , the segment of the reference trajectory corresponding to the interval  $I_\alpha^j \times \{j\}$  is extended using the vector field (2a) with input  $u = \mu(t)$  such that it is defined for all  $t \in [t_0, t_f]$ . This *extended reference trajectory* is denoted  $\bar{\alpha}$  and is thus defined for all  $(t, j) \in [t_0, t_f] \times \{0, 1, \dots, J_\alpha - 1\} =: \bar{I}_\alpha$ . Note that  $\bar{\alpha}$  coincides with  $\alpha$  for  $(t, j) \in \text{dom } \alpha$ . Formally, for each  $j \in \{0, 1, \dots, J_\alpha - 1\}$ , we define  $t \mapsto \bar{\alpha}(t, j)$ ,  $t \in [t_0, t_f]$ , as the solution to

$$\dot{\bar{\alpha}} = f(\bar{\alpha}, \mu(t), t, j), \quad (t, j) \in \bar{I}_\alpha \quad (19)$$

with “initial” condition  $\bar{\alpha}(\tau_j, j) = \alpha(\tau_j, j)$ , where we recall that  $\tau_j$  denotes the  $j$ -th jump time of  $\alpha$  (with the convention  $\tau_0 = t_0$ ). The construction of the extended trajectory requires both forward and backward integration of the vector field to extend the  $j$ -th reference segment outside the original interval  $[\tau_j, \tau_{j+1}]$ . This extension is unique based on the assumption that the vector field  $f$  is locally Lipschitz with respect to state.

**Remark 4.** *Tracking the reference trajectory  $\alpha$  using the notion of extended trajectories as proposed in [19] requires more than just the state of the reference at the current time, it requires knowledge of the future reference. The trajectory  $\alpha$  needs to be known beforehand or, at least, the segment up to the next jump of  $\alpha$ . If the closed-loop system encounters a jump prior to the reference trajectory, for example, it already needs to know what the desired motion is after the jump event. This knowledge comes from the backwards integration of the vector field from the time where the jump is expected to occur (for incremented jump counter), i.e. the reference jump time.  $\triangle$*

Note that when a hybrid trajectory, that is a solution to (2), is projected onto the continuous time domain, it is single valued for all  $t$  in its domain except for the jump times. This is not the case for  $\bar{\alpha}$ , as now for each time  $t$  there are  $J_\alpha$

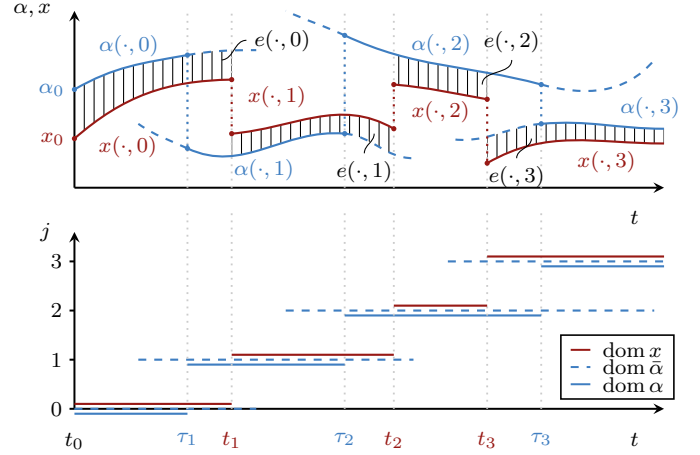


Fig. 2. The extended reference  $\bar{\alpha}$ , the tracking error  $e$  for a trajectory  $x$ , and their corresponding time domains.

extended “reference” trajectories discriminated by the counter  $j$ . This counter will be used to compare the state  $x$  at any given time to the relevant branch of the extended reference  $\bar{\alpha}$ . Note furthermore that the domain  $\bar{I}_\alpha$  is *not* a hybrid time domain as in [1], because the natural ordering of its points is lost, i.e. the boundaries of the time intervals do not form an increasing sequence. Notice also that the construction of the extended trajectory  $\bar{\alpha}(t, j)$  requires the vector field  $f$  in (2a), with  $u = \mu(t)$ , to be defined outside the domain  $C(t, j)$ , ignoring the presence of the jump set and allowing the time integration to continue beyond it. The extended reference trajectory  $\bar{\alpha}$  now allows to define the tracking error as follows (cf. [19]):

$$e(t, j) := x(t, j) - \bar{\alpha}(t, j) \quad (t, j) \in \text{dom } x. \quad (20)$$

In a small neighborhood of  $\text{dom } \alpha$ , this definition of error is similar to the concept of graphical closeness of graphs of solutions as described in [1, Definition 4.11]. A graphical representation of  $\bar{\alpha}$ , its “extended” hybrid time domain  $\bar{I}_\alpha$ , and the tracking error  $e(t, j)$  is given in Fig. 2.

#### B. Stability definition

In order to verify or prove stability of a trajectory  $\alpha$  for the NSTHS in (6), we first define stability and asymptotic stability using the notion of error introduced in the previous section.

**Definition 1 (Stability).** *Given  $t_0$ , a trajectory  $\alpha$  of (6) that is  $t$ -complete is said to be **stable** if for all  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for every trajectory  $x$  of (6) satisfying  $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$ , it holds that a)  $J_x = J_\alpha$ , b) for all  $(t, j) \in \text{dom } x$ ,  $\|x(t, j) - \bar{\alpha}(t, j)\| < \varepsilon$  and, c) for all  $j \in \{1, \dots, J_x - 1\}$ ,  $|t_j - \tau_j| < \varepsilon$ , where  $\bar{\alpha}$  is the extension of  $\alpha$  defined in (19) and  $J_x$  and  $J_\alpha$  are, respectively, the number of time intervals  $I_x^j$  and  $I_\alpha^j$  defined as in (8) for the trajectories  $x(t, j)$  and  $\alpha(t, j)$ , possibly infinite.  $\blacksquare$*

**Remark 5.** *This definition of stability makes use of the error definition in (20), which is the Euclidean distance between the state  $x$  and the extended reference trajectory  $\bar{\alpha}$  for each  $(t, j) \in \text{dom } x$ . Note that  $e(t, j)$  has the same time domain as  $x(t, j)$  and requires  $\text{dom } x \subseteq \bar{I}_\alpha$  for it to be defined. The*

latter is guaranteed when  $J_x \leq J_\alpha$  which is satisfied with strict equality in the case of stability of  $\alpha$ , for sufficiently small  $\delta$  (as in Definition 1). The dependency of the jump times of  $x(t, j)$  on initial condition  $x_0$  will be elaborated on in Lemma 1.  $\triangle$

**Definition 2** (Attractivity). *Given a trajectory  $\alpha$  of (6) that is  $t$ -complete, we say that  $\alpha$  is **attractive** if there exists a  $\delta > 0$  such that  $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$  implies, firstly, that  $J_x = J_\alpha$  and, secondly, that  $\|x(t, j) - \bar{\alpha}(t, j)\| \rightarrow 0$  for  $t \rightarrow \infty$  with  $(t, j) \in \text{dom } x$ , where  $\bar{\alpha}$  is the extension of  $\alpha$  defined in (19) and  $J_x$  and  $J_\alpha$  are, respectively, the number of time intervals  $I_\alpha^j$  and  $I_x^j$  defined as in (8) for the trajectories  $x(t, j)$  and  $\alpha(t, j)$ . If  $J_\alpha = \infty$ , we require that furthermore  $|t_j - \tau_j| \rightarrow 0$  for  $j \rightarrow \infty$ .  $\blacksquare$*

**Definition 3** (Asymptotic stability). *A trajectory  $\alpha$  of the cl-NSTHS (6) is **asymptotically stable** if it is stable and attractive, respectively in the sense of Definitions 1 and 2.  $\blacksquare$*

### C. Linear time-triggered hybrid system (LTTHS)

As mentioned in the previous section, a nonzero tracking error will likely result in a mismatch between the closed-loop jump times  $t_j$  and the reference jump times  $\tau_j$ . The times  $t_j$ , with  $j \in \{1, 2, \dots, J_x - 1\}$  are not known in advance. Instead, as the reference  $\alpha$  is assumed to be known, the event times  $\tau_j$  are known. Next, we construct a linear time-triggered system, that jumps at the reference jump times  $\tau_j$ , and solutions of which can be used to approximate the error in the state evolution for (6) in a neighborhood of  $\alpha$ . We can use this time-triggered linear system to design a stabilizing feedback of the form (5) and we show that asymptotic stability of this closed-loop linear hybrid system *implies* (local) asymptotic stability of the reference trajectory  $\alpha$  for the original cl-NSTHS (6).

We refer to this linear system with jumps at the times  $\tau_j$  as the *Linear Time-Triggered Hybrid System (LTTHS)* associated to the reference trajectory  $\alpha$ . The key feature of this LTTHS is that it converts the state-triggered behavior of (2) to a time-triggered one (cf. [19]) and incorporates a first order approximation of the state-jumps (originally at slightly different times) in the definition of the jump map, see e.g. [19], [22]. We will show that asymptotic stability of  $\alpha$  in the sense of Definition 3 can be assessed by studying the LTTHS corresponding to (2) and the state-input trajectory  $(\alpha(t, j), \mu(t))$ , with  $(t, j) \in \text{dom } \alpha$ . Hence, the stability analysis is significantly simplified as  $\alpha(t, j)$  and the LTTHS jump at the same time. Let us now formally define the LTTHS.

**Definition 4** (LTTHS). *The linear time-triggered hybrid system associated to trajectory  $\alpha$  and NSTHS (2) is given by*

$$\begin{aligned} \dot{z} &= A(t, j)z + B(t, j)v & (t, j) \in \text{dom } \alpha \\ z^+ &= G(j)z^- & (t, j) \in E_\alpha \end{aligned} \quad (21)$$

with initial condition  $z(t_0, 0) = z_0$  and where  $z^+ := z(t, j + 1)$ ,  $z^- := z(t, j)$ ,

$$A(t, j) := \mathbf{D}_1 f(\alpha(t, j), \mu(t), t, j), \quad (22)$$

$$B(t, j) := \mathbf{D}_2 f(\alpha(t, j), \mu(t), t, j), \quad (23)$$

and

$$G(j) := \frac{f^+ - \dot{g}^-}{\dot{\gamma}_\alpha^-} \mathbf{D}_1 \gamma_\alpha^- + \mathbf{D}_1 g^- \quad (24)$$

with

$$f^+ = f(\alpha^+, \mu(\tau), \tau, j + 1) \quad (25)$$

$$f^- = f(\alpha^-, \mu(\tau), \tau, j) \quad (26)$$

$$g^- = g(\alpha^-, \tau, j) \quad (27)$$

$$\dot{g}^- = (\mathbf{D}_1 g^-)f^- + \mathbf{D}_2 g^- \quad (28)$$

$$\gamma_\alpha^- = \gamma_\alpha(\alpha^-, \tau, j) \quad (29)$$

$$\dot{\gamma}_\alpha^- = (\mathbf{D}_1 \gamma_\alpha^-)f^- + \mathbf{D}_2 \gamma_\alpha^- \quad (30)$$

where  $\tau = \tau_{j+1}$ ,  $\alpha^+ = \alpha(\tau, j + 1)$ , and  $\alpha^- = \alpha(\tau, j)$ .  $\blacksquare$

As will be clarified later on, the linear hybrid system (21)-(30) provides an approximation of the NSTHS in the sense that a trajectory of the NSTHS starting at a perturbed initial condition  $x_0 = \alpha_0 + z_0$  with perturbed input  $\mu(t) + v(t, j)$  can be approximated as  $x(t, j) = \bar{\alpha}(t, j) + \bar{z}(t, j) + o(\|z_0\|)$  for  $(t, j) \in \text{dom } x$ . In this,  $\bar{z}$  is the extended trajectory of  $z$  obtained in the same way as  $\bar{\alpha}$ , that is, for each  $j$ , it follows from integrating the vector field  $\dot{z} = A(t)z + B(t)v(t, j)$  with  $t \in [t_0, t_f]$  forward and backward in time from initial condition  $\bar{z}(\tau_j, j) = z(\tau_j, j)$ . The term  $o(\|z_0\|)$  denotes a perturbation that is of order higher than one. The state  $\bar{z}$  is thus a first-order approximation of the error (20) if  $J_x \leq J_\alpha$ .

Given the time-triggered ‘‘linearization’’ of the NSTHS about a trajectory  $\alpha$ , one can attempt to design a control law to make the origin of the LTTHS uniformly asymptotically stable. The uniformity property will be required for showing that  $\alpha$  is an asymptotically stable trajectory of the NSTHS if the origin of the linearization indeed satisfies the posed stability properties. This will be discussed in Section III-D. In many cases, by choosing a feedback control law of the form

$$v(t, j) = -K(t, j)z(t, j) \quad (31)$$

with a suitably designed time-varying feedback gain  $K$ , uniform asymptotic stability of the origin of the closed-loop LTTHS (cl-LTTHS) can be achieved, by which we mean the following (see [23, Section 3.1]):

**Definition 5** (LTTHS: Uniform asymptotic stability). *The origin of the cl-LTTHS (21)-(31) is **uniformly asymptotically stable** if for every  $\varepsilon > 0$  and  $T_0 \geq t_0$  there exists a  $\delta$ , independent of  $T_0$ , such that  $|z(T_0, j_{T_0})| \leq \delta$  implies  $|z(t, j)| \leq \varepsilon$  for all  $(t, j) \in \text{dom } \alpha$  with  $t \geq T_0$  and that  $\lim_{t \rightarrow \infty} |z(t, j)| = 0$ . In this,  $j_{T_0}$  is the counter  $j$  corresponding to the time  $T_0$ , that is, the largest  $j$  such that  $T_0 \geq \tau_j$ .  $\blacksquare$*

The stability assessment of (21) in closed-loop (i.e. with feedback (31)) is well established in literature, see e.g. [23, Sections 3.2 and 6.4] and [24].

Using (31) and the fact that  $\bar{z}(t, j)$  is a local approximation of the error (20), we obtain a cl-NSTHS with input

$$u = \kappa(x, t, j) = \mu(t) - \bar{K}(t, j)(x(t, j) - \bar{\alpha}(t, j)). \quad (32)$$

Note that where  $z$  is defined for all  $(t, j) \in \text{dom } \alpha$ , the state  $x$  and error  $e$  have a different hybrid time domain that is not known in advance. For the feedback law to be well-defined, we thus require the time-varying feedback gain to be defined for a larger time domain than  $\text{dom } \alpha$ , i.e. for all  $(t, j) \in \bar{I}_\alpha = [t_0, t_f] \times \{0, 1, \dots, J_\alpha - 1\}$ . Therefore, in (32), we introduced



$\bar{K}(t, j)$  representing the feedback gain  $K(t, j)$ , but extended such that it is defined for all  $(t, j) \in \bar{I}_\alpha$ . Due to this extension, the feedback control is defined for all  $(t, j) \in \text{dom } x$  (as long as  $J_x \leq J_\alpha$ ). Several approaches are possible in constructing these extensions. However, the extension of the feedback gains  $\bar{K}(t, j)$  does not influence the LTHS since it only depends on the perturbation input  $v(t, j)$  for  $(t, j) \in \text{dom } \alpha$  (see [20]). This property is explained further in Appendix D.

**Remark 6.** Note that, even though we consider stability of the NSTHS for trajectory tracking and more specifically using the feedback law (32), the stability results in the next section about given trajectories apply to any nonlinear state-triggered hybrid system that abides the assumptions in Section II-B. This includes NSTHS without an external input or for which a different control law is considered. Such systems are trivially obtained for the case where there is no  $u$ , respectively,  $v$  in (2) and (21). The choice to consider feedback of the form (32) is based on the fact that it follows naturally from a linear controller that can be designed on the LTHS (21).  $\triangle$

#### D. Main stability result

The problem considered in this paper is that of assessing the stability properties of a jumping reference trajectory  $\alpha(t, j)$  of the cl-NSTHS (6) under Assumptions 1 to 6. As a stepping stone, a key fact that we will exploit is that, for any finite time  $T > t_0$ , the jump times of the cl-NSTHS (6) in  $\text{dom } x \cap [t_0, T] \times \mathbb{N}$  depend in a continuously differentiable fashion on the initial condition  $x_0$  as long as  $x_0$  is chosen in a sufficiently small neighborhood of  $\alpha_0$  dependent on  $T$ . This result follows from Assumptions 1 to 4 and Assumption 6 as well as some minimal regularity assumptions on the vector field  $f_{cl}$  and jump map  $g$ . Note that the size of this neighborhood might vanish for  $T \rightarrow \infty$ . In order to conclude stability, this dependency of jump times on initial condition, Assumption 5, and stability of the linear error dynamics are key. Let us first consider the former property for which we define a jump counter function for the reference trajectory  $\alpha$ .

**Definition 6** (Jump counter function). For a given  $T > t_0$  and hybrid trajectory  $\alpha$ , denote with  $j_\alpha(T)$  the number of encountered jumps of the reference trajectory for  $t \leq T$ .  $\blacksquare$

The jump counter function  $j_\alpha : (\tau_1, \infty) \rightarrow \mathbb{N}$  is right continuous and satisfies the inequality  $\tau_{j_\alpha(T)} \leq T$ . It is equal to the jump counter  $j_{\mathcal{H}}$ , introduced in Section II, when evaluated along the specific trajectory  $\alpha$ , i.e.  $j_\alpha(T) = j_{\mathcal{H}}(T, t_0, \alpha_0)$ . The following lemma now holds.

**Lemma 1.** For the hybrid system (6), assume that the vector field  $f_{cl}$  is locally Lipschitz with respect to  $x$  and continuous and bounded in  $t$ . Let  $\alpha$  denote a reference trajectory of the NSTHS (6) satisfying Assumptions 1, 2, 3, 4, and 6, with initial condition  $\alpha(t_0, 0) = \alpha_0 \in \text{int}(C(t_0, 0))$ . As before,  $\tau_j, j \in \{0, 1, \dots, J_\alpha - 1\}$ , indicates the nominal event times.

There exists a function  $\delta_0 : (\tau_1, \infty) \rightarrow \mathbb{R}_{>0}$  such that, for any  $T > \tau_1$ , a trajectory of the cl-NSTHS (6) with initial condition  $x(t_0, 0) = x_0$  satisfying

$$\|x_0 - \alpha_0\| < \delta_0(T) \quad \text{with} \quad B_{\delta_0(T)}(\alpha_0) \subset C(t_0, 0) \quad (33)$$

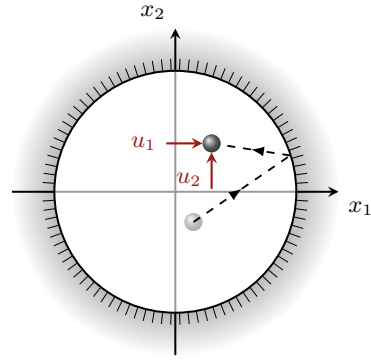


Fig. 3. Schematic representation of a circular billiard.

is defined at least up to time  $T$  and jumps at least  $j_\alpha(T) - 1$  times in the interval  $t \in [t_0, T]$ . Furthermore, when  $j_\alpha(T) \geq 2$ , the function  $\delta_0(\cdot)$  can be chosen such that, in addition, every jump time  $t_j$  except the last is bracketed by the nominal jump times  $\tau_{j-1}$  and  $\tau_{j+1}$ , i.e.

$$\tau_{j-1} \leq t_j \leq \tau_{j+1}, \quad \text{for } j \in \{1, 2, \dots, j_\alpha(T) - 1\}. \quad (34)$$

*Proof.* The proof of Lemma 1 is presented in Appendix A.  $\square$

Stability of the reference trajectory  $\alpha$  for the cl-NSTHS (6), (32) can now be related to the stability of the cl-LTHS (21)-(31), resulting in the main result of this work given below.

**Theorem 1.** Adopt Assumptions 1-6. Let a state-input trajectory  $(\alpha, \mu)$  be a solution to the cl-NSTHS (6), (32). If the origin of the associated LTHS (21)-(30) in closed-loop with control law (31) is uniformly asymptotically stable, then,  $\alpha(t, j)$  is a (locally) asymptotically stable trajectory of the cl-NSTHS (6), (32) in the sense of Definition 3.

*Proof.* The proof of Theorem 1 is given in Appendix B.  $\square$

To bridge the gap between this work and other approaches in literature for analyzing stability of state-triggered hybrid systems, in Appendix C we show that uniform asymptotic stability of the cl-LTHS (21)-(31) also implies asymptotic stability of the reference trajectory  $\alpha$  for (a variant of the) cl-NSTHS (6), (32) in terms of the distance function defined in [13], [16]. Mind that, in this implication, the hybrid system class that is considered is reduced to one where the jump map  $g$ , and the flow and jump sets  $C$  and  $D$ , respectively, do not depend on time explicitly as the distance function in [13], [16] does not accommodate such time-varying nature.

#### IV. ILLUSTRATIVE EXAMPLE

In this section, a trajectory tracking example for mechanical systems with a unilateral constraint is presented. A circular billiard with periodic reference trajectory is considered. In this, the number of jumps  $J_\alpha - 1$  becomes infinite as  $t \rightarrow \infty$ . We only consider a periodic reference here for simplicity, but for the theory presented in this work no periodicity is required.

Consider the time-invariant system depicted in Fig. 3 (that is also considered in [11] for fully elastic restitution) consisting of an actuated point mass moving in a plane that is confined by a circular boundary. Such a system is commonly referred to in literature as a billiard.



The position of the point mass in the plane, at a particular time, is given by the coordinates  $x_1$  and  $x_2$  (see Fig. 3). The velocity components of the mass in the  $x_1$  and  $x_2$  direction are denoted by  $x_3 = \dot{x}_1$  and  $x_4 = \dot{x}_2$ , respectively. The state of the point mass thus is  $x = [x_1 \ x_2 \ x_3 \ x_4]^T$  and accelerations can be imposed in  $x_1$  and  $x_2$  direction (denoted  $u_1$ , respectively  $u_2$ , such that  $u = [u_1 \ u_2]^T$ ). A rigid object confines the space in which the mass can move such that the jump set is defined as  $D = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1, x_1 x_3 + x_2 x_4 > 0\}$ . The flow set therefore becomes  $C = \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 \leq 1\}$  and a suitable guard function for the system is  $\gamma_\alpha = 1 - x_1^2 - x_2^2$  (satisfying Assumption 2). Whenever the mass impacts the boundary, partially elastic restitution occurs with a coefficient of restitution  $e$ . The system can be described by (2), with

$$f(x, u, t, j) = [x_3 \ x_4 \ u_1 \ u_2]^T =: Ax + Bu, \quad (35)$$

$$A = \begin{bmatrix} 0_{2 \times 2} & I_2 \\ 0_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{2 \times 2} \\ I_2 \end{bmatrix},$$

and

$$g(x, t, j) = \begin{bmatrix} x_1 \\ x_2 \\ (x_2^2 - ex_1^2)x_3 - (1+e)x_1x_2x_4 \\ (x_1^2 - ex_2^2)x_4 - (1+e)x_1x_2x_3 \end{bmatrix}, \quad (36)$$

cf. [11]. In this example, it is assumed that the interaction between actuated mass and obstacle can fully be modeled using the impact law, that is, periods of persistent unilateral contact do not occur and (finite) contact forces thus need not be included in the vector field  $f$ . Furthermore, the pair  $(\alpha, \mu)$  is considered to be such that grazing incidence of the point mass on the obstacle is avoided (cf. Assumption 3). Note that, in this example, the flow set  $C$ , jump set  $D$ , vector field  $f$ , and jump map  $g$  do not depend explicitly on the hybrid time  $(t, j)$ . This time-invariance is chosen here for the sake of simplicity of the analysis only, and is not required for the applicability of the theory in this work.

Consider the reference trajectory shown in Fig. 4a. It is a periodic solution to the system description above where the state returns back to its initial condition after five impacts with coefficient of restitution  $e = 0.3$ . The impacts are separated in time by a period  $\tau = 1$  (cf. Assumption 1) and since the trajectory starts at the boundary it follows that  $\tau_j = j\tau$  for  $j = 0, 1, \dots$  (taking  $t_0 = 0$ ). As can be discerned from the fact that the trajectory segments between impacts are curved, the input  $\mu(t)$  is not zero for all time. As a consequence, constructing the extended trajectory  $\bar{\alpha}$  as described in Section III will be different from the strategy of ‘‘mirroring’’ as is used for billiards in [15], [25], [26]. Furthermore, note that the coefficient of restitution is significantly below one. If the input would be zero and  $e$  equal to one, for this particular system, the two approaches would result in the same behavior. The strategy taken in this paper however is more generic and applicable to a larger class of systems. It allows to exploit knowledge of the complete reference trajectory, therewith making optimal control approaches within reach.

It is straightforward, but tedious, to show that for the specific example Assumptions 1-6 are satisfied. This proof is not included here due to space restrictions and is left to the reader.

To design the stabilizing control law, by using Theorem 1, we first consider stability of the LTTHS. The desired trajectory is periodic with a period of  $5\tau$ , but due to its point symmetry with respect to the origin, it is possible to assess stability of the LTTHS corresponding to the considered system by only looking at the evolution from one post-impact position and velocity to the next. When applying a feedback input of the form (31) (with constant gain  $K = K(t, j)$ ) to this linearized system, the state after the  $j$ -th impact and that just after the next are related to each other by  $z(\tau_{j+1}, j+1) = G(j) \exp((A+BK)\tau) z(\tau_j, j)$ , with  $G(j)$  given by (24), combined with  $\dot{\gamma}_\alpha^- = \mathbf{D}_1 \gamma_\alpha^- \cdot f^-, \dot{g}^- = \mathbf{D}_1 g^- \cdot f^-$ ,

$$\begin{aligned} f^+ &= [\alpha_3^+ \ \alpha_4^+ \ \mu_1^+ \ \mu_2^+]^T, \\ f^- &= [\alpha_3^- \ \alpha_4^- \ \mu_1^- \ \mu_2^-]^T, \\ \mathbf{D}_1 \gamma_\alpha^- &= [-2\alpha_1^- \ -2\alpha_2^- \ 0 \ 0], \\ \mathbf{D}_1 g^- &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ dg_{31} & dg_{32} & dg_{33} & dg_{34} \\ dg_{41} & dg_{42} & dg_{43} & dg_{44} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} dg_{31} &= -2e\alpha_1^- \alpha_3^- - (1+e)\alpha_2^- \alpha_4^- \\ dg_{32} &= 2\alpha_2^- \alpha_3^- - (1+e)\alpha_1^- \alpha_4^- \\ dg_{33} &= (\alpha_2^-)^2 - e(\alpha_1^-)^2 \\ dg_{34} &= -(1+e)\alpha_1^- \alpha_2^- \\ dg_{41} &= 2\alpha_1^- \alpha_4^- - (1+e)\alpha_2^- \alpha_3^- \\ dg_{42} &= -(1+e)\alpha_1^- \alpha_3^- - 2e\alpha_2^- \alpha_4^- \\ dg_{43} &= -(1+e)\alpha_1^- \alpha_2^- \\ dg_{44} &= (\alpha_1^-)^2 - e(\alpha_2^-)^2 \end{aligned}$$

and in which  $\alpha_s^+ = \alpha_s((j+1)\tau, j+1)$  and  $\alpha_s^- = \alpha_s((j+1)\tau, j)$  denote the right, respectively, left limit of the  $s$ -th state of the reference trajectory at time  $t = (j+1)\tau$ . Similarly,  $\mu_s^+$  and  $\mu_s^-$  denote the right and left limits (that are the same due to continuity of  $\mu$ ), respectively, of the  $s$ -th reference input at that time. It follows that the LTTHS is asymptotically stable if all eigenvalues of the matrix  $G(j) \exp((A+BK)\tau)$  are within the unit circle in the complex plane. When taking a feedback gain of the form  $K = [\beta \ 0 \ 2\sqrt{\beta} \ 0; \ 0 \ \beta \ 0 \ 2\sqrt{\beta}]$ , it is found that the eigenvalues are within the unit circle when  $\beta > 0.393$ . Applying Theorem 1, we therefore conclude that  $\alpha$  is asymptotically stable for the cl-NSTHS (2), (32), (35), (36) for such choice of feedback gain. A solution to the cl-NSTHS (with constant gain extensions) for  $\beta = 3$  and initial condition  $x_0 = 0$  is depicted in Fig. 4. The figure shows that the solution indeed converges towards the reference state-input trajectory  $(\alpha(t, j), \mu(t))$ , even for this large initial error.

## V. CONCLUSIONS

In this work, a notion of asymptotic stability and an associated stability analysis for discontinuous trajectories of hybrid systems with state-triggered jumps are detailed. The results have also a direct applicability to the related problem of trajectory tracking.

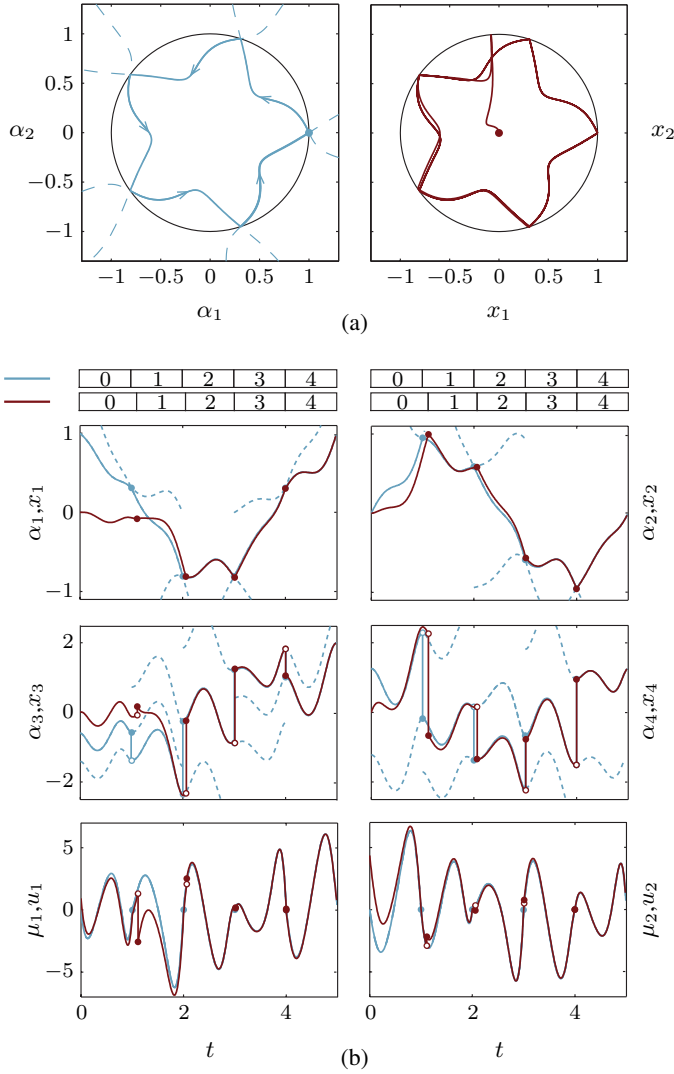


Fig. 4. The reference trajectory  $\alpha(t, j)$  (light blue) with extensions (dashed) and a tracking solution  $x(t, j)$  (dark red) to the cl-NSTHS in Fig. 3 in the  $x_1, x_2$ -plane (where the dot indicates the initial position) (a) and the evolution of state and input over time (b).

Asymptotic stability is defined by making use of a notion of error that allows for the comparison of two nearby discontinuous trajectories, even when there is a time mismatch between the jumps of both trajectories. It is shown that asymptotic stability of a discontinuous trajectory of the hybrid system with state-triggered jumps is guaranteed when an associate time-triggered linear system is uniformly asymptotically stable. As this linear system jumps at the same times as the reference trajectory, the design of a stabilizing feedback and stability analysis is greatly simplified. The results are illustrated by means of a tracking example for a mechanical system with unilateral constraint and partially elastic impacts.

In a series of related publications, the tracking strategy based on reference extensions has been given the name of reference spreading control and has been applied to more complex systems (such as, e.g., a multi-body humanoid model) and even for the case where the constrained state space after each jump has a different dimension, representing an example of effective tracking for a multi-domain hybrid system.

## APPENDIX A PROOF OF LEMMA 1

The proof of Lemma 1 is split in two parts. First, we show that the solution  $x(t, j)$  to the NSTHS (6) from initial condition  $x(t_0, 0) = x_0$  is defined for all  $t \in [t_0, T]$  as long as  $x_0$  is sufficiently close to  $\alpha_0$ . Then, we show that the bracketing condition (34) is satisfied as long as this neighborhood is chosen small enough, due to a continuity argument. This sufficiently small neighborhood is, in essence, what defines the function  $\delta_0$  at  $T$ .

Due to Assumption 1,  $j_\alpha(T)$  is finite for any  $T > t_0$ . The existence of a solution to (6) up to time  $T$  is straightforwardly guaranteed if, for every  $j \in \{1, 2, \dots, j_\alpha(T)\}$  and in a neighborhood of  $\alpha_0$ , we can define an event-time function  $x_0 \mapsto t_j$  that is continuously differentiable, where  $t_j(x_0) = \tau_j$  whenever  $x_0 = \alpha_0$ . Indeed, if these functions  $x_0 \mapsto t_j$ ,  $j \in \{1, 2, \dots, j_\alpha(T)\}$  exist, the flow of the NSTHS (6) is just a composition of continuously differentiable jump maps (due to Assumption 4) with continuously differentiable continuous-time flows on the time intervals  $[t_{j-1}(x_0), t_j(x_0)]$ ,  $j \in \{1, 2, \dots, j_\alpha(T) - 1\}$ , terminated by a continuous-time flow over  $[t_{j_\alpha(T)-1}(x_0), T]$ , when  $t_{j_\alpha(T)}(x_0) \geq T$ , or over  $[t_{j_\alpha(T)-1}(x_0), t_{j_\alpha(T)}(x_0)]$  followed by a jump and another flow phase over  $[t_{j_\alpha(T)}(x_0), T]$ , when  $t_{j_\alpha(T)}(x_0) < T$ .

Due to Assumption 2, there exists a guard function  $\gamma_\alpha$  that implicitly defines the jump set in a neighborhood of  $\alpha$ . Therefore, if the event time functions  $x_0 \mapsto t_j$  exist, they have to satisfy the implicit conditions

$$\gamma_\alpha(x(t_{j+1}(x_0), j), t_{j+1}(x_0), j) = 0 \quad (37)$$

for  $j \in \{0, 1, \dots, j_\alpha(T) - 1\}$ , where

$$\begin{aligned} x(t_{j+1}(x_0), j) &:= \varphi_j(t_{j+1}(x_0), t_j(x_0), \phi_{\mathcal{H}}(t_j(x_0), t_0, x_0)) \\ &= \varphi_j(t_{j+1}(x_0), t_j(x_0), x(t_j(x_0), j)). \end{aligned} \quad (38)$$

Note that Assumption 6 guarantees that the flow  $\varphi_j$  as used above is defined whenever  $x_0$  is sufficiently close to  $\alpha_0$ . By definition of  $\alpha$  and  $\gamma_\alpha$ , we know that (37) holds at least when we pose  $x_0 = \alpha_0$  and  $t_j(\alpha_0) = \tau_j$ . Using (38), the implicit condition (37) can equivalently be rewritten as

$$M_{j+1}(x_0, t) := \gamma_\alpha(\varphi_j(t, t_j(x_0), x(t_j(x_0), j)), t, j) = 0. \quad (39)$$

We aim to prove that, for each  $j$ ,  $t$  in (39) is a function of  $x_0$ , i.e., that  $t = t_{j+1}(x_0)$ . The transversality condition provided in Assumption 3 guarantees that one can apply the implicit function theorem for each of the implicit conditions in (39) and conclude that all the functions  $x_0 \mapsto t_j$  are continuously differentiable for a sufficiently small neighborhood of  $\alpha_0$ .

More precisely, one can employ a proof by induction showing that  $x_0 \mapsto t_1$  is continuously differentiable (base induction) and that  $x_0 \mapsto t_j$  being continuously differentiable implies  $x_0 \mapsto t_{j+1}$  to be continuously differentiable (induction step). The base induction has been proven in [19], while it is straightforward to show, that  $x_0 \mapsto t_j$  being continuously differentiable implies that  $M_{j+1}$  is continuously differentiable in a neighborhood of  $(\alpha_0, \tau_{j+1})$  being the composition of continuously differentiable functions. Furthermore, as the partial derivative of  $M_{j+1}$  in (39) with respect to  $t$  evaluated

at  $(x_0, t) = (\alpha_0, \tau_{j+1})$  is equivalent to the left-hand side of (16) and therefore nonzero by Assumption 3, the implicit function theorem can be applied to conclude that  $t$  in (39) is indeed a function of  $x_0$ . There is however a fundamental limitation in carrying out this induction reasoning for an infinite number of jumps. In the induction step mentioned above, the neighborhood of  $\alpha_0$  for which  $t_j$  is defined can, in principle, become smaller and smaller as  $j$  is increased and, in the worst case, ceases to exist if no other conditions are imposed (this corresponds to a situation where the intersection of an infinite number of open sets containing  $\alpha_0$  just ends up in the closed set containing just the point itself). This is why, with the given assumptions, the statement of this lemma holds just for any finite value of  $T$ , but not for  $T = \infty$ . We will explain in the proof of Theorem 1 how to overcome this limitation by adding an extra condition on the solutions of the cl-LTTHS. This concludes the first part of the proof.

We now prove the existence of  $\delta_0(T) > 0$  to satisfy the bracketing condition (34). As the functions  $x_0 \mapsto t_j$  are continuous and we are considering the *finite* set  $j \in \{1, \dots, j_\alpha(T) - 1\}$ , we can find a closed ball around  $\alpha_0$  with radius  $\delta_0(T)$  that is contained in each domain of definition of the functions  $x_0 \mapsto t_j$  and such that each function  $x_0 \mapsto t_j$  is contained in the interval  $[\tau_{j-1}, \tau_{j+1}]$ .

#### APPENDIX B PROOF OF THEOREM 1

The proof of Theorem 1 is divided in two steps. First, a *Nonlinear Time-Triggered Hybrid System* (NTTHS) related to the NSTHS will be introduced. The NTTHS has  $\alpha(t, j)$  as a solution and the LTTHS is its linearization as shown in Appendix D. Fig. 5 schematically depicts the structure of the proof and the role of the NTTHS therein.

##### A. Nonlinear time-triggered hybrid system (NTTHS)

The NTTHS, introduced below, always jumps at the times  $\tau_j$  of the reference  $\alpha$  and, as will be clarified below, its solutions coincide with solutions of the NSTHS for continuous-times sufficiently far away from the reference jump times.

When we start with a state-input trajectory  $(\alpha(t, j), \mu(t))$  of (2) and slightly change its initial condition or input, we typically obtain a trajectory  $x(t, j)$  that jumps at different times than the reference jump times  $\tau_j$ . This has been shown already schematically in Fig. 2. We illustrate this phenomenon in more detail in Fig. 6 focussing on a single jump. Note that we allow to design the input differently for each jump counter  $j$ , so that instead of  $\mu(t)$ , we use  $u(t, j)$ . In constructing the NTTHS, the procedure is to replace the trajectory of (2) from initial condition  $x(t_0, 0) = x_0$  between the time instances  $t_j$  and  $\tau_j$  with a new trajectory that always jumps at time  $\tau_j$ , as illustrated in Fig. 6, where the trajectory of the NTTHS is denoted as  $x_{TT}$ . This construction is related to the concept of zero-time discontinuity mapping (ZDM) and Poincaré discontinuity mapping (PDM) in [27, Section 6.2]. The trajectory  $x_{TT}$  near  $\tau_{j+1}$  is attained by flowing according to the vector field  $f(x, u, t, j)$  similarly as the reference trajectory up to  $\tau_{j+1}$  and after the time  $\tau_{j+1}$  by flowing

according to the vector field  $f(x, u, t, j + 1)$ . A suitable jump map is applied at the nominal event time  $\tau_{j+1}$  such that it maps the trajectory  $x_{TT}(t, j)$  back to the trajectory  $x(t, j)$  at the end of this time mismatch period ( $[t_{j+1}, \tau_{j+1}]$  or  $[\tau_{j+1}, t_{j+1}]$ ).

In order to define this jump map, we denote by  $\varphi_j^u(t, \tau, x)$  the state evolution according to vector field  $f$  for jump counter  $j$  at time  $t$  with initial condition  $x$  at time  $\tau$  and a given input curve  $u(t, j)$ . Note that  $t \leq \tau$  implies integration backwards in time and that this operator is different for different input curves. The NTTHS, with state  $x_{TT}$ , is defined as follows.

**Definition 7 (NTTHS).** *The nonlinear time-triggered hybrid system is given by*

$$\begin{aligned} \dot{x}_{TT} &= f(x_{TT}, u(t, j), t, j) & (t, j) \in \text{dom } \alpha \\ x_{TT}^+ &= g_{TT}^u(x_{TT}^-, t, j) & (t, j) \in E_\alpha, \end{aligned} \quad (40)$$

with initial condition  $x_{TT}(t_0, 0) = x_0$ , where  $x_{TT}^+ = x_{TT}(t, j + 1)$ ,  $x_{TT}^- = x_{TT}(t, j)$ , and the jump map  $g_{TT}^u(x_{TT}^-, t, j)$ , with  $(t, j) \in E_\alpha$ , is given by

$$\varphi_{j+1}^u(t, t_{j+1}, g(\varphi_j^u(t_{j+1}, t, x_{TT}), t_{j+1}, j)), \quad (41)$$

where  $t_{j+1}$  is the  $(j + 1)$ -th jump time of the solution  $x(t, j)$  of the NSTHS (2) starting from the initial condition  $x_0$ . ■

The jump map  $g_{TT}^u(x_{TT}^-, \tau_{j+1}, j)$  can be defined whenever  $t_{j+1}$  is defined, i.e. when the trajectory  $x$  of the NSTHS with initial condition  $x(t_0) = x_0$  and chosen input  $u(t, j)$  will experience the  $(j+1)$ -th jump (see Fig. 6). This property is satisfied if  $x_0$  is close enough to  $\alpha_0$  and follows from Lemma 1. More details are provided in the proof of Proposition 1.

Furthermore, since the NTTHS jumps at the same times as the reference, the time domain of its solution  $x_{TT}(t, j)$  is the same as that of the reference trajectory, i.e.  $\text{dom } x_{TT} =$

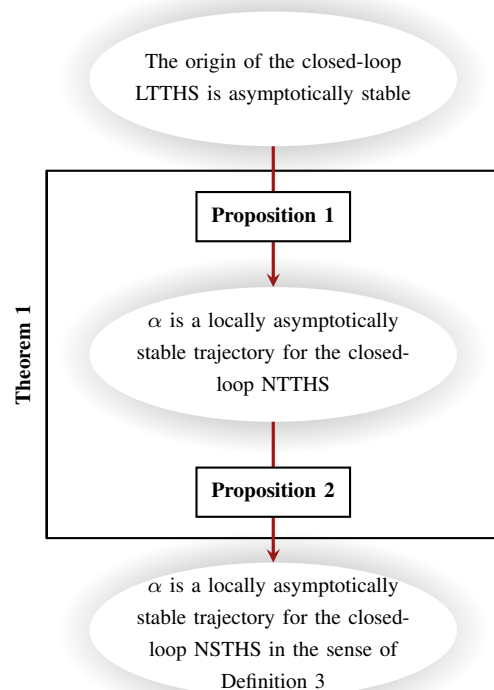


Fig. 5. Subdivision of Theorem 1 in two implication steps, by introducing the NTTHS and Propositions 1 and 2.

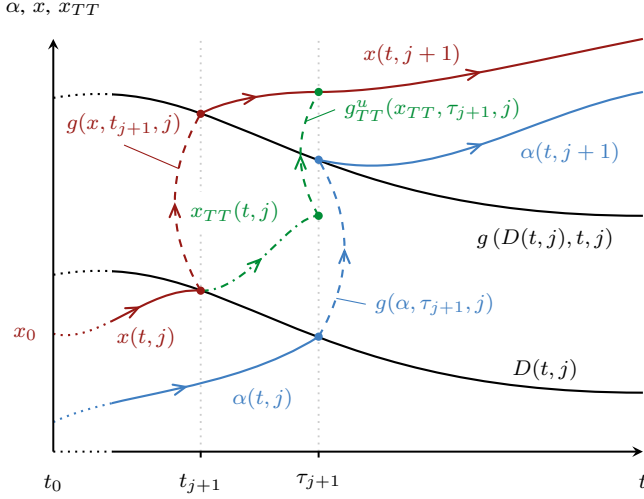


Fig. 6. Relation between trajectory  $x(t, j)$  of the NSTHS, corresponding reference  $\alpha(t, j)$ , and trajectory  $x_{TT}(t, j)$  of the NTTHS for jump  $(j+1)$ .

dom  $\alpha$ , as illustrated in Fig. 7. This figure also illustrates the time-triggered error  $e_{TT}(t, j) := x_{TT}(t, j) - \alpha(t, j)$  (compare with Fig. 2) and their hybrid time domains,  $e_{TT}$  being the error for trajectories with jumps at fixed time instants.

When the input  $u$  is given by

$$u = \kappa(x_{TT}, t, j) \quad (42)$$

as in (5), we obtain the closed-loop NTTHS (or cl-NTTHS)

$$\begin{aligned} \dot{x}_{TT} &= f_{cl}(x_{TT}, t, j) & (t, j) \in \text{dom } \alpha, \\ x_{TT}^+ &= g_{TT}(x_{TT}^-, t, j) & (t, j) \in E_\alpha, \end{aligned} \quad (43)$$

with  $x_{TT}(t_0, 0) = x_0$  and where the jump map  $g_{TT}(x_{TT}, t, j)$  is given by

$$\varphi_{j+1}(t, t_{j+1}, g(\varphi_j(t_{j+1}, t, x_{TT}(t, j)), t_{j+1}, j)) \quad (44)$$

with  $\varphi_j$  the closed-loop flow as in (41) with  $u$  as in (42). More specifically, the cl-NTTHS that we will use in the proof of Theorem 1 is the one employing the affine feedback law.

$$u(t, j) = \mu(t) + K(t, j)(x_{TT}(t, j) - \alpha(t, j)). \quad (45)$$

Note that (45) is the same as (32), but that in the latter no “bars” are needed on top of  $K$  and  $\alpha$  as the controller will only be used for the hybrid times  $(t, j) \in \text{dom } x_{TT} = \text{dom } \alpha$ .

### B. Proposition 1 and its proof

As mentioned at the beginning of this appendix, a key property in the proof of Theorem 1 is that the linearization of the NTTHS in Definition 7 is the LTHS provided by Definition 4. Given  $\alpha$ , solutions to the NTTHS and LTHS clearly have the same hybrid time domain. Here we prove that uniform asymptotic stability of the closed-loop linearization implies that the reference trajectory  $\alpha(t, j)$  is a locally asymptotically stable solution to the cl-NTTHS.

**Proposition 1.** *A trajectory  $(\alpha(t, j), \mu(t))$  of the NSTHS (6) satisfying Assumptions 1, 2, 3, 4, 5, and 6, is an asymptotically stable trajectory of the cl-NTTHS (43), (45), if the associated cl-LTHS (21)-(31) is uniformly asymptotically stable.*

*Proof.* As both cl-LTHS and cl-NTTHS are time-triggered and each jump event corresponds to a jump event of  $\alpha$ , we can employ the jump counter  $j_\alpha : \mathbb{R} \rightarrow \mathbb{N}$  of Definition 6 to simplify the notation within this proof. To this end, with a slight abuse of notation, we will write  $\alpha(t)$ ,  $z(t)$ ,  $x_{TT}(t)$ , etc. to mean  $\alpha(t, j_\alpha(t))$ ,  $z(t, j_\alpha(t))$ ,  $x_{TT}(t, j_\alpha(t))$ , etc. At the event times  $\tau_j$ , we will write  $\alpha^+(\tau_j)$  and  $\alpha^-(\tau_j)$  to indicate  $\alpha(\tau_j, j_\alpha(\tau_j))$  and  $\alpha(\tau_j, j_\alpha(\tau_j) - 1)$ , respectively. Similarly, we will employ the  $+$  and  $-$  notation for other signals.

Our goal is to conclude local asymptotic stability of  $\alpha$  for the cl-NTTHS. To this end, let us consider the time-triggered error  $e_{TT}(t)$  between the state of the cl-NTTHS and the reference  $\alpha$ . The error  $e_{TT}$  satisfies the hybrid dynamics

$$\begin{aligned} \dot{e}_{TT} &= A_{cl}(t)e_{TT} + r_1(e_{TT}, t), \\ e_{TT}^+ &= G(j)e_{TT}^- + r_2(e_{TT}^-, j), \quad \text{for } t = \tau_1, \tau_2, \dots, \end{aligned} \quad (46)$$

with  $j = j_\alpha(t) - 1$ ,  $e_{TT}(t_0) = z_0$ , and where the matrices  $A_{cl}$  and  $G$  and the residuals  $r_1$  and  $r_2$  are defined below. In (46),  $e_{TT}$  is obtained by alternating state resets according to the jump map with integrations of the ODE until  $t$  equals  $\tau_{j+1}$ . In (46),  $A_{cl}(t) := A(t, j_\alpha(t)) + B(t, j_\alpha(t))K(t, j_\alpha(t))$  and  $G$  is given by (24), therefore corresponding to the cl-LTHS. The residuals  $r_1$  and  $r_2$  are the higher-order terms of the vector field and jump map of the time-triggered error dynamics associated to the cl-NTTHS, namely

$$r_1(e_{TT}, t) := f_{cl}(\alpha(t) + e_{TT}, t, j_\alpha(t)) \quad (47)$$

$$- f_{cl}(\alpha(t), t, j_\alpha(t)) - A_{cl}(t)e_{TT}$$

$$r_2(e_{TT}, j) := g_{TT}(\alpha^-(\tau_{j+1}) + e_{TT}^-, \tau_{j+1}, j) \quad (48)$$

$$- g_{TT}(\alpha^-(\tau_{j+1}), \tau_{j+1}, j) - G(j)e_{TT}^-.$$

The origin  $e_{TT} = 0$  is an equilibrium point for (46) and for the cl-LTHS. Both these hybrid systems jump at the same fixed time instants  $\tau_j$ , known in advance. For such class of systems, denoting generically the system state with  $y$ , uniform asymptotic stability implies that for an arbitrary  $\varepsilon_{TT} > 0$  and  $t_S \geq t_0$  there exists a  $\delta_{TT}$  such that  $\|y(t_S)\| < \delta_{TT}$  implies  $\|y(t)\| < \varepsilon_{TT}$  for all  $t \geq t_S$  and that, furthermore,

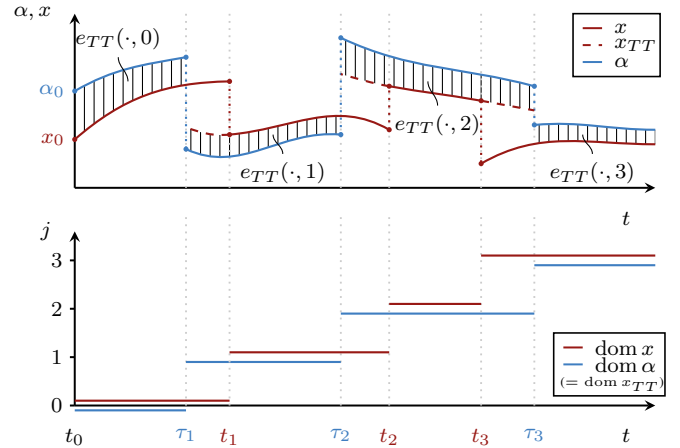


Fig. 7. The classical error for the time-triggered hybrid system ( $e_{TT}(t, j) = x_{TT}(t, j) - \alpha(t, j)$ ) and the hybrid time domains of the corresponding required trajectories.



$\lim_{t \rightarrow \infty} |y(t)| = 0$  (see, e.g., [23, Section 3.1]). We aim to show that uniform asymptotic stability of the cl-LTTHS implies that the origin  $e_{TT} = 0$  is uniformly locally asymptotically stable for (46). Let us now consider two cases:

**CASE 1:** Consider first the case where the number of events  $J_\alpha - 1$  is infinite. From Assumption 5, for each time interval  $I_\alpha^j = [\tau_j, \tau_{j+1}]$ ,  $j = 0, 1, 2, \dots$ , the growth of the solution can be bounded (see, e.g., [28, Corollary 6.4]) according to

$$\|y(t)\| \leq \exp(L(t - \tau_j)) \|y^+(\tau_j)\|, \quad \tau_j \leq t \leq \tau_{j+1}. \quad (49)$$

Clearly, we require  $\|y^+(\tau_j)\|$  to be also sufficiently small to keep  $\|y(t)\|$  within the region where the vector field  $f$  is uniformly Lipschitz. When the continuous dynamics satisfies an exponential bound as in (49), asymptotic stability of a time-triggered hybrid system is equivalent to the asymptotic stability of the discrete time system that originates from only considering the state at the event times  $\tau_j$  [29]. Recall that  $\tau_{j+1} - \tau_j$  is uniformly bounded from below and from above (bounded dwell time) due to Assumption 1.

In our case, the local asymptotic stability of the cl-NTTHS follows directly from uniform asymptotic stability (for linear systems, equivalent to exponential stability) of the cl-LTTHS, by considering their respective associated discrete time systems at the nominal event times. This statement is based on the discrete time version of [30, Theorem 4.13]. This concludes the proof for the case where  $J_\alpha = \infty$ .

**CASE 2:** When the number of events  $J_\alpha - 1$  is finite, it suffices to consider the stability of the continuous time dynamics after the last event. This is valid because, from (49) and continuity of  $g_{TT}$  with respect to  $x_{TT}$ , an exponential bound on the error  $\|e_{TT}\|$  of the cl-NTTHS in the finite time interval  $[t_0, \tau_{J_\alpha-1}]$  can be found (in particular, for every  $\varepsilon > 0$  and  $t_0 \leq t_S \leq \tau_{J_\alpha-1}$  there is a  $\delta > 0$  such that  $\|e_{TT}(t_S)\| < \delta$  implies  $\|e_{TT}(\tau_{J_\alpha-1})\| < \varepsilon$ ). It thus suffices to consider the last flow phase when  $t \geq \tau_{J_\alpha-1}$ .

As the cl-LTTHS is, by assumption, exponentially stable and it is the linearization of the error system (46), specifically in the time interval  $[\tau_{J_\alpha-1}, \infty)$ , the origin of the error system is (locally) asymptotically stable [30, Theorem 4.13] and, equivalently,  $\alpha$  is an asymptotically stable trajectory of the cl-NTTHS. This concludes the proof for finite  $J_\alpha$ .

So far, it has been tacitly assumed that the solution of the NTTHS exists for all  $t > t_0$ . However, one has actually to prove this result based on the assumptions in the proposition statement. For the NTTHS to be defined, the jump map  $g_{TT}$  needs to be defined for all jumps, which in turn requires that the NSTHS corresponding to the NTTHS experiences those jumps as well as we discussed after (41).

From the proof of Lemma 1 and, in particular, from the continuity of the map  $t_j(x_0)$ , the  $j$ -th jump of the NSTHS occurs close to the  $j$ -th jump time of  $\alpha$  as long as  $\|e_{TT}(t_0)\|$  is chosen in a sufficiently small neighborhood of zero (equivalently,  $x_{TT}(t_0, 0)$  is chosen sufficiently close to  $\alpha_0$ ). This property is illustrated in Fig. 8. However, a uniform (i.e., independent of  $j$ ) neighborhood is not guaranteed to exist when  $\alpha$  has an infinite number of jumps (cf. proof of Lemma 1). Assuming the exponential stability of the cl-LTTHS, though, guarantees the existence of such a uniform neighborhood.

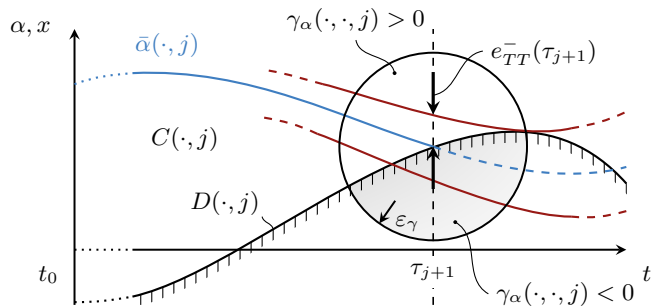


Fig. 8. Illustration of the fact that the transversality assumption (Assumption 3) implies a relation between state error and event time mismatch.

We start by noting that the  $j$ -th time-triggered jump map of (46) is defined as long as  $e_{TT}^-(\tau_j)$  is sufficiently small, following from the fact that the  $j$ -th jump occurs for the original NSTHS if  $x(t, j)$  is sufficiently close to  $\alpha(t, j)$  as discussed in the proof of Lemma 1.  $\|e_{TT}^-(\tau_j)\|$  is required to be smaller than  $\varepsilon_\gamma$  for example (cf. Assumption 2). Equivalently, the  $j$ -th time-triggered jump map  $g_{TT}(\cdot, j)$  of the cl-NTTHS is defined as long as  $x_{TT}^-(\tau_j)$  is sufficiently close to  $\alpha^-(\tau_j) := \alpha(\tau_j, j - 1)$ . The proof for the existence of a neighborhood about the origin for  $e_{TT}^-(\tau_j)$  where the time-triggered jump map is defined follows again from a straightforward application of the implicit function theorem employing the guard function  $\gamma_\alpha(\cdot, \cdot, j)$ . In particular, one can show that when  $\|e_{TT}^-(\tau_j)\| = \|x_{TT}^-(\tau_j) - \alpha^-(\tau_j)\|$  is sufficiently small then  $|t_{j,TT} - \tau_j| < S_j \|e_{TT}^-(\tau_j)\|$  where the function  $t_{j,TT} := t_{j,TT}(e)$  represents the event time of the cl-NSTHS corresponding to the time-triggered error trajectory with value  $e = e_{TT}^-(\tau_j)$  at  $\tau_j$  and  $S_j$  a constant that depends on the continuous-time closed-loop vector field  $f_{cl}$ , guard function  $\gamma_\alpha$ , and the reference trajectory  $\alpha$ .

In details, the function  $t_{j,TT}(e)$  is implicitly defined as  $H_j(t, e) := \gamma_\alpha(\varphi_{j-1}(t, \tau_j, \alpha(\tau_j, j - 1) + e), t, j - 1) = 0$  and Assumption 3 provides the sufficient condition for the validity of the application of the implicit function theorem. Again, note that due to Assumption 6 the flow  $\varphi_{j-1}$  in  $H_j$  is defined as long as it is sufficiently close to the reference trajectory. Finally, because  $t_{j,TT}(e) = \tau_j + \mathbf{D}t_{j,TT}(0) \cdot e + o(\|e\|)$ , by choosing  $S_j > \|\mathbf{D}t_{j,TT}(0)\| > 0$ , one obtains  $|t_{j,TT} - \tau_j| < S_j \|e\|$  for sufficiently small  $e$ . A uniform bound on the maximum time shift of the form  $|t_{j,TT} - \tau_j| < S \|e\|$ , valid for all (possibly infinite)  $j$ , follows from the uniform bound on the derivative of  $\gamma_\alpha$  in Assumption 2. It is straightforward to show that  $\|\mathbf{D}_1 t_{j,TT}(0)\| = \|\mathbf{D}_1 H_j(\tau_j, 0)\|^{-1} \|\mathbf{D}_2 H_j(\tau_j, 0)\| \leq c_1/c_2$  (see Assumptions 2 and 3).

From the above, one concludes that as long as  $e_{TT}(t)$  remains within a predetermined sufficiently small neighborhood of the origin, then the (possibly infinite number of) jump maps  $g_{TT}$  are all defined (implying  $J_x \geq J_\alpha$ ), with the difference between the jump times  $t_j$  of the cl-NSTHS and the nominal jump times  $\tau_j$  becoming smaller as this neighborhood is chosen smaller. Uniform asymptotic stability of the LTTHS allows then to conclude that  $e_{TT}(t)$ , will remain within such a predetermined, sufficiently small, neighborhood where all jump maps  $g_{TT}$  are defined as long as the initial condition  $e_{TT}(t_0)$  is chosen sufficiently close to zero, ensuring

the existence of  $e_{TT}(t)$  for  $t \rightarrow \infty$ . This concludes the proof of the proposition.  $\square$

### C. Proposition 2 and its proof

**Proposition 2.** *Let  $(\alpha(t, j), \mu(t))$  be a trajectory of the NSTHS (6) satisfying Assumptions 1, 2, 3, 4, 5, and 6. If  $\alpha(t, j)$  is an asymptotically stable solution of the cl-NTTHS (43), (45), then it is also an asymptotically stable solution of the cl-NSTHS (6), (32), in the sense of Definition 3.*

*Proof.* By construction, except in suitable neighborhoods of the event times  $\tau_j$ , the solution of the cl-NTTHS is identical, when it exists, to the solution of the cl-NSTHS. As  $\alpha$  is assumed to be asymptotically stable for the cl-NTTHS, the solution of the cl-NTTHS exists for all  $t \geq t_0$  as long as the corresponding initial condition is taken sufficiently close to  $\alpha(t_0, 0)$ . This follows, in particular, from the uniform bound

$$\begin{aligned} |t_j - \tau_j| &< S \|e_{TT}(\tau_j, j - 1)\| \\ &= S \|x_{TT}(\tau_j, j - 1) - \alpha(\tau_j, j - 1)\| \end{aligned} \quad (50)$$

obtained in the proof of Proposition 1, with  $S$  a suitable strictly positive constant. From the proof of Proposition 1 and Assumption 6 furthermore follows that, if the initial condition of the cl-NSTHS is taken sufficiently close to  $\alpha(t_0, 0)$ , it is guaranteed that the number of events  $J_x - 1$  of the cl-NSTHS (equivalently, of the cl-NTTHS) equals the number of events  $J_\alpha - 1$  of the reference. The equality  $J_x = J_\alpha$  is the first condition that  $\alpha$  has to satisfy for being locally asymptotically stable for the cl-NSTHS, in the sense of Definition 3.

What is left to be shown in this proof is therefore that the other two conditions of Definition 3 are fulfilled. Namely, guaranteeing that for an arbitrary  $\varepsilon > 0$ , there exists a  $\delta = \delta(\varepsilon)$  such that  $\|x(t_0, 0) - \alpha(t_0, 0)\| < \delta$  implies  $|t_j - \tau_j| < \varepsilon$  and  $\|x(t, j) - \bar{\alpha}(t, j)\| < \varepsilon$  for all  $(t, j) \in \text{dom } x$  and that both quantities converge to zero as  $t \rightarrow \infty$ . We will show that taking  $\varepsilon_{TT} = \varepsilon_{TT}(\varepsilon)$  such that

$$\max(\varepsilon_{TT}, S\varepsilon_{TT}, \exp(LS\varepsilon_{TT})\varepsilon_{TT}) \leq \varepsilon \quad (51)$$

with  $S > 0$  as in (50) and  $L > 0$  as in Assumption 5 and  $\delta := \delta_{TT}(\varepsilon_{TT}(\varepsilon))$  will indeed satisfy the above stability conditions.

First, for a given  $\varepsilon$ , if  $\varepsilon_{TT}$  is chosen according to (51) then local asymptotic stability of  $\alpha$  for the cl-NTTHS ensures the existence of a neighborhood of  $\alpha(t_0, 0)$  containing a ball of radius  $\delta = \delta(\varepsilon)$  such that the corresponding trajectories of the cl-NSTHS satisfy the bound  $|t_j - \tau_j| < S\varepsilon_{TT} \leq \varepsilon$ , as requested by Definition 3.

Second, from (50), (51), and local asymptotic stability of the cl-NTTHS, one concludes immediately that  $\|x_{TT}(t, j) - \alpha(t, j)\| < \varepsilon_{TT} \leq \varepsilon$  for every time interval  $I_j^\alpha$ . However, in fact, we need to prove something stronger, namely that  $\|x(t, j) - \bar{\alpha}(t, j)\| < \varepsilon$ , for  $t \in I_x^j = [t_{j-1}, t_j]$ . In the simplest case, namely when  $\tau_j \leq t_j$  and  $t_{j+1} \leq \tau_{j+1}$ , the condition is trivially satisfied on the  $j$ -th time interval  $I_j^x$ , because there  $x(t, j) = x_{TT}(t, j)$  and  $\bar{\alpha}(t, j) = \alpha(t, j)$ . In general, one will have  $t_j \leq \tau_j$  and/or  $\tau_{j+1} \leq t_{j+1}$ : indeed, if the simplest case occurs in  $j$ -th interval, by construction, it will not occur in the  $(j - 1)$ -th and  $(j + 1)$ -th intervals. Fortunately, as in

the proof of Proposition 1, we can make use of a uniform exponential bound on the growth of the solutions of locally Lipschitz vector fields, cf. (49). For a given initial condition  $x(t_0, 0)$  within the ball of radius  $\delta = \delta_{TT}(\varepsilon_{TT}(\varepsilon))$  centered at  $\alpha(t_0, 0)$ , if  $\tau_{j+1} \leq t_{j+1}$ , then we can bound the solutions of the cl-NSTHS for  $t \in [\tau_{j+1}, t_{j+1}]$  as

$$\begin{aligned} \|x(t, j) - \bar{\alpha}(t, j)\| &\leq \exp(L(t - \tau_{j+1})) \|e_{TT}(\tau_{j+1}, j)\| \\ &\leq \exp(L(t - \tau_{j+1})) \varepsilon_{TT}, \end{aligned} \quad (52)$$

where  $L$  is the upper bound on the Lipschitz constants as in Assumption 5. Similarly, if  $t_j \leq \tau_j$ , then we can bound the solutions of cl-NSTHS for  $t \in [t_j, \tau_j]$  as

$$\begin{aligned} \|x(t, j) - \bar{\alpha}(t, j)\| &\leq \exp(L(\tau_j - t)) \|e_{TT}(\tau_j, j)\| \\ &\leq \exp(L(\tau_j - t)) \varepsilon_{TT}. \end{aligned} \quad (53)$$

Recalling (50) and for each  $j$ , the two equations above, due to (51), finally lead to  $\|x(t, j) - \bar{\alpha}(t, j)\| < \exp(LS\varepsilon_{TT})\varepsilon_{TT} \leq \varepsilon$  for  $t \in I_x^j = [t_{j-1}, t_j]$ , as required in Definition 3.

Convergence to zero of  $\|t_j - \tau_j\|$  and  $\|x(t, j) - \bar{\alpha}(t, j)\|$  for  $t \rightarrow \infty$  follows from the convergence to zero of  $\|x_{TT}(\tau_j, j) - \alpha(\tau_j, j)\|$  as  $j = j_x(t) \rightarrow \infty$ . This concludes the proof.  $\square$

It can be concluded that the trajectory  $\alpha(t, j)$  is an asymptotically stable solution of the cl-NSTHS (6), (32) if it is an asymptotically stable trajectory for the cl-NTTHS (43) (Prop. 2), which is the case if the origin of the closed-loop linearization (21)-(31) about that trajectory, also referred to as the cl-LTTHS, is uniformly asymptotically stable (Prop. 1). These two steps together form the proof of Theorem 1.

## APPENDIX C

### RELATION TO EXISTING DEFINITION OF STABILITY

In this appendix, we will present a link with existing literature on hybrid system stability analysis. More precisely, we will show that uniform asymptotic stability of the origin of the cl-LTTHS (21)-(31) implies asymptotic stability (as defined in [13, Definition 2] and given below) of the reference trajectory  $\alpha$  employing the distance function given in [13, Definition 1]. As this distance function is not designed to accommodate non-autonomous jump maps, flow sets, and jump sets, in this appendix, we consider state-triggered hybrid systems of the following form:

$$\dot{x} = f(x, u, t, j) \quad x \in C \quad (54a)$$

$$x^+ = g(x^-) \quad x \in D, \quad (54b)$$

with vector field  $f(x, u, t, j) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}^n$ , jump map  $g : D \rightarrow \mathbb{R}^n$ , and constant flow and jump sets  $C \subseteq \mathbb{R}^n$  and  $D \subseteq \partial C$ , respectively. Moreover, for the sake of brevity, when we refer to  $x(t)$  and  $\alpha(t)$  in this appendix, one should read  $x(t, j_x(t))$  and  $\alpha(t, j_\alpha(t))$ , respectively, where  $j_x$  and  $j_\alpha$  are given by Definition 6 in Section III-D for the trajectories  $x$  and  $\alpha$ . In the following, by (asymptotic) stability in terms of the distance function  $d$  we mean (cf. [13, Definition 2])

**Definition 8.** *The trajectory  $\alpha$  is stable with respect to the distance  $d$  if for all  $\varepsilon_d > 0$ , there exists a  $\delta_d(\varepsilon_d) > 0$  such that for every trajectory  $x$  of (32), (54) satisfying  $d(x_0, \alpha_0) <$*



$\delta_d(\varepsilon_d)$ , it holds that  $d(x(t), \alpha(t)) < \varepsilon_d$  for all  $t \geq t_0$ . If it moreover holds that  $d(x(t), \alpha(t)) \rightarrow 0$  for  $t \rightarrow \infty$ , then the trajectory  $\alpha$  is asymptotically stable with respect to  $d$ . ■

We can now formulate the following result.

**Corollary 1.** *Adopt Assumptions 1-6. Let a state-input trajectory  $(\alpha, \mu)$  be a solution to the cl-NSTHS (32), (54), starting from the initial condition  $\alpha(t_0) = \alpha_0 \in \text{int } C \setminus g(D)$ . Let the closed-loop vector field furthermore satisfy  $\|f_{cl}(x, t)\| < F$  for some  $F > 0$  and for all  $(x, t)$  in a neighborhood of the graph of  $\alpha$ . If the origin of the associated cl-LTTHS (21)-(31) is uniformly asymptotically stable, then, given any uniformly continuous distance function  $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  that is compatible with (32), (54) in the sense of [13, Definition 1], the trajectory  $\alpha$  of (32), (54) is asymptotically stable with respect to the distance  $d$  as given in Definition 8.*

*Proof.* Stability of the trajectory  $\alpha$  is guaranteed if A) for every  $\varepsilon_d > 0$ , there exist  $\varepsilon_{TT}(\varepsilon_d) > 0$  and  $\delta_{TT}(\varepsilon_d)$  such that, for all trajectories  $x, x_{TT}$  (to (32), (54), respectively, the autonomous equivalent of (43)) with initial condition  $x(t_0) = x_{TT}(t_0) = x_0$  satisfying  $\|x_0 - \alpha_0\| < \delta_{TT}(\varepsilon_d)$  it holds that

$$\|x_{TT}(t) - \alpha(t)\| < \varepsilon_{TT}(\varepsilon_d) \implies d(x(t), \alpha(t)) < \varepsilon_d, \quad (55)$$

and B), that for this  $\delta_{TT}(\varepsilon_d)$ , there exists a  $\delta_d(\varepsilon_d) > 0$  such that

$$d(x_0, \alpha_0) < \delta_d(\varepsilon_d) \implies \|x_0 - \alpha_0\| < \delta_{TT}(\varepsilon_d). \quad (56)$$

In proving that condition A) is satisfied, we start by pointing out that, from Proposition 1,  $\alpha$  is an asymptotically stable solution to the cl-NTTHS (43), (45) since, by hypothesis in the theorem, the cl-LTTHS (40)-(31) is uniformly asymptotically stable. From this, we can conclude that for every  $\varepsilon_{TT} > 0$  there exists a  $\delta_{TT} > 0$  such that  $\|x_{TT}(t_0) - \alpha(t_0)\| < \delta_{TT}$  implies that  $\|x_{TT}(t) - \alpha(t)\| < \varepsilon_{TT}$  for all  $t \geq t_0$ . What remains to be proven is that if we arbitrarily fix  $\varepsilon_d$ , we can indeed find a  $\varepsilon_{TT}$  such that (55) is satisfied. In doing so, we make a distinction between the times  $t$  for which  $(t, j_\alpha(t)) \in \text{dom } x \cap \text{dom } \alpha$ , i.e. the times for which the trajectory  $x$  and the reference  $\alpha$  have encountered the same number of jumps, and the times  $t$  for which this is not the case. In the former situation, we exploit uniform continuity of  $d$ , as well as the fact that  $d(x, y) = 0$  if  $x = y$  [13, Definition 1], to infer that there exists a  $\varepsilon'_{TT} > 0$  such that  $\|x - y\| < \varepsilon'_{TT}$  implies  $d(x, y) < \varepsilon_d$ . Exploiting that in this situation, for those specific times, as by construction,  $x_{TT}(t) = x(t)$ , we conclude that (55) clearly is satisfied whenever  $\varepsilon_{TT}(\varepsilon_d) \leq \varepsilon'_{TT}$ .

Let us now look at the times for which  $x$  has experienced more or less jumps than  $\alpha$  has (i.e. the times  $t$  for which  $j_x(t) \neq j_\alpha(t)$ ) and consider an arbitrary event time  $\tau_j$  of the reference. The  $j$ -th jump time ‘‘mismatch’’ interval we denote by  $I_\Delta^j$  and define it as  $I_\Delta^j := (t_j, \tau_j)$  when  $t_j < \tau_j$  and as  $I_\Delta^j := (\tau_j, t_j)$  in the case that  $\tau_j < t_j$  (where we exploit the bracketing condition in Lemma 1). We remark that  $d$  is invariant to jumps in  $x$  as well as to jumps in  $\alpha$  and it is a continuous function. It follows that the distance also is continuous with respect to time when evaluated along the two trajectories. Selecting a positive number  $\varepsilon'_d < \varepsilon_d$ ,

as discussed above, there exist  $\varepsilon'_{TT}(\varepsilon'_d) > 0$  such that  $\|e_{TT}(t, j)\| < \varepsilon'_{TT}$  implies  $d(x(t, j), \alpha(t, j)) < \varepsilon'_d$  for all  $(t, j) \in \text{dom } x \cap \text{dom } \alpha$ . With continuity of  $d$  over jumps, we deduce that  $d(x(t), \alpha(t)) < \varepsilon'_d$  at the beginning and end of the intervals  $I_\Delta^j$ , i.e. for  $t = t_j$  and for  $t = \tau_j$ . In the intervals  $I_\Delta^j$ , the distance  $d$  can increase (continuously) however and we now provide a bound for such increase. Select  $\varepsilon''_{TT} \in (0, \varepsilon'_{TT})$  such that  $C_d(\sqrt{2}FS\varepsilon''_{TT}) < \varepsilon_d - \varepsilon'_d$ , with  $S$  as in the bound  $|t_j - \tau_j| < S\varepsilon''_{TT}$ , shown in the proof of Proposition 1, and  $C_d$  the modulus of continuity of  $d$ . We now exploit that  $d(x(t), \alpha(t))$  equals

$$\begin{aligned} & d(x(t_j), \alpha(t_j)) + d(x(t), \alpha(t)) - d(x(t_j), \alpha(t_j)) \\ & \leq \varepsilon'_d + C_d \left( \left\| \begin{array}{c} x(t) - x(t_j) \\ \alpha(t) - \alpha(t_j) \end{array} \right\| \right) \end{aligned}$$

and that, by overapproximation of the second term, follows

$$\begin{aligned} d(x(t), \alpha(t)) & \leq \varepsilon'_d + C_d \left( \left\| \begin{array}{c} F|t - t_j| \\ F|t - t_j| \end{array} \right\| \right) \\ & \leq \varepsilon'_d + C_d(\sqrt{2}FS\varepsilon''_{TT}). \end{aligned} \quad (57)$$

Selecting  $\varepsilon_{TT}(\varepsilon_d) < \min(\varepsilon'_{TT}, \varepsilon''_{TT})$ , we conclude that (55) holds, and with that condition A), is satisfied.

For proving condition B), we note that the set of points  $x_0$  where  $d(x_0, \alpha_0) = 0$  is the singleton  $\{x_0 = \alpha_0\}$  since, by hypothesis,  $\alpha_0 \notin D \cup g(D)$ . With continuity of  $d$  and [13, equation (6b)], this implies that indeed for all  $\delta_{TT} > 0$ , there exist  $\delta_d > 0$  such that (56) is satisfied. Combining now conditions A) and B), it follows that if the cl-LTTHS (21)-(31) is uniformly asymptotically stable, then, the trajectory  $\alpha$  is stable with respect to  $d$ .

To prove asymptotic stability, we recall that  $\alpha$  is an asymptotically stable trajectory of the cl-NTTHS by hypothesis (and Proposition 1) and consequently that, for sufficiently small  $\delta_{TT} > 0$ , solutions  $x_{TT}$  to the cl-NTTHS starting from initial conditions  $x_0$ , satisfying  $\|x_0 - \alpha_0\| < \delta_{TT}$ , it holds that  $\|x_{TT}(t) - \alpha(t)\| \rightarrow 0$  for  $t \rightarrow \infty$ . Moreover, as per Theorem 1, we know that  $|t_j - \tau_j| \rightarrow 0$  for  $t \rightarrow \infty$ . Again exploiting uniform continuity of  $d$  and the upper bound on  $d$  given by (57), these convergence aspects allow to conclude asymptotic stability of  $\alpha$  with respect to the distance function  $d$ .

This concludes the proof of Corollary 1. □

## APPENDIX D

### DERIVATION OF THE LTTHS FROM THE NTTHS

In this appendix, we show that the LTTHS (21)-(30) is the linearization of the NTTHS (40), (41). The hybrid dynamics of the NTTHS are described in terms of the vector field  $f$  of the original NSTHS (2) and the input dependent reset map  $g_{TT}^u$  given in (41), also graphically represented in Fig. 6. We now perturb the initial condition and input slightly from the reference state-input trajectory  $(\alpha(t, j), \mu(t))$ , that is, we take  $x_{TT, \varepsilon}(t_0, 0) = \alpha_0 + \varepsilon z_0$  and  $u_\varepsilon(t, j) = \mu(t) + \varepsilon v(t, j)$  for some initial state perturbation  $z_0$ , input perturbation  $v(t, j)$  (that may be different for different  $j$ ) and scalar perturbation parameter  $\varepsilon$ . Standard results can be used [30] to show that the trajectory of the NTTHS can be expanded in series with respect to  $\varepsilon$  as  $x_{TT, \varepsilon}(t, j) = \alpha(t, j) + \varepsilon z(t, j) + o(\varepsilon)$ . It follows

that  $\dot{x}_{TT,\epsilon} = f(\alpha + \epsilon z + o(\epsilon), \mu + \epsilon v, t, j)$ , for  $(t, j) \in \text{dom } \alpha$ , which can be expanded, in each of the intervals  $[\tau_j, \tau_{j+1}]$ , as

$$\begin{aligned} \dot{\alpha} + \epsilon \dot{z} + o(\epsilon) &= f(\alpha(t, j), \mu(t), t, j) + \\ &\epsilon [\mathbf{D}_1 f(\alpha(t, j), \mu(t), t, j) z \\ &+ \mathbf{D}_2 f(\alpha(t, j), \mu(t), t, j) v] + o(\epsilon). \end{aligned}$$

Matching terms in the expansion allows to conclude that the flow dynamics of the linear approximation about the state-input trajectory  $(\alpha(t, j), \mu(t))$  for each continuous time interval  $[\tau_j, \tau_{j+1}]$  is given by

$$\begin{aligned} \dot{z} &= \mathbf{D}_1 f(\alpha(t, j), \mu(t), t, j) z + \mathbf{D}_2 f(\alpha(t, j), \mu(t), t, j) v \\ &=: A(t, j)z + B(t, j)v, \end{aligned}$$

that matches the expressions given for  $A$  and  $B$  in (22) and (23), respectively.

Next, for each jump time  $\tau_{j+1}$ , we seek a relation between the states  $z^-(\tau_{j+1}) := z(\tau_{j+1}, j)$  and  $z^+(\tau_{j+1}) := z(\tau_{j+1}, j+1)$  that will eventually be equal to (24). Roughly speaking, this corresponds to the linearization of the jump map  $g_{TT}^u$  in (41). We follow a similar strategy in deriving the linear jump map as detailed in [19], but now fitted in the framework of hybrid time.

Consider the  $(j+1)$ -th event with reference event time  $\tau_{j+1}$  and corresponding event time  $t_{j+1,\epsilon}$  of the NSTHS. Define  $\Delta_\epsilon = t_{j+1,\epsilon} - \tau_{j+1}$ , which is assumed to be small based on the fact that  $x_{TT,\epsilon}$  and  $\alpha$  are close to each other and Assumption 3. To make the derivation of (24) concise, below we denote the event time  $\tau_{j+1}$  simply by  $\tau$ . Furthermore, when we refer to  $x_{TT,\epsilon}$ , one should read  $x_{TT,\epsilon}(\tau_{j+1}, j)$  and similarly  $\alpha^- = \alpha(\tau_{j+1}, j)$  and  $z^- = z(\tau_{j+1}, j)$ . Analogously, for the right limits at the time  $\tau$ , we employ the notation  $x_{TT,\epsilon}^+ = x_{TT,\epsilon}(\tau_{j+1}, j+1)$ ,  $\alpha^+ = \alpha(\tau_{j+1}, j+1)$ , and  $z^+ = z(\tau_{j+1}, j+1)$ .

The jump map  $g_{TT}^u$  in (41) for the event  $j+1$  constitutes phases of flow to and from the event time  $t_{j+1,\epsilon}$  that depends on  $\epsilon$  and the choice of  $v(t, j)$ . These flows can be captured by extending the solution  $z$ , using the continuous time part of (21), to form  $\bar{z}$  (where states for consecutive counter are related via a ‘to be defined’ jump map) in a similar fashion as done for the reference trajectory  $\alpha$  as explained in Section III-A. This allows us to form the extended trajectory of the NTTHS which, due to the locally Lipschitz property of the vector field  $f$ , is the same as that of the original NSTHS, i.e.  $\bar{x}_\epsilon(t, j) := \bar{x}_{TT,\epsilon}(t, j) = \bar{\alpha}(t, j) + \epsilon \bar{z}(t, j) + o(\epsilon)$ . It follows that, instead of the jump map  $x_{TT,\epsilon}^+ = g_{TT}^u(x_{TT,\epsilon}^-, \tau, j)$ , we can consider

$$\bar{x}_\epsilon(\tau + \Delta_\epsilon, j+1) = g(\bar{x}_\epsilon(\tau + \Delta_\epsilon, j), \tau + \Delta_\epsilon, j). \quad (58)$$

and use a series expansion to account for the difference in time. Again to keep notation concise, we append the states  $\bar{\alpha}$ ,  $\bar{x}_\epsilon$ , and  $\bar{z}$  with a superscript, i.e.  $(\cdot)^\triangleright$  to denote that it is the left limit for the event time  $t_{j+1,\epsilon} = \tau_{j+1} + \Delta_\epsilon$ , e.g.  $\bar{z}^\triangleright = \bar{z}(t_{j+1,\epsilon}, j)$ . For the right limit we employ the superscript  $(\cdot)^\triangleleft$ , e.g.  $\bar{z}^\triangleleft = \bar{z}(t_{j+1,\epsilon}, j+1)$ . For the state  $\bar{x}_\epsilon^\triangleright$  we find

$$\begin{aligned} \bar{x}_\epsilon^\triangleright &= \bar{\alpha}^\triangleright + \epsilon \bar{z}^\triangleright + o(\epsilon) \\ &= \alpha^- + \dot{\alpha}^- \Delta'_0 \epsilon + (z^- + \dot{z}^- \Delta'_0 \epsilon) \epsilon + o(\epsilon) \\ &= \alpha^- + (\dot{\alpha}^- \Delta'_0 + z^-) \epsilon + o(\epsilon) \end{aligned} \quad (59)$$

where  $\Delta'_0 = \frac{\partial \Delta_\epsilon}{\partial \epsilon} |_{\epsilon=0}$  and  $\dot{\alpha}^- = f(\alpha^-, \mu(\tau), \tau, j)$  and similarly that

$$\begin{aligned} \bar{x}_\epsilon^\triangleleft &= \bar{\alpha}^\triangleleft + \epsilon \bar{z}^\triangleleft + o(\epsilon) \\ &= \alpha^+ + (\dot{\alpha}^+ \Delta'_0 + z^+) \epsilon + o(\epsilon) \end{aligned} \quad (60)$$

where  $\dot{\alpha}^+ = f(\alpha^+, \mu(\tau), \tau, j+1)$ . These expansions make use of the dependence of the jump time difference  $\Delta_\epsilon$  on  $\epsilon$ . More precisely, they contain the derivative of  $\Delta_\epsilon$  with respect to  $\epsilon$  evaluated at zero. From Assumption 2, it follows that for the reference jump at time  $\tau$  holds that  $\gamma_\alpha(\alpha^-, \tau, j) = 0$  since, by definition, the state will be in the jump set  $D$ . Similarly, the definition of  $t_{j+1,\epsilon}$  implies  $\gamma_\alpha(\bar{x}_\epsilon(t_{j+1,\epsilon}, j), t_{j+1,\epsilon}, j) = 0$ , or formulated differently that  $\gamma_\alpha(x_\epsilon^\triangleright, \tau + \Delta_\epsilon, j) = 0$ . Using (59), this can be rewritten as

$$\begin{aligned} \gamma_\alpha(\alpha^-, \tau, j) + \mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j) (\dot{\alpha}^- \Delta'_0 + z^-) \epsilon + \\ \mathbf{D}_2 \gamma_\alpha(\alpha^-, \tau, j) \Delta'_0 \epsilon + o(\epsilon) = 0. \end{aligned}$$

Since this needs to hold for all  $\epsilon$  and since  $\gamma_\alpha(\alpha^-, \tau, j) = 0$ , it follows that

$$\Delta'_0 = - \frac{\mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j)}{\mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j) \dot{\alpha}^- + \mathbf{D}_2 \gamma_\alpha(\alpha^-, \tau, j)} z^-. \quad (61)$$

Note that the transversality assumption (Assumption 3) implies that the denominator in (61) is nonzero.

Incorporating (59) and (60) in (58) and expanding in series with respect to  $\epsilon$  gives

$$\begin{aligned} \bar{x}_\epsilon^\triangleleft &= \alpha^+ + (\dot{\alpha}^+ \Delta'_0 + z^+) \epsilon + o(\epsilon) \\ &= g(\alpha^- + (\dot{\alpha}^- \Delta'_0 + z^-) \epsilon + o(\epsilon), \tau + \Delta_t(\epsilon), j) \\ &= g(\alpha^-, \tau, j) + \mathbf{D}_1 g(\alpha^-, \tau, j) (\dot{\alpha}^- \Delta'_0 + z^-) \epsilon + \\ &\quad \mathbf{D}_2 g(\alpha^-, \tau, j) \Delta'_0 \epsilon + o(\epsilon). \end{aligned}$$

Now we recall that  $\alpha^+ = g(\alpha^-, \tau, j)$  and match terms of  $\epsilon$  to obtain  $z^+ = (-\dot{\alpha}^+ + \dot{g}^-) \Delta'_0 + \mathbf{D}_1 g(\alpha^-, \tau, j) z^-$ , where  $\dot{g}^- = \mathbf{D}_1 g(\alpha^-, \tau, j) \dot{\alpha}^- + \mathbf{D}_2 g(\alpha^-, \tau, j)$ . After incorporating (61), it follows that the linearized jump map satisfies

$$z^+ = \left[ (\dot{\alpha}^+ - \dot{g}^-) \frac{\mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j)}{\dot{\gamma}_\alpha^-} + \mathbf{D}_1 g(\alpha^-, \tau, j) \right] z^-,$$

with  $\dot{\gamma}_\alpha^- = \mathbf{D}_1 \gamma_\alpha(\alpha^-, \tau, j) \cdot \dot{\alpha}^- + \mathbf{D}_2 \gamma_\alpha(\alpha^-, \tau, j) \cdot 1$ , which is equivalent to (24).

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